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Existence and asymptotics of travelling waves in a thermo-diffusive model in half cylinders. Part I: Neumann boundary conditions

Yannick Sire

Abstract. The aim of this work is to prove existence results and derive asymptotic limits for some nonlinear elliptic problems arising in flame propagation and set in unbounded cylinders. These problems are involved in the modelling of burner flames. The existence proof is a combination of topological degree arguments and estimates that are specific to the problems under consideration. We also derive some asymptotic limits for our model. We emphasize on the fact that the model under consideration is a system of reaction-diffusion equations.

1. Introduction

1.1. Physical context and setting of the problem

The physical situation is the following: we consider infinite tubes filled with premixed gases. We assume the case of one step irreversible chemistry $A \to B$. We denote by $T(x,y)$ the temperature of the mixture and by $Y(x,y)$ the mass fraction of the reactant. We study a model describing burner flame propagation. To understand the multi-dimensional effects involved in such systems, we introduce a shear flow in the equations, making clear the dependence with respect to the transversal variable of the cylinder. In the present case, we consider Neumann boundary conditions.

In our models, gases move from the left semi-cylinder to the right semi-cylinder at constant temperature. We refer the reader to the monograph of Sivashinsky [14] or the book of Williams [17], where extensive studies of combustion theory are made. We also refer the reader to the appendix of [3] for a formal derivation of combustion models from the full physical system. We also refer the reader to the book [2]. The model under consideration in the present paper is related to standard models in [3], [4] and [6]. In these last papers, the authors consider scalar

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reaction-diffusion equations set in unbounded cylinders. In our case, we deal with a system of reaction-diffusion equations.

Together with purely theoretical issues concerning existence problems for nonlinear elliptic problems set in cylinders, our study is also motivated by physical considerations. Indeed, another interesting aspect of this type of model comes from the observation of oscillatory instabilities in the associated parabolic problems (see [9]). In particular, Joulin [9] derived, via an asymptotic analysis, an evolution equation for a planar flame held downstream of an isothermal flat burner, and the numerical study of this equation, which has delay terms, leads to front oscillations for flames held around the minimum steady stand off distance. Our model is Joulin's, and our goal is to understand the instabilities.

We first introduce some notation. The problems under consideration are set in unbounded cylinders. We write

$$\Sigma^+ = \mathbb{R}^+ \times \omega, \quad \Sigma^- = \mathbb{R}^- \times \omega, \quad \text{and} \quad \Sigma = \mathbb{R} \times \omega,$$

where ω is a smooth bounded connected open set of $\mathbb{R}^{N-1}$, $N \geq 2$.

We will consider a nonlinearity f satisfying: $f \in C^2(0,1)$, its derivatives are bounded, and there exists $\theta \in (0,1)$ such that $f \equiv 0$ on $[0,\theta]$ and $f > 0$ in $(\theta, 1)$. We will also consider the following modified nonlinearity (of Kolmogorov–Petrovski–Piskunov type):

$$g_{T^*}(x) = (T^* - x) f(x)\,,$$

where $T^* \in (\theta, 1)$.

We introduce a smooth function $\beta(y)$ on $\overline{\omega}$ in $\mathbb{R}$ satisfying

$$\beta_0 = \min_{y \in \overline{\omega}} \beta(y) > 0.$$

We consider the following model, where the unknowns are T, the temperature of the mixture; Y, the mass fraction of the reactant; and c, the velocity of the flame front. Since the velocity c is a free parameter, the following systems can be seen as generalized nonlinear eigenvalue problems. We look for a triple (c, T, Y) satisfying

$$\begin{cases} -\Delta T + c\beta(y) T_x = Y f(T) & \text{in } \Sigma^+, \\ -\Delta Y + c\beta(y) Y_x = -Y f(T) & \text{in } \Sigma, \end{cases} \tag{1.1}$$

with the boundary conditions

$$\begin{cases} T(0,.) = 0, & T(+\infty,.) = T^* \in (\theta, 1), \\ Y(-\infty,.) = 1, & Y(+\infty,.) = 0, \\ \partial_\nu Y = 0 \quad \text{on } \mathbb{R} \times \partial\omega, & \partial_\nu T = 0 \quad \text{on } \mathbb{R}^+ \times \partial\omega. \end{cases} \tag{1.2}$$

We extend T to be zero in Σ^- and $\nu(y)$ is the normal vector to the boundary $\partial\omega$.

Remark 1.1. Notice here that the transport terms $c\beta(y)T_x$ and $c\beta(y)Y_x$ are not of the form $c + \beta(y)T_x$ and $c + \beta(y)Y_x$, which would be the real physical case. Indeed, the parabolic equation (satisfied by T for instance) is

$$T_t - \Delta T + \beta(y)\partial_x T = f(T)Y \quad \text{in } \Sigma^+,$$

with additional boundary conditions. In this case, searching for a planar wave amounts to the ansatz (still denoted T for the sake of simplicity) $T(x+ct,y)$. In this latter case, the analysis is more involved due to sign changes in the transport term. However, our model still remains suitable when the velocity field β is small enough.

1.2. Results

The aim of this paper is to prove the following two theorems. Our first theorem is an existence result.

Theorem 1.1. *Let f be as described above. Then for any $T^* \in (\theta, 1)$, the problem (1.1)–(1.2) has a solution (T, Y, c).*

Note that the models under consideration involve heat loss since the value $T^* = 1$ is forbidden, and it is therefore natural to study the behaviour of our models in the limit $T^* \to 1$. Indeed, one can prove an *a priori* estimate on the temperature:

$$< \partial_x T > (0) = c(1 - T^*) < \beta >,$$

where the brackets stand for the mean value with respect to the variable y. Consequently, if $T^* = 1$, using the maximum principle together with a Hopf argument, we end up with an impossibility and our model does not have a solution. From a physical point of view, these asymptotics mean that the flame front moves away from the cold wall. Precisely, one can state the following result:

Theorem 1.2. *Let $(T_{T^*}, Y_{T^*}, c_{T^*})$ be a family of solutions of (1.1)–(1.2). Then the speed c_{T^*} satisfies*

$$\lim_{T^* \to 1} c_{T^*} = c^*,$$

where (U, c^*) are the unique solutions (up to translations for U) of

$$\begin{cases} -\Delta U + c\beta(y)U_x = g_1(U) & \text{in } \Sigma, \\ \partial_\nu U = 0 & \text{on } \mathbb{R} \times \partial\omega, \\ U(-\infty, .) = 0, \quad U(+\infty, .) = 1. \end{cases}$$

Problems with Neumann boundary conditions on the cylinder have been treated in [3], [4], [15] for instance.

There are now a number of methods to deal with elliptic problems in unbounded cylinders. One way, for instance, is to approximate the problem in a bounded cylinder, then send the cylinder's size to infinity; this is done in [3], [4], [6]. For our purposes, the *standard* Leray–Schauder degree theory is sufficient. However, a crucial step is to derive suitable estimates, the main quantities to be estimated being the speed c and the L^2 norm of $T^* - T$. A more elaborate topological degree theory for unbounded domains can be found in [16].

2. *A priori* estimates

These estimates are crucial for the existence result and are twofold: bounds for the velocity c and an L^2 estimate for the temperature $T^* - T$.

2.1. Control of the enthalpy

The following results deal with a control of the enthalpy term $T^* - Y - T$. This forms the cornerstone of the estimates. The following lemma gives an existence result for a linear problem in the left half cylinder.

Lemma 2.1. *Let $c > 0$ and $h \in L^\infty(\omega)$. The problem*

$$\begin{cases} -\Delta u + c\beta(y)\partial_x u = 0 & \text{in } \Sigma^-, \\ \partial_\nu u = 0 & \text{in } \mathbb{R}^- \times \partial\omega, \\ u(0,y) = h(y) & \text{in } \omega, \\ u(-\infty, y) = 0 & \text{in } \omega, \end{cases} \tag{2.1}$$

admits a solution u.

Proof. The constant function $-\|h\|_{L^\infty}$ is subsolution in Σ^-, and since $c\beta(y) > c\beta_0 > 0$, the function $Ce^{\eta x}$ is a supersolution for suitable $\eta > 0$ and $C > 0$. The existence of u follows from a standard Perron's argument. □

Proposition 2.2. *Assume the existence of a bounded W^+ satisfying the following linear problem in Σ^+, and $c > 0$:*

$$\begin{cases} -\Delta W^+ + c\beta(y)\partial_x W^+ = 0 & \text{in } \Sigma^+ \\ \partial_\nu W^+ = 0 & \text{on } \mathbb{R}^+ \times \partial\omega, \\ W^+(+\infty, y) = 0 & \text{in } \omega. \end{cases}$$

Call W^- the solution of (2.1) with $h = W^+(0,.)$, i.e.,

$$\begin{cases} -\Delta W^- + c\beta(y)\partial_x W^- = 0 & \text{in } \Sigma^-, \\ \partial_\nu W^- = 0 & \text{in } \mathbb{R}^- \times \partial\omega, \\ W^-(0,y) = W^+(0,y) \in L^\infty(\omega) & \text{in } \omega, \\ W^-(-\infty, y) = 0 & \text{in } \omega. \end{cases} \tag{2.2}$$

Consider the function W defined by

$$W = \begin{cases} W^+ & \text{in } \Sigma^+, \\ W^- & \text{in } \Sigma^- \end{cases}.$$

Then there exists $C > 0$, independent of the parameter c, such that

$$\|W\|_{L^2(\Sigma)} \le C\|\phi\|_{L^2(\omega)},$$

where $\phi(y) = \partial_x W^+(0,y) - \partial_x W^-(0,y)$.

Proof. The existence of the function W^- follows from Lemma 2.1. Direct computations show

$$\begin{cases} -\Delta W + c\beta(y)\partial_x W = \phi(y)\,\delta_0 & \text{in } \Sigma, \\ \partial_\nu W = 0 & \text{on } \mathbb{R}\times\partial\omega, \\ W(\pm\infty, y) = 0 & \text{in } \omega, \end{cases}$$

where $\phi(y) = \partial_x W^+(0,y) - \partial_x W^-(0,y)$. We now consider its Fourier transform in the variable x, i.e.,

$$\hat{W}(\xi, y) = \int_{-\infty}^{+\infty} e^{-ix\xi}\, W(x,y)\, dx.$$

The function $\hat{W}$ solves the problem

$$\begin{cases} -\Delta_y \hat{W} + (ic\beta(y)\xi + \xi^2)\hat{W} = \phi(y) & \text{in } \Sigma, \\ \partial_\nu \hat{W} = 0 & \text{on } \mathbb{R}\times\partial\omega, \end{cases} \tag{2.3}$$

where Δ_y stands for the laplacian in the y variable. We set $\hat{W} =< \hat{W} > +\hat{W}^\perp$, where $< u >$ is the y-mean value of the function u (we will frequently use this notation throughout this paper). Integration of equation (2.3) on ω yields

$$< \phi >= i\xi c < \beta\hat{W} > +\xi^2 < \hat{W} > . \tag{2.4}$$

Then, taking $\xi = 0$, we have

$$< \phi >= 0. \tag{2.5}$$

From equation (2.4), we get

$$| < \hat{W} > |(\xi) = \frac{c| < \beta\hat{W}^\perp > |}{|ic < \beta > +\xi|}.$$

By the Cauchy–Shwarz inequality, we infer the existence of $C > 0$, independent of c, such that

$$| < \hat{W} > |(\xi) \le C\,\|\hat{W}(\xi,.)^\perp\|_{L^2(\omega)}. \tag{2.6}$$

We now multiply equation (2.3) by $\bar{\hat{W}}$ and integrate over ω. Taking the real part and using the Cauchy–Schwarz inequality leads to

$$\int_\omega |\nabla_y \hat{W}^\perp(\xi,y)|^2 dy + \xi^2 \int_\omega |\hat{W}(\xi,y)|^2 dy \le \|\phi\|_{L^2}\|\hat{W}(\xi,.)\|_{L^2}. \tag{2.7}$$

By the Poincaré inequality, equation (2.7) implies the estimate

$$\|\hat{W}^\perp(\xi,.)\|_{L^2}^2 \le \frac{\|\phi\|_{L^2}\|\hat{W}(\xi,.)\|_{L^2}}{C_1 + \xi^2}, \tag{2.8}$$

where C_1 is a positive constant. Combining (2.6) and (2.8), and using

$$\int_\omega |\hat{W}(\xi,y)|^2 dy = |\omega|| < \hat{W} > |(\xi)^2 + \int_\omega |\hat{W}^\perp(\xi,y)|^2 dy, \tag{2.9}$$

we infer

$$\|\hat{W}(\xi,.)\|_{L^2}^2 \leq C \frac{\|\phi\|_{L^2}^2}{(C_1+\xi^2)^2}.$$

Integrating over $\mathbb{R}$, we get

$$\|\hat{W}\|_{L^2}^2 \leq C' \|\phi\|_{L^2}^2 ,$$

and we infer the result by the Parseval–Plancherel theorem. □

As a consequence of the previous proposition, we have the following results:

Proposition 2.3. *Let $(c_n)_{n\geq 0}$ be a bounded sequence of positive numbers and consider a sequence W_n^+ of bounded solutions of*

$$\begin{cases} -\Delta W_n^+ + c_n\beta(y)\partial_x W_n^+ = 0 & \textit{in } \Sigma^+, \\ \partial_\nu W_n^+ = 0 & \textit{on } \mathbb{R}^+\times\partial\omega, \\ W_n^+(+\infty,y) = 0 & \textit{in } \omega. \end{cases}$$

If $c_n \to 0$ and W_n^+ locally converges to a constant in $C^{2,\beta}(\overline{\Sigma+})$, $0<\beta<1$, then

$$\lim_{n\to+\infty} \|W_n^+\|_{L^2(\Sigma^+)} = 0.$$

Proof. Introduce

$$W_n = \begin{cases} W_n^+ & \text{in } \Sigma^+, \\ W_n^- & \text{in } \Sigma^- \end{cases},$$

where W_n^- solves

$$\begin{cases} -\Delta W_n^- + c_n\beta(y)\partial_x W_n^- = 0 & \text{in } \Sigma^-, \\ \partial_\nu W_n^- = 0 & \text{in } \mathbb{R}^-\times\partial\omega, \\ W_n^-(0,y) = W_n^+(0,y) \in L^\infty(\omega) & \text{in } \omega, \\ W_n^-(-\infty,y) = 0 & \text{in } \omega. \end{cases}$$

From Proposition 2.2, we have

$$\|W_n\|_{L^2(\Sigma)} \leq C\,\|\partial_x W_n^+(0,.) - \partial_x W_n^-(0,.)\|_{L^2(\omega)},$$

where the constant C is independent of c_n. Therefore, we get

$$\|W_n^+\|_{L^2(\Sigma^+)} \leq C\,\|\partial_x W_n^+(0,.) - \partial_x W_n^-(0,.)\|_{L^2(\omega)}.$$

To prove the proposition, it is enough to have

$$\lim_{n\to+\infty} \|\partial_x W_n^\pm(0,.)\|_\infty = 0.$$

Up to extracting a subsequence, and thanks to standard elliptic estimates (see [8]), W_n^+ (resp. W_n^-) converges in $C^2_{\rm loc}(\Sigma^+)$ (resp. $C^2_{\rm loc}(\Sigma^-)$) to a function W_∞^+ (resp. W_∞^-) and we have

$$\begin{cases} -\Delta W_\infty^\pm = 0 & \text{in } \Sigma^\pm, \\ \partial_\nu W_\infty^\pm = 0 & \text{on } \mathbb{R}^\pm \times \partial\omega, \\ W_\infty^-(0,y) = W_\infty^+(0,y) \in L^\infty(\omega) & \text{in } \omega. \end{cases}$$

The function W_∞^+ is then a constant. Then, the function W_∞^- is also a constant. Consequently, we have

$$\partial_x W_\infty^+ = \partial_x W_\infty^- = 0\,,$$

and the proposition is proved. □

2.2. Bounds for the speed

First we prove an integral identity, accounting for a compatibility condition on c.

Lemma 2.4. *Let* (T, Y, c) *be a solution of* (1.1)–(1.2). *Then*

$$< T_x > (0) = c\,(1 - T^*) < \beta > . \tag{2.10}$$

Proof. The integration over the whole cylinder Σ of the equation satisfied by T in (1.1) leads to

$$\int_\Sigma f(T)Y = |\omega|\, c < \beta > T^* + \int_\omega T_x(0,y)\, dy.$$

Similarly, the integration of the equation satisfied by Y in (1.1) yields

$$\int_\Sigma f(T)Y = |\omega|\, c < \beta > .$$

The desired identity is then a combination of these two equations. □

We now prove that the solutions T and Y are uniformly bounded.

Lemma 2.5. *Let* (T, Y, c) *be a solution of* (1.1)–(1.2). *Then*

$$0 \le T, Y \le 1.$$

Proof. The bounds for Y come from a direct application of the maximum principle. Suppose that the minimum of T is negative. Then it is achieved on $\{x = 0\}$, a contradiction. We derive the upper bound in a similar way. □

As a consequence we have:

Lemma 2.6. *Let* (T, Y, c) *be a solution of* (1.1)–(1.2). *Then, we have* $c > 0$.

Proof. Since $T \ge 0$ and $T = 0$ at $x = 0$, the Hopf lemma gives that $T_x(0, y) > 0$ for all $y \in \omega$. Consequently, using the integral identity (2.10), we get the desired result. □

We now derive upper and lower bounds for the speed $c > 0$.

Proposition 2.7. *There exists $\bar{c} > 0$ such that, for all solutions (T, Y, c) of problem (1.1)–(1.2), we have $c \leq \bar{c}$.*

Proof. Set $M = \max_{x \in (\theta,1)} f(x)$. Let h be the solution of the following one-dimensional Cauchy problem:

$$\begin{cases} -h'' + c\beta_0 h' = M & \text{in } \mathbb{R}^+, \\ h(0) = 0, \\ h'(0) = c\,(1 - T^*)\beta_0. \end{cases}$$

We have readily

$$h(x) = \frac{M}{c\beta_0} x + C\left(e^{c\beta_0 x} - 1\right),$$

where $C = 1 - T^* - \frac{M}{c^2\beta_0^2}$. Then there are two cases: if $C \leq 0$, we get the desired bound. Otherwise, the function h is a supersolution for T in Σ^+. By the Hopf lemma, we get

$$h'(0) - \partial_x T(0, y) > 0,$$

which leads to the contradiction $< \beta >< \beta_0$ by using identity (2.10). □

The main result of this section is the following lower bound for the speed:

Proposition 2.8. *There exists $\underline{c} > 0$ such that for all solutions (T, Y, c) of problem (1.1)–(1.2), we have $c \geq \underline{c}$.*

Proof. Assume there exists a sequence (T_n, Y_n, c_n) such that $c_n > 0$, (T_n, Y_n, c_n) solves (1.1)–(1.2) and $\lim_{n\to\infty} c_n = 0$. Due to the fact that c_n is bounded and also the term $f(T_n)Y_n$ is in $L^\infty(\Sigma^+)$, by elliptic regularity, the (T_n, Y_n) are bounded in $C^{2,\beta}_{\text{loc}}(\mathbb{R}^+ \times \overline{\omega})$ with $0 < \beta < 1$. Then from Hölder elliptic estimates (see [8]), the sequences T_n and Y_n (up to extracting a subsequence) converge to T_∞ and Y_∞ over all compact sets, and the limits satisfy

$$\text{(2.11)} \qquad \begin{cases} -\Delta T_\infty = Y_\infty f(T_\infty) & \text{in } \Sigma^+, \\ \partial_\nu T_\infty = 0 & \text{on } \mathbb{R}^+ \times \partial\omega, \\ T_\infty(0, .) = 0\,. \end{cases}$$

Two cases must then be distinguished:

- T_∞ is not constant, i.e., problem (2.11) admits a non trivial solution. By the integral identity (2.10), one gets $< \partial_x T_\infty > (0) = 0$. Because $T > 0$ in $\overline{\Sigma^+}$, the Hopf lemma then leads to a contradiction.

- T_∞ is constant; then we have $T_\infty \equiv 0$. The function Y_∞ satisfies

$$\begin{cases} -\Delta Y_\infty = 0 & \text{in } \Sigma, \\ \partial_\tau Y_\infty = 0 & \text{on } \mathbb{R} \times \partial\omega. \end{cases}$$

Since Y_∞ is bounded, Y_∞ is also a constant. Applying Proposition 2.3 to $W_n^+ = T^* - T_n - Y_n$ gives $Y_\infty \equiv T^*$. We now choose a sequence x_n such that $x_n \to +\infty$ (x_n cannot be bounded) and $\max_{y\in\omega} T_n(x_n, y) = \gamma$, with $\gamma \in (\theta, T^*)$. We want to reach a contradiction. To this end, we set

$$U_n(x,y) = T_n(x + x_n, y) \quad \text{and} \quad V_n(x,y) = Y_n(x + x_n, y). \tag{2.12}$$

Due to elliptic estimates [8], we have $(U_n, V_n, c_n) \to (U_\infty, V_\infty, 0)$ on all compact sets in $C^{2,\beta}_{\text{loc}}(\mathbb{R}\times\bar{\omega})$ with $0 < \beta < 1$. Invoking once again Proposition 2.3, we have $V_\infty = T^* - U_\infty$ and U_∞ satisfies

$$\begin{cases} -\Delta U_\infty = g_{T^*}(U_\infty) & \text{in } \Sigma, \\ \partial_\nu U_\infty = 0 & \text{on } \mathbb{R}\times\partial\omega, \\ \max\limits_{y\in\omega} U_\infty(0,y) = \gamma\,. \end{cases} \tag{2.13}$$

Note that $0 \le U_\infty \le T^*$ since $V_\infty \ge 0$. Integrating the equation over a bounded cylinder and using the fact that U_∞ is bounded in the C^2 norm, one can see that in fact $g_{T^*}(U_\infty) \in L^1(\Sigma)$. Furthermore, multiplying by U_∞ and integrating, we see that $\nabla U_\infty \in L^2(\Sigma)$. Therefore, by elliptic regularity, we get $\nabla U_\infty(\pm\infty, y) = 0$, uniformly in y. We denote by $u_\pm$ the limits at $\pm\infty$. We have $u_\pm \ge 0$. Integrating the equation (2.13) over the whole cylinder, we obtain $\int_\Sigma g_{T^*}(U_\infty) = 0$. Since $g_{T^*}(U_\infty) \ge 0$ by $0 \le U_\infty \le T^*$, we infer that U_∞ is either such that $U_\infty \le \theta$ (which is impossible thanks to the boundary condition), or $U_\infty \equiv T^*$, which contradicts $g_{T^*}(\gamma) \ne 0$. □

2.3. L^2 uniform estimates for the temperature

We set $W = T^* - T - Y$. With this change of unknowns, the system (1.1)–(1.2) can be rewritten as

$$\begin{cases} -\Delta T + c\beta(y)T_x = g_{T^*}(T) - f(T)W & \text{in } \Sigma^+, \\ -\Delta W + c\beta(y)W_x = 0 & \text{in } \Sigma^+, \\ \partial_\nu W = \partial_\nu T = 0 & \text{on } \mathbb{R}^+\times\partial\omega. \end{cases} \tag{2.14}$$

Notice that $W(+\infty, .) = 0$. We first prove the following proposition:

Proposition 2.9. *Let (c, T, Y) be a solution of (2.14). There exists $K_1 > 0$ such that*

$$\|W\|_{L^2(\Sigma^+)} \le K_1\,.$$

Proof. Recall that there exist two nonnegative constants $\overline{c}$ and $\underline{c}$ such that $\underline{c} \le c \le \overline{c}$ and that W is bounded. By Proposition 2.2, we have

$$\|W\|_{L^2(\Sigma)} \le C\|\phi\|_{L^2(\omega)}\,,$$

where $\phi(y) = \partial_x W(0,y) - \partial_x W^-(0,y)$. Using the elliptic Hölder estimates up to the boundary (see [8]), we get the desired result. □

We now derive an L^2 estimate for $U = T^* - T$. Consider first the problem

$$\begin{cases} -\Delta U + c\beta(y)U_x = g_1(U) & \text{in } \Sigma, \\ \partial_\nu U = 0 & \text{on } \mathbb{R}\times\partial\omega, \\ U(-\infty,.) = 0,\ U(+\infty,.) = 1. \end{cases} \tag{2.15}$$

Recall that $g_1(u) = (1-u)f(u)$. Bérestycki *et al.* proved in [4], [6] the following theorem:

Theorem 2.1. *Problem* (2.15) *admits a solution* (U,c) *unique up to translations. Furthermore, we have the following estimate in* Σ*:*

$$\partial_x U > 0.$$

We can now state our lemma.

Lemma 2.10. *Let* $(\theta_1, \delta_0) \in (\theta, T^*) \times (0,+\infty)$ *be such that* $g'_{T^*}(x) \le -\delta_0$ *for all* $x \in [\theta_1, T^*]$*. Then there exists a constant* $C' > 0$ *such that, for every solution* (T, Y, c) *of* (1.1)–(1.2)*, there holds*

$$\Omega' := \{0 \le T \le \theta_1\} \subset [0, C'] \times \omega.$$

Proof. We suppose the contrary. This implies the existence of a sequence of numbers a_n satisfying

$$\max_{y\in\omega} T(a_n, y) = \frac{\theta_1}{2}, \quad \lim_{n\to+\infty} a_n = +\infty \quad \text{and} \quad \partial_x T(a_n,.) \le 0.$$

We then define the sequences

$$T_n(x,y) = T(x+a_n, y), \quad \text{and} \quad W_n(x,y) = W(x+a_n,y),$$

where T and W satisfy (2.14). Up to extracting a subsequence and on all compact sets in $C^{2,\beta}_{\text{loc}}(\Sigma)$ $(0 < \beta < 1)$, T_n and W_n converge to T_∞ and 0, respectively. Then T_∞ satisfies

$$\begin{cases} -\Delta T_\infty + c\beta(y)\partial_x T_\infty = g_{T^*}(T_\infty) & \text{in } \Sigma, \\ \partial_\nu T_\infty = 0 & \text{on } \mathbb{R}\times\partial\omega. \end{cases}$$

We divide the rest of the proof into several steps:

- We multiply the equation by T_∞ and integrate on a finite cylinder. This gives $\nabla T_\infty \in L^2(\Sigma)$. By a standard compactness argument, the function T_∞ admits limits at $\pm\infty$, denoted by $T^\pm$.
- If $T^+ \le T^-$, then integration of the equation yields

$$\int_\Sigma g_{T^*}(T_\infty) = c\,(T^+ - T^-)|\omega| < \beta >,$$

which leads to a contradiction since the left hand side is positive (from the definitions of θ_1 and Ω'), whereas the right hand side is positive.

- If $T^+ > T^-$, we have another contradiction since one has $\partial_x T_\infty > 0$ from the monotonicity property of Theorem 2.1, contradicting $\partial_x T_\infty(0, y) \leq 0$.

This proves the lemma. □

We now can prove the following proposition, providing the desired bound for the unknown $T^* - T$.

Proposition 2.11. *Let $U = T^* - T$, where T solves the system (2.14). There exists $K_2 > 0$ such that*

$$\|U\|_{L^2(\Sigma^+)} \leq K_2. \tag{2.16}$$

Proof. The function U satisfies, by the Taylor formula,

$$-\Delta U + c\beta(y)U_x - \int_0^1 g'_{T^*}(T^* - tU)\,dt\,U = f(T^* - U)W.$$

Note that due to Proposition 2.9, since f is bounded, we have that $f(T^* - U)W \in L^2(\Sigma^+)$. Let θ_1 and δ_0 defined as in Lemma 2.10 and note that on $[T > \theta_1]$, we have

$$-\int_0^1 g'_{T^*}(T^* - tU)\,dt \geq \delta_0 > 0$$

since $T^* - tU \in (\theta_1, T^*)$. We write

$$\|U\|^2_{L^2(\Sigma^+)} = \|U\|^2_{L^2(0 \leq T \leq \theta_1)} + \|U\|^2_{L^2(T > \theta_1)}.$$

The first term is uniformly controlled, by Lemma 2.10. For the second term, since

$$-\Delta + c\beta(y)\partial_x - \int_0^1 g'_{T^*}(T^* - t\cdot)\,dt$$

is invertible from $L^2(\Sigma^+)$ into itself and from the fact that $f(T^* - U)W$ is controlled in $L^2(\Sigma^+)$, we infer that $U \in L^2(T > \theta_1)$, and the proposition is proved. □

The L^2 bound on U allows to derive the following corollary:

Corollary 2.12. *There exist controlled constants $\lambda, C > 0$ such that, for all $x \in \Sigma^+$,*

$$\sup_{y \in \omega} |U(x, y)| \leq Ce^{-\lambda x}.$$

Proof. We already know that U is uniformly controlled in L^2 by Proposition 2.11. From Lemma 2.10, we have, for $x \in [0 \leq T \leq \theta_1]$,

$$\sup_{y \in \omega} |U(x, y)| \leq C.$$

For $x \in [T > \theta_1]$, the zero order term in the equation satisfied by U being uniformly controlled from below, we deduce that there exist $C, \eta > 0$ (depending on δ_0 and the L^2 norm of U) such that $Ce^{-\eta x}$ is a supersolution for U, hence the result. □

3. Topological degree argument and construction of the solutions

The method is based on Leray–Schauder degree arguments, and we refer the reader to [11] for definitions and properties related to the Leray–Schauder degree. We first write our problem as a fixed point equation in appropriate functional spaces. Secondly, using homotopy invariance, we prove that these fixed point equations have non trivial solutions. More precisely, we prove that there exist two operators $\mathcal{L}_1$ and $\mathcal{L}_2$ acting on suitable functional spaces such that (1.1)–(1.2) can be formulated as a problem of the form $(\mathcal{L}_1 + \mathcal{L}_2)(T, Y, c) = 0$, where $\mathcal{L}_1$ is invertible with bounded inverse and $\mathcal{L}_2$ is a compact operator. Consequently, solving our problem is equivalent to solving the fixed point equation $(I + \mathcal{L}_1^{-1}\mathcal{L}_2)(T, Y, c) = 0$, where $\mathcal{L}_1^{-1}\mathcal{L}_2$ is a compact perturbation of the identity, and to which the Leray–Schauder degree theory can be applied.

3.1. Formulation as a fixed point equation

We set

$$U = T^* - T, \quad W = T^* - T - Y,$$

and

$$h(U) = -\int_0^1 (1-t)\, g''_{T^*}(T^* - tU)\, dt.$$

With this change of unknowns, the function U in the problem (1.1)–(1.2) satisfies the elliptic equation

$$-\Delta U + c\beta(y)U_x - g'_{T^*}(T^*)U = h(U)U^2 + f(T^* - U)W \quad \text{in } \Sigma^+.$$

As the domain Σ^+ is unbounded, the compactness of the operators (which is necessary to use Leray–Schauder degree arguments) cannot be ensured. Consequently we use weighted spaces. Fix $r > 0$ and consider the scale of Banach spaces

$$X_r = \left\{u \in C_0(\overline{\Sigma^+}) : e^{rx}u \in C_0(\overline{\Sigma^+})\right\},$$

where $C_0(\overline{\Sigma^+})$ is the set of all continuous functions in $\overline{\Sigma^+}$ vanishing when $x \to +\infty$. We endow X_r with the norm

$$\|u\|_r = \sup_{\Sigma^+} \left| e^{rx}\, u(x, y)\right|.$$

The function U satisfies the equation

$$\text{(3.1)} \qquad \begin{cases} -\Delta U + c\beta(y)U_x - g'_{T^*}(T^*)U = h(U)U^2 + f(T^* - U)W \quad \text{in } \Sigma^+ \\ \partial_\nu U = 0 \ \text{ on } \mathbb{R}^+ \times \partial\omega, \\ U(0, .) = T^*. \end{cases}$$

We want to derive a single equation for the temperature $U = T^* - T$; we therefore have to perform an intermediate step where we express W in terms of U.

3.1.1. Computation of W in terms of U. The function Y satisfies the linear equation

$$\begin{cases} -\Delta Y + c\beta(y)Y_x + f(T^* - U)Y = 0 & \text{in } \Sigma, \\ \partial_\nu Y = 0 & \text{on } \mathbb{R}\times\omega, \\ Y(-\infty,.) = 1, \quad Y(+\infty,.) = 0\,. \end{cases} \tag{3.2}$$

We have the following lemma:

Lemma 3.1. *Given $c > 0$ and a function $U \in X_{r_0}$ for some $r_0 > 0$, the problem (3.2) admits a solution Y. Furthermore, there exists a bounded operator $\mathcal{L}_1 \in \mathcal{L}(X_{r_0}, X_{r_0})$ mapping U into the restriction of Y to $\overline{\Sigma^+}$.*

Proof. We set $Y = e^{\rho x}Z$, where $\rho > 0$ will be suitably chosen. Therefore, the function Z satisfies

$$\begin{cases} -\Delta Z + (c-2\rho)\partial_x Z + (f(T^*-U) - \rho^2 + c\rho)Z = 0 & \text{in } \Sigma, \\ \partial_\nu Z = 0 & \text{on } \mathbb{R}\times\omega. \end{cases}$$

Denote by $\mathcal{T}$ the operator

$$\mathcal{T} = -\Delta + (c-2\rho)\partial_x + (f(T^*-U) - \rho^2 + c\rho)\,, \tag{3.3}$$

with domain

$$D(\mathcal{T}) = \Big\{u \in C(\Sigma) : u \in \bigcap_{p>1} W^{2,p}_{\text{loc}}(\Sigma),\ \partial_\nu u|_{\partial\Sigma} = 0\Big\}.$$

Since $U \in X_{r_0}$, there are $\kappa > 0$ and $(x_0, y_0) \in \Sigma^+$ bounded uniquely by $\|U\|_{L^2(\Sigma^+)}$ such that for all $x \geq x_0$, $f(T^* - U) \geq \kappa$. Hence, the operator $\mathcal{T}$ is coercive (at infinity) for ρ small enough (note that c is uniformly bounded from below). Furthermore, $\mathcal{T}$ is an isomorphism from $\mathcal{D}(\Sigma^+)$ to $C_0(\Sigma^+)$, where $\mathcal{D}(\Sigma^+)$ is the space of C^∞ functions compactly supported on Σ^+. Consequently we infer the existence of a unique Z by the Lax–Milgram Theorem. The function Z decays exponentially to 0 at $+\infty$ at a rate r_1, depending on ρ, chosen such that $r_1 < r_0$. This gives the existence of a function Y satisfying

$$\begin{cases} -\Delta Y + c\beta(y)Y_x + f(T^* - U)Y = 0 & \text{in } \Sigma, \\ \partial_\nu Y = 0 & \text{on } \mathbb{R}\times\omega. \end{cases} \tag{3.4}$$

We now have to check the limits at $\pm\infty$. For the limit at $-\infty$, it is easy to see that one can use supersolutions of the type $1 - e^{\eta x}$. The limit at $+\infty$ follows from the previous argument. □

To summarize, we have proved that the function W can be written as

$$W = T^* - T - Y = U - \mathcal{L}_1(U)\,,$$

where the bounded operator $\mathcal{L}_1$ maps X_{r_0} into X_{r_0} for some $r_0 > 0$.

3.1.2. Fixed point equation. Let $\mathcal{L}_2$ be the operator

$$\mathcal{L}_2 U = -\Delta U + c\beta(y) U_x - g'_{T^*}(T^*) U,$$

with domain

$$D(\mathcal{L}_2) = \Big\{ u \in C_0(\Sigma^+) : \, u \in \bigcap_{p>1} W^{2,p}_{\text{loc}}(\Sigma^+), \, \partial_\nu u|_{\mathbb{R}^+ \times \partial\omega} = 0, \, u(0,.) = T^* \Big\}.$$

Note that we have $D(\mathcal{L}_2) \subset X_{r_1}$ since $-g'_{T^*}(T^*) > 0$ and $\mathcal{L}_2$ is invertible from X_{r_1} into itself with a bounded inverse for some $r_1 > 0$. It follows that, for some $r_2 > 0$, our problem can be written as a fixed point equation:

$$(U, c) - F(U, c) = 0, \tag{3.5}$$

where F is the operator

$$\begin{aligned} F : \; & X_{r_2} \times \mathbb{R} \to X_{r_2} \times \mathbb{R} \\ & (U, c) \mapsto (F_1(U, c), F_2(U, c)) \end{aligned}$$

with

$$F_1(U, c) = \mathcal{L}_2^{-1}(h(U)\, U^2 + f(T^* - U)(U - \mathcal{L}_1(U)))$$

and

$$F_2(U, c) = c + < \partial_x F_1(U, c) > (0) - c < \beta > (1 - T^*).$$

It has to be noticed here that, since we are in a half cylinder Σ^+, the boundary of the cylinder at $\{x = 0\}$ (i.e., $\{x = 0\} \times \partial\omega$) is not smooth and involves mixed Dirichlet/Neumann conditions. Note that the term $< \partial_x F_1(U, c) > (0)$ is well defined when $U \in X_{r_2}$ since $F_1(U, c)$ involves the inverse operator $\mathcal{L}_2^{-1}$, which is regularizing.

Lemma 3.2. *The operator* $F : \; X_{r_2} \times \mathbb{R} \to X_{r_2} \times \mathbb{R}$ *is compact.*

Proof. We write $F_1 = T_1 + T_2$, where

$$T_1(U) = \mathcal{L}_2^{-1} h(U)\, U^2 \quad \text{and} \quad T_2(U) = \mathcal{L}_2^{-1} f(T^* - U)(U - \mathcal{L}_1(U)).$$

We divide the proof into several steps. First, we deal with the easy part, the compactness of F_2.

Compactness of F_2. Since the term $< \partial_x F_1(U, c) > (0)$ is well defined and bounded since U and c are bounded, the compactness of $F_2(U, c)$ is obvious.

Compactness of T_1 *and* T_2. First notice the following facts: since U is bounded, we have that $h(U) \in L^\infty$ and $h(U)U^2 \in X_{r_2}$ if $U \in X_{r_2}$. Similarly, we have that $f(T^* - U)(U - \mathcal{L}_1(U)) \in X_{r_2}$. Therefore, we can treat in a unified way both operators T_1 and T_2. Let U_n be a bounded sequence in X_{r_2}. Then $V_n = T_1(U_n)$ (or $T_2(U_n)$) satisfies the problem

$$\mathcal{L}_2 V_n = f_n \in X_{r_2}$$

with appropriate boundary conditions. The idea of the proof is to distinguish interior regularity and estimates up to the boundary, which is not smooth at $\{x = 0\}$, as mentioned previously.

We start by regularizing the domain by considering $\Sigma_\varepsilon^+ \subset \Sigma^+$ for $\varepsilon > 0$ such that $\partial\Sigma_\varepsilon^+$ is smooth. By standard elliptic estimates on smooth domains, we have that there exists a constant C depending on ε such that

$$\|V_n\|_{C^{2,\alpha}(K)} \le C \tag{3.6}$$

for every $K \subset\subset \Sigma_\varepsilon^+$, and, by regularity, we have also

$$\|V_n\|_{r_2} \le C\,. \tag{3.7}$$

We now estimate the function V_n close to the boundary. Let $z \in \{x = 0\} \cap \partial\omega$ and consider the ball of radius $\rho > 0$ centered at z. One can find, by means of generalized spherical harmonic functions, a function continuous up to the boundary $\bar{v}_n$ such that

$$\forall\ (x, y) \in \Sigma_\varepsilon^+ \cap B(z, \rho), \quad V_n(x, y) \le \mu\bar{v}_n(x, y),$$

for some $\mu > 0$. By means of a partition of the unity, we then get the existence of a function $\overline{V}_n$ which by continuity extends V_n to $\{x = 0\} \cap \partial\omega$. Therefore the sequence of functions V_n satisfies

$$V_n \in C^{2,\alpha}_{\mathrm{loc}}(\Sigma_\varepsilon^+) \cap C_0(\overline{\Sigma^+}).$$

We now conclude the argument. On the one hand, from the estimate (3.6), there exists a subsequence (still denoted V_n) such that V_n converges to V^ϵ in the C^2_{loc} topology. Furthermore, V^ϵ is bounded in the C^1 norm. On the other hand, from the last argument, the function V^ϵ extends by continuity to $\partial\Sigma^+$. Letting ε go to zero, we conclude the convergence of a subsequence of V_n to a function V in X_{r_2}, hence the compactness. □

3.2. Computation of the Leray–Schauder degree and construction of the solutions

The aim of the section is to prove the following theorem:

Theorem 3.1. *Let f be as previously described. Then for every $T^* \in (\theta, 1)$, equation (3.5) admits at least one solution (U, c).*

Proof. As is classical in degree theory, since F is a compact perturbation of the identity, we just have to prove that the Leray–Schauder degree of the operator $I - F$ with respect to 0 in a suitable open set of $X_{r_2} \times \mathbb{R}$ is well defined and not equal to zero. This leads directly to the existence of a solution for the problem (3.5). To this end, we use the homotopy invariance of the degree. We consider the set

$$\mathcal{O} = \Big\{(U, c) \in X_{r_2} \times \mathbb{R} \,|\, \|U\|_{L^2(\Sigma^+)} < K + 1,\ \frac{\underline{c}}{2} < c < \bar{c} + 1\Big\}. \tag{3.8}$$

One has $0 \notin (I - F)(\partial\mathcal{O})$ by our estimates, which proves that $\deg(I - F, \mathcal{O}, 0)$ is well defined. We now use the basic properties of the Leray–Schauder degree (see [11]) to get the desired result. Under this homotopy, the bounds for the speed c and U

are changed, but we have that $\forall\, \nu \in [0,1]$, $0 \notin (I-F)(\partial\mathcal{O})$ and the degree is still well defined. We perform the homotopy consisting in replacing the term $c\beta(y)$ by $c(1-\mu+\mu\beta(y))$, where $\mu \in [0,1]$. We end up with a problem of the type

$$-\Delta U + c\partial_x U - g'_{T^*}(T^*)U = h(U)U^2 + f(T^*-U)(U-\mathcal{L}_1(U)),$$

with Neumann boundary conditions on the cylinder. We do not know any uniqueness property for this type of system. To overcome this, we perform a second type of homotopy replacing the term $U-\mathcal{L}_1(U)$ by $\mu'(U-\mathcal{L}_1(U))$. Under this homotopy, when $\mu'=0$, we get the following problem for the function U:

$$\begin{cases} -\Delta U + cU_x = g_{T^*}(U) & \text{in } \Sigma^+, \\ \partial_\nu U = 0 & \text{on } \mathbb{R}^+ \times \partial\omega, \\ U(0,.) = T^*, \;\; U(+\infty,.) = 0, & \\ < \partial_x U > (0) = c < \beta > (T^*-1). & \end{cases} \tag{3.9}$$

Note that one has to add the compatibility equation coming from F_2 to (3.5). The system (3.9) admits a one-dimensional solution given by

$$\begin{cases} -u'' + cu' = g_{T^*}(u) \quad \text{in } \mathbb{R}^+, \\ u(0) = T^*, \;\; u'(0) = c(T^*-1), \\ u(+\infty) = 0. \end{cases} \tag{3.10}$$

The following theorem gives the main properties of this one-dimensional front, and we postpone its proof to Appendix A.

Theorem 3.2. *Problem (3.10) has a (unique) solution (u,c) satisfying:*

1. *There exist $\underline{c}, \overline{c} > 0$ such that $\underline{c} \le c \le \overline{c}$.*
2. *$u' < 0$ and $0 \le u \le T^*$.*
3. *There is no non-trivial solution of*

$$\begin{cases} -\Delta w + c\partial_x w - g'_{T^*}(u)w = 0 & \text{in } \Sigma^+, \\ w(0,.) = 0, \;\; \partial_\nu w = 0 & \text{on } \mathbb{R}^+ \times \partial\omega, \\ w \in W^{1,\infty}(\Sigma^+). & \end{cases} \tag{3.11}$$

We now come to the topological degree argument to conclude the existence of a solution. By the invariance of the degree under homotopy, we have

$$\deg(I-F,\mathcal{O},0) = \deg(\mathcal{F},\mathcal{O},0)\,, \tag{3.12}$$

where $\mathcal{F}(U,c)=0$ is equivalent to the solution (U,c) of problem (3.9). One has now to prove that $\deg(\mathcal{F},\mathcal{O},0) \neq 0$ to get the desired result. On the one hand, problem (3.9) admits at most one solution (see [6]). On the other hand, thanks to Theorem 3.2, problem (3.9) admits a one-dimensional solution and the linearized operator around it is not degenerate. This implies that $\deg(\mathcal{F},\mathcal{O},0) \neq 0$ (see [11]). This ends the proof of the theorem. □

4. The limit $T^* \to 1$

We consider the behaviour of the problem (1.1)–(1.2) under the limit $T^* \to 1$. In a first part, we derive uniform estimates for the speed with respect to the parameter T^*. Then we provide the proof of Theorem 1.2.

4.1. Uniform estimates for c

We have the following result:

Proposition 4.1. *Let (Y, T, c) be a solution of (1.1)–(1.2). There exists c_1 independent of T^* such that*

$$c_1 < c < c^*, \tag{4.1}$$

where c^* is the speed involved in Theorem 2.1.

Proof. The proof of the upper bound relies on a variant of the sliding method together with the strong maximum principle. Suppose that $c \geq c^*$ and let U be a solution of (2.15). For x large enough, we have $T < U$ since $T^* < 1$. We set $U_t(x, y) = U(x + t, y)$, the variable t being chosen such that the inequality $T \leq U_t$ holds in Σ^+ with equality somewhere. If we denote $v = U_t - T$, we have

$$\begin{aligned} -\Delta v + c\beta(y)v_x &= g_1(U_t) - f(T)Y + (c - c^*)\beta(y)\partial_x U_t \\ &\geq g_1(U_t) - g_1(T) + (c - c^*)\beta_0 \partial_x U_t. \end{aligned}$$

Consequently, we get (recall that $\partial_x U_t > 0$)

$$\begin{cases} -\Delta v + c\beta(y)v_x - d(U_t, T)v \geq 0 & \text{in } \Sigma^+, \\ \partial_\nu v = 0 & \text{on } \mathbb{R}^+ \times \partial\omega, \\ v(0, .) > 0, \ v(+\infty, .) = 1 - T^* > 0, \end{cases} \tag{4.2}$$

where $d(U_t, T) = \frac{g_1(U_t) - g_1(T)}{U_t - T}$ whenever $U_t \neq T$ and $d(U_t, T) = g_1'(U_t)T$ when $U_t = T$. From the strong maximum principle and the Hopf lemma, we have $v \equiv 0$ since $v = 0$ somewhere in the cylinder. This is a contradiction and one then gets $c < c^*$. The existence of c_1 can be shown as in Proposition 2.8. We sum up the steps. Consider a sequence $T_n^* \in (\theta, 1)$ and a sequence c_n converging respectively to 1 and 0. Up to the extraction of a subsequence, the associated solutions T_n and Y_n converge over all compact set in $C^2_{\text{loc}}(\Sigma^+)$ to T_∞ and Y_∞, which satisfy

$$\begin{cases} -\Delta T_\infty = f(T_\infty)Y_\infty & \text{in } \Sigma^+, \\ \partial_\nu T_\infty = 0 & \text{on } \mathbb{R} \times \partial\omega, \\ T_\infty(0, .) = 0, \ \ < \partial_x T_\infty >= 0. \end{cases} \tag{4.3}$$

By the Hopf lemma, this implies that $T_\infty \equiv 0$. We infer then that Y_∞ is also a constant. By translating the functions (T_n, Y_n) and using the same arguments as in Proposition 2.8, we reach a contradiction.

Notice that both bounds are uniform in $1 - T^*$. This is clear for the upper bound since it is the velocity involved in problem (2.15), which is independent of T^*. For the lower bound, the proof shows that the sequence c_{T^*} is bounded from below by a constant independent of T^*. □

We now come to the proof of Theorem 1.2. The main steps are the following: we prove first that in the limit $T^* \to 1$, the solutions Y and T converge to constants. In order to avoid this triviality, we renormalize the temperature such that it reaches the ignition temperature on the wall $\{x = 0\}$. The convergence is no longer to constants and one can reach the desired conclusion.

4.2. Proof of Theorem 1.2

We divide the proof into several steps.

Definition and properties of limit functions. Let (Y_n, T_n, c_n) be a solution of problem (1.1)–(1.2) indexed by a sequence $T_n^* \in (0,1)$ converging to 1. Up to extracting a subsequence and due to elliptic estimates (see [8]), we get that

$$(Y_n, T_n, c_n) \to (Y_\infty, T_\infty, c_\infty) \tag{4.4}$$

on all compact sets in $C^{2,\beta}_{\mathrm{loc}}(\Sigma^+)$ with $0 < \beta < 1$, and that T_∞ satisfies

$$\begin{cases} -\Delta T_\infty + c_\infty \beta(y) \partial_x T_\infty = f(T_\infty) Y_\infty & \text{in } \Sigma^+, \\ \partial_\nu T_\infty = 0 & \text{on } \mathbb{R}^+ \times \partial\omega, \\ T_\infty(0,.) = 0\,. \end{cases} \tag{4.5}$$

Moreover, from (2.10), we have $< \partial_x T_\infty > (0) = 0$. From the maximum principle and the Hopf lemma, we infer $T_\infty \equiv 0$. We now prove that $Y_\infty \equiv 1$. We have

$$\begin{cases} -\Delta Y_\infty + c_\infty \beta(y) \partial_x Y_\infty = 0 & \text{in } \Sigma, \\ \partial_\nu Y_\infty = 0 & \text{on } \mathbb{R} \times \partial\omega. \end{cases} \tag{4.6}$$

We take the Fourier transform of the equation with respect to the variable x. Multiplying the resulting equation by $\bar{Y}_\infty$ and integrating over ω, one easily gets that Y_∞ is a constant. Furthermore, since c is bounded from below independently of T_n^*, there exist constants $\rho > 0$ and $C > 0$ such that we have

$$1 - Y_n \le C e^{\rho x}$$

in Σ^-, for C and ρ independent of n. This leads to $Y_\infty \equiv 1$. It follows that the function $W_n^+ = T_n^* - T_n - Y_n$ converges locally to 0.

Translation of functions. Since $W_n^+ \to 0$ locally in $C^2(\Sigma^+)$, applying Proposition 2.3 yields $W_n^+ \to 0$ in $L^2(\Sigma^+)$. Consider now the sequence x_n such that $\max_{y \in \omega} T_n(x_n, y) = \theta$. This sequence is unbounded by the strong maximum principle (and the Hopf lemma). So we have $x_n \to +\infty$ as $n \to +\infty$ and we define

$$U_n(x,y) = T_n(x + x_n, y) \quad \text{and} \quad W_n(x,y) = T_n^* - U_n(x,y) - Y_n(x,y).$$

Once again from Proposition 2.3, we have that $W_n \to 0$ in $L^2(\Sigma^+)$ and then $U_n \to U_\infty$ and $Y_n \to 1 - U_\infty$ on all compact sets in $C^{2,\beta}_{\text{loc}}(\Sigma)$ with $0 < \beta < 1$. The function U_∞ solves the problem

$$\left\{\begin{aligned} -\Delta U_\infty + c_\infty \beta(y)\partial_x U_\infty = g_1(U_\infty) \quad & \text{in } \Sigma, \\ \partial_\nu U_\infty = 0 \quad & \text{on } \mathbb{R} \times \partial\omega. \end{aligned}\right. \tag{4.7}$$

By regularity, U_∞ admits constant limits at $\pm\infty$. Furthermore, using the super-solution $\theta e^{c\beta_0 x}$ in Σ^-, we get $U_\infty(-\infty, .) = 0$. We have $U_\infty(+\infty, y) = u^+ \geq 0$. This implies that $\partial_x U_\infty > 0$, yielding $u^+ = 1$. The conclusion follows from the uniqueness statement in Theorem 2.1 and Proposition 4.1. □

Appendix A: One-dimensional fronts

This section is devoted to proofs of the existence and the properties of the one-dimensional front associated to our problem. Recall the equation under consideration:

$$\left\{\begin{array}{l} -u'' + cu' = g_{T^*}(u) \quad \text{in } \mathbb{R}^+, \\ u(0) = T^*, \ u'(0) = c(T^* - 1), \\ u(+\infty) = 0. \end{array}\right. \tag{4.8}$$

We want to prove the following theorem:

Theorem 4.1. *Problem (4.8) has a (unique) solution (u, c) satisfying:*

1. *There exist $\underline{c}, \overline{c} > 0$ such that $\underline{c} \leq c \leq \overline{c}$.*
2. *$u' < 0$ and $0 \leq u \leq T^*$.*
3. *There is no non-trivial solution of*

$$\left\{\begin{aligned} -\Delta w + c\partial_x w - g'_{T^*}(u)w = 0 \quad & \text{in } \Sigma^+, \\ w(0, .) = 0, \ \partial_\nu w = 0 \quad & \text{on } \mathbb{R}^+ \times \partial\omega, \\ w \in W^{1,\infty}(\Sigma^+). \quad & \end{aligned}\right. \tag{4.9}$$

Proof. Step 1: Existence. There are several ways to tackle the existence part of this problem. We use a shooting method whose parameter is the slope of u at the origin. To this end, we introduce the Cauchy problem

$$\left\{\begin{array}{l} -u'' + cu' = g_{T^*}(u) \quad \text{in } \mathbb{R}^+, \\ u(0) = T^*, \quad u'(0) = c(T^* - 1). \end{array}\right. \tag{4.10}$$

Lemma 4.2. *Let u be a solution of (4.10) and suppose $c > 0$. Then,*

- *$0 \leq u \leq T^*$.*
- *If there exists $x_0 > 0$ such that $u(x_0) = T^*$ then, for all $x \geq x_0$, $u'(x) < 0$.*
- *If there exists $x_0 > 0$ such that $u(x_0) = 0$ then, for all $x \geq x_0$, $u'(x) > 0$.*

Proof. The proof of the first point is standard. We prove only the second point since the proof of the third one is similar. We first prove that $u'(x_0) < 0$. Suppose the contrary. If $u'(x_0) = 0$, then by uniqueness of the Cauchy problem, we have $u \equiv T^*$, a contradiction with $u'(0) = c(T^* - 1) \neq 0$. If $u'(x_0) > 0$, by the continuity of u, there exists $x_1 \in (0, x_0)$ such that $u(x_1) < T^*$ and $u'(x_1) < 0$. We integrate the equation over (x_1, x_0) to obtain

$$u'(x_1) - u'(x_0) = \int_{x_1}^{x_0} g_{T^*}(u(s))\, ds. \tag{4.11}$$

The left hand side is strictly negative whereas the right one is ≥ 0, a contradiction. Repeating the same argument for $x \geq x_0$ leads to the desired result. □

Lemma 4.3. *Let u be a solution of (4.10). Then we have:*

- *If there exists x_0 such that $u(x_0) = T^*$, then $\lim_{x\to+\infty} u(x) = +\infty$.*
- *If there exists x_0 such that $u(x_0) = 0$, then $\lim_{x\to+\infty} u(x) = +\infty$.*

Proof. We only prove the first point since the proof of the second one is similar. From Lemma 4.2, u is strictly decreasing and bounded, hence admits a limit at $+\infty$. Since $u \leq T^*$ for $x \geq x_0$, this limit cannot be finite. □

We now introduce the sets

$$\Gamma_0 = \{c > 0 \,|\, \exists x_0 > 0 \text{ such that } u(x_0) = 0\},$$
$$\Gamma_{T^*} = \{c > 0 \,|\, \exists x_0 > 0 \text{ such that } u(x_0) = T^*\}.$$

We have:

Lemma 4.4. *The sets Γ_0 and Γ_{T^*} have the following properties:*

- *Γ_0 and Γ_{T^*} are open.*
- *$\Gamma_0 \cap \Gamma_{T^*} = \emptyset$.*
- *$\Gamma_0 \neq \emptyset$ and $\Gamma_{T^*} \neq \emptyset$. More precisely, there exist $\underline{c}$ and $\overline{c}$ such that $[\overline{c}, +\infty) \subset \Gamma_{T^*}$ and $[\underline{c}, +\infty) \subset \Gamma_0$.*

Proof. The first point follows from the continuity with respect to the parameter c. The second point follows from Lemma 4.2. The third point requires estimates on c which can be obtained by multiplying the equation by 1, u, u', and integrating. □

We now come to the existence part of Theorem 3.2. By Lemma 4.4, we choose $c > 0$ such that $c \notin \Gamma_0 \bigcup \Gamma_{T^*}$ and consider the solution u associated with this speed c. We have to show that $u(+\infty) = 0$. From Lemmas 4.2 and 4.3, we have $u' < 0$ and $0 < u < T^*$. This implies that $\lim_{x\to+\infty} u(x) = \ell \in (0, \theta)$ exists. By continuity, we have $g_{T^*}(\ell) = 0$ and then $\ell = 0$. This ends the proof of the first two points of Theorem 3.2.

Step 2: Properties of the one-dimensional front. Assume there exists a non-trivial solution of (4.9). We reach a contradiction. The proof is based on exponential estimates for solutions of scalar elliptic equations on half cylinders. More precisely, we have (see [5])

$$\|w(x,.) - e^{-\lambda_+ x} w_+\|_{C^{1,\delta}(\overline{\omega})} \le Ce^{-(\lambda_+ + \epsilon)x}, \tag{4.12}$$

$$\|\partial_x w(x,.) - \lambda_+ e^{-\lambda_+ x} w_+\|_{C^{0,\delta}(\overline{\omega})} \le Ce^{-(\lambda_+ + \epsilon)x}, \tag{4.13}$$

where $\lambda_+, \epsilon > 0$ and $e^{-\lambda_+ x} w_+(y)$ is a positive exponential solution of

$$\begin{cases} -\Delta w + c\partial_x w - g'_{T^*}(u)w = 0 & \text{in } \Sigma^+, \\ \partial_\nu w = 0 & \text{on } \mathbb{R}^+ \times \partial\omega. \end{cases} \tag{4.14}$$

We shall also need the following additional result from [5] on exponential solutions of (4.14). We denote by $\mu_k(\mathcal{L})$ the k-th eigenvalue for a second order elliptic operator $\mathcal{L}$ in Ω with Neumann boundary conditions on $\partial\Omega$.

Theorem 4.2. *Any exponential solution of problem* (4.14) *has the form* $e^{\lambda x}\psi(y)$, *with* $\lambda \in \mathbb{R}$. *Moreover, there exists* $k \in \mathbb{N}$ *such that* λ *is a solution of*

$$\mu_k(-\Delta_y + \lambda\beta - g'_{T^*}(u)) = \lambda^2, \tag{4.15}$$

and $-\lambda^+$ *is the only negative eigenvalue with* $k = 1$.

From the theory in [1], Theorem 4.2 and estimates (4.12)–(4.13), we infer the existence of $\alpha \in \mathbb{R}$ such that

$$w(x,y) = \alpha w_+(y) e^{-\lambda_+ x} + O(e^{-(\lambda_+ + \epsilon)x}) \quad x \to +\infty,\ y \in \omega. \tag{4.16}$$

We set $v(x,y) = \frac{w(x,y)}{u'(x)}$ (recall that $u' \neq 0$). Then v satisfies

$$\begin{cases} -\Delta v + (c - 2\frac{u''}{u'})\partial_x v = 0 & \text{in } \Sigma^+, \\ \partial_\nu v = 0 & \text{on } \mathbb{R}^+ \times \partial\omega, \\ v(0,.) = 0, \ \ v(+\infty,.) = \alpha. \end{cases} \tag{4.17}$$

Note that the coefficients in (4.17) are bounded and Hölder continuous in $\overline{\Sigma^+}$. We now have two cases: $\alpha = 0$ and $\alpha \neq 0$.

Suppose first that $\alpha = 0$. Let

$$M = \sup_{\overline{\Sigma^+}} v \quad \text{and} \quad m = \inf_{\overline{\Sigma^+}} v.$$

M and m cannot be both zero. Suppose $M \neq 0$. Since v vanishes at infinity, M is attained at some point $(x_0, y_0) \in \overline{\Sigma^+}$. If $(x_0, y_0) \in \Sigma^+$, then by the strong maximum principle, $v \equiv 0$, which is a contradiction. By the Hopf lemma, M cannot be attained on $\mathbb{R}^+ \times \partial\omega$. Consequently, the supremum of v is necessarily achieved on $\{x = 0\}$. We now use the following version of the Serrin Lemma (see [13], [7]) to reach a contradiction with the compatibility condition.

Lemma 4.5. *Let Ω be a domain in $\mathbb{R}^n$ with the origin 0 on its boundary. Assume that near 0 the boundary consists of two transversally intersecting C^2 hypersurfaces $\rho = 0$ and $\sigma = 0$. Suppose $\rho, \sigma < 0$ in Ω. Let w be a function in $C^2(\bar{\Omega})$, with $w < 0$ in Ω, $w(0) = 0$, satisfying in Ω the elliptic inequality*

$$a_{ij}(x)\partial_{ij}w + b_i(x)\partial_i w + c(x)w \geq 0\,.$$

Assume that

$$a_{ij}\rho_i\sigma_j \geq 0 \tag{4.18}$$

at 0. If this is zero, assume furthermore that $a_{ij} \in C^2$ in $\bar{\Omega}$ near 0 and that $D(a_{ij}\rho_i\sigma_j) = 0$ at 0 for any first derivative D at 0 tangent to the submanifold $\{\rho = 0\} \cap \{\sigma = 0\}$. Then, for every direction s at 0 which enters Ω transversally to each hypersurface,

- $\dfrac{\partial w}{\partial s} < 0$ *at 0 in the case of strict inequality in (4.18),*
- $\dfrac{\partial w}{\partial s} < 0$ *or* $\dfrac{\partial^2 w}{\partial s^2} < 0$ *at 0 in the case of equality in (4.18).*

Suppose now that $\alpha \neq 0$. One can assume that $\alpha > 0$. By the strong maximum principle and the Hopf lemma, we have $0 < v(x,y) < \alpha$, which leads to

$$0 < w(x,y) < \alpha u'(x) \quad \text{in } \Sigma^+. \tag{4.19}$$

On the other hand, following the proof of Theorem 2 in [12] for example, one can prove that in fact u' decays faster than w as $x \to +\infty$. Consequently, we get

$$\lim_{x\to+\infty} \frac{w(x,y)}{u'(x)} = +\infty. \tag{4.20}$$

This leads to another contradiction.

In conclusion, we have proven that necessarily $v \equiv 0$, a contradiction with the assumed non-triviality of w. As a consequence, 0 is not an eigenvalue of the linearized operator. □

Remark 4.6. It has to be noticed that a hint that 0 is not an eigenvalue is that the derivative of the wave $\partial_x T$ does not satisfy the linearized problem.

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Yannick Sire: Université Aix-Marseille Paul Cézanne, LATP, UMR 7353, 9, rue F. Joliot Curie, 13453 Marseille, France.
E-mail: sire@cmi.univ-mrs.fr