



On Hardy spaces associated with certain Schrödinger operators in dimension 2

Jacek Dziubański and Jacek Zienkiewicz

Abstract. We study the Hardy space H^1 associated with the Schrödinger operator $L = -\Delta + V$ on \mathbb{R}^2 , where $V \geq 0$ is a compactly supported non-zero C^2 -potential. We prove that this space, which is originally defined by means of the maximal function associated with the semigroup generated by $-L$, admits a special atomic decomposition with atoms satisfying a weighted cancellation condition with a weight of logarithmic growth.

1. Introduction

The aim of this paper is to prove a special atomic characterization of the Hardy spaces H^1 associated with Schrödinger operators in \mathbb{R}^2 with compactly supported non-negative smooth potentials. In other dimensions such characterizations were obtained in [4] and [10], and only the 2-dimensional situation remained open.

Let $K_t(x, y)$ be the integral kernels of the semigroup of linear operators $\{K_t\}_{t>0}$ on \mathbb{R}^2 , generated by a Schrödinger operator $-L = \Delta - V(x)$, where $V(x)$ is a non-zero nonnegative C^2 -function supported by the unit open ball $B(0, 1)$. The Feynman–Kac formula

$$(1.1) \quad K_t f(x) = \mathbb{E}^x \left(e^{-\int_0^t V(X_s) ds} f(X_t) \right),$$

where X_t is the Brownian motion associated with the heat semigroup $P_t := e^{t\Delta}$ (see, e.g., Chapter V of [16]), implies that

$$(1.2) \quad 0 \leq K_t(x, y) \leq (4\pi t)^{-1} \exp(-|x - y|^2/4t) := P_t(x - y).$$

Clearly, for every $1 \leq p < \infty$, the family $\{K_t\}_{t>0}$ forms a semigroup of linear contractions on $L^p(\mathbb{R}^2)$.

The Hardy space H_L^1 is defined by means of the maximal function for the semigroup $\{K_t\}_{t>0}$, namely,

$$(1.3) \quad H_L^1 = \left\{ f \in L^1(\mathbb{R}^2) : \mathcal{M}_L f \in L^1(\mathbb{R}^2) \right\},$$

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where

$$(1.4) \quad \mathcal{M}_L f(x) = \sup_{t>0} |K_t f(x)|.$$

The norm $\|f\|_{H_L^1}$ on the space H_L^1 is given by the formula

$$(1.5) \quad \|f\|_{H_L^1} = \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^2)}.$$

Let $w(x) > 0$ be a locally integrable function. We say that a measurable function b defined on \mathbb{R}^2 is an atom for the Hardy space $H_{w,at}^1$ if there exists a ball $B = B(x_0, r)$ such that

$$(1.6) \quad \text{supp } b \subset B, \quad \|b\|_{L^\infty} \leq |B|^{-1},$$

$$(1.7) \quad \int_{\mathbb{R}^2} b(x)w(x) dx = 0.$$

The atomic norm $\|f\|_{H_{w,at}^1}$ is given by the formula

$$(1.8) \quad \|f\|_{H_{w,at}^1} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j b_j, \lambda_j \in \mathbb{C}, b_j \text{ are atoms for } H_{w,at}^1 \right\}.$$

Clearly, if $w \equiv 1$, then the $H_{w,at}^1$ -atoms coincide with the classical $(1, \infty)$ -atoms for the Hardy space $H^1(\mathbb{R}^2)$. For the theory of the classical real Hardy spaces $H^p(\mathbb{R}^d)$ we refer the reader to [3], [11], [12], [17], and the references therein.

Our goal in this paper is to prove the following theorem:

Theorem 1.1. *Assume that $L = -\Delta + V$ is a Schrödinger operator in \mathbb{R}^2 , where V is a non-zero nonnegative C^2 -function such that $\text{supp } V$ is contained in the unit ball $B(0, 1)$. Then there exists a regular L -harmonic weight w such that*

$$(1.9) \quad C^{-1} \ln(2 + |x|) \leq w(x) \leq C \ln(2 + |x|),$$

$$H_L^1 = H_{w,at}^1, \text{ and}$$

$$(1.10) \quad C^{-1} \|f\|_{H_{w,at}^1} \leq \|f\|_{H_L^1} \leq C \|f\|_{H_{w,at}^1}$$

The construction and the properties of the weight w are given in Section 3.

Let us finally emphasize the differences which occur in atomic decompositions of the Hardy spaces associated with Schrödinger operators with compactly supported potentials in dimensions different than 2. It was proved in [10] that for $d \geq 3$ and any compactly supported nonnegative $V \in L^q(\mathbb{R}^d)$, $q > d/2$, the Hardy space $H_{-\Delta+V}^1$ coincides with $H_{w,at}^1$, where

$$(1.11) \quad w(x) = \lim_{t \rightarrow \infty} \int_{\mathbb{R}^d} K_t(x, y) dy.$$

Here $K_t(x, y)$ are the integral kernels of the semigroup generated by $\Delta - V$ in \mathbb{R}^d . The function w defined by (1.11) satisfies

$$0 < c \leq w(x) \leq 1, \quad \lim_{|x| \rightarrow \infty} w(x) = 1, \quad |w(x) - w(x')| \leq C|x - x'|.$$

See [10] for details and proofs.

The one-dimensional situation is different and was studied in [4]. Let $V \in L^1_{\text{loc}}(\mathbb{R})$ be a nonnegative potential. In order to define the notion of an atom for the space $H^1_{-\Delta+V}$, one defines the auxiliary function

$$\rho(y) := \inf \left\{ r > 0 : r \int_{y-r}^{y+r} V(x) dx \geq 1 \right\}.$$

Then atoms for $H^1_{-\Delta+V}$ are either the classical $(1, \infty)$ -atoms for the Hardy space $H^1(\mathbb{R})$ or $|I|^{-1}\chi_I(x)$, where $I = (y - \rho(y), y + \rho(y))$. Hence, in this case, the Hardy space is local, where the scale of localization is adapted to the behavior of the potential (see [4]).

The reader interested in results concerning Hardy spaces associated with semigroups of linear operators, and in particular semigroups generated by Schrödinger operators, is referred to [1], [2], [4], [6], [7], [8], [9], [10] and [15].

The paper is organized as follows. In Section 2 we derive estimates for the integral kernels $K_t(x, y)$ of the semigroup generated by $-L = \Delta - V$, where $V \in C^2(B(0, 1)) \subset \mathbb{R}^2$, $V \geq 0$, $V \not\equiv 0$. Then we prove some properties of the operator VL^{-1} . Section 3 is devoted for the construction of the weight function w (see Theorem 1.1). In Section 4 we provide a relation between the classical atoms and the $H^1_{w, \text{at}}$ -atoms (see Proposition 4.2). The proofs of the inequalities in (1.10) are presented in Sections 5 and 6.

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2. Estimates of kernels in \mathbb{R}^2

Henceforth, unless otherwise stated, we assume that V is a non-zero nonnegative C^2 -potential supported in the unit ball $B(0, 1)$ of \mathbb{R}^2 , $K_t(x, y)$ are the integral kernels of the semigroup $\{K_t\}_{t>0}$ generated by $-L = \Delta - V(x)$.

For $C_0 > 0$ (big enough) we define

$$G(x) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x - y|V(y) dy + C_0.$$

Then $G \in C^\infty(\mathbb{R}^2)$ and, if we take C_0 large,

$$(2.1) \quad G(x) \geq 2 \quad \text{and} \quad G(x) \sim \ln(2 + 2|x|) \quad \text{for } x \in \mathbb{R}^2.$$

The function $G(x)$ is L -subharmonic, that is, $LG(x) \geq 0$. Indeed,

$$\begin{aligned} LG(x) &= (-\Delta + V(x))\left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x - y| V(y) dy + C_0\right) \\ &= -V(x) + V(x)G(x) \geq 0. \end{aligned}$$

Hence,

$$\frac{d}{dt} \int_{\mathbb{R}^2} K_t(z, y) G(y) dy = - \int_{\mathbb{R}^2} K_t(z, y) LG(y) dy \leq 0,$$

and, consequently,

$$(2.2) \quad \int_{\mathbb{R}^2} K_t(z, y) G(y) dy \leq G(z).$$

Since $G(x)$ is bounded on compact sets,

$$(2.3) \quad \int K_t(z, y) G(y) dy \leq C \quad \text{for } t > 0 \text{ and } |z| \leq 3.$$

Lemma 2.1. *There is $C > 0$ such that*

$$\int_{\mathbb{R}^2} K_t(z, y) dy \leq \begin{cases} 1 & \text{for } t \leq 2(1 + |z|), \\ C \frac{\ln(2+2|z|)}{\ln t} & \text{for } t > 2(1 + |z|). \end{cases}$$

Proof. Thanks to (1.2), it suffices to consider $t > 2(1 + |z|)$. Using (2.2) together with (2.1), we have that

$$\begin{aligned} \int_{\mathbb{R}^2} K_t(z, y) dy &= \int_{|y| \leq t^{1/4}} K_t(z, y) dy + \int_{|y| > t^{1/4}} K_t(z, y) dy \\ &\leq C t^{-1} \int_{|y| \leq t^{1/4}} dy + C \int_{|y| > t^{1/4}} K_t(z, y) \frac{\ln|y|}{\ln t^{1/4}} dy \\ &\leq C t^{-1/2} + C \frac{G(z)}{\ln t^{1/4}} \leq C \frac{\ln(2 + 2|z|)}{\ln t}. \end{aligned}$$

□

Lemma 2.2. *There exists a constant $\delta > 0$ such that*

$$(2.4) \quad \int_{\mathbb{R}^2} K_t(x, y) dy \geq \delta \frac{\ln(2 + |x|)}{\ln t} \quad \text{for } t > 2(1 + |x|).$$

Proof. The Feynman–Kac formula (1.1) implies that

$$(2.5) \quad e^{-t\|V\|_\infty} P_t(x - y) \leq K_t(x, y).$$

Set $\phi(x, t) = \int_{\mathbb{R}^2} K_t(x, y) dy$. Clearly, by (1.2) and the semigroup property,

$$(2.6) \quad 0 < \phi(x, t_1) \leq \phi(x, t_2) \quad \text{for } t_1 > t_2 > 0.$$

Let $\Omega = \{x \in \mathbb{R}^2 : |x| > 1\}$. We have

$$0 \geq \frac{d}{dt} \phi(x, t) = \Delta \phi(x, t) - V(x)\phi(x, t) = \Delta \phi(x, t) \quad \text{for } x \in \Omega,$$

and so for fixed t , $\phi(x, t)$ is superharmonic on Ω . Moreover, from the Feynman–Kac formula we conclude that there exists a constant $\delta_0 > 0$ such that $\phi(x, t) \geq \delta_0$ for $|x| = \sqrt{t} > 2$. For fixed $t > 4$, let

$$u(x) = \delta_0 \frac{\ln |x|}{\ln \sqrt{t}}.$$

Then

$$0 = u(x) \leq \phi(x, t) \text{ for } |x| = 1 \quad \text{and} \quad \delta_0 = u(x) \leq \phi(x, t) \text{ for } |x| = \sqrt{t},$$

and, consequently, since u is harmonic,

$$(2.7) \quad \delta_0 \frac{\ln |x|}{\ln \sqrt{t}} = u(x) \leq \phi(x, t) = \int_{\mathbb{R}^2} K_t(x, y) dy \quad \text{for } 1 \leq |x| \leq \sqrt{t}.$$

Now we consider $|x| \leq 2$. Let $t > 5$. Then,

$$\begin{aligned} \int_{\mathbb{R}^2} K_t(x, y) dy &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_1(x, z) K_{t-1}(z, y) dz dy \\ &\geq \int_{\mathbb{R}^2} \int_{2 < |z| < 3} K_1(x, z) K_{t-1}(z, y) dz dy \\ &\geq \inf_{\{|x| \leq 2, 2 < |z| < 3\}} \{K_1(x, z)\} \int_{\mathbb{R}^2} \int_{2 < |z| < 3} K_{t-1}(z, y) dz dy \\ &\geq \frac{c \delta_0}{\ln \sqrt{t} - 1}, \end{aligned}$$

where in the last inequality we have used (2.5) and (2.7). Thus, thanks to (2.6) and (2.7), the lemma is proved for $t > 2(1 + |x|)^2$. Assume now that $t > 2(1 + |x|)$. Since $t^2 > t$,

$$\int_{\mathbb{R}^2} K_t(x, y) dy \geq \int_{\mathbb{R}^2} K_{t^2}(x, y) dy \geq \delta \frac{\ln(2 + |x|)}{\ln(t^2)} = \frac{\delta \ln(2 + |x|)}{2 \ln t}.$$

□

The perturbation formula (see, e.g., Chapter 3 of [5]) asserts that

$$(2.8) \quad P_t = K_t + \int_0^t P_{t-s} V K_s ds.$$

Thus

$$(2.9) \quad \begin{aligned} \int_0^t \int_{\mathbb{R}^2} V(z) K_s(z, y) dz ds &= \int_{\mathbb{R}^2} (P_t(x - y) - K_t(x, y)) dx \\ &= 1 - \int_{\mathbb{R}^2} K_t(x, y) dx. \end{aligned}$$

Hence, using Lemma 2.1, we get

$$(2.10) \quad \int_0^\infty \int_{\mathbb{R}^2} V(z) K_s(z, y) dz ds = 1.$$

Lemma 2.3. *There exists a constant $C > 0$ such that for $x, y \in \mathbb{R}^2$ we have*

$$(2.11) \quad \int_2^\infty K_t(x, y) dt \leq C \min(\ln(2 + |x|), \ln(2 + |y|)).$$

Proof. Observe first that for $x, y \in \mathbb{R}^2$, and $t \geq 2$ one has

$$(2.12) \quad K_t(x, y) \leq C \frac{\ln(2 + |x|) \ln(2 + |y|)}{t (\ln t)^2}.$$

Indeed, if $u \in \mathbb{R}^2$, then from Lemma 2.1 we conclude

$$(2.13) \quad \begin{aligned} K_t(x, u) &= \int_{\mathbb{R}^2} K_{t/2}(x, z) K_{t/2}(z, u) dz \leq C t^{-1} \int_{\mathbb{R}^2} K_{t/2}(x, z) dz \\ &\leq C \frac{\ln(2 + |x|)}{t \ln t}. \end{aligned}$$

Applying again Lemma 2.1 combined with (2.13), we obtain

$$\begin{aligned} K_t(x, y) &= \int_{\mathbb{R}^2} K_{t/2}(x, z) K_{t/2}(z, y) dz \\ &\leq \frac{C \ln(2 + |x|)}{t \ln t} \int_{\mathbb{R}^2} K_{t/2}(z, y) dz \leq \frac{C \ln(2 + |x|) \ln(2 + |y|)}{t (\ln t)^2}. \end{aligned}$$

Clearly, (2.12) implies (2.11) for $|x|, |y| \leq 2$. Thus, to complete the proof of the lemma, we can assume that $|y| \geq |x|$ and $|y| > 2$. If additionally $|y| > 2|x|$, then by (2.12) and (1.2),

$$\begin{aligned} \int_2^\infty K_t(x, y) dt &\leq \int_2^{|y|^2} K_t(x, y) dt + \int_{|y|^2}^\infty K_t(x, y) dt \\ &\leq C \int_2^{|y|^2} t^{-1} e^{-|y|^2/16t} dt + C \int_{|y|^2}^\infty \frac{\ln(2 + |x|) \ln(2 + |y|)}{t (\ln t)^2} dt \leq C \ln(2 + |x|). \end{aligned}$$

If $2|x| \geq |y| \geq |x|$, then from (2.12) and (1.2) we get

$$\int_2^\infty K_t(x, y) dt \leq C \int_2^{|y|^2} t^{-1} dt + C \int_{|y|^2}^\infty \frac{\ln(2 + |x|) \ln(2 + |y|)}{t (\ln t)^2} dt \leq C \ln(2 + |x|).$$

□

For $\varepsilon > 0$ we set

$$\Delta_\varepsilon := \Delta - \varepsilon I, \quad L_\varepsilon := -\Delta + \varepsilon I + V = -\Delta_\varepsilon + V.$$

We have

$$\Delta_\varepsilon^{-1} = - \int_0^\infty e^{-\varepsilon t} P_t dt, \quad L_\varepsilon^{-1} = \int_0^\infty e^{-\varepsilon t} K_t dt.$$

Since P_t and K_t are contractions on every $L^p(\mathbb{R}^2)$, the operators Δ_ε^{-1} and L_ε^{-1} are bounded on every $L^p(\mathbb{R}^2)$, $1 \leq p < \infty$. Moreover,

$$(I - V \Delta_\varepsilon^{-1})(I - V L_\varepsilon^{-1}) = (I - V L_\varepsilon^{-1})(I - V \Delta_\varepsilon^{-1}) = I.$$

These equalities can be proved by direct computation if we substitute $V = \Delta_\varepsilon + L_\varepsilon$.

For $f \in L^1(\mathbb{R}^2)$ let

$$L^{-1}f(x) = \int_0^\infty \int_{\mathbb{R}^2} K_t(x, y)f(y) dy dt.$$

Formula (2.10) implies that the operator VL^{-1} is bounded on $L^1(\mathbb{R}^2)$ (see also Lemma 2.3). Clearly,

$$(2.14) \quad I = (I - V\Delta_\varepsilon^{-1})(I - VL^{-1}) + (I - V\Delta_\varepsilon^{-1})(VL^{-1} - VL_\varepsilon^{-1}) \quad \text{on } L^1(\mathbb{R}^2).$$

Finally let $\Delta^{-1}f(x) = f * E(x)$, with $E(x) = (2\pi)^{-1} \ln|x|$. Since E belongs to $BMO(\mathbb{R}^2)$, there is a constant $C > 0$ such that for any classical $H^1(\mathbb{R}^2)$ -atom a one has $|\Delta^{-1}a(x)| \leq C$ and

$$(2.15) \quad \Delta^{-1}a(x) = - \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^{1/\varepsilon} P_t a(x) dt$$

(see also (5.2) and (5.3)).

As a consequence of (2.10) we have the following lemma:

Lemma 2.4. *For every $f \in L^1(\mathbb{R}^2)$*

$$\int (I - VL^{-1})f(x) dx = 0.$$

For $\varepsilon > 0$ let

$$W_\varepsilon(x, y) := V(x) \int_0^\infty K_t(x, y)(1 - e^{-\varepsilon t}) dt.$$

It is clear that $W_\varepsilon(x, y)$ is the integral kernel of the operator $VL^{-1} - VL_\varepsilon^{-1}$.

Lemma 2.5. *As $\varepsilon \rightarrow 0$, $W_\varepsilon(x, y) \rightarrow 0$ uniformly on sets of the form $\mathbb{R}^2 \times B(0, R)$.*

Proof. Recall that V is a C^2 function supported in $B(0, 1)$. Hence, $W_\varepsilon(x, y) = 0$ for $|x| \geq 1$. Fix $R > 2$. If $|y| \leq R$ and $|x| < 1$, then by (1.2) and (2.12) we obtain

$$(2.16) \quad \begin{aligned} W_\varepsilon(x, y) &\leq V(x) \int_0^R \|K_t\|_\infty \varepsilon t dt + V(x) \int_R^\infty K_t(x, y)(1 - e^{-\varepsilon t}) dt \\ &\leq C \|V\|_\infty R\varepsilon + C \|V\|_\infty \int_R^\infty \frac{\ln(2 + R)}{t(\ln t)^2} (1 - e^{-\varepsilon t}) dt. \end{aligned}$$

The lemma follows by applying the Lebesgue dominated convergence theorem. \square

The following corollary can be concluded from Lemmas 2.3 and 2.5:

Corollary 2.6. $\|VL^{-1} - VL_\varepsilon^{-1}\|_{L^1 \rightarrow L^1} \leq 1$. *Moreover, for every $f \in L^1(\mathbb{R}^2)$,*

$$\lim_{\varepsilon \rightarrow 0} \|(VL^{-1} - VL_\varepsilon^{-1})f\|_{L^1(\mathbb{R}^2)} = 0.$$

3. Auxiliary weight function

Let

$$\theta(\varepsilon) = \frac{1}{4\pi} \int_1^\infty \exp(-\varepsilon u) \frac{du}{u} = \frac{1}{4\pi} \int_\varepsilon^\infty \exp(-u) \frac{du}{u}.$$

Note that

$$\theta(\varepsilon) - \frac{1}{4\pi} \ln \frac{1}{\varepsilon} = c_1 + O(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0^+.$$

For $0 < \varepsilon \leq 1$ we define the weight function

$$w_\varepsilon(y) = \theta(\varepsilon) \int_{\mathbb{R}^2} \int_0^\infty V(z) K_s(z, y) (1 - e^{-\varepsilon s}) ds dz.$$

Proposition 3.1. *There are constants $C, c > 0$ such that*

$$\begin{aligned} w_\varepsilon(y) &\leq C \ln(2 + 2|y|), \\ c \ln(2 + 2|y|) &\leq w_\varepsilon(y) \quad \text{for } |y| < \varepsilon^{-1}. \end{aligned}$$

Proof. We split the integral which defines w_ε into two parts:

$$\begin{aligned} (3.1) \quad w_\varepsilon(y) &= \theta(\varepsilon) \int_0^{\varepsilon^{-1}} \int_{\mathbb{R}^2} V(z) K_s(z, y) (1 - e^{-\varepsilon s}) dz ds \\ &\quad + \theta(\varepsilon) \int_{\varepsilon^{-1}}^\infty \int_{\mathbb{R}^2} V(z) K_s(z, y) (1 - e^{-\varepsilon s}) dz ds \\ &:= J_1^\varepsilon(y) + J_2^\varepsilon(y). \end{aligned}$$

Using (2.10) and Lemma 2.1, we get

$$\begin{aligned} (3.2) \quad J_2^\varepsilon(y) &\sim \theta(\varepsilon) \int_{\varepsilon^{-1}}^\infty \int_{\mathbb{R}^2} V(z) K_s(z, y) dz ds \\ &= \theta(\varepsilon) \left(\int_0^\infty \int_{\mathbb{R}^2} V(z) K_s(z, y) dz ds - \int_0^{\varepsilon^{-1}} \int_{\mathbb{R}^2} V(z) K_s(z, y) dz ds \right) \\ &= \theta(\varepsilon) \left(1 - \left(1 - \int_{\mathbb{R}^2} K_{\varepsilon^{-1}}(z, y) dz \right) \right) \leq C \theta(\varepsilon) \frac{\ln(2 + 2|y|)}{\ln 1/\varepsilon} \\ &\leq C \ln(2 + 2|y|). \end{aligned}$$

Observe that if $|y| < \varepsilon^{-1}$, then from Lemma 2.2 we obtain

$$(3.3) \quad J_2^\varepsilon(y) \geq c \ln(2 + |y|).$$

We now turn to estimating $J_1^\varepsilon(y)$. By the perturbation formula (see (2.9))

and Lemma 2.1 we have

$$\begin{aligned}
 J_1^\varepsilon(y) &= \theta(\varepsilon) \left(\int_0^{\varepsilon^{-1}} \int_{\mathbb{R}^2} V(z) K_s(z, y) \, dz \, ds \right. \\
 &\quad \left. - \int_0^{\varepsilon^{-1}} \int_{\mathbb{R}^2} (\varepsilon + V(z)) K_s(z, y) e^{-\varepsilon s} \, dz \, ds + \int_0^{\varepsilon^{-1}} \int_{\mathbb{R}^2} \varepsilon K_s(z, y) e^{-\varepsilon s} \, dz \, ds \right) \\
 &= \theta(\varepsilon) \left(- \int_{\mathbb{R}^2} K_{\varepsilon^{-1}}(z, y) \, dz \right. \\
 &\quad \left. + \int_{\mathbb{R}^2} K_{\varepsilon^{-1}}(z, y) e^{-1} \, dz + \int_0^{\varepsilon^{-1}} \int_{\mathbb{R}^2} \varepsilon K_s(z, y) e^{-\varepsilon s} \, dz \, ds \right) \\
 (3.4) \quad &= \theta(\varepsilon) (e^{-1} - 1) \int_{\mathbb{R}^2} K_{\varepsilon^{-1}}(z, y) \, dz + \theta(\varepsilon) \varepsilon \int_0^{\varepsilon^{-1}} \int_{\mathbb{R}^2} K_s(z, y) e^{-\varepsilon s} \, dz \, ds \\
 &\leq C \theta(\varepsilon) \frac{\ln(2 + 2|y|)}{\ln(1/\varepsilon)} + \theta(\varepsilon) \varepsilon \int_0^{\varepsilon^{-1/2}} \int_{\mathbb{R}^2} K_s(z, y) \, dz \, ds \\
 &\quad + \theta(\varepsilon) \varepsilon \int_{\varepsilon^{-1/2}}^{\varepsilon^{-1}} e^{-\varepsilon s} \ln(2 + 2|y|) (\ln \sqrt{1/\varepsilon})^{-1} \, ds \\
 &\leq C \ln(2 + 2|y|) + C \theta(\varepsilon) \varepsilon^{1/2} + C \theta(\varepsilon) \varepsilon \ln(2 + 2|y|) (\ln 1/\varepsilon)^{-1} \int_{\varepsilon^{-1/2}}^{\varepsilon^{-1}} \, ds \\
 &\leq C \ln(2 + 2|y|).
 \end{aligned}$$

Thus from (3.1)–(3.4) we get the proposition. □

From Proposition 3.1 we conclude that there is a subsequence $\varepsilon_j \rightarrow 0^+$ such that $w_{\varepsilon_j}(x)$ converges to a function $w(x)$ in the weak* sense and (1.9) holds.

Since for every $y \in \mathbb{R}^2$ and $t > 0$ the function $x \mapsto K_t(x, y)$ is continuous and has Gaussian decay, we have

$$(3.5) \quad \lim_{j \rightarrow \infty} \int_{\mathbb{R}^2} w_{\varepsilon_j}(x) K_t(x, y) \, dx = \int_{\mathbb{R}^2} w(x) K_t(x, y) \, dx.$$

Proposition 3.2. *The function w is L -harmonic, that is,*

$$K_t w(y) = \int_{\mathbb{R}^2} w(x) K_t(x, y) \, dx = w(y) \quad \text{for every } t > 0.$$

In order to prove the proposition we need the following lemma:

Lemma 3.3. *For every $t > 0$, there is a constant $C > 0$ such that, for every $0 < \varepsilon < 1$, one has*

$$|K_t w_\varepsilon(x) - w_\varepsilon(x)| \leq C \theta(\varepsilon) \varepsilon.$$

Proof of Lemma 3.3. We have:

$$\begin{aligned}
 K_t w_\varepsilon(x) &= \theta(\varepsilon) \int_{\mathbb{R}^2} \int_0^\infty \int_{\mathbb{R}^2} K_t(x, y) V(z) K_s(z, y) (1 - e^{-\varepsilon s}) dz ds dy \\
 &= \theta(\varepsilon) \int_0^\infty \int_{\mathbb{R}^2} V(z) K_{s+t}(x, z) (1 - e^{-\varepsilon s}) dz ds \\
 &= \theta(\varepsilon) \int_t^\infty \int_{\mathbb{R}^2} V(z) K_s(x, z) (1 - e^{\varepsilon t} e^{-\varepsilon s}) dz ds \\
 &= \theta(\varepsilon) \int_t^\infty \int_{\mathbb{R}^2} V(z) K_s(x, z) (1 - e^{-\varepsilon s}) dz ds \\
 &\quad + \theta(\varepsilon) \int_t^\infty \int_{\mathbb{R}^2} V(z) K_s(x, z) (e^{-\varepsilon s} - e^{\varepsilon t} e^{-\varepsilon s}) dz ds \\
 &= \theta(\varepsilon) \int_0^\infty \int_{\mathbb{R}^2} V(z) K_s(x, z) (1 - e^{-\varepsilon s}) dz ds \\
 &\quad - \theta(\varepsilon) \int_0^t \int_{\mathbb{R}^2} V(z) K_s(x, z) (1 - e^{-\varepsilon s}) dz ds \\
 &\quad + \theta(\varepsilon) \int_t^\infty \int_{\mathbb{R}^2} V(z) K_s(x, z) e^{-\varepsilon s} (1 - e^{\varepsilon t}) dz ds \\
 &= w_\varepsilon(x) - \theta(\varepsilon) \int_0^t \int_{\mathbb{R}^2} V(z) K_s(x, z) (1 - e^{-\varepsilon s}) dz ds \\
 &\quad + \theta(\varepsilon) \int_t^\infty \int_{\mathbb{R}^2} V(z) K_s(x, z) e^{-\varepsilon s} (1 - e^{\varepsilon t}) dz ds,
 \end{aligned}$$

which, by using (2.10) and the mean value theorem, implies the lemma. □

Proof of Proposition 3.2. Write $v(x) = K_t w(x)$. Since $w(y)$ is the weak* limit of $w_{\varepsilon_j}(y)$, we get

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^2} w_{\varepsilon_j}(y) K_t(x, y) dy = \int_{\mathbb{R}^2} w(y) K_t(x, y) dy = v(x).$$

On the other hand, by Lemma 3.3, we have

$$v(x) = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^2} w_{\varepsilon_j}(y) K_t(x, y) dy = \lim_{j \rightarrow \infty} w_{\varepsilon_j}(x) + O(\theta(\varepsilon_j)\varepsilon_j).$$

Thus, $v(x) = w(x)$ for every $x \in \mathbb{R}^2$. □

Clearly, $\Delta w(x) = V(x)w(x)$, because w is L -harmonic. Since w has logarithmic growth (see (1.9)), there is $C_1 > 0$ such that

$$(3.6) \quad w(x) = (2\pi)^{-1} \int_{\mathbb{R}^2} V(y)w(y) \ln|x - y| dy + C_1.$$

It follows from (3.6) that

$$(3.7) \quad |\nabla w(x)| \leq C(1 + |x|)^{-1}$$

and

$$(3.8) \quad |w(x) - w(x')| \leq C \quad \text{provided} \quad |x|/2 \leq |x'| \leq 2|x|.$$

Another direct consequence of (3.6) is the following lemma:

Lemma 3.4. *Let a be a classical $H^1(\mathbb{R}^2)$ -atom. Set $g = (I - V\Delta^{-1})a$. Then*

$$\int g(x)w(x) dx = 0.$$

4. Atoms

We start this section by proving the following lemma:

Lemma 4.1. *Let ε_j be a sequence as in the definition of the function $w(x)$. Let b be an atom for $H^1_{w,at}$. Then*

$$(4.1) \quad \lim_{j \rightarrow \infty} (V\Delta_{\varepsilon_j}^{-1}(VL^{-1} - VL_{\varepsilon_j}^{-1}))b(x) = 0$$

uniformly in x .

Proof. Since $\text{supp } V \subset B(0, 1)$, it is enough to consider $|x| \leq 1$. Let $U_\varepsilon(x, y)$ denote the integral kernel of the operator $-V\Delta_\varepsilon^{-1}(VL^{-1} - VL_\varepsilon^{-1})$. Fix an atom b for $H^1_{w,at}$ (see (1.6)–(1.7)). Let r be such that $\text{supp } b \subset B(0, r)$. Then for $|x| \leq 1$ and $|y| \leq r$ we have

$$(4.2) \quad \begin{aligned} U_\varepsilon(x, y) &= \int_0^\infty \int_{\mathbb{R}^2} \int_0^\infty V(x)P_u(x - z)e^{-\varepsilon u}V(z)K_t(z, y)(1 - e^{-\varepsilon t}) dt dz du \\ &= \int_0^1 \int_{\mathbb{R}^2} \int_0^3 \dots dt dz du + \int_0^1 \int_{\mathbb{R}^2} \int_3^\infty \dots dt dz du \\ &\quad + \int_1^\infty \int_{\mathbb{R}^2} \int_0^\infty \dots dt dz du \\ &:= I_1(x, y) + I_2(x, y) + I_3(x, y). \end{aligned}$$

By the mean value theorem and (1.2) we obtain that

$$(4.3) \quad \begin{aligned} I_1(x, y) &\leq CV(x) \int_0^1 \int_{\mathbb{R}^2} \int_0^3 P_u(x - z)V(z)\|K_t\|_\infty \varepsilon t dt dz du \\ &\leq C\varepsilon V(x)\|V\|_\infty. \end{aligned}$$

To estimate I_2 we apply (2.12) and obtain

$$(4.4) \quad \begin{aligned} I_2(x, y) &\leq CV(x) \int_0^1 \int_{\mathbb{R}^2} \int_3^\infty P_u(x - z)V(z) \frac{\ln(2+r)}{t(\ln t)^2} (1 - e^{-\varepsilon t}) dt dz du \\ &\leq CV(x)\|V\|_\infty \int_1^\infty \frac{\ln(2+r)}{t(\ln t)^2} (1 - e^{-\varepsilon t}) dt \\ &\leq CV(x) \ln(2+r) \|V\|_\infty (\ln \varepsilon^{-1})^{-1}. \end{aligned}$$

In order to deal with I_3 we split the integral as follows:

$$\begin{aligned}
 & I_3(x, y) \\
 (4.5) \quad &= \int_1^\infty \int_{\mathbb{R}^2} \int_0^\infty V(x)(P_u(x - z) - P_u(0)) e^{-\varepsilon u} V(z) K_t(z, y) (1 - e^{-\varepsilon t}) dt dz du \\
 &+ \int_1^\infty \int_{\mathbb{R}^2} \int_0^\infty V(x) P_u(0) e^{-\varepsilon u} V(z) K_t(z, y) (1 - e^{-\varepsilon t}) dt dz du \\
 &= I'_3(x, y) + V(x) w_\varepsilon(y).
 \end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
 (4.6) \quad |I'_3(x, y)| &\leq C \int_1^\infty \int_{\mathbb{R}^2} \int_0^\infty V(x) u^{-3/2} V(z) K_t(z, y) (1 - e^{-\varepsilon t}) dt dz du \\
 &\leq C V(x) \int_{\mathbb{R}^2} \int_0^{1/\sqrt{\varepsilon}} V(z) K_t(z, y) \varepsilon t dt dz \\
 &\quad + C V(x) \int_{\mathbb{R}^2} \int_{1/\sqrt{\varepsilon}}^\infty V(z) K_t(z, y) dt dz \\
 &\leq C V(x) \sqrt{\varepsilon} \|V\|_{L^1} + C V(x) \|V\|_{L^1} (\ln \varepsilon^{-1})^{-1},
 \end{aligned}$$

where in the last inequality we have used (2.12). Hence we obtain the lemma from (4.2)–(4.6) and (1.7), because

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^2} V(x) w_{\varepsilon_j}(y) b(y) dy = V(x) \int_{\mathbb{R}^2} w(y) b(y) dy = 0.$$

□

The goal of this section is to prove the following proposition, which will be used in the proof of the second inequality in (1.10).

Proposition 4.2. *There is a constant $C > 0$ such that if b is an atom for the Hardy space $H^1_{w, \text{at}}$ associated with a ball $B(x, r)$, then there exist a finite sequence a_k of classical $H^1(\mathbb{R}^2)$ -atoms and a sequence $\lambda_k \in \mathbb{C}$ such that*

$$(4.7) \quad \text{each atom } a_k \text{ is associated with a ball } B(x_k, r_k) \text{ with } r_m \leq \max(1, r),$$

$$(4.8) \quad \sum_k |\lambda_k| \leq C, \quad \text{and} \quad b = \sum_k \lambda_k (I - V \Delta^{-1}) a_k.$$

The proof of the proposition consists of Lemmas 4.3, 4.7, 4.8, 4.9, and Corollary 4.6 below.

For $n, m \in \mathbb{N} \cup \{0\}$, $n < m$, let $R_{n,m} = \{x \in \mathbb{R}^2 : 2^{n/2} \leq |x| \leq 2^{m/2}\}$. The following decomposition will be frequently used here:

$$(4.9) \quad |R_{n,n+1}|^{-1} \chi_{R_{n,n+1}} = |R_{0,1}|^{-1} \chi_{R_{0,1}} + \sum_{k=1}^n f_k,$$

where

$$f_k = |R_{k,k+1}|^{-1}\chi_{R_{k,k+1}} - |R_{k-1,k}|^{-1}\chi_{R_{k-1,k}}.$$

Observe that for $k = 1, 2, \dots, n$ we have

$$(4.10) \quad \begin{aligned} \text{supp } f_k &\subset R_{k-1,k+1}, \quad \int_{\mathbb{R}^2} f_k(y) dy = 0, \\ \|f_k\|_\infty &\leq \frac{C}{|R_{k-1,k+1}|}, \quad \|f_k\|_{H^1(\mathbb{R}^2)} \leq C, \end{aligned}$$

with a constant C independent of n and k . Consequently,

$$(4.11) \quad |R_{n,n+1}|^{-1}\chi_{R_{n,n+1}} = |R_{0,1}|^{-1}\chi_{R_{0,1}} + \sum_{k=1}^n V\Delta^{-1}f_k + \sum_{k=1}^n (I - V\Delta^{-1})f_k,$$

$$(4.12) \quad \left\| |R_{0,1}|^{-1}\chi_{R_{0,1}} + \sum_{k=1}^n V\Delta^{-1}f_k \right\|_\infty \leq C(n+1).$$

Lemma 4.3. *Assume that b is an $H^1_{w,\text{at}}$ -atom such that $\text{supp } b \subset B(0, 3)$ and $\|b\|_\infty \leq |B(0, 3)|^{-1}$. Then there is an $H^1(\mathbb{R}^2)$ -atom a such that $\text{supp } a \subset B(0, 3)$, $\|a\|_\infty \leq C$, and $b = (I - V\Delta^{-1})a$.*

Proof. Set $a = (I - VL^{-1})b$. Clearly, $\text{supp } a \subset B(0, 3)$, $\|a\|_\infty \leq C$ and, by Lemma 2.4, $\int a = 0$. Moreover, using (2.14) combined with (2.15), and Lemmas 2.5 and 4.1, we get $b = (I - V\Delta^{-1})a$. \square

Lemma 4.4. *Let b be a function such that $\text{supp } b \subset R_{n,n+1}$, $\|b\|_\infty \leq 2^{-n}$, and $\int bw = 0$. Then there exists a finite sequence $\{a_j\}_{j=0}^{n+1}$ of classical $H^1(\mathbb{R}^2)$ -atoms and a sequence $\{\lambda_j\}_{j=0}^{n+1}$ of complex numbers such that*

$$b = \sum_{j=0}^{n+1} \lambda_j (I - V\Delta^{-1})a_j, \quad \sum_{j=0}^{n+1} |\lambda_j| \leq C.$$

The constant C is independent of b .

Proof. Set

$$\kappa = \int_{\mathbb{R}^2} b(x) dx.$$

Fix $x_0 \in R_{n,n+1}$. By the assumptions on b and (3.8), we obtain

$$(4.13) \quad |\kappa| = w(x_0)^{-1} \left| \int_{\mathbb{R}^2} b(x)(w(x_0) - w(x)) dx \right| \leq Cw(x_0)^{-1} \leq Cn^{-1},$$

where in the last inequality we have used (1.9). Applying (4.9) (see also (4.11)),

we have

$$\begin{aligned}
 (4.14) \quad b &= \frac{\kappa}{|R_{n,n+1}|} \chi_{R_{n,n+1}} + \left(b - \frac{\kappa}{|R_{n,n+1}|} \chi_{R_{n,n+1}} \right) \\
 &= \kappa |R_{0,1}|^{-1} \chi_{R_{0,1}} + \kappa \sum_{k=1}^n V \Delta^{-1} f_k + V \Delta^{-1} \left(b - \frac{\kappa}{|R_{n,n+1}|} \chi_{R_{n,n+1}} \right) \\
 &\quad + (I - V \Delta^{-1}) \left(b - \frac{\kappa}{|R_{n,n+1}|} \chi_{R_{n,n+1}} \right) + \kappa \sum_{k=1}^n (I - V \Delta^{-1}) f_k.
 \end{aligned}$$

Clearly, $b - \kappa |R_{n,n+1}|^{-1} \chi_{R_{n,n+1}}$ and $f_k, k = 1, 2, \dots, n$, are multiples of $H^1(\mathbb{R}^2)$ -atoms with a universal constant factor independent of b (see (4.10) and (4.13)). Moreover, the function

$$g(x) = \kappa |R_{0,1}|^{-1} \chi_{R_{0,1}} + \kappa \sum_{k=1}^n V \Delta^{-1} f_k + V \Delta^{-1} \left(b - \frac{\kappa}{|R_{n,n+1}|} \chi_{R_{n,n+1}} \right)$$

is supported in $B(0, \sqrt{2})$, and, thanks to (4.12) and (4.13), we have that $\|g\|_\infty \leq C$. Since $\int bw = 0$, we conclude from (4.14) and Lemma 3.4 that $\int gw = 0$. Therefore, by Lemma 4.3, $g = (I - V \Delta^{-1})f_0$, where f_0 is a multiple of a classical $H^1(\mathbb{R}^2)$ -atom. Hence, Lemma 4.4 follows from (4.14), (4.12), and (4.13). \square

Lemma 4.5. *Assume that b is an $H^1_{w, \text{at}}$ -atom associated with $B(0, 2^{n/2})$, $n > 1$. Then there is a sequence of functions $b_k, k = 0, 1, \dots, n - 1$, such that*

$$\begin{aligned}
 \|b_k\|_\infty &\leq C2^{-n}, \quad \int b_k w = 0, \\
 \text{supp } b_0 &\subset B(0, \sqrt{2}), \quad \text{supp } b_k \subset R_{k-1, k+1} \quad \text{for } k = 1, 2, \dots, n - 1, \\
 b &= \sum_{k=0}^{n-1} b_k.
 \end{aligned}$$

Proof. Let $B_0 = B(0, 1)$ and $A = |B(0, 2^{n/2})|^{-1} = \pi^{-1}2^{-n}$. By the assumptions, $\|b\|_\infty \leq A$. Define

$$s_0 = 1 + \frac{w(B(0, 1))}{w(R_{0,1})}, \quad s_k = 1 + s_{k-1} \frac{w(R_{k-1, k})}{w(R_{k, k+1})} \quad \text{for } k = 1, 2, \dots$$

where $w(U) = \int_U w(x) dx$ for a measurable set $U \subset \mathbb{R}^2$. Formula (3.6) implies that

$$0 < w(R_{k-1, k})/w(R_{k, k+1}) \leq r < 1 \quad \text{for } k \geq 2.$$

Hence the sequence s_k is bounded. Set

$$b = (b \chi_{B_0} - c_0 \chi_{R_{0,1}}) + (c_0 \chi_{R_{0,1}} + b \chi_{R_{0,n}}) := b_0 + \tilde{b}_0,$$

where

$$c_0 = \frac{1}{w(R_{0,1})} \int_{B_0} bw.$$

Obviously, $|c_0| \leq Aw(B_0)w(R_{0,1})^{-1}$. We easily see that

$$\int b_0 w = \int \tilde{b}_0 w = 0,$$

$$\text{supp } b_0 = B(0, \sqrt{2}), \quad \text{supp } \tilde{b}_0 \subset R_{0,n},$$

$$|b_0(x)| \leq A s_0,$$

$$|\tilde{b}_0(x)| \leq \begin{cases} A s_0 & \text{for } x \in R_{0,1}, \\ A & \text{for } x \in R_{1,n}. \end{cases}$$

Now we decompose \tilde{b}_0 :

$$\tilde{b}_0 = (\tilde{b}_0 \chi_{R_{0,1}} - c_1 \chi_{R_{1,2}}) + (c_1 \chi_{R_{1,2}} + b \chi_{R_{1,n}}) := b_1 + \tilde{b}_1,$$

where $c_1 = w(R_{1,2})^{-1} \int_{R_{0,1}} \tilde{b}_0 w$. Clearly, $\int b_1 w = \int \tilde{b}_1 w = 0$. Moreover,

$$|c_1| \leq A s_0 \frac{w(R_{0,1})}{w(R_{1,2})},$$

$\text{supp } b_1 \subset R_{0,2}$, $\text{supp } \tilde{b}_1 \subset R_{1,n}$, and

$$|b_1(x)| \leq \begin{cases} A s_0 & \text{for } x \in R_{0,1}, \\ A s_0 \frac{w(R_{0,1})}{w(R_{1,2})} & \text{for } x \in R_{1,2}, \end{cases}$$

$$|\tilde{b}_1(x)| \leq \begin{cases} A s_1 & \text{for } x \in R_{1,2}, \\ A & \text{for } x \in R_{2,n}. \end{cases}$$

Then we decompose \tilde{b}_1 :

$$\tilde{b}_1 = (\tilde{b}_1 \chi_{R_{1,2}} - c_2 \chi_{R_{2,3}}) + (c_2 \chi_{R_{2,3}} + b \chi_{R_{2,n}}) := b_2 + \tilde{b}_2,$$

where $c_2 = w(R_{2,3})^{-1} \int_{R_{1,2}} \tilde{b}_1 w$. We have $\int b_2 w = \int \tilde{b}_2 w = 0$, and

$$|c_2| \leq A s_1 \frac{w(R_{1,2})}{w(R_{2,3})}.$$

In addition, $\text{supp } b_2 \subset R_{1,3}$, $\text{supp } \tilde{b}_2 \subset R_{2,n}$, and

$$|b_2(x)| \leq \begin{cases} A s_1 & \text{for } x \in R_{1,2}, \\ A s_1 \frac{w(R_{1,2})}{w(R_{2,3})} & \text{for } x \in R_{2,3}, \end{cases}$$

$$|\tilde{b}_2(x)| \leq \begin{cases} A s_2 & \text{for } x \in R_{2,3}, \\ A & \text{for } x \in R_{3,n}. \end{cases}$$

We continue the procedure and get functions b_k and \tilde{b}_k , $k = 1, 2, \dots, n - 2$, such that $\text{supp } b_k \subset R_{k-1, k+1}$, $\text{supp } \tilde{b}_k \subset R_{k, n}$, $\int b_k w = \int \tilde{b}_k w = 0$, $b = \tilde{b}_{n-2} + \sum_{k=0}^{n-2} b_k$ and

$$|b_k(x)| \leq \begin{cases} A s_{k-1} & \text{for } x \in R_{k-1, k}, \\ A s_{k-1} \frac{w(R_{k-1, k})}{w(R_{k, k+1})} & \text{for } x \in R_{k, k+1}, \end{cases}$$

$$|\tilde{b}_k(x)| \leq \begin{cases} A s_k & \text{for } x \in R_{k, k+1}, \\ A & \text{for } x \in R_{k+1, n}. \end{cases}$$

Set $b_{n-1} = \tilde{b}_{n-2}$. From the above we see that $|b_k| \leq C 2^{-n}$, with C independent of b . □

As a direct consequence of Lemmas 4.4 and 4.5, we obtain the following corollary:

Corollary 4.6. *The conclusion of Proposition 4.2 holds for every $H^1_{w, \text{at}}$ -atom b associated with a ball $B(0, 2^{n/2})$, $n > 1$.*

Lemma 4.7. *The conclusion of Proposition 4.2 holds for every $H^1_{w, \text{at}}$ -atom b associated with a ball $B(x_0, r)$, $|x_0| < 2$, $r < 1$.*

Proof. Set $c_0 = |B(x_0, r)|^{-1} \int b(x) dx$. By (1.6)–(1.7) and the mean value theorem,

$$(4.15) \quad |c_0| = |B(x_0, r)|^{-1} w(x_0)^{-1} \left| \int_{\mathbb{R}^2} b(x)(w(x_0) - w(x)) dx \right| \leq Cr^{-1}.$$

Define $B_j = B(x_0, 2^j r)$, $c_j = c_0 \frac{|B_0|}{|B_j|}$, $j = 0, 1, \dots, [\log_2 r^{-1}] = k$. Then,

$$b = (b - c_0 \chi_{B_0}) + c_0 |B_0| \sum_{j=0}^k (|B_j|^{-1} \chi_{B_j} - |B_{j+1}|^{-1} \chi_{B_{j+1}}) + c_0 |B_0| |B_{k+1}|^{-1} \chi_{B_{k+1}}.$$

Set $a_0 = (b - c_0 \chi_{B_0})$ and $a_{j+1} = |B_j|^{-1} \chi_{B_j} - |B_{j+1}|^{-1} \chi_{B_{j+1}}$, for $j = 0, 1, \dots, k$. Clearly a_j , $j = 0, 1, \dots, k$, are multiples of classical $H^1(\mathbb{R}^2)$ -atoms with a constant factor independent of b . So,

$$(4.16) \quad b = (I - V\Delta^{-1}) a_0 + (I - V\Delta^{-1}) \sum_{j=1}^k c_0 |B_0| a_j + V\Delta^{-1} a_0 + V\Delta^{-1} \sum_{j=1}^k c_0 |B_0| a_j + c_0 \frac{|B_0|}{|B_{k+1}|} \chi_{B_{k+1}}.$$

Note that $\sum_{j=1}^k c_0 |B_0| \leq C$. Further, from Lemma 3.4 we deduce that

$$V\Delta^{-1} \left(a_0 + \sum_{j=1}^k c_0 |B_0| a_j \right) + c_0 \frac{|B_0|}{|B_{k+1}|} \chi_{B_{k+1}}$$

is a multiple of an $H^1_{w, \text{at}}$ -atom associated with the ball $B(0, 3)$. Hence, the lemma follows by applying (4.16) and Lemma 4.3. □

Lemma 4.8. *Assume that b is an $H^1_{w,at}$ -atom associated with $B(x_0, r)$, $|x_0| > 1$, $r \leq 1$. Then the conclusion of Proposition 4.2 about the decomposition holds.*

Proof. Let $n = \lceil \log_2 r^{-1} \rceil$, $s = \lceil |x_0| \rceil$, $B_k = B(x_0, 2^k r)$, for $k = 0, 1, \dots, n$, and $B_{n+j} = B(\frac{s-j}{s}x_0, 2^n r)$, for $j = 1, \dots, s$. Put $c_0 = |B_0|^{-1} \int b(x) dx$,

$$\begin{aligned} a_0 &= b - c_0 \chi_{B_0}, \quad \lambda_0 = 1; \\ a_k &= |B_{k-1}|^{-1} \chi_{B_{k-1}} - |B_k|^{-1} \chi_{B_k}, \quad \lambda_k = c_0 |B_0|, \quad k = 1, 2, \dots, n; \\ a_{n+j} &= |B_n|^{-1} (\chi_{B_{n+j-1}} - \chi_{B_{n+j}}), \quad \lambda_{n+j} = c_0 |B_0|, \quad j = 1, 2, \dots, s. \end{aligned}$$

Then, from (1.6), (1.7), (3.7), and the mean value theorem we obtain that

$$|c_0| w(x_0) |B_0| |x_0| \leq Cr.$$

Moreover, the a_m are multiples of classical $H^1(\mathbb{R}^2)$ -atoms (with a constant factor independent of b) and

$$b = \sum_{m=0}^{n+s} \lambda_m a_m + c_0 \frac{|B_0|}{|B_n|} \chi_{B_{n+s}}.$$

Clearly, $\sum_{m=0}^{n+s} |\lambda_m| \leq C$, and

$$(4.17) \quad b = \sum_{m=0}^{n+s} \lambda_m (I - V \Delta^{-1}) a_m + \sum_{m=0}^{n+s} \lambda_m V \Delta^{-1} a_m + c_0 \frac{|B_0|}{|B_{n+s}|} \chi_{B_{n+s}}.$$

The function $g = \sum_{m=0}^{n+s} \lambda_m V \Delta^{-1} a_m + c_0 |B_0| |B_{n+s+1}|^{-1} \chi_{B_{n+s+1}}$ is supported in $B(0, 2)$, $\|g\|_\infty \leq C$, and, by Lemma 3.4, $\int g(x) w(x) dx = 0$. The proof is completed by the use of (4.17) and Lemma 4.3. \square

Lemma 4.9. *The conclusion of Proposition 4.2 holds for every $H^1_{w,at}$ -atom b associated with a ball $B(x_0, r)$, $|x_0| > 2r > 2$.*

Proof. Let $s = \lceil |x_0|/r \rceil$, $n = \lceil \log_2 r \rceil + 1$, $B_k = B(\frac{s-k}{s}x_0, r)$, $k = 0, 1, \dots, s$, and $B_{s+j} = B(0, 2^{-j}r)$, $j = 1, 2, \dots, n$. Set $c_0 = |B_0|^{-1} \int b(x) dx$. We have

$$|c_0| w(x_0) |B_0| |x_0| \leq r.$$

Define

$$\begin{aligned} a_0 &= b - c_0 \chi_{B_0}, \quad \lambda_0 = 1; \\ a_k &= |B_k|^{-1} (\chi_{B_{k-1}} - \chi_{B_k}), \quad \lambda_k = c_0 |B_0|, \quad k = 1, \dots, s; \\ a_{s+j} &= |B_{s+j-1}|^{-1} \chi_{B_{s+j-1}} - |B_{s+j}|^{-1} \chi_{B_{s+j}}, \quad \lambda_{s+j} = c_0 |B_s|, \quad j = 1, 2, \dots, n. \end{aligned}$$

The functions a_k , $k = 0, 1, \dots, s+n$, are multiples of the classical $H^1(\mathbb{R}^2)$ -atoms. By the same arguments as in the proof of the previous lemma,

$$(4.18) \quad b = \sum_{m=0}^{s+n} (I - V \Delta^{-1}) \lambda_m a_m + \sum_{m=0}^{s+n} V \Delta^{-1} \lambda_m a_m + c_0 \frac{|B_s|}{|B_{s+n}|} \chi_{B_{s+n}}$$

with $\sum_{m=0}^{s+n} |\lambda_m| \leq C$, the function

$$g = \sum_{m=0}^{s+n} V\Delta^{-1}\lambda_m a_m + c_0|B_s||B_{s+n}|^{-1}\chi_{B_{s+n}}$$

being a multiple of an $H^1_{w,at}$ -atom supported in $B(0, 2)$. From equation (4.18) and Lemma 4.3, we get the decomposition. □

5. Boundedness of maximal functions

The main goal of this section is to prove the second inequality in (1.10) of Theorem 1.1, that is, there exists a constant $C > 0$ such that

$$(5.1) \quad \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^2)} \leq C\|f\|_{H^1_{w,at}}.$$

Let us first note that if a is a classical $H^1(\mathbb{R}^2)$ -atom associated with a ball of radius $r > 0$, then

$$(5.2) \quad |P_s a(x)| \leq \begin{cases} Cs^{-3/2}r & \text{for } s > r^2, \\ Cr^{-2} & \text{for } s \leq r^2. \end{cases}$$

Hence, there is $C > 0$ such that

$$(5.3) \quad \int_0^\infty |P_s a(x)| ds \leq C$$

for every classical $H^1(\mathbb{R}^2)$ -atom a .

The perturbation formula asserts that

$$(5.4) \quad K_t = P_t - \int_0^t (K_{t-s} - K_t)VP_s ds - K_t \int_0^t VP_s ds.$$

Let $f \in H^1(\mathbb{R}^2)$. Then

$$(5.5) \quad \begin{aligned} K_t(I - V\Delta^{-1})f &= P_t f + K_t \int_t^\infty VP_s f ds + \int_0^t (K_t - K_{t-s})VP_s f ds \\ &= P_t f + Q_t f + R_t f + S_t f, \end{aligned}$$

where

$$\begin{aligned} Q_t &= K_t \int_t^\infty VP_s ds, \\ R_t &= \int_0^{t/2} (K_t - K_{t-s})VP_s ds, \\ S_t &= \int_{t/2}^t (K_t - K_{t-s})VP_s ds. \end{aligned}$$

Lemma 5.1. *There exists a constant $C > 0$ such that, for every $H^1(\mathbb{R}^2)$ -atom a , one has*

$$(5.6) \quad \left\| \sup_{0 < t \leq 4} |Q_t a| \right\|_{L^1} \leq C, \quad \left\| \sup_{0 < t \leq 4} |R_t a| \right\|_{L^1} \leq C, \quad \left\| \sup_{0 < t \leq 4} |S_t a| \right\|_{L^1} \leq C.$$

Proof. Let a be a classical $H^1(\mathbb{R}^2)$ -atom associated with a ball $B(x_0, r)$. Denote

$$\mathbf{M}_0 f(x) = \sup_{0 < t \leq 4} P_t |f|(x).$$

Clearly, $\sup_{0 < t \leq 4} K_t |f|(x) \leq \mathbf{M}_0 f(x)$. Using (5.3), we get

$$(5.7) \quad \sup_{0 < t \leq 4} |Q_t a(x)| \leq C \mathbf{M}_0 \chi_{B(0,1)}(x).$$

Similarly, for $0 < t \leq 4$, we have

$$(5.8) \quad \begin{aligned} |R_t a(x)| &\leq K_t \left(\int_0^{t/2} V |P_s a| ds \right)(x) + \int_0^{t/2} K_{t-s} V |P_s a|(x) ds \\ &\leq C \mathbf{M}_0 \chi_{B(0,1)}(x) + C \int_0^{t/2} K_{t/2} V |P_s a|(x) ds \\ &\leq C' \mathbf{M}_0 \chi_{B(0,1)}(x). \end{aligned}$$

To deal with S_t we apply (5.2) and obtain

$$(5.9) \quad \left| \int_{t/2}^t K_{t-s} V P_s a(x) ds \right| \leq \begin{cases} C \int_{t/2}^t (K_{t-s} V)(x) r^{-2} ds & \text{if } t < r^2, \\ C \int_{t/2}^t (K_{t-s} V)(x) s^{-3/2} r ds & \text{if } t \geq r^2, \end{cases} \\ \leq C \mathbf{M}_0 \chi_{B(0,1)}(x).$$

Similarly to (5.8), by the use of (5.3), we get

$$(5.10) \quad \left| \int_{t/2}^t K_t V P_s a(x) ds \right| \leq C \mathbf{M}_0 \chi_{B(0,1)}(x).$$

Now the lemma follows from (5.7)–(5.10) and from the fact that \mathbf{M}_0 maps $\chi_{B(0,1)}$ into L^1 . □

Lemma 5.2. *There exists a constant $C > 0$ such that, for every classical $H^1(\mathbb{R}^2)$ -atom a associated with a ball $B(x_0, r)$, one has*

$$(5.11) \quad \begin{aligned} \left\| \sup_{t > \max(r^2, 4)} |Q_t a| \right\|_{L^1} &\leq C, \\ \left\| \sup_{t > \max(r^2, 4)} |R_t a| \right\|_{L^1} &\leq C, \\ \left\| \sup_{t > \max(r^2, 4)} |S_t a| \right\|_{L^1} &\leq C. \end{aligned}$$

Proof. From (5.2), for $t > \max(r^2, 4)$, we have

$$(5.12) \quad |Q_t a(x)| \leq CK_t \left(\int_0^\infty Vrs^{-3/2} ds \right) (x) \leq Cr t^{-1/2} K_t V(x).$$

In order to deal with R_t we note that for $0 < s < t/2$ one has

$$(5.13) \quad |K_t(x, z) - K_{t-s}(x, z)| \leq C \frac{s}{t^2} \exp(-c|x - z|/t^{1/2}) = \frac{s}{t} \phi_t(x - z).$$

Hence, by (5.2), we obtain

$$(5.14) \quad \begin{aligned} |R_t a(x)| &\leq \int_0^{r^2/2} \int_{\mathbb{R}^2} \frac{s}{t} \phi_t(x - z) V(z) |P_s a(z)| dz ds \\ &\quad + \int_{r^2/2}^{t/2} \int_{\mathbb{R}^2} \frac{s}{t} \phi_t(x - z) V(z) |P_s a(z)| dz ds \\ &\leq \int_0^{r^2/2} \int_{\mathbb{R}^2} \frac{s}{t} \phi_t(x - z) V(z) r^{-2} dz ds \\ &\quad + \int_{r^2/2}^{t/2} \int_{\mathbb{R}^2} \frac{s}{t} \phi_t(x - z) V(z) r s^{-3/2} dz ds \\ &\leq Cr t^{-1/2} \phi_t * V(x) + Ct^{-1/2} \phi_t * V(x). \end{aligned}$$

We now turn to estimating $S_t a$. Applying (5.2) we get

$$(5.15) \quad \begin{aligned} \left| \int_{t/2}^t K_{t-s} V P_s a(x) \right| &\leq C \int_{t/2}^t \int_{\mathbb{R}^2} K_{t-s}(x, z) V(z) r s^{-3/2} dz ds \\ &\leq C \frac{r}{t^{3/2}} \int_0^{t/2} \int_{\mathbb{R}^2} K_s(x, z) V(z) dz ds \\ &= C \frac{r}{t^{3/2}} \int_0^2 (K_s V)(x) ds + C \frac{r}{t^{3/2}} \int_2^{t/2} (K_s V)(x) ds \\ &\leq Ct^{-1} M_0 V(x) + C \frac{r}{t^{3/2}} \int_2^{t/2} (K_s V)(x) ds. \end{aligned}$$

Set

$$S(x) = \sup_{t > \max(r^2, 4)} \frac{r}{t^{3/2}} \int_2^{t/2} (K_s V)(x) ds.$$

If $|x| \leq 2$, then we use (2.12) and get

$$(5.16) \quad S(x) \leq C \sup_{t > \max(r^2, 4)} \frac{r}{t^{3/2}} \int_2^{t/2} s^{-1} (\ln s)^{-2} ds \leq Cr(1 + r)^{-3}.$$

If $2 \leq |x| \leq \max(r, 2)$, then

$$\begin{aligned}
 (5.17) \quad S(x) &\leq C \sup_{t > \max(r^2, 4)} \left(\frac{r}{t^{3/2}} \int_2^{|x|} (K_s V)(x) ds + \frac{r}{t^{3/2}} \int_{|x|}^{t/2} (K_s V)(x) ds \right) \\
 &\leq C \sup_{t > \max(r^2, 4)} \left(\frac{r}{t^{3/2}} \int_2^{|x|} s^{-1} e^{-c|x|^2/s} ds + \frac{r}{t^{3/2}} \int_{|x|}^{t/2} \frac{\ln |x|}{s(\ln s)^2} ds \right) \\
 &\leq C e^{-c|x|} + C \sup_{t > \max(r^2, 4)} \frac{r}{t^{3/2}} \ln |x| \left(\frac{1}{\ln |x|} - \frac{1}{\ln t} \right) \\
 &\leq C e^{-c|x|} + C (1+r)^{-2}.
 \end{aligned}$$

Finally, if $|x| > \max(r, 2)$, then

$$\begin{aligned}
 (5.18) \quad S(x) &\leq C \sup_{|x|^2 > t > \max(r^2, 4)} \frac{r}{t^{3/2}} \int_2^{t/2} K_s V(x) ds + C \sup_{t > |x|^2} \frac{r}{t^{3/2}} \int_2^{t/2} K_s V(x) ds \\
 &\leq C \sup_{|x|^2 > t > \max(r^2, 4)} C \frac{r}{t^{3/2}} \int_2^t \frac{s^{N-1}}{|x|^{2N}} ds \\
 &\quad + C \sup_{t > |x|^2} \frac{r}{t^{3/2}} \int_2^{|x|} K_s V(x) ds + C \sup_{t > |x|^2} \frac{r}{t^{3/2}} \int_{|x|}^{t/2} K_s V(x) ds \\
 &\leq C \frac{r}{|x|^3} + C \sup_{t > |x|^2} \frac{r}{t^{3/2}} \int_2^{|x|} \frac{s^{N-1}}{|x|^{2N}} ds + C \sup_{t > |x|^2} \frac{r}{t^{3/2}} \int_{|x|}^{t/2} \frac{\ln |x|}{s(\ln s)^2} ds \\
 &\leq C \frac{r}{|x|^3}.
 \end{aligned}$$

The estimates for $\sup_{t > \max(r^2, 4)} |\int_{t/2}^t K_t V P_s a ds|$ are similar to those we have provided for Q_t (see (5.12)). Observe that

$$\left\| \sup_{t > \max(r^2, 4)} r t^{-1/2} \phi_t * V(x) \right\|_{L^1(\mathbb{R}^2)} \leq C.$$

Hence, taking together (5.12)–(5.18) we obtain the lemma. □

Proof of (5.1). It suffices to prove that $\|\mathcal{M}_L b\|_{L^1} \leq C$ for b being any $H_{w, \text{at}}^1$ -atom. If an atom b is associated with a ball $B(x_0, r)$, then, from (1.2) one easily concludes that $\|\sup_{0 < t \leq r^2} |K_t b|\|_{L^1} \leq C$. Thus there remains only the case $t > r^2$. To this end, we note that by Proposition 4.2 it is enough to prove that $\|\sup_{t > r^2} |K_t(I - V\Delta^{-1})a|\|_{L^1} \leq C$ for every classical $H^1(\mathbb{R}^2)$ -atom a . This is a direct consequence of (5.5), and Lemmas 5.1 and 5.2. □

6. Atomic decomposition

The goal of this section is to prove the first inequality in (1.10) of Theorem 1.1,

$$(6.1) \quad \|f\|_{H_{w, \text{at}}^1} \leq C \|\mathcal{M}_L f\|_{L^1},$$

with a constant $C > 0$ independent of f . For this purpose we shall use results about Hardy spaces associated with Schrödinger operators proved in [15].

Let $\{T_t\}_{t>0}$ be a semigroup of linear operators generated by a Schrödinger operator $\mathcal{L} = \Delta - \mathcal{V}$ on \mathbb{R}^d , where \mathcal{V} is a nonnegative locally integrable potential. Let $M \geq 1$ be a positive integer. A function $\tilde{a} \in L^1(\mathbb{R}^d)$ is called a $(1, 2, M)$ -atom associated with \mathcal{L} (see Section 2 of [15]) if there exists a function \tilde{b} which belongs to the domain of the operator \mathcal{L}^M considered on $L^2(\mathbb{R}^d)$ and a ball $B = B(x_0, r_B)$ such that

- (i) $\tilde{a} = L^M \tilde{b}$;
- (ii) $\text{supp } \mathcal{L}^k \tilde{b} \subset B, k = 0, 1, \dots, M$;
- (iii) $\|\mathcal{L}^k \tilde{b}\|_{L^2(\mathbb{R}^d)} \leq r_B^{2M-2k} |B|^{-1/2}, k = 0, 1, \dots, M$.

The atomic norm $\|f\|_{H^1_{\mathcal{L}, \text{at}, M}}$ is defined by

$$\|f\|_{H^1_{\mathcal{L}, \text{at}, M}} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \right\},$$

where the infimum is taken over all representations $f = \sum_{j=1}^{\infty} \lambda_j \tilde{a}_j$, where $\lambda_j \in \mathbb{C}$ and the \tilde{a}_j are $(1, 2, M)$ -atoms. The following atomic decomposition was shown in Theorem 8.2 of [15]. For every integer $M \geq 1$ there is a constant $C_M > 0$ which depends on M and d such that

$$(6.2) \quad C_M^{-1} \|f\|_{H^1_{\mathcal{L}, \text{at}, M}} \leq \|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^d)} \leq C_M \|f\|_{H^1_{\mathcal{L}, \text{at}, M}}.$$

Using (6.2) we shall prove another atomic decomposition for elements of the Hardy space H^1 associated with \mathcal{L} . We say that a function \mathbf{a} is a generalized $(1, q)$ -atom ($1 < q \leq \infty$) for $\mathbf{H}^1_{q, \mathcal{L}, \text{at}}$ if there exists a function \mathbf{b} and a ball $B = B(x_0, r_B)$ such that

- (i) $\text{supp } \mathbf{b} \subset B$;
- (ii) $\|\mathbf{b}\|_{L^q(\mathbb{R}^d)} \leq |B|^{1/q-1}$;
- (iii) $\mathbf{a} = (I - T_{r_B^2})\mathbf{b}$.

The norm $\|f\|_{\mathbf{H}^1_{q, \mathcal{L}, \text{at}}}$ is defined by

$$\|f\|_{\mathbf{H}^1_{q, \mathcal{L}, \text{at}}} = \inf \left\{ \sum_j |\lambda_j| : f = \sum_j \lambda_j \mathbf{a}_j \right\},$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j \mathbf{a}_j, \lambda_j \in \mathbb{C}$, and the \mathbf{a}_j are generalized $(1, q)$ -atoms.

Proposition 6.1. *There is a constant C_q such that*

$$(6.3) \quad \|\mathcal{M}_{\mathcal{L}} \mathbf{a}\|_{L^1(\mathbb{R}^d)} \leq C_q.$$

for every generalized $(1, q)$ -atom \mathbf{a} .

Proof. Using standard arguments (see, e.g., [13]) we have that there is a constant C_q such that

$$\left\| \sup_{0 < t < r_B^2} |T_t \mathbf{a}(x)| \right\|_{L^1(dx)} \leq 2 \left\| \sup_{0 < t < 2r_B^2} |T_t \mathbf{b}(x)| \right\|_{L^1(dx)} \leq C_q$$

for every generalized $(1, q)$ -atom \mathbf{a} .

We now consider $\sup_{t > r_B^2} |K_t \mathbf{a}(x)|$. Using the functional calculus for Schrödinger operators, see e.g. [14], one gets $\|\mathcal{L}^2 T_t\|_{L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d)} \leq Ct^{-2}$. Hence,

$$\begin{aligned} \sup_{t \geq r_B^2} |T_t \mathbf{a}(x)| &= \sup_{t \geq r_B^2} |(T_{t+r_B^2} - T_t) \mathbf{b}(x)| = \sup_{t \geq r_B^2} \left| \int_t^{t+r_B^2} \mathcal{L} K_s \mathbf{b}(x) ds \right| \\ &= \sup_{t \geq r_B^2} \left| \int_t^{t+r_B^2} \int_s^\infty \mathcal{L}^2 K_u \mathbf{b}(x) du ds \right| \\ &\leq \sup_{t \geq r_B^2} \int_t^{t+r_B^2} \int_{r_B^2}^\infty |\mathcal{L}^2 K_u \mathbf{b}(x)| du ds = r_B^2 \int_{r_B^2}^\infty |\mathcal{L}^2 K_u \mathbf{b}(x)| du \end{aligned}$$

and, consequently,

$$\left\| \sup_{t \geq r_B^2} |T_t \mathbf{a}(x)| \right\|_{L^1(dx)} \leq C.$$

□

Lemma 6.2. *There is a constant $C > 0$ such that*

$$(6.4) \quad \|f\|_{\mathbf{H}_{\infty, \mathcal{L}, \mathbf{a}t}^1} \leq C \|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^d)}.$$

Proof. Assume that $\|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^d)} < \infty$. Then, by virtue of (6.2),

$$(6.5) \quad f = \sum_j \lambda_j \tilde{a}_j, \quad \sum_j |\lambda_j| \leq C_0 \|\mathcal{M}_{\mathcal{L}} f\|_{L^1(\mathbb{R}^d)},$$

where the \tilde{a}_j are $(1, 2, 1)$ -atoms.

Let \tilde{a} be a $(1, 2, 1)$ -atom. By definition there is a function \tilde{b} and a ball $B = B(x_0, r_B)$ such that

$$\tilde{a} = \mathcal{L} \tilde{b}, \quad \text{supp } \tilde{b} \subset B(x_0, r_B), \quad \text{and} \quad \|\mathcal{L}^k \tilde{b}\|_{L^2(\mathbb{R}^d)} \leq r_B^{2-2k} |B|^{-1/2}, \quad k = 0, 1.$$

Set

$$\mathbf{g} = r_B^{-2} \varphi(r_B^2 \mathcal{L}) \tilde{b},$$

where $\varphi(\lambda) = \lambda e^{-\lambda} (1 - e^{-\lambda})^{-1}$. Obviously, $\varphi \in \mathcal{S}([0, \infty))$ and $\tilde{a} = (I - T_{r_B^2})(\tilde{a} + \mathbf{g})$. By [14] the integral kernel $\varphi(r_B^2 \mathcal{L})(x, y)$ of the operator $\varphi(r_B^2 \mathcal{L})$ satisfies

$$(6.6) \quad |\varphi(r_B^2 \mathcal{L})(x, y)| \leq c_N r_B^{-d} \left(1 + \frac{|x - y|}{r_B}\right)^{-N}.$$

Hence,

$$(6.7) \quad |\mathbf{g}(x)| \leq C_N r_B^{-d} \left(1 + \frac{|x - x_0|}{r_B}\right)^{-N}.$$

One can easily conclude from (6.7) that \mathbf{g} can be written as

$$(6.8) \quad \mathbf{g} = \sum_i \mu_i \mathbf{b}_i, \quad \sum_i |\mu_i| \leq C'_0,$$

where each \mathbf{b}_i is supported by a ball $B_i = B(y_i, r_B)$, $\|\mathbf{b}_i\|_{L^\infty} \leq |B_i|^{-1}$ and the constant C'_0 is independent of \tilde{a} . Now let $A = \{x \in B : |\tilde{a}(x)| > \varepsilon^{-1}|B|^{-1}\}$, where $\varepsilon > 0$ is a small constant which will be determined later. By the Tchebychev inequality, $|A| \leq \varepsilon^2|B|$. Set $a^0 = \tilde{a}(x)\chi_{B \setminus A}(x)$ and $a^\infty(x) = \tilde{a}(x)\chi_A(x)$. We have that $\|a^0\|_{L^\infty} \leq \varepsilon^{-1}|B|^{-1}$ and $\|a^\infty\|_{L^{3/2}(\mathbb{R}^d)} \leq \varepsilon^{1/3}|B|^{-1/3}$. Proposition 6.1 implies that $\|\mathcal{M}_L(I - T_{r_B^2})a^\infty\|_{L^1} \leq C_{3/2}\varepsilon^{1/3}$. Thus, using (6.8), we get

$$(6.9) \quad \tilde{a} = \left((I - T_{r_B^2})a^0 + \sum_i \mu_i (I - T_{r_B^2})\tilde{\mathbf{b}}_i \right) + (I - T_{r_B^2})a^\infty := \mathbf{h}_1 + \mathbf{h}_2,$$

with

$$\|\mathbf{h}_1\|_{\mathbf{H}^1_{\infty, \mathcal{L}, \text{at}}} \leq \max(\varepsilon^{-1}, C'_0), \quad \|\mathcal{M}_L \mathbf{h}_2\|_{L^1(\mathbb{R}^d)} \leq C_{3/2}\varepsilon^{1/3}.$$

Thanks to (6.5) and (6.9), we decompose

$$(6.10) \quad f = \mathbf{f}_1 + f_1$$

with

$$\begin{aligned} \|\mathbf{f}_1\|_{\mathbf{H}^1_{\infty, \mathcal{L}, \text{at}}} &\leq C_0 \max(\varepsilon^{-1}, C'_0) \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^d)}, \\ \|\mathcal{M}_L f_1\|_{L^1(\mathbb{R}^d)} &\leq C_0 C_{3/2} \varepsilon^{1/3} \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Taking ε small enough we guarantee that $C_0 C_{3/2} \varepsilon^{1/3} < 1/2$. Now we use the decomposition (6.10) for f_1 instead of f and obtain $f_1 = \mathbf{f}_2 + f_2$ with

$$\begin{aligned} \|\mathbf{f}_2\|_{\mathbf{H}^1_{\infty, \mathcal{L}, \text{at}}} &\leq C_0 \max(\varepsilon^{-1}, C'_0) \|\mathcal{M}_L f_1\|_{L^1(\mathbb{R}^d)} \\ &\leq C_0 \max(\varepsilon^{-1}, C'_0) 2^{-1} \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^d)}, \\ \|\mathcal{M}_L f_2\|_{L^1(\mathbb{R}^d)} &\leq C_0 C_{3/2} \varepsilon^{1/3} \|\mathcal{M}_L f_1\|_{L^1(\mathbb{R}^d)} \leq 2^{-2} \|\mathcal{M}_L f\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Iterating this procedure we get the desired decomposition (6.4). □

Proof of (6.1). According to Lemma 6.2 it suffices to prove that every generalized $(1, \infty)$ -atom $\mathbf{a} = (I - T_{r^2})\mathbf{b}$ can be written as

$$\mathbf{a} = \sum_j \lambda_j b_j,$$

where the b_j are $H^1_{w, \text{at}}$ -atoms and $\sum_j |\lambda_j| \leq C$.

Let $B = B(x_0, r)$ be a ball such that $\text{supp } \mathbf{b} \subset B$, $\|\mathbf{b}\|_{L^\infty} \leq |B|^{-1}$. Then

$$\int_{\mathbb{R}^2} \mathbf{a}(x)w(x) dx = \int_{\mathbb{R}^2} \mathbf{b}(x)(I - T_{r^2})w(x) dx = 0,$$

because w is L -harmonic. Clearly,

$$|\mathbf{a}(x)| \leq Cr^{-2} \exp(-|x - x_0|/r).$$

Write $2^k B = B(x_0, 2^k r)$ and set

$$c_k = -w(2^k B)^{-1} \int_{(2^k B)^c} \mathbf{a}(x)w(x) dx, \quad k = 0, 1, 2, \dots;$$

$$b_0 = (\mathbf{a} - c_0)\chi_B, \quad b_k = c_{k-1}\chi_{2^{k-1}B} - c_k\chi_{2^k B} + \mathbf{a}\chi_{2^k B \setminus 2^{k-1}B}.$$

Then $\mathbf{a} = \sum_{j=0}^\infty b_j$. One can easily see, using the fact that $w(x) dx$ is a doubling measure, that each b_j is a multiple of an $H^1_{w, \text{at}}$ -atom and $\|b_j\|_{H^1_{w, \text{at}}} \leq Cj^{-2}$, which finishes the proof. \square

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JACEK DZIUBAŃSKI: Instytut Matematyczny, Uniwersytet Wrocławski, 50-384 Wrocław, Pl. Grunwaldzki 2/4, Poland.

E-mail: jdziuban@math.uni.wroc.pl

JACEK ZIENKIEWICZ: Instytut Matematyczny, Uniwersytet Wrocławski, 50-384 Wrocław, Pl. Grunwaldzki 2/4, Poland.

E-mail: zenek@math.uni.wroc.pl