Quasi-geostrophic equations, nonlinear Bernstein inequalities and α-stable processes

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Abstract. We prove some functional inequalities for the fractional differentiation operator $(-\Delta)^{\alpha}$ through the formalism of semi-groups. This gives us an estimate of the regularity of Marchand's weak solutions for the dissipative quasi–geostrophic equation.

1. Introduction

In this paper, we are interested in the regularity of weak solutions of the dissipative quasi-geostrophic equation (QG_{α}), a generalization of the quasi-geostrophic equation (QG) which is related to fluid mechanics, [13], and whose mathematical study was initiated by Constantin, Majda and Tabak in 1994 ([5]). The quasi-geostrophic equation (QG) describes the evolution of a function $\theta(t,x)$, $t > 0$, $x \in \mathbb{R}^2$, as

$$(1.1) \qquad \begin{cases} \partial_t \theta + \vec{u}.\vec{\nabla}\theta = 0\,, \\ \qquad\quad \vec{u} = (-R_2\theta, R_1\theta)\,, \\ \quad\theta(0,.) = \theta_0\,, \end{cases}$$

where R_i is the Riesz transform, $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ (so that the vector field $\vec{u}$ is divergence-free: $\operatorname{div}\vec{u} = 0$).

Throughout the paper, we will denote $\sqrt{-\Delta}$ by Λ (this is Calderón's operator). For $0 < \alpha \le 1$, the dissipative quasi-geostrophic equation (QG_{α}) is the equation (QG) penalized by a dissipative term $-\Lambda^{2\alpha}\theta$:

$$(1.2) \qquad \begin{cases} \partial_t \theta + \vec{u}.\vec{\nabla}\theta = -\Lambda^{2\alpha}\theta\,, \\ \qquad\quad \vec{u} = (-R_2\theta, R_1\theta)\,, \\ \quad\theta(0,.) = \theta_0\,. \end{cases}$$

In order to deal with irregular solutions, we rewrite the advection term $\vec{u}.\vec{\nabla}\theta$ as $\operatorname{div}(\theta\ \vec{u})$:

$$(1.3) \qquad \begin{cases} \partial_t\theta + \operatorname{div}(\theta\ \vec{u}) = -\Lambda^{2\alpha}\theta\,, \\ \qquad\qquad \vec{u} = (-R_2\theta, R_1\theta)\,, \\ \quad \theta(0,.) = \theta_0\,. \end{cases}$$

In 1995, Resnick [15] proved the existence of weak solutions of the equation (1.3) for $\theta_0 \in L^2(\mathbb{R}^2)$; these solutions satisfy the inequality

$$(1.4) \qquad \text{for } t > 0, \quad \|\theta(t,.)\|_2^2 + 2\int_0^t \int |\Lambda^\alpha\theta|^2\ dx\ ds \le \|\theta_0\|_2^2\,,$$

so that $\theta \in L_t^\infty L^2 \cap L_t^2 \dot{H}^\alpha$, where $\dot{H}^\alpha$ is a homogeneous Sobolev space.

In 2008, Marchand [12] studied the case of an initial value $\theta_0 \in L^p$; he proved the existence of weak solutions to equation (1.3) when $p \ge 4/3$. Moreover, when $p \ge 2$, Marchand's solutions satisfy the inequality

$$(1.5) \qquad \text{for } t > 0, \quad \|\theta(t,.)\|_p^p + p\int_0^t \int \theta|\theta|^{p-2}\Lambda^{2\alpha}\theta\ dx\ ds \le \|\theta_0\|_p^p\,,$$

where the double integral gives a nonnegative contribution, as shown by Córdoba's inequality [6], [10]:

$$(1.6) \qquad 2\int |\Lambda^\alpha(|\theta|^{p/2})|^2\ dx \le p\int \theta|\theta|^{p-2}\Lambda^{2\alpha}\theta\ dx.$$

However, the regularity of Marchand's solutions remained unclear.

In this paper, we will establish the regularity of Marchand's solutions in terms of a norm in a Besov space. More precisely, we shall establish a variant of Córdoba's inequality and get that, for $2 \le p < \infty$ and $0 < \alpha < 1$,

$$(1.7) \qquad \|\theta\|_{\dot{B}_p^{2\alpha/p,p}}^p \le C_p \int \theta|\theta|^{p-2}\Lambda^{2\alpha}\theta\ dx\,;$$

and for $2 \le p < \infty$,

$$(1.8) \qquad \|\theta\|_{\dot{B}_p^{2/p,\infty}}^p \le C_p \int \theta|\theta|^{p-2}(-\Delta)\theta\ dx\,,$$

where $\dot{B}_p^{2\alpha/p,p}$ and $\dot{B}_p^{2/p,\infty}$ are homogeneous Besov spaces. Our method gives us a new proof of a nonlinear Bernstein inequality given by Danchin [8]: for $\theta \in L^p(\mathbb{R}^n)$ such that its Fourier transform $\hat{\theta}(\xi)$ is supported in the annulus $1/2 \le |\xi| \le 2$, we have, for $1 < p < \infty$,

$$(1.9) \qquad A\|\theta\|_p^p \le \|\vec{\nabla}(|\theta|^{p/2})\|_2^2 \le B\|\theta\|_p^p\,,$$

where the constants A and B are positive and depend only on p and on the dimension n.

We will mainly apply general results from the theory of semi-groups to the semi-group $e^{-t\Lambda^{2\alpha}}$. This is a symmetric diffusion semi-group (in the sense of Stein [19]), and we will use a representation of the semi-group as a barycentric mean of heat kernels through a formula derived from the theory of α-stable processes [22]. For instance, when $\alpha = 1$, we have $e^{-t\Lambda^2} = e^{t\Delta}$ (the heat kernel); for $\alpha = 1/2$, we have $e^{-t\Lambda} = P_t$, the Poisson semi-group. In dimension 1, $e^{-|\xi|}$ is the Fourier transform of $\frac{1}{\pi}\frac{1}{1+x^2}$; we write

$$(1.10) \qquad \frac{1}{\pi}\frac{1}{1+x^2} = \frac{1}{\pi}\int_0^\infty e^{-\sigma}e^{-\sigma x^2}\,d\sigma = \frac{1}{2\pi}\int_0^\infty e^{-\frac{1}{2\sigma}}e^{-\frac{x^2}{2\sigma}}\frac{d\sigma}{\sigma^2}$$

and we get

$$(1.11) \qquad e^{-|\xi|} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2\sigma}}e^{-\sigma\frac{\xi^2}{2}}\frac{d\sigma}{\sigma^{3/2}},$$

and finally

$$(1.12) \qquad e^{-t\Lambda} = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2\sigma}}e^{\sigma\frac{t^2}{2}\Delta}\frac{d\sigma}{\sigma^{3/2}}.$$

We shall use a generalization of (1.12) to the case of $e^{-t\Lambda^{2\alpha}}$.

2. One-dimensional stable distributions

The aim of this section is to recall the useful following representation:

Proposition 2.1. *For $0 < \alpha < 1$, there exists a probability measure $d\mu_\alpha$ concentrated on $[0, +\infty)$ such that for all $x \in \mathbb{R}$ we have*

$$(2.1) \qquad e^{-|x|^{2\alpha}} = \int_0^{+\infty} e^{-\sigma x^2}d\mu_\alpha(\sigma).$$

Corollary 2.2. *Let $\Lambda = \sqrt{-\Delta}$ be the Calderón operator on $\mathbb{R}^n$ and let $e^{t\Delta}$ be the heat kernel on $\mathbb{R}^n$. Then the operator $e^{-t\Lambda^{2\alpha}}$ ($t \geq 0$, $0 < \alpha < 1$) may be represented as*

$$(2.2) \qquad e^{-t\Lambda^{2\alpha}} = \int_0^{+\infty} e^{\sigma t^{1/\alpha}\Delta}d\mu_\alpha(\sigma).$$

Formula (2.1) is well known. See for instance Proposition 1.2.12 in [18]. Due to a celebrated theorem of Bernstein, [2], it amounts to say that the function $x > 0 \mapsto e^{-|x|^\alpha}$ is completely monotone, which is easily checked.

Formula (2.1) is linked to the theory of one-dimensional stable processes. The probability density function $d\mu$ of a random variable X is called α-stable [22] if its characteristic function $\chi(\xi) = E(e^{iX\xi}) = \int e^{ix\xi}\,d\mu(x)$ is of the form

$$(2.3) \qquad \chi(\xi) = \begin{cases} e^{im\xi - \sigma^\alpha|\xi|^\alpha - i\beta\sigma^\alpha\xi|\xi|^{\alpha-1}\tan(\pi\alpha/2)} & \text{if } \alpha \neq 1, \\ e^{im\xi - \sigma|\xi| + i\beta\sigma\xi\ln|\xi|} & \text{if } \alpha = 1. \end{cases}$$

The admissible values for the parameters are: $0 < \alpha \leq 2$ for the stability index α, $m \in \mathbb{R}$ for the position parameter m, $\sigma \geq 0$ for the scale parameter σ, and $-1 \leq \beta \leq 1$ for the bias parameter β. We will write $X \sim S_\alpha(m, \sigma, \beta)$.

For $X \sim S_\alpha(0, \sigma, 1)$ with $0 < \alpha < 1$, the function χ is given by

$$\chi(\xi) = e^{-\sigma^\alpha |\xi|^\alpha (1 + i\,\mathrm{sgn}(\xi)\,\tan(\pi\alpha/2))}.$$

If z^α is the holomorphic function defined on $\mathbb{C} \backslash \mathbb{R}^-$ as $z^\alpha = |z|^\alpha e^{i\alpha\,\mathrm{Arg}(z)}$ where the argument of z is taken in $(-\pi, \pi)$, we have

$$(i\xi)^\alpha = |\xi|^\alpha e^{i\alpha\,\mathrm{sgn}(\xi)\pi/2} = \cos(\alpha\pi/2)|\xi|^\alpha(1 + i\,\mathrm{sgn}(\xi)\tan(\alpha\pi/2)).$$

Thus, when $X \sim S_\alpha((\cos(\alpha\pi/2))^{-1/\alpha}, 0, 1)$, we have $\chi(\xi) = e^{-(i\xi)^\alpha}$. For $z = \eta + i\xi$ with $\eta \geq 0$, we have $|e^{-z^\alpha}| = e^{-|z|^\alpha \cos(\alpha\,\mathrm{Arg}(z))} \leq 1$. The Paley–Wiener–Schwartz theorem ensures that the probability density function $d\mu_\alpha$ of X is supported on $\mathbb{R}^+$ and that, for $z = \xi + i\eta$ with $\eta \geq 0$, we have $e^{(iz)^\alpha} = \int_0^{+\infty} e^{i\sigma z}\, d\mu_\alpha(\sigma)$. When $z = ix^2$, we obtain $e^{-|x|^{2\alpha}} = \int_0^{+\infty} e^{-\sigma x^2}\, d\mu_\alpha(\sigma)$.

3. Diffusion semi-groups

In this section, we consider a symmetric diffusion semi-group as considered by Stein in [19]:

Definition 3.1. A symmetric diffusion semi-group with infinitesimal generator L is a family of operators $(e^{tL})_{t \geq 0}$ such that:

  i) e^{tL} is self-adjoint for $t \geq 0$.

  ii) e^{tL} is the convolution operator with a probability density function $p_t(x)$ ($p_t(x) \geq 0$ and $\int p_t(x)\, dx = 1$).

  iii) $e^{tL}e^{sL} = e^{(t+s)L}$ and, for $f \in L^2$, $\lim_{t \to 0^+} \|e^{tL}f - f\|_2 = 0$.

We then have

  iv) $Lf = \lim_{t \to 0} \frac{1}{t}(e^{tL}f - f)$ on a dense subspace of L^2.

  v) $\partial_t e^{tL}f = L(e^{tL}f)$.

For classical results on such semi-groups, we refer to the survey of Bakry [1]. A crucial result is that, for a convex function ϕ, we have Jensen's inequality

$$(3.1) \qquad\qquad \phi(e^{tL}f) \leq e^{tL}\phi(f),$$

and, by looking at the derivatives of both terms at $t = 0$,

$$(3.2) \qquad\qquad \phi'(f)Lf \leq L(\phi f).$$

When $\phi(t) = t^2$, we get $2fL(f) \leq L(f^2)$: this is the positivity of the square field operator

$$(3.3) \qquad\qquad \Gamma(f, g) = \frac{1}{2}(L(f, g) - fL(g) - gL(f)).$$

For $\phi(t) = |t|^\gamma$ with $\gamma > 1$, we find $\gamma f |f|^{\gamma-2} L(f) \leq L(|f|^\gamma)$. For $\gamma = p/2$ with $2 < p < +\infty$, we multiply the inequality by $|f|^{p/2}$ and we integrate. We thus get

$$(3.4) \quad p \int f|f|^{p-2} \, Lf \, dx \leq 2 \int |f|^{p/2} L(|f|^{p/2}) \, dx = -2 \int |\sqrt{-L}(|f|^{p/2})|^2 \, dx.$$

We are now going to generalize (3.4) by taking into account the sign of f in the RHS of the inequality:

Theorem 3.2. *Let $(e^{tL})_{t \geq 0}$ be a symmetric diffusion semi-group. Then:*

i) *For $2 \leq p < +\infty$, we have the inequality*

$$(3.5) \quad p \int f|f|^{p-2} L(f) \, dx \leq \int f|f|^{\frac{p}{2}-1} L(f|f|^{\frac{p}{2}-1}) \, dx = - \int |\sqrt{-L}(f|f|^{\frac{p}{2}-1})|^2 \, dx.$$

ii) *For $1 \leq p \leq 2$, we have the inequality*

$$(3.6) \quad 4 \int f|f|^{\frac{p}{2}-1} L(f|f|^{\frac{p}{2}-1}) \, dx = -4 \int |\sqrt{-L}(f|f|^{\frac{p}{2}-1})|^2 \, dx \leq p \int f|f|^{p-2} L(f) \, dx \, ,$$

where, moreover, $p \int f|f|^{p-2} L(f) \, dx \leq 0$.

Proof. We use the convex function $\phi(t) = |t|$, and we find $\operatorname{sgn}(f) \, L(f) \leq L(|f|)$, hence $fL(f) \leq |f|L(|f|)$. We decompose f into $f = f^+ - f^-$ with $f^+ = \frac{f+|f|}{2}$, and we get

$$(3.7) \qquad\qquad f^+ L(f^-) + f^- L(f^+) \geq 0$$

Integrating (3.7) and using the self-adjointness of L gives $\int f^+ L(f^-) \, dx \geq 0$. The case of f^+ and f^- approximating two Dirac masses at separate points gives then that the distribution kernel K of L satisfies $K(x,y) \geq 0$ away from the diagonal $x = y$, and we get finally that

$$(3.8) \qquad\qquad f^+ L(f^-) = \int_{x \neq y} K(x,y) f^+(x) f^-(y) \, dy \geq 0 \, ,$$

and similarly $f^- L(f^+) \geq 0$. In particular, we get that, for $1 \leq p < +\infty$, we have

$$(3.9) \qquad\qquad \int (f^+)^{p-1} L(f^-) + (f^-)^{p-1} L(f^+) \, dx \geq 0 \, .$$

On the other hand, we have that $t \mapsto \|e^{tL}\|_p^p$ is nonincreasing, so that (by looking at the derivative at $t = 0$) we have $p \int f|f|^{p-2} Lf \, dx \leq 0$. This inequality together with (3.9) gives

$$2 \int (f^+)^{p-1} L(f^+) + (f^-)^{p-1} L(f^-) dx \leq \int f|f|^{p-2} L(f) dx$$

$$(3.10) \qquad\qquad\qquad\qquad \leq \int (f^+)^{p-1} L(f^+) + (f^-)^{p-1} L(f^-) dx \, ,$$

and, similarly, we have for $g = f|f|^{\frac{p}{2}-1}$, $g^+ = (f^+)^{p/2}$ and $g^- = (f^-)^{p/2}$,

$$(3.11) \quad 2\int g^+ L(g^+) + g^- L(g^-) \, dx \leq \int g \, L(g) \, dx \leq \int g^+ L(g^+) + g^- L(g^-) \, dx.$$

When $p \geq 2$, we write $\|e^{tL} f^+\|_p^p \leq \|e^{tL}(g^+)\|_2^2$ and we get (by looking at the derivative at $t = 0$) that $p\int (f^+)^{p-1} L(f^+) \, dx \leq 2\int g^+ L(g^+) \, dx$; we have the same inequality for f^- and g^-. Thus, (3.10) and (3.11) give (3.5).

When $p \leq 2$, we write $\|e^{tL} g^+\|_2^2 \leq \|e^{tL}(f^+)\|_p^p$ and get that $2\int g^+ L(g^+) \, dx \leq p\int (f^+)^{p-1} L(f^+) \, dx$; we have the same inequality for f^- and g^-. Thus, (3.10) and (3.11) give (3.6). □

4. A. and D. Córdoba's inequality and Besov norms

The semi-group $(e^{-t\Lambda^{2\alpha}})_{t\geq 0}$ is a symmetric diffusion semi-group on $\mathbb{R}^n$. The positivity of its kernel is a consequence of the positivity of the heat kernel $e^{t\Delta}$ and of the representation formula given by Corollary 2.2. Thus, Córdoba's inequality (1.6) is just a special case of inequality (3.4). In this section, we shall apply Theorem 3.2 (generalization of (3.4)) to the semi-group $(e^{-t\Lambda^{2\alpha}})_{t\geq 0}$. Our application will be based on the following easy lemma:

Lemma 4.1. *Let* $0 < \gamma \leq 1$. *Then for all a and b in $\mathbb{R}$ we have*

$$(4.1) \quad \left| a\,|a|^{\gamma-1} - b\,|b|^{\gamma-1} \right| \leq 2|a-b|^\gamma$$

Proof. This is obvious if $ab < 0$: if $uv < 0$ then $\max(|u|, |v|) \leq |u - v| \leq 2\max(|u|, |v|)$. If $ab \geq 0$, we use the fact that $d_\gamma(x, y) = |x - y|^\gamma$ is a distance on $\mathbb{R}$ and we write $|d_\gamma(a, 0) - d_\gamma(b, 0)| \leq d_\gamma(a, b)|$. □

We may now prove the following extension of Córdoba's inequality, using norms in homogeneous Sobolev and Besov spaces:

Theorem 4.2. (A) *Let* $0 < \alpha < 1$ *and* $2 \leq p < +\infty$. *Then there is a positive constant* $c_{\alpha,p,n} > 0$ *such that*

$$(4.2) \quad \begin{aligned} c_{\alpha,p,n}\|f\|_{\dot{B}_p^{2\alpha/p,p}}^p &\leq \|f|f|^{\frac{p}{2}-1}\|_{\dot{H}^\alpha}^2 = \int |\Lambda^\alpha(f|f|^{\frac{p}{2}-1})|^2 \, dx \\ &\leq p\int f|f|^{p-2}\Lambda^{2\alpha}(f) \, dx \,. \end{aligned}$$

(B) *Let* $2 \leq p < +\infty$. *Then there is a positive constant* $c_{p,n} > 0$ *such that*

$$(4.3) \quad \begin{aligned} c_{p,n}\|f\|_{\dot{B}_p^{2\alpha/p,\infty}}^p &\leq \|f|f|^{\frac{p}{2}-1}\|_{\dot{H}^1}^2 = \int |\vec{\nabla}(f|f|^{\frac{p}{2}-1})|^2 \, dx \\ &\leq p\int f|f|^{p-2}(-\Delta f) \, dx \,. \end{aligned}$$

(C) *Let $0 < \alpha < 1$ and $\max(1, 2\alpha) < p < 2$. Then there is a positive constant $C_{\alpha,p,n} > 0$ such that*

(4.4)
$$0 \leq p \int f|f|^{p-2}\Lambda^{2\alpha}(f) \, dx \leq 4\|f|f|^{\frac{p}{2}-1}\|_{\dot{H}^\alpha}^2 = 4 \int |\Lambda^\alpha(f|f|^{\frac{p}{2}-1})|^2 \, dx$$
$$\leq C_{\alpha,p,n}\|f\|_{\dot{B}_p^{2\alpha/p,p}}^p.$$

Proof. First, we will apply Theorem 3.2 to the symmetric diffusion semi-group $(e^{-t\Lambda^{2\alpha}})_{t\geq 0}$: (3.5) gives the RHS inequalities in (4.2) and (4.3), while (3.6) gives the LHS inequality in (4.4). Thus, the proof of Theorem 4.2 is reduced to a comparison between a Besov norm and a Sobolev norm.

Besov norms may be defined in various (more or less) equivalent ways. We shall use the characterization of Besov spaces through moduli of continuity. For $\beta \in (0, 1)$ and $1 \leq p < \infty$, the norms of $\dot{B}_p^{\beta,p}$ may be defined as

(4.5)
$$\begin{cases} \|f\|_{\dot{B}_p^{\beta,p}} = \left(\int\int \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\beta}} \, dx \, dy\right)^{\frac{1}{p}} \\ \text{and} \\ \|f\|_{\dot{B}_p^{\beta,\infty}} = \sup_{h \in \mathbb{R}^n, \, h \neq 0} \frac{\|f(x) - f(x+h)\|_p}{|h|^\beta} \end{cases}$$

Moreover, we have $\dot{H}^\alpha = \dot{B}_2^{\alpha,2}$. Thus, the Sobolev norm $\|f\|_{\dot{H}^\alpha}$ is equivalent, for $\alpha \in (0,1)$, to $\|f\|_{\dot{B}_2^{\alpha,2}} = \sqrt{\int\int \frac{|f(x)-f(y)|^2}{|x-y|^{n+2\alpha}} \, dx \, dy}$. For $\alpha = 1$, the Sobolev norm $\|f\|_{\dot{H}^1}$ is equivalent to $\sup_{h \in \mathbb{R}^n, \, h \neq 0} \frac{\|f(x)-f(x+h)\|_2}{|h|}$.

To finish the proof, we use Lemma 4.1. For $p \geq 2$, we take $\gamma = 2/p$, $a = f(x)|f(x)|^{\frac{p}{2}-1}$, and $b = f(y)|f(y)|^{\frac{p}{2}-1}$, and we get

(4.6)
$$|f(x) - f(y)|^p \leq 2^p \left|f(x)|f(x)|^{\frac{p}{2}-1} - f(y)|f(y)|^{\frac{p}{2}-1}\right|^2$$

Using (4.6) and (4.5), we then get the LHS inequalities of (4.2) and (4.3).

For $p < 2$, we take $\gamma = p/2$, $a = f(x)$ and $b = f(y)$, and we get

(4.7)
$$|f(x)|f(x)|^{\frac{p}{2}-1} - f(y)|f(y)|^{\frac{p}{2}-1}|^2 \leq 4|f(x) - f(y)|^p.$$

Using (4.7) and (4.5), for $2\alpha/p < 1$, we then get the RHS inequality of (4.4). □

5. Frequency gaps

Let $1 < p < +\infty$ and let $f \in L^p(\mathbb{R}^n)$ be such that the Fourier transform $\hat{f}$ has no low frequency: $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then it is well known that the norm of $e^{t\Delta}f$ decays exponentially:

(5.1)
$$\|e^{t\Delta}f\|_p \leq \frac{1}{c_p} e^{-c_p t A^2} \|f\|_p$$

(see for instance Chemin [3]). But (5.1) contains no information for small t's: if $t \leq A^{-2}\frac{1}{c_p}\ln\frac{1}{c_p}$ we have $\|e^{t\Delta}f\|_p \leq \|f\|_p$ and $1 \leq \frac{1}{c_p}e^{-c_p t A^2}$.

In this section, we want to prove a more precise estimate:

$$\|e^{t\Delta}f\|_p \leq e^{-c_p t A^2}\|f\|_p. \tag{5.2}$$

We begin with two classical lemmas:

Lemma 5.1. (A) *Let $1 \leq p \leq +\infty$ and let $g \in L^p(\mathbb{R}^n)$ be such that the Fourier transform $\hat{g}$ has no low frequency: $\hat{g}(\xi) = 0$ for $|\xi| \leq A$. Then, for $1 \leq j \leq n$, $\|\frac{\partial_j}{\Delta}g\|_p \leq cA^{-1}\|g\|_p$.*

(B) *Let $1 \leq p \leq +\infty$ and let $f \in L^p(\mathbb{R}^n)$ be such that the Fourier transform $\hat{f}$ has no low frequency: $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then $\|f\|_p \leq cA^{-1}\|\vec{\nabla}f\|_p$.*

(C) *Let $1 \leq p \leq +\infty$ and let $f \in L^p(\mathbb{R}^n)$ be such that the Fourier transform $\hat{f}$ has no low frequency: $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then there exists $F_j \in L^p$ such that $f = \sum_{j=1}^n \partial_j F_j$, with $\|F_j\|_p \leq cA^{-1}\|f\|_p$.*

Proof. (A) is obvious: if $\omega \in \mathcal{D}(\mathbb{R}^n)$ is equal to 1 on the ball $B(0, 1/4)$ and to 0 outside from the ball $B(0, 1/2)$, then the function k_j whose Fourier transform $\hat{k}_j$ is equal to $\hat{k}_j(\xi) = -\frac{i\xi_j}{\|\xi\|^2}(1 - \omega(\xi))$ satisfies $k_j \in L^1$. We have $\frac{\partial_j}{\Delta}g = A^{n-1}k_j(Ax) * g$, so that $\|\frac{\partial_j}{\Delta}g\|_p \leq A^{-1}\|k_j\|_1\|g\|_p$.

For (B) and (C), we just write $f = -\sum_{j=1}^n \frac{\partial_j}{\Delta}\partial_j f = -\sum_{j=1}^n \partial_j \frac{\partial_j}{\Delta}f$. $\square$

The following lemma can be found in [11]:

Lemma 5.2. *Let $1 < p < +\infty$ and let f be a $\mathcal{C}^1$ function. If $f \in W^{2,p}(\mathbb{R}^n)$, then we have*

$$-\int f|f|^{p-2}\Delta f \ dx = (p-1)\int_{f(x)\neq 0} |\vec{\nabla}f|^2 \ |f|^{p-2} \ dx \tag{5.3}$$

Proof. For $p \geq 2$, this is obvious. $f|f|^{p-2}$ is $\mathcal{C}^1$ and $\partial_j(f|f|^{p-2}) = (p-1)|f|^{p-2}\partial_j f$. Thus, (5.3) is a direct consequence of integration by parts.

For $1 < p < 2$, we approximate $f|f|^{p-2}$ by $g_\epsilon = f|f^2 + \epsilon^2|^{\frac{p-2}{2}}$ with $\epsilon > 0$. By dominated convergence, we have $-\int f|f|^{p-2}\Delta f \ dx = \lim_{\epsilon \to 0}\int g_\epsilon(-\Delta f) \ dx$. We have $\partial_j(g_\epsilon) = \partial_j f|f^2 + \epsilon^2|^{\frac{p-2}{2}}(1 + (p-2)\frac{f^2}{f^2+\epsilon^2})$. We consider $\omega \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \omega \leq 1$ and $\omega = 1$ on $B(0, 1)$. Then we have

$$-\int \partial_j^2 f g_\epsilon = \lim_{R \to +\infty}\int \partial_j f \ (\omega(x/R)\partial_j g_\epsilon + \frac{1}{R}\partial_j\omega(x/R)g_\epsilon) \ dx$$
$$= \int |\partial_j f|^2|f^2 + \epsilon^2|^{\frac{p-2}{2}}\left(1 + (p-2)\frac{f^2}{f^2+\epsilon^2}\right) \ dx \tag{5.4}$$

since $|\int |\partial_j f \frac{1}{R}\partial_j\omega(x/R)g_\epsilon \ dx| \leq R^{-1}\|\partial_j\omega\|_\infty\|f\|_{W^{2,p}}^p$ (and thus goes to 0 as R goes to $+\infty$), and since $\partial_j f \partial_j g_\epsilon \geq 0$ (note that $p-1 \leq 1 + (p-2)\frac{f^2}{x^2+\epsilon^2} \leq 1$), so that we may apply monotone convergence to $\int \partial_j f \partial_j g_\epsilon \omega(x/R) \ dx$. We may

restrict the domain of the integral on the RHS of (5.4) to the set of x such that $f(x) \neq 0$, since the set of x such that $f(x) = 0$ and $\partial_j f(x) \neq 0$ has Lebesgue measure 0. Thus, we have

$$(5.5) \quad -\int f|f|^{p-2}\Delta f \, dx = \lim_{\epsilon \to 0^+} \int_{f(x) \neq 0} |\vec{\nabla} f|^2 \, |f^2 + \epsilon^2|^{\frac{p-2}{2}} \left(1 + (p-2)\frac{f^2}{f^2 + \epsilon^2}\right) dx \, .$$

Moreover $\epsilon \mapsto |r^2 + \epsilon^2|^{\frac{p-2}{2}}$ is nonincreasing function of $\epsilon \in [0, +\infty)$ and we may apply again monotone convergence to see that

$$(5.6) \quad \lim_{\epsilon \to 0} \int_{f(x) \neq 0} |\partial_j f|^2 |f^2 + \epsilon^2|^{\frac{p-2}{2}} \, dx = \int_{f(x) \neq 0} |\partial_j f|^2 |f|^{p-2} \, dx \, .$$

The inequality $|\vec{\nabla} f|^2 \, |f^2 + \epsilon^2|^{\frac{p-2}{2}} \, (1 + (p-2)\frac{f^2}{f^2 + \epsilon^2}) \geq (p-1)|\vec{\nabla} f|^2 \, |f^2 + \epsilon^2|^{\frac{p-2}{2}}$, together with (5.5) and (5.6), gives us that the limit in (5.6) is finite. The inequality $|\vec{\nabla} f|^2 \, |f^2 + \epsilon^2|^{\frac{p-2}{2}} \, (1 + (p-2)\frac{f^2}{f^2 + \epsilon^2}) \leq |\vec{\nabla} f|^2 \, |f^2 + \epsilon^2|^{\frac{p-2}{2}}$, together with (5.5), gives us, by dominated convergence, the equality (5.3). $\qquad\square$

We may now prove our theorem on frequency gaps:

Theorem 5.3. *Let $1 < p < +\infty$ and let $f \in L^p(\mathbb{R}^n)$ be such that the Fourier transform $\hat{f}$ has no low frequency: $\hat{f}(\xi) = 0$ for $|\xi| \leq A$. Then:*

(A) If $f \in W^{2,p}$, we have the inequality

$$(5.7) \qquad c_p \|f\|_p^p \leq A^{-2} p \int f|f|^{p-2}(-\Delta f) \, dx \, ,$$

where the constant $c_p > 0$ depends only on n and p.

(B) We have the inequality, for all $t \geq 0$,

$$(5.8) \qquad \|e^{t\Delta} f\|_p \leq e^{-c_p A^2 t} \|f\|_p \, ,$$

where the constant $c_p > 0$ depends only on n and p.

(C) For $0 < \alpha < 1$ and $t \geq 0$, we have the inequality

$$(5.9) \qquad \|e^{-t\Lambda^{2\alpha}} f\|_p \leq e^{-c_{\alpha,p} A^{2\alpha} t} \|f\|_p \, ,$$

where the constant $c_{\alpha,p} > 0$ depends only on n, α and p.

Proof. We may assume (by a density argument) that f is smooth. In order to prove (A), we shall consider separately the cases $p \geq 2$ and $p < 2$.

Case $p \geq 2$. We use Lemma 5.1 and write $f = \sum_{j=1}^n \partial_j F_j$. Then we have

$$(5.10) \qquad \|f\|_p^p = \sum_{j=1}^n \int \partial_j F_j \, f|f|^{p-2} \, dx = -(p-1) \sum_{j=1}^n \int \partial_j f F_j |f|^{p-2} \, dx \, ,$$

and by Cauchy–Schwarz,

$$(5.11) \qquad \|f\|_p^p \leq (p-1) \Big(\int |\vec{\nabla} f|^2 |f|^{p-2} \, dx \Big)^{1/2} \Big(\int \sum_j |F_j|^2 |f|^{p-2} \, dx \Big)^{1/2}$$

We conclude with Lemma 5.1 (C) and Lemma 5.2.

Case $p < 2$. We use Lemma 5.1 (B) and write $\|f\|_p \leq cA^{-1}\|\vec{\nabla} f\|_p$. Moreover, when computing the integral $\int |\vec{\nabla} f|^p \, dx$, we may restrict the domain of integration to the set of x such that $f(x) \neq 0$. Then we use Hölder's inequality to get

$$(5.12) \qquad \int |\vec{\nabla} f|^p \, dx \leq \Big(\int_{f(x) \neq 0} |\vec{\nabla} f|^2 |f|^{p-2} \, dx \Big)^{p/2} \Big(\int_{f(x) \neq 0} |f|^p \, dx \Big)^{1-p/2},$$

and we conclude with Lemma 5.1 (B) and Lemma 5.2.

Thus, (A) is proved. (B) is a direct consequence of (A): the derivative of $H(t) = \|e^{t\Delta} f\|_p^p$ is equal to $p \int e^{t\Delta} f |e^{t\Delta} f|^{p-2} \Delta(e^{t\Delta} f) \, dx$, and the the derivative of $K(t) = e^{-c_p A^2 t}\|f\|_p^p$ is $-c_p A^2 e^{-c_p A^2 t}\|f\|_p^p$. (A) gives that $H'(t) \leq -c_p A^2 H(t)$; thus, we get, for $J(t) = H(t) - K(t)$, $J'(t) \leq -c_p A^2 J(t)$ and $J(t) \leq J(0)e^{-c_p A^2 t} = 0$. Thus, $H(t) \leq K(t)$ and (B) is proved.

Finally, (C) is a consequence of (B) and of the representation formulae (2.1) and (2.2):

$$(5.13) \qquad \begin{aligned} \|e^{-t\Lambda^{2\alpha}} f\|_p &\leq \int_0^\infty \|e^{\sigma t^{1/\alpha} \Delta} f\|_p^p \, d\mu_\alpha(\sigma) \leq \int_0^\infty e^{-c_p A^2 \sigma t^{1/\alpha}} \|f\|_p \, d\mu_\alpha(\sigma) \\ &= e^{-(c_p A^2 t^{1/\alpha})^\alpha} \|f\|_p = e^{-c_p^\alpha A^{2\alpha} t} \|f\|_p \,. \end{aligned}$$

Thus, (C) is proved.   □

6. Band limited functions

In this section, we shall estimate the decay of $\|e^{-t\Lambda^{2\alpha}} f\|_p$ by below:

Theorem 6.1. *Let $1 < p < +\infty$ and let $f \in L^p(\mathbb{R}^n)$ be such that the Fourier transform $\hat{f}$ has no high frequency: $\hat{f}(\xi) = 0$ for $|\xi| \geq A$. Then:*

(A) *For $0 < \alpha \leq 1$, we have the inequality*

$$(6.1) \qquad A^{-2\alpha} p \int f |f|^{p-2} \Lambda^{2\alpha} f \, dx \leq c_{\alpha,p} \|f\|_p^p,$$

where the constant $c_{\alpha,p} > 0$ depends only on n and p.

(B) *For $0 < \alpha < 1$ and $t \geq 0$, we have the inequality*

$$(6.2) \qquad \|e^{-t\Lambda^{2\alpha}} f\|_p \geq e^{-c_{\alpha,p} A^{2\alpha} t} \|f\|_p \,,$$

where the constant $c_{\alpha,p} > 0$ depends only on n, α and p.

Proof. The case $p \geq 2$ is easy. The Bernstein inequalities give us that $\|\Lambda^{2\alpha}(\theta)\|_p \leq cA^{2\alpha}\|\theta\|_p$, and thus (6.1) is obvious.

When $p < 2$, we use Theorem 3.2 (equation (3.6)) (or the LHS of Theorem 4.2, equation (4.4), which is valid for $1 < p < 2$), and get that

$$(6.3) \quad p \int f|f|^{p-2}\Lambda^{2\alpha}(f)\ dx \le 4\|f|f|^{\frac{p}{2}-1}\|_{\dot{H}^\alpha}^2 \le 4\|f\|_p^{(1-\alpha)p/2}\|\vec{\nabla}(f|f|^{\frac{p}{2}-1})\|_2^{2\alpha}\,.$$

We approximate $f|f|^{\frac{p-2}{2}}$ by $g_\epsilon = f|f^2 + \epsilon^2|^{\frac{p-2}{4}}$, with $\epsilon > 0$. We have $\partial_j g_\epsilon = \partial_j f|f^2 + \epsilon^2|^{\frac{p-2}{4}}\left(1 + \frac{p-2}{2}\frac{f^2}{f^2+\epsilon^2}\right)$. We have that

$$(6.4)
\begin{aligned}
\|\vec{\nabla}g_\epsilon\|_2^2 &= \int_{f(x)\ne 0} |\vec{\nabla}f|^2|f^2 + \epsilon^2|^{\frac{p-2}{2}}(1 + \frac{p-2}{2}\frac{f^2}{f^2+\epsilon^2})^2\ dx \\
&\xrightarrow[\epsilon>0]{} (p-1)^2 \int_{f(x)\ne 0} |\vec{\nabla}f|^2\ |f|^{p-2}\ dx\,.
\end{aligned}$$

We use Lemma 5.2 to get that the limit in (6.4) is finite. Thus, we get that $\vec{\nabla}(f|f|^{\frac{p}{2}-1}) \in L^2$ and that (using Bernstein's inequality)

$$(6.5) \qquad \|\vec{\nabla}(f|f|^{\frac{p}{2}-1})\|_2^2 = -(p-1)\int f|f|^{p-2}\Delta f\ dx \le cA^2\|f\|_p^p.$$

Thus (A) is proved.

(B) is a direct consequence of (A): the derivative of $H(t) = \|e^{-t\Lambda^{2\alpha}}f\|_p^p$ is equal to $-p\int e^{-t\Lambda^{2\alpha}}f|e^{-t\Lambda^{2\alpha}}f|^{p-2}\Lambda^{2\alpha}(e^{-t\Lambda^{2\alpha}}f)\,dx$, and the derivative of $K(t) = e^{-c_{\alpha,p}A^2 t}\|f\|_p^p$ is $-c_{\alpha,p}A^2 e^{-c_{\alpha,p}A^2 t}\|f\|_p^p$. (A) gives that $H'(t) \ge -c_{\alpha,p}A^2 H(t)$; thus, we get, for $J(t) = H(t) - K(t)$, the inequalities $J'(t) \ge -c_{\alpha,p}A^2 J(t)$ and $J(t) \ge J(0)e^{-c_p A^2 t} = 0$. Thus, $H(t) \ge K(t)$, and (B) is proved. □

7. Danchin's inequality

In this section, we shall discuss the nonlinear Bernstein inequality given by Danchin in [8] and [9]: for $\theta \in L^p(\mathbb{R}^n)$ such that its Fourier transform $\hat{\theta}(\xi)$ is supported in the annulus $1/2 \le |\xi| \le 2$, we have, for $1 < p < \infty$,

$$(7.1) \qquad\qquad A\|\theta\|_p^p \le \|\vec{\nabla}(|\theta|^{p/2})\|_2^2 \le B\|\theta\|_p^p\,,$$

where the constants A and B are positive and depend only on p and on the dimension n. Danchin [8] proved it for $p \in 2\mathbb{N}^*$, then Planchon [14] proved it for $p \ge 2$, and finally Danchin gave a proof for $p > 1$ in [9]. We shall use our previous results to prove it and generalize it:

Theorem 7.1. *Let $1 < p < +\infty$. Let $\theta \in L^p(\mathbb{R}^n)$ be such that its Fourier transform $\hat{\theta}(\xi)$ is supported in the annulus $1/2 \le |\xi| \le 2$. Then, for $0 < \alpha \le 1$, we have*

$$(7.2) \qquad\qquad A\|\theta\|_p^p \le \|\Lambda^\alpha(\theta|\theta|^{p/2-1})\|_2^2 \le B\|\theta\|_p^p\,,$$

where the constants A and B are positive and depend only on p, on α, and on the dimension n.

Proof. Due to the spectral localization of θ, we have

$$(7.3) \qquad \|\theta\|_p \sim \|\theta\|_{\dot{B}_p^{2\alpha/p,p}} \sim \|\theta\|_{\dot{B}_p^{2\alpha/p,\infty}} \,.$$

The case $p \geq 2$ is easy: (7.3) and Theorem 4.2 give that $A\|\theta\|_p^p \leq \|\Lambda^\alpha(\theta|\theta|^{p/2-1})\|_2^2$. On the other hand, the Bernstein inequalities give us that $\|\Lambda^{2\alpha}(\theta)\|_p \leq Bp^{-1}\|\theta\|_p$ so that, using Theorem 4.2 again, we have $\|\Lambda^\alpha(\theta|\theta|^{p/2-1})\|_2^2 \leq p \int \theta|\theta|^{p-2}\Lambda^{2\alpha}(\theta)\,dx$ $\leq B\|\theta\|_p^p$.

When $p \leq 2$, we use Theorem 5.3. We have $\|e^{-t\Lambda^{2\alpha}}f\|_p^p \leq e^{-c_{\alpha,p}t}\|f\|_p^p$. Looking at the derivatives at $t = 0$ (and using Theorem 4.2), we get

$$(7.4) \qquad c_{\alpha,p}\|f\|_p^p \leq p\int f|f|^{p-2}\Lambda^{2\alpha}f\,dx \leq 4\int |\Lambda^\alpha(f|f|^{\frac{p}{2}-1})|^2\,dx\,.$$

On the other hand, (6.3) and (6.5) give us the converse inequality. $\qquad\square$

Remark. Theorem 7.1 has been proved for $p \geq 2$ by Wu in [21], and Chen, Miao and Zhang in [4].

8. Lie groups of polynomial growth

Since our method is mainly based on the use of symmetric diffusion semigroups, our results may be adapted to various settings. In this section, we consider a connected Lie group G and its Lie algebra $\mathcal{G}$, generated from a set of left-invariant vector fields $(X_i)_{1 \leq i \leq N}$ (in the sense of Hörmander: $\mathcal{G}$ is generated by the fields X_i and their successive Lie brackets). We consider dx a left-invariant Haar measure on G.

We have a Carnot–Carathéodory metric $\rho(x,y) = |y^{-1}.x|_G$ on G associated to the vector fields X_i, [7]. We write $B(x,r)$ for the radius $r > 0$ ball centered at $x \in G$, and $V(r)$ for the volume of the ball, $V(r) = \int_{|y|_G < r} dy$. The volume obeys to two dimensional orders: for $r < 1$, we have $ar^d \leq V(r) \leq br^d$ for some local dimension $d > 0$ and positive constants a, b; for $r \geq 1$, either V has a finite dimensional behaviour, $ar^D \leq V(r) \leq br^D$ for some $D > 0$ (the dimension at infinity), or V grows exponentially, $e^{ar} \leq V(r) \leq e^{br}$. In the first case, G is called a group with polynomial growth (versus exponential growth in the second case).

The sublaplacian on G is the operator $\mathcal{J} = -\sum_{i=1}^N X_i^2$. We define the convolution on G by $f * h(x) = \int_G f(xy^{-1})h(y)\,dy = \int_G f(y)h(y^{-1}x)\,dy$. Then $(e^{-t\mathcal{J}})_{t\geq 0}$ is a semi-group of positive self-adjoint convolution operators on G, so that the theory of symmetric diffusion semigroups can be applied.

We can define Sobolev and Besov spaces on G, [20]. When $p = 2$, the Besov space $\dot{B}_2^{s,2}$ coincides with the Sobolev space $\dot{H}^s = \mathcal{D}(\mathcal{J}^{s/2})$ (normed by $\|f\|_{\dot{H}^s} = \|\mathcal{J}^{s/2}f\|_2$). It is easy to check that Saka's characterization of Besov spaces [16] on stratified Lie groups can be extended to the setting of Lie groups with polynomial growth. More precisely, L. Saloff-Coste [17] proved the following result:

Proposition 8.1. *Let G be a connected Lie group with polynomial growth. For $0 < s < 1$ and $1 \leq p < +\infty$, the norm of the Besov space $\dot{B}_p^{s,p}$ is equivalent to*

$$(8.1) \qquad \|f\|_{\dot{B}_p^{s,p}} = \Big(\iint \frac{|f(x.y) - f(y)|^p}{|y|_G^{sp} \, V(|y|_G)} \, dx \, dy \Big)^{1/p}$$

Now, a direct adaptation of Theorem 4.2 gives:

Theorem 8.2. *Let $\mathcal{J}$ be the sublaplacian operator on a connected Lie group G with polynomial growth. Let $0 < \alpha < 1$ and $2 \leq p < +\infty$. Then there is a positive constant $c_{\alpha,p,G} > 0$ such that*

$$(8.2) \qquad \begin{aligned} c_{\alpha,p,G} \|f\|_{\dot{B}_p^{2\alpha/p,p}}^p &\leq \| |f|^{\frac{p}{2}-1} \|_{\dot{H}^\alpha}^2 = \int |\mathcal{J}^{\alpha/2}(f|f|^{\frac{p}{2}-1})|^2 \, dx \\ &\leq p \int f|f|^{p-2} \mathcal{J}^\alpha(f) \, dx \,. \end{aligned}$$

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