Real-variable characterizations of Orlicz–Hardy spaces on strongly Lipschitz domains of $\mathbb{R}^n$

Dachun Yang and Sibei Yang

Abstract. Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$, whose complement in $\mathbb{R}^n$ is unbounded. Let L be a second order divergence form elliptic operator on $L^2(\Omega)$ with the Dirichlet boundary condition, and the heat semigroup generated by L having the Gaussian property $(G_{\mathrm{diam}(\Omega)})$ with the regularity of its kernels measured by $\mu \in (0, 1]$, where $\mathrm{diam}(\Omega)$ denotes the diameter of Ω. Let Φ be a continuous, strictly increasing, subadditive and positive function on $(0, \infty)$ of upper type 1 and of strictly critical lower type $p_\Phi \in (n/(n + \mu), 1]$. In this paper, the authors introduce the Orlicz–Hardy space $H_{\Phi, r}(\Omega)$ by restricting arbitrary elements of the Orlicz–Hardy space $H_\Phi(\mathbb{R}^n)$ to Ω and establish its atomic decomposition by means of the Lusin area function associated with $\{e^{-tL}\}_{t \geq 0}$. Applying this, the authors obtain two equivalent characterizations of $H_{\Phi, r}(\Omega)$ in terms of the nontangential maximal function and the Lusin area function associated with the heat semigroup generated by L.

1. Introduction

The theory of Hardy spaces on the n-dimensional Euclidean space $\mathbb{R}^n$, was originally initiated by Stein and Weiss in [48]. Later, Fefferman and Stein [20] systematically developed a real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, which plays an important role in various fields of analysis; see, for example, [47], [11], [40], and [46]. It is well known that the Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ is a good substitute for $L^p(\mathbb{R}^n)$ in the study of the boundedness of operators; for example, the classical Riesz transform is bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$. An important feature of the $H^p(\mathbb{R}^n)$ is their characterizations in terms of atomic decompositions, which were established by Coifman [12] when $n = 1$ and Latter [34] when $n > 1$ (see also [51]).

On the other hand, as a generalization of $L^p(\mathbb{R}^n)$, the Orlicz space was introduced by Birnbaum–Orlicz in [7] and Orlicz in [41]; since then, the theory of the Orlicz spaces themselves has been well developed and these spaces have been widely used in probability, statistics, potential theory, and partial differential equations, as well as harmonic analysis and some other fields of analysis (see, for example, [43], [44], [8], [37] and [26]). Moreover, Orlicz–Hardy spaces are also suitable substitutes for the Orlicz spaces in the study of boundedness of operators (see, for example, [27], [50], [29], [31] and [28]). Recall that Orlicz–Hardy spaces and their dual spaces were studied by Janson [27] on $\mathbb{R}^n$ and Viviani [50] on spaces of homogeneous type in the sense of Coifman and Weiss [14].

It is known that the Hardy spaces $H^p(\mathbb{R}^n)$ are essentially related to the Laplacian

$$\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

In recent years, the study of the real-variable theory of various function spaces associated with different differential operators has inspired great interests (see, for example, [2], [3], [18], [52], [16], [22], [21], [28], [29], [31], [30], and [53]). In particular, Orlicz–Hardy spaces associated with some differential operators and their dual spaces were introduced and studied in [31], [29] and [28].

One important aspect of the development in the theory of Hardy spaces is the study of Hardy spaces on domains of $\mathbb{R}^n$ (see, for example, [39], [10], [9], [49], [4], [17], [25] and [24]). Especially, Chang, Krantz and Stein [10] introduced the Hardy spaces $H_r^p(\Omega)$ and $H_z^p(\Omega)$ on the domain Ω for $p \in (0,1]$, respectively, by restricting arbitrary elements of $H^p(\mathbb{R}^n)$ to Ω, and restricting elements of $H^p(\mathbb{R}^n)$ which are zero outside $\overline{\Omega}$ to Ω, where here and in what follows, $\overline{\Omega}$ denotes the *closure* of Ω in $\mathbb{R}^n$. We point out that the Hardy spaces $H_r^p(\Omega)$ and $H_z^p(\Omega)$, when Ω is a bounded smooth domain of $\mathbb{R}^n$ and $p \in (0,1]$, appear naturally in the study of the regularity of the Green operators, respectively, for the Dirichlet boundary problem and the Neumann boundary problem in [10] and [9]. For these Hardy spaces, atomic decompositions have been obtained in [10] when Ω is a special Lipschitz domain or a bounded Lipschitz domain of $\mathbb{R}^n$. Let Ω be a strongly Lipschitz domain, and let $H_r^1(\Omega)$ and $H_z^1(\Omega)$ be defined as in [10]. Auscher and Russ [4] proved that $H_r^1(\Omega)$ and $H_z^1(\Omega)$ can be characterized by the nontangential maximal function and the Lusin area function associated with $\{e^{-t\sqrt{L}}\}_{t \geq 0}$, respectively, under the so-called Dirichlet and the Neumann boundary conditions, where L is an elliptic second-order divergence operator such that for all $t \in (0, \infty)$, the kernel of e^{-tL} has the Gaussian property (G_∞) in the sense of Auscher and Russ in Definition 3 of [4] (see also Definition 2.1 below). Moreover, for these Hardy spaces, Huang [25] established a characterization in terms of the Littlewood–Paley–Stein function associated with L. Assume that the regularity of the kernel of the heat semigroup generated by L is measured by $\mu \in (0,1]$. When Ω is a special Lipschitz domain of $\mathbb{R}^n$, $p \in (n/(n+\mu), 1]$ and L satisfies the Neumann boundary condition, Duong and Yan [17] gave a simple proof of the atomic decomposition for elements in $H_z^p(\Omega)$ via the nontangential maximal function associated with the Poisson semigroup generated by L.

Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$, whose complement in $\mathbb{R}^n$ is unbounded. Let L be a second order divergence form elliptic operator on $L^2(\Omega)$ with the Dirichlet boundary condition, and the heat semigroup generated by L having the Gaussian property $(G_{\mathrm{diam}(\Omega)})$ with the regularity of its kernels measured by $\mu \in (0, 1]$ (see Definition 2.1 below for the definition), where $\mathrm{diam}(\Omega)$ denotes the *diameter* of Ω. Let Φ be a continuous, strictly increasing, subadditive and positive function on $(0, \infty)$ of upper type 1 and of strictly critical lower type $p_\Phi \in (n/(n+\mu), 1]$ (see (2.4) below for the definition of p_Φ). A typical example of such a function is

$$\Phi(t) := t^p$$

for $t \in (0, \infty)$ and $p \in (n/(n+\mu), 1]$. Motivated by [4], [10], [31], [29] and [50], in this paper, we introduce the Orlicz–Hardy space $H_{\Phi, r}(\Omega)$ by restricting elements of the classical Orlicz–Hardy space $H_\Phi(\mathbb{R}^n)$ to Ω, and give its atomic decomposition by means of the Lusin area function associated with the heat semigroup generated by L. Applying this, we obtain two equivalent characterizations of $H_{\Phi, r}(\Omega)$ in terms of the nontangential maximal function and the Lusin area function associated with the heat semigroup generated by L. Let $H^1_{S_P}(\Omega)$ be the Hardy space defined by the Lusin area function associated with the Poisson semigroup generated by L. As a byproduct, by applying the method used in this paper for the atomic decomposition of elements in $H_{\Phi, r}(\Omega)$ via the Lusin area function associated with the heat semigroup generated by L (see Proposition 3.14 below), we also give a direct proof of the atomic decomposition for all $f \in H^1_{S_P}(\Omega)$ in Proposition 3.17 below, which answers the question asked by Duong and Yan in [17] (see Remarks (iii), page 485) in the case that $p = 1$.

To state the main result of this paper, we first recall some necessary notions. Throughout the whole paper, we always assume that Ω is a *strongly Lipschitz domain* of $\mathbb{R}^n$; namely, Ω is a proper open connected set in $\mathbb{R}^n$ whose boundary is a finite union of parts of rotated graphs of Lipschitz maps, at most one of these parts possibly unbounded. It is well known that strongly Lipschitz domains include special Lipschitz domains, bounded Lipschitz domains and exterior domains (see, for example, [4] and [6] for their definitions and properties).

Throughout the whole paper, for the sake of convenience, we choose the norm on $\mathbb{R}^n$ to be the *supremum norm*; namely, for any

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad |x| := \max\{|x_1|, \ldots, |x_n|\}.$$

Balls determined by this norm are cubes associated with the usual Euclidean norm with sides parallel to the axes.

Remark 1.1. Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Then Ω is a space of homogeneous type in the sense of Coifman and Weiss [14]. Furthermore, as a space of homogeneous type, the collection of all balls of Ω is given by the set

$$\{Q \cap \Omega : \text{ cube } Q \subset \mathbb{R}^n \text{ satisfies } x_Q \in \Omega \text{ and } l(Q) \leq 2\mathrm{diam}(\Omega)\},$$

where x_Q denotes the *center* of Q, $l(Q)$ the *sidelength* of Q, and $\mathrm{diam}(\Omega)$ the *diameter* of Ω, namely, $\mathrm{diam}(\Omega) := \sup\{|x - y| : x, y \in \Omega\}$ (see, for example, [4]).

Motivated by [10], we introduce the Orlicz–Hardy space $H_{\Phi,\,r}(\Omega)$ as follows. We first recall the definition of the Orlicz–Hardy space $H_{\Phi}(\mathbb{R}^n)$ introduced by Viviani [50]. Let $\mathcal{S}(\mathbb{R}^n)$ denote the *space of all Schwartz functions* with the classical topology and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual* with the weak $*$-topology. For all $f \in \mathcal{S}'(\mathbb{R}^n)$, let $\mathcal{G}(f)$ denote its *grand maximal function* (see page 90 of [47]).

Definition 1.2. Let Φ be a function of type (p_0, p_1), where $0 < p_0 \le p_1 \le 1$ (see Section 2.2 below for the definition of type (p_0, p_1)). Define

$$H_{\Phi}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\mathcal{G}(f)(x))\, dx < \infty \right\}$$

and

$$\|f\|_{H_{\Phi}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi\Big(\frac{\mathcal{G}(f)(x)}{\lambda} \Big) dx \le 1 \right\}.$$

In what follows, we shall denote by $\mathcal{D}(\Omega)$ the *space of all infinitely differentiable functions with compact support in Ω* endowed with the inductive topology, and by $\mathcal{D}'(\Omega)$ its *topological dual* with the weak $*$-topology, which is called the *space of distributions on Ω*.

Definition 1.3. Let Φ be as in Definition 1.2 and let Ω be a subdomain in $\mathbb{R}^n$. A distribution f on Ω is said to be in the *Orlicz–Hardy space $H_{\Phi,\,r}(\Omega)$* if f is the restriction to Ω of a distribution F in $H_{\Phi}(\mathbb{R}^n)$; namely,

$$H_{\Phi,\,r}(\Omega) := \{ f \in \mathcal{D}'(\Omega) : \text{ there exists an } F \in H_{\Phi}(\mathbb{R}^n) \text{ such that } F|_{\Omega} = f \}$$
$$= H_{\Phi}(\mathbb{R}^n) / \{ F \in H_{\Phi}(\mathbb{R}^n) : F = 0 \text{ on } \Omega \}.$$

Moreover, for all $f \in H_{\Phi,\,r}(\Omega)$, the *quasi-norm* of f in $H_{\Phi,\,r}(\Omega)$ is defined by

$$\|f\|_{H_{\Phi,\,r}(\Omega)} := \inf \left\{ \|F\|_{H_{\Phi}(\mathbb{R}^n)} : F \in H_{\Phi}(\mathbb{R}^n) \text{ and } F|_{\Omega} = f \right\},$$

where the infimum is taken over all $F \in H_{\Phi}(\mathbb{R}^n)$ satisfying $F = f$ on Ω.

Remark 1.4. Let $p \in (0, 1]$. When $\Phi(t) := t^p$ for all $t \in (0, \infty)$, the space $H_{\Phi,\,r}(\Omega)$ was introduced by Chang, Krantz and Stein [10]. In this case, we denote the Orlicz–Hardy spaces $H_{\Phi}(\mathbb{R}^n)$ and $H_{\Phi,\,r}(\Omega)$, respectively, by $H^p(\mathbb{R}^n)$ and $H_r^p(\Omega)$.

We now describe the divergence form elliptic operators considered in this paper. The most typical example is the Laplace operator on the Lipschitz domain of $\mathbb{R}^n$ with the Dirichlet boundary condition. If Ω is a strongly Lipschitz domain of $\mathbb{R}^n$, we denote by $W^{1,\,2}(\Omega)$ the usual *Sobolev space on Ω* equipped with the norm

$$\left(\|f\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)}^2 \right)^{1/2},$$

where ∇f denotes the *distributional gradient* of f. In what follows, $W_0^{1,\,2}(\Omega)$ stands for the *closure of $C_c^{\infty}(\Omega)$ in $W^{1,\,2}(\Omega)$*, where $C_c^{\infty}(\Omega)$ denotes the *set of all $C^{\infty}(\mathbb{R}^n)$ functions on Ω with compact support*.

If $A : \mathbb{R}^n \to M_n(\mathbb{C})$ is a measurable function, define

$$\|A\|_{\infty} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n,\, |\xi| = |\eta| = 1} |A(x)\xi \cdot \overline{\eta}|,$$

where $M_n(\mathbb{C})$ denotes the *set of all $n \times n$ complex-valued matrixes*, ξ, $\eta \in \mathbb{C}^n$ and $\overline{\eta}$ denotes the *conjugate vector* of η. For all $\delta \in (0, 1]$, denote by $\mathcal{A}(\delta)$ the *class of all measurable functions* $A : \mathbb{R}^n \to M_n(\mathbb{C})$ *satisfying the ellipticity condition*; namely, for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{C}^n$,

$$(1.1) \qquad \|A\|_\infty \leq \delta^{-1} \quad \text{and} \quad \Re(A(x)\xi \cdot \xi) \geq \delta|\xi|^2,$$

where above and in what follows, $\Re(A(x)\xi \cdot \xi)$ denotes the *real part* of $A(x)\xi \cdot \xi$. Denote by $\mathcal{A}$ the *union of all $\mathcal{A}(\delta)$ for $\delta \in (0, 1]$.*

When $A \in \mathcal{A}$ and V is a *closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$*, denote by L the *maximal-accretive operator* (see Def. 1.46 in [42] for the definition) on $L^2(\Omega)$ with $D(L) \subset V$ the largest domain such that for all $f \in D(L)$ and $g \in V$,

$$(1.2) \qquad \langle Lf, g \rangle = \int_\Omega A(x)\nabla f(x) \cdot \overline{\nabla g(x)} \, dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the *interior product* in $L^2(\Omega)$. In this sense, for all $f \in D(L)$, we write

$$(1.3) \qquad Lf = -\text{div}(A\nabla f).$$

We recall the following Dirichlet and Neumann boundary conditions of L from page 152 of [4]:

Definition 1.5. Let Ω be a strongly Lipschitz domain and L as in (1.3). The operator L is said to satisfy the *Dirichlet boundary condition* (for simplicity, DBC) if $V = W_0^{1,2}(\Omega)$ and the *Neumann boundary condition* (for simplicity, NBC) if $V = W^{1,2}(\Omega)$.

Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Recall that for an Orlicz function Φ on $(0, \infty)$, a measurable function f on Ω is said to be in the *space $L^\Phi(\Omega)$* if

$$\int_\Omega \Phi(|f(x)|) \, dx < \infty.$$

Moreover, for any $f \in L^\Phi(\Omega)$, define

$$\|f\|_{L^\Phi(\Omega)} := \inf \left\{ \lambda \in (0, \infty) : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

If $p \in (0, 1]$ and $\Phi(t) = t^p$ for $t \in (0, \infty)$, we then *denote $L^\Phi(\Omega)$ simply by $L^p(\Omega)$.*

Definition 1.6. Let Φ satisfy Assumption (A) (see Section 2.2 for the definition of Assumption (A)), let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and let L be as in (1.3). For all $f \in L^2(\Omega)$ and $x \in \Omega$, let

$$\mathcal{N}_h(f)(x) := \sup_{y \in \Omega, \, t \in (0, 2\text{diam}(\Omega)), \, |y-x|<t} \left| e^{-t^2 L}(f)(y) \right|.$$

A function $f \in L^2(\Omega)$ is said to be in $\widetilde{H}_{\Phi, \mathcal{N}_h}(\Omega)$ if $\mathcal{N}_h(f) \in L^\Phi(\Omega)$; moreover, define

$$\|f\|_{H_{\Phi, \mathcal{N}_h}(\Omega)} := \|\mathcal{N}_h(f)\|_{L^\Phi(\Omega)} := \inf \Big\{ \lambda \in (0, \infty) : \int_\Omega \Phi\Big(\frac{\mathcal{N}_h(f)(x)}{\lambda} \Big) \, dx \leq 1 \Big\}.$$

The *Orlicz–Hardy space* $H_{\Phi, \mathcal{N}_h}(\Omega)$ is defined to be the completion of the space $\widetilde{H}_{\Phi, \mathcal{N}_h}(\Omega)$ in the quasi-norm $\| \cdot \|_{H_{\Phi, \mathcal{N}_h}(\Omega)}$.

Remark 1.7. (i) Since Φ is of strictly lower type p_Φ (see (2.4) for its definition), we see that for all $f_1, f_2 \in H_{\Phi, \mathcal{N}_h}(\Omega)$,

$$\big\| f_1 + f_2 \big\|_{H_{\Phi, \mathcal{N}_h}(\Omega)}^{p_\Phi} \leq \big\| f_1 \big\|_{H_{\Phi, \mathcal{N}_h}(\Omega)}^{p_\Phi} + \big\| f_2 \big\|_{H_{\Phi, \mathcal{N}_h}(\Omega)}^{p_\Phi}.$$

(ii) From the completion theorem of Yosida (see page 56 of [55]), it follows that $\widetilde{H}_{\Phi, \mathcal{N}_h}(\Omega)$ is dense in $H_{\Phi, \mathcal{N}_h}(\Omega)$; namely, for any $f \in H_{\Phi, \mathcal{N}_h}(\Omega)$, there exists a Cauchy sequence $\{f_k\}_{k=1}^\infty \subset \widetilde{H}_{\Phi, \mathcal{N}_h}(\Omega)$ such that

$$\lim_{k \to \infty} \|f_k - f\|_{H_{\Phi, \mathcal{N}_h}(\Omega)} = 0.$$

Moreover, if $\{f_k\}_{k=1}^\infty$ is a Cauchy sequence in $\widetilde{H}_{\Phi, \mathcal{N}_h}(\Omega)$, then there exists a unique $f \in H_{\Phi, \mathcal{N}_h}(\Omega)$ such that

$$\lim_{k \to \infty} \|f_k - f\|_{H_{\Phi, \mathcal{N}_h}(\Omega)} = 0.$$

In what follows, $Q(x, t)$ denotes the *closed cube of $\mathbb{R}^n$ centered at x and of the sidelength t with sides parallel to the axes*. Similarly, given $Q := Q(x, t)$ and $\lambda \in (0, \infty)$, we write λQ for the λ-*dilated cube*, which is the cube with the same center x and with sidelength λt. For any $f \in L^2(\Omega)$ and $x \in \Omega$, the *Lusin area functions S_h and $\widetilde{S}_h$ associated with* $\{e^{-t^2 L}\}_{t \geq 0}$ are respectively defined by

$$S_h(f)(x) := \Big\{ \int_{\Gamma(x)} \big| t^2 L e^{-t^2 L}(f)(y) \big|^2 \, \frac{dy \, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/2}$$

and

$$\widetilde{S}_h(f)(x) := \Big\{ \int_{\Gamma(x)} \big| t \nabla e^{-t^2 L}(f)(y) \big|^2 \, \frac{dy \, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/2},$$

where $\Gamma(x)$ is the *cone* defined by

$$\Gamma(x) := \{ (y, t) \in \Omega \times (0, 2\mathrm{diam}(\Omega)) : |y - x| < t \}.$$

Definition 1.8. Let Φ satisfy Assumption (A), let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and let L be as in (1.3). Assume that L satisfies DBC and that the semigroup generated by L has the Gaussian property $(G_{\mathrm{diam}(\Omega)})$. A function $f \in L^2(\Omega)$ is said to be in $\widetilde{H}_{\Phi, S_h}(\Omega)$ if $S_h(f) \in L^\Phi(\Omega)$. Recall that

$$(1.4) \qquad \|S_h(f)\|_{L^\Phi(\Omega)} := \inf \Big\{ \lambda \in (0, \infty) : \int_\Omega \Phi\Big(\frac{S_h(f)(x)}{\lambda} \Big) \, dx \leq 1 \Big\}.$$

Furthermore, define

$$\|f\|_{H_{\Phi, S_h}(\Omega)} := \|S_h(f)\|_{L^{\Phi}(\Omega)}.$$

The *Orlicz–Hardy space* $H_{\Phi, S_h}(\Omega)$ is defined to be the completion of $\widetilde{H}_{\Phi, S_h}(\Omega)$ in the quasi-norm $\|\cdot\|_{H_{\Phi, S_h}(\Omega)}$.

If Ω is bounded, a function $f \in L^2(\Omega)$ is said to be in $\widetilde{H}_{\Phi, S_h, d_{\Omega}}(\Omega)$ if $S_h(f) \in L^{\Phi}(\Omega)$; moreover, define

(1.5) $\quad \|f\|_{H_{\Phi, S_h, d_{\Omega}}(\Omega)}$

$$:= \|S_h(f)\|_{L^{\Phi}(\Omega)} + \inf\left\{ \lambda \in (0, \infty) : \ \Phi\left(\frac{\|e^{-d_{\Omega}^2 L}(f)\|_{L^1(\Omega)}}{\lambda} \right) \leq 1 \right\},$$

where above and in what follows, $d_{\Omega} := 2\mathrm{diam}(\Omega)$ and $\|S_h(f)\|_{L^{\Phi}(\Omega)}$ is as in (1.4). The *Orlicz–Hardy space* $H_{\Phi, S_h, d_{\Omega}}(\Omega)$ is defined to be the completion of the space $\widetilde{H}_{\Phi, S_h, d_{\Omega}}(\Omega)$ in the quasi-norm $\|\cdot\|_{H_{\Phi, S_h, d_{\Omega}}(\Omega)}$.

When Ω is bounded, the *Orlicz–Hardy spaces* $H_{\Phi, \widetilde{S}_h}(\Omega)$ and $H_{\Phi, \widetilde{S}_h, d_{\Omega}}(\Omega)$ are defined, respectively, via replacing S_h by $\widetilde{S}_h$ in the definitions of $H_{\Phi, S_h}(\Omega)$ and $H_{\Phi, S_h, d_{\Omega}}(\Omega)$.

If Ω is bounded, by $|\Omega| < \infty$, we know that $L^2(\Omega) \subset L^1(\Omega)$, which, together with the Gaussian property $(G_{\mathrm{diam}(\Omega)})$ and Fubini's theorem, implies that for all $f \in L^2(\Omega)$, $e^{-d_{\Omega}^2 L}(f) \in L^1(\Omega)$. Thus, if $f \in L^2(\Omega)$ and $S_h(f) \in L^{\Phi}(\Omega)$, then $\|f\|_{H_{\Phi, S_h, d_{\Omega}}(\Omega)}$ and $\|f\|_{H_{\Phi, \widetilde{S}_h, d_{\Omega}}(\Omega)}$ make sense.

In what follows, we denote by Ω^{c} the *complement of Ω in $\mathbb{R}^n$*. The main result of this paper is as follows:

Theorem 1.9. *Let Φ satisfy Assumption* (A) *and let L be as in* (1.3)*. Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ such that Ω^{c} is unbounded. Assume that L satisfies* DBC *and the semigroup generated by L has the Gaussian property $(G_{\mathrm{diam}(\Omega)})$.*

(i) *If Ω is unbounded, then the spaces*

$$H_{\Phi, r}(\Omega), \quad H_{\Phi, \mathcal{N}_h}(\Omega), \quad H_{\Phi, \widetilde{S}_h}(\Omega) \quad and \quad H_{\Phi, S_h}(\Omega)$$

coincide with equivalent norms.

(ii) *If Ω is bounded, then the spaces*

$$H_{\Phi, r}(\Omega), \quad H_{\Phi, \mathcal{N}_h}(\Omega), \quad H_{\Phi, \widetilde{S}_h, d_{\Omega}}(\Omega) \quad and \quad H_{\Phi, S_h, d_{\Omega}}(\Omega)$$

coincide with equivalent norms. Moreover, if, in addition, $n \geq 3$ and (G_{∞}) holds, then the spaces

$$H_{\Phi, \widetilde{S}_h, d_{\Omega}}(\Omega), \quad H_{\Phi, S_h, d_{\Omega}}(\Omega), \quad H_{\Phi, \widetilde{S}_h}(\Omega) \quad and \quad H_{\Phi, S_h}(\Omega)$$

coincide with equivalent norms.

We first point out that the coincidence between $H_{\Phi,\,r}(\Omega)$ and $H_{\Phi,\,\mathcal{N}_h}(\Omega)$ of Theorem 1.9 when $\Phi(t) = t$ for $t \in (0,\infty)$ was already obtained by Auscher and Russ (see Proposition 19, and Theorems 1 and 20 in [4]).

We also remark that although a strongly Lipschitz domain can be regarded as a space of homogeneous type, Theorem 1.9 can not be deduced from a general theory of Hardy spaces on spaces of homogeneous type, since its proof depends on the geometrical properties of strongly Lipschitz domains and the divergence structure of the operator L considered.

The following chains of inequalities give the strategy of the proof of Theorem 1.9 (i). For all $f \in H_{\Phi,\,r}(\Omega) \cap L^2(\Omega)$, we have

$$(1.6) \quad \|f\|_{H_{\Phi,\,r}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,\mathcal{N}_h}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,\widetilde{S}_h}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,S_h}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,r}(\Omega)},$$

where the implicit constants are independent of f. The proof of the first inequality in (1.6) is standard by applying the atomic decomposition of $H_\Phi(\mathbb{R}^n)$ established by Viviani [50] and the relation between $H_{\Phi,\,r}(\Omega)$ and $H_\Phi(\mathbb{R}^n)$ (see Proposition 3.1 below). We prove the second and the third inequalities, respectively, in Propositions 3.5 and 3.13 below. We point out that Proposition 3.5 plays an important role in the proof of Theorem 1.9, and the key step in the proof of Proposition 3.5 is to establish a "good-λ inequality" concerning $\mathcal{N}_h(f)$ and $\widetilde{S}_h(f)$; see Lemma 3.10 below. To show the last inequality of (1.6) in Proposition 3.14 (i) below, for all $f \in H_{\Phi,\,S_h}(\Omega) \cap L^2(\Omega)$, we establish its atomic decomposition by using a Calderón reproducing formula on $L^2(\Omega)$ associated with L (see (3.42) below), the atomic decomposition of functions in the tent space on Ω, and the reflection technology related to Lipschitz domains on $\mathbb{R}^n$ which was proved by Auscher and Russ (see page 183 of [4]) and plays a key role in the proof of Theorem 1.9 (see also Lemma 3.16 below). However, this reflection technology was not necessary in the study of the Orlicz–Hardy space $H_{\Phi,\,z}(\Omega)$ in [54] (see also [4]).

Similar to the proof of Theorem 1.9 (i), the following chains of inequalities give the strategy of the proof of Theorem 1.9 (ii). Namely, we show that for all $f \in H_{\Phi,\,r}(\Omega) \cap L^2(\Omega)$,

$$\|f\|_{H_{\Phi,\,r}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,\mathcal{N}_h}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,S_h,\,d_\Omega}(\Omega)} \gtrsim \|f\|_{H_{\Phi,\,r}(\Omega)},$$

where the implicit constants are independent of f. In this case that Ω is bounded, the Calderón reproducing formula (3.42) on $L^2(\Omega)$ associated with L used in the proof of Theorem 1.9 (i) is never valid. Thus, instead of (3.42), we use a local Calderón reproducing formula on $L^2(\Omega)$ associated with L (see (3.72) below). Moreover, if Ω is bounded, $n \geq 3$ and (G_∞) holds, using the fact that the operator L^{-1} is bounded from $L^p(\Omega)$ into $L^q(\Omega)$ for some $p, q \in (1,\infty)$ satisfying $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{2}{n}$, which can be proved in a manner similar to the proof of Proposition 5.3 in [1], we further show that the second term in (1.5) can be controlled by the Orlicz norm of the Lusin area function $S_h(f)$, which implies the second part of Theorem 1.9 (ii).

Let Φ satisfy Assumption (A), let Ω be an unbounded strongly Lipschitz domain of $\mathbb{R}^n$, and let L be an elliptic second order divergence operator on $L^2(\Omega)$

satisfying the Neumann boundary condition and the Gaussian property (G_∞). As mentioned above, the Orlicz–Hardy space $H_{\Phi,z}(\Omega)$ was introduced in [54] and its several equivalent characterizations, including the nontangential maximal function characterization and the Lusin area function characterization associated with $\{e^{-t\sqrt{L}}\}_{t\geq 0}$, the vertical and the nontangential maximal function characterizations associated with $\{e^{-tL}\}_{t\geq 0}$, and the Lusin area function characterization associated with $\{e^{-tL}\}_{t\geq 0}$, were also obtained therein.

For all $f \in L^2(\Omega)$ and $x \in \Omega$, let

$$S_P(f)(x) := \Big\{ \int_{\widetilde{\Gamma}(x)} \Big| t\, \partial_t e^{-t\sqrt{L}}(f)(y) \Big|^2 \frac{dy\,dt}{t\,|Q(x,t)\cap\Omega|} \Big\}^{1/2},$$

where

$$\widetilde{\Gamma}(x) := \{(y,t) \in \Omega \times (0,\infty): \, |x-y| < t\}.$$

Let

$$\widetilde{H}^1_{S_P}(\Omega) := \big\{ f \in L^2(\Omega): \, \|f\|_{H^1_{S_P}(\Omega)} := \|S_P(f)\|_{L^1(\Omega)} < \infty \big\}.$$

The *Hardy space* $H^1_{S_P}(\Omega)$ is defined to be the completion of $\widetilde{H}^1_{S_P}(\Omega)$ in the norm $\|\cdot\|_{H^1_{S_P}(\Omega)}$. By applying the method used in the proof of Proposition 3.14 (i) below, we also give a direct proof for the atomic decomposition of elements in $H^1_{S_P}(\Omega)$ in Proposition 3.17 below, which gives an answer to the question asked by Duong and Yan in [17] (see Remarks (iii), page 485) in the case $p = 1$. (We point out that the Lusin area function S_P was also given in page 154 of [4] by replacing $|Q(x,t)\cap\Omega|$ by t^n. This may be problematic for obtaining some estimates, like the estimate in line 1 from the bottom of page 164 in [4], by regarding Ω as a space of homogeneous type when Ω is *bounded*.)

The layout of this paper is as follows. In Section 2, we first recall some properties of the divergence form elliptic operator L on $\mathbb{R}^n$ or a strongly Lipschitz domain Ω, and then describe some basic assumptions on L; then we describe some basic assumptions on Orlicz functions and present some properties of these functions. In Section 3, we give the proof of Theorem 1.9.

Finally we make some conventions on notation. Throughout the paper, L always denotes the *second order divergence form elliptic operator* as in (1.3). We denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. We also use $C(\gamma, \beta, \ldots)$ to denote a *positive constant* depending on the indicated parameters $\gamma, \beta, \ldots$ The *symbol* $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The *symbol* $\lfloor s \rfloor$ for $s \in \mathbb{R}$ denotes the maximal integer not more than s; $Q(x,t)$ denotes a *closed cube* in $\mathbb{R}^n$ with *center* $x \in \mathbb{R}^n$ and *sidelength* $l(Q) := t$ and

$$CQ(x,t) := Q(x,Ct).$$

For any given normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For any subset G of $\mathbb{R}^n$, we denote by G^c the *set* $\mathbb{R}^n \setminus G$; for a measurable set E,

denote by χ_E the *characteristic function* of E. We also set $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For any $\theta := (\theta_1, \dots, \theta_n) \in \mathbb{Z}_+^n$, let

$$|\theta| := \theta_1 + \cdots + \theta_n \quad \text{and} \quad \partial_x^\theta := \frac{\partial^{|\theta|}}{\partial x_1^{\theta_1} \cdots \partial x_n^{\theta_n}}.$$

For any sets E, $F \subset \mathbb{R}^n$ and $z \in \mathbb{R}^n$, let

$$\operatorname{dist}(E, F) := \inf_{x \in E,\, y \in F} |x - y| \quad \text{and} \quad \operatorname{dist}(z, E) := \inf_{x \in E} |x - z|.$$

2. Preliminaries

In Subsection 2.1, we first recall some properties of the divergence form elliptic operator L on $\mathbb{R}^n$ on a strongly Lipschitz domain Ω, and then describe some basic assumptions about L. In Subsection 2.2, we describe some basic assumptions about Orlicz functions and then present some properties of these functions.

2.1. The divergence form elliptic operator L

Let L be as in (1.3). Then L generates a semigroup $\{e^{-tL}\}_{t \geq 0}$ of operators that is *analytic* (namely, it has an extension to the complex half cone $|\arg z| < \mu$ for some $\mu \in (0, \pi/2)$) and *contracting* on $L^2(\Omega)$, namely

$$\text{for all } f \in L^2(\Omega) \text{ and } t \in (0, \infty), \ \|e^{-tL}f\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$$

(see, for example, [42] for the details). Also, L has a unique *maximal accretive square root* $\sqrt{L}$ such that $-\sqrt{L}$ generates an analytic and $L^2(\Omega)$-contracting semigroup $\{P_t\}_{t \geq 0}$ with $P_t := e^{-t\sqrt{L}}$, the *Poisson semigroup* for L (see, for example, [32] for details).

Now we recall the Gaussian property of $\{e^{-tL}\}_{t \geq 0}$, introduced by Auscher and Russ (see Definition 3 in [4]), on a strongly Lipschitz domain (see also [5] and [6]).

Definition 2.1. Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and let L be as in (1.3). Let $\beta \in (0, \infty]$. The semigroup generated by L is said to have the *Gaussian property* (G_β), if the following hold:

(i) The *kernel* of e^{-tL}, denoted by K_t, is a measurable function on $\Omega \times \Omega$, and there exist positive constants C and α such that for all $t \in (0, \beta)$ and all x, $y \in \Omega$,

$$(2.1) \qquad\qquad |K_t(x, y)| \leq \frac{C}{t^{n/2}}\, e^{-\alpha \frac{|x-y|^2}{t}};$$

(ii) For all $x \in \Omega$ and $t \in (0, \beta)$, the functions

$$y \mapsto K_t(x, y) \quad \text{and} \quad y \mapsto K_t(y, x)$$

are Hölder continuous in Ω and there exist positive constants C and $\mu \in (0, 1]$ such that for all $t \in (0, \beta)$ and x, y_1, $y_2 \in \Omega$,

$$(2.2) \quad |K_t(x, y_1) - K_t(x, y_2)| + |K_t(y_1, x) - K_t(y_2, x)| \leq \frac{C}{t^{n/2}} \frac{|y_1 - y_2|^\mu}{t^{\mu/2}}.$$

Remark 2.2. (i) The assumption (G_∞) is always satisfied if L is the Laplacian or a real symmetric operator (under DBC or NBC) on $\mathbb{R}^n$ or on a Lipschitz domain except under NBC with Ω bounded (see, for example, [6]).

(ii) The assumption (G_∞) implies that for all $\beta \in (0, \infty)$, (G_β) holds. If β is finite, by Lemma A.1 in page 178 of [4] and the properties of semigroups, we know that (G_β) and (G_1) are equivalent.

The following well-known fact is a simple corollary of the analyticity of the semigroup $\{e^{-tL}\}_{t \geq 0}$. We omit the details.

Lemma 2.3. *Let $\beta \in (0, \infty]$. Assume that L has the Gaussian property (G_β). Then the estimate (2.1) also holds for $t \partial_t K_t$.*

2.2. Orlicz functions

Let Φ be a positive function on $\mathbb{R}_+ := (0, \infty)$. The function Φ is said to be of *upper type p* (resp. *lower type p*) for some $p \in [0, \infty)$, if there exists a positive constant C such that for all $t \in [1, \infty)$ (resp. $t \in (0, 1]$) and $s \in (0, \infty)$,

$$(2.3) \qquad\qquad\qquad \Phi(st) \leq C\,t^p\,\Phi(s).$$

Obviously, if Φ is of lower type p for some $p \in (0, \infty)$, then

$$\lim_{t \to 0^+} \Phi(t) = 0.$$

Thus, for the sake of convenience, if it is necessary, we may assume that $\Phi(0) = 0$. If Φ is of both upper type p_1 and lower type p_0, then Φ is said to be of *type (p_0, p_1)*. The function Φ is said to be of *strictly lower type p* if for all $t \in (0, 1)$ and $s \in (0, \infty)$,

$$\Phi(st) \leq t^p\,\Phi(s),$$

and we define

$$(2.4) \quad p_\Phi := \sup\{p \in (0, \infty)\colon \Phi(st) \leq t^p \Phi(s) \text{ holds for all } t \in (0, 1) \text{ and } s \in (0, \infty)\}.$$

In what follows, p_Φ is called the *strictly critical lower type index of* Φ. We point out that if p_Φ is defined as in (2.4), then Φ is also of strictly lower type p_Φ (see [29] for the proof).

Throughout the paper, we always assume that Φ satisfies the following assumptions:

Assumption (A). *Let μ be as in (2.2), and let Φ be a positive function defined on $\mathbb{R}_+$ which is of upper type 1 and strictly critical lower type $p_\Phi \in (\frac{n}{n+\mu}, 1]$. Also assume that Φ is continuous, strictly increasing, and subadditive.*

Let
$$p \in \Big(\frac{n}{n+\mu}, 1 \Big] \quad \text{and} \quad \Phi(t) = t^p \ \text{ or } \ \Phi(t) = t^p \ln(e^4 + t)$$

for all $t \in (0, \infty)$. Then Φ satisfies Assumption (A) with $p_\Phi = p$ (see [29] and [35] for some other examples).

Notice that if Φ satisfies Assumption (A), then $\Phi(0) = 0$. For any positive function $\widetilde{\Phi}$ of upper type 1 and $p_{\widetilde{\Phi}} \in (\frac{n}{n+\mu}, 1]$, if we set

$$\Phi(t) := \int_0^t \frac{\widetilde{\Phi}(s)}{s} \, ds$$

for all $t \in [0, \infty)$, then by Proposition 3.1 in [50], Φ is equivalent to $\widetilde{\Phi}$; namely, there exists a positive constant C such that

$$C^{-1} \widetilde{\Phi}(t) \le \Phi(t) \le C \widetilde{\Phi}(t)$$

for all $t \in [0, \infty)$; moreover, Φ is a strictly increasing, subadditive and continuous function of upper type 1 and strictly critical lower type

$$p_\Phi \equiv p_{\widetilde{\Phi}} \in \Big(\frac{n}{n+\mu}, 1 \Big].$$

Notice that all our results are the same for equivalent functions satisfying Assumption (A). From this, we deduce that all results with Φ as in Assumption (A) also hold for all positive functions $\widetilde{\Phi}$ of type 1 and strictly critical lower type $p_{\widetilde{\Phi}} \in (\frac{n}{n+\mu}, 1]$.

Since Φ is strictly increasing, we define the function $\rho(t)$ on $\mathbb{R}_+$ by setting, for all $t \in (0, \infty)$,

$$(2.5) \qquad\qquad\qquad \rho(t) := \frac{t^{-1}}{\Phi^{-1}(t^{-1})},$$

where Φ^{-1} is the inverse function of Φ. Then the types of Φ and ρ have the following relation: *If $0 < p_0 \le p_1 \le 1$ and Φ is an increasing function, then Φ is of type (p_0, p_1) if and only if ρ is of type $(p_1^{-1} - 1, p_0^{-1} - 1)$* (see [50] for the proof).

3. Proof of Theorem 1.9

In this section, we present the proof of Theorem 1.9. To this end, we need some auxiliary area functions defined as follows. Recall that $d_\Omega := 2\text{diam}(\Omega)$. Let $\alpha \in (0, \infty)$, ϵ, $R \in (0, d_\Omega)$ and $\epsilon < R$. For all given $f \in L^2(\Omega)$ and $x \in \Omega$, let

$$\widetilde{S}_h^\alpha(f)(x) := \Big\{ \int_{\Gamma_\alpha(x)} \Big| t \, \nabla e^{-t^2 L}(f)(y) \Big|^2 \, \frac{dy \, dt}{t \, |Q(x,t) \cap \Omega|} \Big\}^{1/2}$$

and

$$\widetilde{S}_h^{\epsilon, R, \alpha}(f)(x) := \Big\{ \int_{\Gamma_\alpha^{\epsilon, R}(x)} \Big| t \, \nabla e^{-t^2 L}(f)(y) \Big|^2 \, \frac{dy \, dt}{t \, |Q(x,t) \cap \Omega|} \Big\}^{1/2},$$

where above and in what follows, for all $x \in \Omega$, $\Gamma_\alpha(x)$ and $\Gamma_\alpha^{\epsilon, R}(x)$ are the *cone* and the *truncated cone*, respectively, defined by

$$\Gamma_\alpha(x) := \{(y, t) \in \Omega \times (0, d_\Omega) : |y - x| < \alpha t\}$$

and

$$\Gamma_\alpha^{\epsilon, R}(x) := \{(y, t) \in \Omega \times (\epsilon, R) : |y - x| < \alpha t\}$$

for $\alpha \in (0, \infty)$ and $0 < \epsilon < R < d_\Omega$. *When* $\alpha = 1$, *denote* $\widetilde{S}_h^\alpha(f)$, $\widetilde{S}_h^{\epsilon, R, \alpha}(f)$ *and* $\Gamma_\alpha(x)$ *simply by* $\widetilde{S}_h(f)$, $\widetilde{S}_h^{\epsilon, R}(f)$ *and* $\Gamma(x)$, *respectively.*

To show Theorem 1.9, we first establish the following proposition:

Proposition 3.1. *Let Φ satisfy Assumption* (A), *let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and let L be as in* (1.3). *Assume that the semigroup generated by L has the Gaussian property* $(G_{\mathrm{diam}(\Omega)})$. *Then under* DBC,

$$(H_{\Phi, r}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi, \mathcal{N}_h}(\Omega) \cap L^2(\Omega))$$

and there exists a positive constant C such that for all $f \in H_{\Phi, r}(\Omega) \cap L^2(\Omega)$,

$$\|f\|_{H_{\Phi, \mathcal{N}_h}(\Omega)} \leq C \|f\|_{H_{\Phi, r}(\Omega)}.$$

To show Proposition 3.1, we need the atomic decomposition characterization of the Orlicz–Hardy space $H_\Phi(\mathbb{R}^n)$ established by Viviani in [50]. To state it, we begin with the notions of (ρ, q, s)-atoms and the atomic Orlicz–Hardy space $H^{\rho, q, s}(\mathbb{R}^n)$.

Definition 3.2. Let Φ be as in Definition 1.2, let ρ be as in (2.5), and let $q \in (0, \infty]$ and $s \in \mathbb{Z}_+$. A function a is called a (ρ, q, s)-*atom* if

 (i) $\mathrm{supp}\, a \subset Q$, where Q is a closed cube of $\mathbb{R}^n$;

 (ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |Q|^{1/q-1}[\rho(|Q|)]^{-1}$;

 (iii) for all $\beta := (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}_+^n$ with $|\beta| \leq s$, $\int_{\mathbb{R}^n} a(x) x^\beta \, dx = 0$.

Obviously, when $\Phi(t) := t$ for all $t \in (0, \infty)$, a (ρ, q, s)-atom is just a classical $(1, q, s)$-atom (see, for example, [47]).

Definition 3.3. Let p_0 be as in Definition 1.2, let Φ, q and ρ be as in Definition 3.2, and set

$$s := \lfloor n(1/p_0 - 1) \rfloor.$$

The *atomic Orlicz–Hardy space* $H^{\rho, q, s}(\mathbb{R}^n)$ is defined to be the space of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ that can be written as $f = \sum_j b_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{b_j\}_j$ is a sequence of constant multiples of (ρ, q, s)-atoms, with the constant depending on j, such that for each j, $\mathrm{supp}\, b_j \subset Q_j$ and

$$\sum_j |Q_j| \Phi\left(\frac{\|b_j\|_{L^q(\mathbb{R}^n)}}{|Q_j|^{1/q}}\right) < \infty.$$

Define

$$\Lambda_q(\{b_j\}_j) := \inf \Big\{ \lambda \in (0, \infty) : \ \sum_j |Q_j| \, \Phi\Big(\frac{\|b_j\|_{L^q(\mathbb{R}^n)}}{\lambda |Q_j|^{1/q}} \Big) \leq 1 \Big\}$$

and

$$\|f\|_{H^{\rho, q, s}(\mathbb{R}^n)} := \inf\{\Lambda_q(\{b_j\}_j)\},$$

where the infimum is taken over all decompositions of f as above.

The notion of (ρ, q, s)-atom and the atomic Orlicz–Hardy space $H^{\rho, q, s}(\mathbb{R}^n)$ were introduced by Viviani in [50], in which the following lemma was also obtained (see Theorem 2.1 in [50]):

Lemma 3.4. *Let p_0 be as in Definition* 1.2, *let Φ, q and ρ be as in Definition* 3.2, *and set*

$$s := \lfloor n(1/p_0 - 1) \rfloor.$$

Then the spaces $H_\Phi(\mathbb{R}^n)$ and $H^{\rho, q, s}(\mathbb{R}^n)$ coincide with equivalent norms.

Now we prove Proposition 3.1 by applying Lemma 3.4.

Proof of Proposition 3.1. Let $f \in H_{\Phi, r}(\Omega) \cap L^2(\Omega)$. By the definition of $H_{\Phi, r}(\Omega)$, we know that there exists $\widetilde{f} \in H_\Phi(\mathbb{R}^n)$ such that $\widetilde{f}\big|_\Omega = f$ and

$$(3.1) \qquad \qquad \big\| \widetilde{f} \big\|_{H_\Phi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\Phi, r}(\Omega)}.$$

To show Proposition 3.1, we only need to prove that for any constant multiple of a $(\rho, \infty, 0)$-atom b supported in the closed cube $Q_0 := Q(x_0, r_0)$,

$$(3.2) \qquad \qquad \int_\Omega \Phi(\mathcal{N}_h(b)(x)) \, dx \lesssim |Q_0| \, \Phi\big(\|b\|_{L^\infty(\mathbb{R}^n)} \big).$$

Indeed, for $\widetilde{f} \in H_\Phi(\mathbb{R}^n)$, by Lemma 3.4, there exists a sequence $\{b_i\}_i$ of constant multiples of $(\rho, \infty, 0)$-atoms, with the constant depending on i, such that $\widetilde{f} = \sum_i b_i$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\Lambda_\infty(\{b_i\}_i) \sim \|\widetilde{f}\|_{H_\Phi(\mathbb{R}^n)}.$$

Moreover, by the proof of Theorem 2.1 in [50] and (2.15) in Lemma (2.9) of [36], we know that the supports of $\{b_i\}_i$ have the *finite intersection property*. By this, $f \in L^2(\Omega)$, $\widetilde{f} = \sum_i b_i$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\widetilde{f}\big|_\Omega = f$, we conclude that $f = \sum_i b_i$ almost everywhere on Ω, which further implies that

$$\int_\Omega K_{t^2}(x, y) f(y) \, dy = \sum_i \int_\Omega K_{t^2}(x, y) b_i(y) \, dy.$$

From this, we deduce that for all $x \in \Omega$,

$$\mathcal{N}_h(f)(x) \leq \sum_i \mathcal{N}_h(b_i)(x).$$

By this and the fact that Φ is strictly increasing, continuous and subadditive, if (3.2) holds, we then have

$$\int_{\Omega} \Phi\left(\mathcal{N}_h(f)(x)\right) dx \leq \sum_i \int_{\Omega} \Phi\left(\mathcal{N}_h(b_i)(x)\right) dx \lesssim \sum_i |Q_i| \Phi\left(\|b_i\|_{L^{\infty}(\mathbb{R}^n)}\right),$$

where for each i, $\operatorname{supp} b_i \subset Q_i$. This, together with the facts that for all $\lambda \in (0, \infty)$,

$$\mathcal{N}_h(f/\lambda) = \mathcal{N}_h(f)/\lambda$$

and, for each i,

$$\|b_i/\lambda\|_{L^{\infty}(\mathbb{R}^n)} = \|b_i\|_{L^{\infty}(\mathbb{R}^n)}/\lambda,$$

implies that for all $\lambda \in (0, \infty)$,

$$\int_{\Omega} \Phi\left(\frac{\mathcal{N}_h(f)(x)}{\lambda}\right) dx \lesssim \sum_i |Q_i| \Phi\left(\frac{\|b_i\|_{L^{\infty}(\mathbb{R}^n)}}{\lambda}\right).$$

By this and (3.1), we find that

$$\|f\|_{H_{\Phi, \mathcal{N}_h}(\Omega)} \lesssim \Lambda_{\infty}(\{b_i\}_i) \sim \|\widetilde{f}\|_{H_{\Phi}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\Phi, r}(\Omega)},$$

which, together with the arbitrariness of $f \in H_{\Phi, r}(\Omega) \cap L^2(\Omega)$, yields the conclusions of Proposition 3.1.

It is easy to see that for all $x \in \Omega$,

$$(3.3) \qquad e^{-t^2 L}(b)(x) = \int_{Q_0 \cap \Omega} K_{t^2}(x, y) b(y) \, dy.$$

Now we show (3.2) by considering the following three cases for Q_0:

Case 1) $Q_0 \cap \Omega = \emptyset$. In this case, by (3.3), we know that for all $x \in \Omega$, $\mathcal{N}_h(b)(x) = 0$. From this, it follows that (3.2) holds.

Case 2) $Q_0 \subset \Omega$. In this case, let $\widetilde{Q}_0 := 8Q_0$. Then we have

$$(3.4) \qquad \int_{\Omega} \Phi(\mathcal{N}_h(b)(x)) \, dx = \int_{\widetilde{Q}_0 \cap \Omega} \Phi(\mathcal{N}_h(b)(x)) \, dx + \int_{(\widetilde{Q}_0)^c \cap \Omega} \cdots =: I_1 + I_2.$$

We first estimate I_1. For any $x \in \widetilde{Q}_0$, by (3.3) and (2.1), we see that

$$\mathcal{N}_h(b)(x) \leq \sup_{y \in \Omega, \, t \in (0, d_{\Omega}), \, |x-y| < t} \int_{\Omega} |K_{t^2}(y, z)| \, |b(z)| \, dz \lesssim \|b\|_{L^{\infty}(\mathbb{R}^n)},$$

which, together with the upper type 1 property of Φ, implies that

$$(3.5) \qquad I_1 \lesssim \int_{\widetilde{Q}_0} \Phi(\|b\|_{L^{\infty}(\mathbb{R}^n)}) \, dx \lesssim |Q_0| \, \Phi\left(\|b\|_{L^{\infty}(\mathbb{R}^n)}\right).$$

Now we estimate I$_2$. Let $x \in (\widetilde{Q}_0)^c \cap \Omega$, $t \in (0, d_\Omega)$ and $y \in \Omega$ satisfy $|x-y| < t$. By the moment condition for b and (3.3), we have

$$(3.6) \qquad e^{-t^2 L}(b)(y) = \int_{Q_0} [K_{t^2}(y, z) - K_{t^2}(y, x_0)] \, b(z) \, dz.$$

Since $p_\Phi \in (\frac{n}{n+\mu}, 1]$, there exists $\widetilde{\mu} \in (0, \mu)$ such that $p_\Phi > \frac{n}{n+\widetilde{\mu}}$. Now we estimate $e^{-t^2 L}(b)(y)$ by considering the following two cases for t:

(i) $t < \frac{1}{4}|x - x_0|$. In this case, let $z \in Q_0$. Then

$$|x - x_0| \le |x - y| + |y - x_0| < \frac{1}{4}|x - x_0| + |y - x_0|,$$

which yields $|x - x_0| < \frac{4}{3}|y - x_0|$. Moreover,

$$|x - x_0| \ge 4r_0 \ge 4|z - x_0|.$$

Thus, we have

$$|y - x_0| \ge \frac{3}{4}|x - x_0| \ge 3|z - x_0|,$$

which implies that

$$(3.7) \qquad |y - z| \ge |y - x_0| - |z - x_0| \ge \frac{2}{3}|y - x_0| \ge \frac{1}{2}|x - x_0|.$$

Thus, by (3.7), (2.1) and (2.2), we obtain

$$|K_{t^2}(y, z) - K_{t^2}(y, x_0)| \lesssim \frac{|z - x_0|^{\widetilde{\mu}}}{|x - x_0|^{n+\widetilde{\mu}}},$$

which, together with (3.6), implies that

$$(3.8) \qquad \left| e^{-t^2 L}(b)(y) \right| \lesssim \frac{r_0^{n+\widetilde{\mu}}}{|x - x_0|^{n+\widetilde{\mu}}} \, \|b\|_{L^\infty(\mathbb{R}^n)}.$$

(ii) $t \ge \frac{1}{4}|x - x_0|$. In this case, by (2.2), we obtain

$$|K_{t^2}(y, z) - K_{t^2}(y, x_0)| \lesssim \frac{|z - x_0|^\mu}{t^{n+\mu}} \lesssim \frac{|z - x_0|^\mu}{|x - x_0|^{n+\mu}} \lesssim \frac{|z - x_0|^{\widetilde{\mu}}}{|x - x_0|^{n+\widetilde{\mu}}},$$

which, together with (3.6), implies that (3.8) also holds in this case.

By the estimates obtained in (i) and (ii), and the arbitrariness of $y \in \Omega$ satisfying $|x - y| < t$, we see that

$$\mathcal{N}_h(b)(x) \lesssim \frac{r_0^{n+\widetilde{\mu}}}{|x - x_0|^{n+\widetilde{\mu}}} \, \|b\|_{L^\infty(\mathbb{R}^n)},$$

which, together with the lower type p_Φ property of Φ and $p_\Phi > \frac{n}{n+\widetilde{\mu}}$, implies that

$$(3.9) \quad I_2 \lesssim \int_{4r_0}^{d_\Omega} \Phi\Big(\frac{r_0^{n+\widetilde{\mu}}}{s^{n+\widetilde{\mu}}} \|b\|_{L^\infty(\mathbb{R}^n)}\Big) s^{n-1} \, ds$$

$$\lesssim \Phi(\|b\|_{L^\infty(\mathbb{R}^n)}) \, r_0^{(n+\widetilde{\mu})p_\Phi} \int_{4r_0}^{\infty} s^{n-(n+\widetilde{\mu})p_\Phi-1} \, ds \sim |Q_0| \, \Phi(\|b\|_{L^\infty(\mathbb{R}^n)}).$$

Thus, by (3.4), (3.5) and (3.9), we know that (3.2) holds in this case.

Case 3) $Q_0 \cap \partial\Omega \neq \emptyset$. In this case, recall that for any $x \in \Omega$, $t \in (0,\infty)$ and $y \in \partial\Omega$, $K_t(x,y) = 0$ (see, for example, page 156 of [4]). Take $y_0 \in Q_0 \cap \partial\Omega$. Then we find that for any $x \in \Omega$ and $t \in (0, d_\Omega)$, $K_{t^2}(x, y_0) = 0$, which further implies that for any $x \in \Omega$,

$$e^{-t^2 L}(b)(x) = \int_{Q_0 \cap \Omega} [K_{t^2}(x,y) - K_{t^2}(x,y_0)] \, b(y) \, dy.$$

The remaining estimates are similar to those of Case 2). We omit the details. This completes the proof of Proposition 3.1. □

To show Theorem 1.9, we need the following key proposition:

Proposition 3.5. *Let* Φ, Ω *and* L *be as in Proposition* 3.1. *Then under* DBC, *there exists a positive constant* C *such that for all* $f \in H_{\Phi, \mathcal{N}_h}(\Omega) \cap L^2(\Omega)$,

$$\big\|\widetilde{S}_h(f)\big\|_{L^\Phi(\Omega)} \leq C \, \|f\|_{H_{\Phi, \mathcal{N}_h}(\Omega)}.$$

To show Proposition 3.5, we need Lemmas 3.6 through 3.12.

Lemma 3.6. *Let* Ω *be a strongly Lipschitz domain of* $\mathbb{R}^n$, *let* L *be as in* (1.3), *and set*

$$I_r(x_0, t_0) := (Q(x_0, r) \cap \Omega) \times [t_0 - cr^2, t_0],$$

where $(x_0, t_0) \in \Omega \times (4cr^2, \infty)$, $r \in (0,\infty)$ *and* c *is a positive constant. If*

$$\partial_t u_t = -L u_t$$

in $I_{2r}(x_0, t_0)$, *then there exists a positive constant* C, *depending only on* Ω, c *and* δ *in* (1.1), *such that*

$$(3.10) \qquad \int_{I_r(x_0,t_0)} |\nabla u_t(x)|^2 \, dx \, dt \leq \frac{C}{r^2} \int_{I_{2r}(x_0,t_0)} |u_t(x)|^2 \, dx \, dt.$$

Lemma 3.6 is usually called the *Caccioppoli inequality*. Its proof is similar to that of Lemma 3 (a) in [33]. We omit the details.

Remark 3.7. Let Ω, L, x_0, t_0, r, c and u_t be as in Lemma 3.4 but with $t_0^2 \in (4cr^2, \infty)$. Then, by making a change of variables in (3.10), we see that

$$\int_{\sqrt{t_0^2-cr^2}}^{t_0} \int_{Q(x_0,r)\cap\Omega} t|\nabla u_{t^2}(x)|^2 \, dx \, dt \lesssim \frac{1}{r^2} \int_{\sqrt{t_0^2-4cr^2}}^{t_0} \int_{Q(x_0,2r)\cap\Omega} t|u_{t^2}(x)|^2 \, dx \, dt.$$

In [4] (see page 183), Auscher and Russ proved the following geometric property of strongly Lipschitz domains, which plays an important role in this paper:

Lemma 3.8. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Then there exists a constant $C \in (0,1]$ such that for all cubes Q centered in Ω with $l(Q) \in (0,\infty) \cap (0, d_\Omega]$, it holds that $|Q \cap \Omega| \geq C|Q|$.*

In what follows, we denote by $B((z,\tau), r)$ the *ball in $\mathbb{R}^n \times (0,\infty)$ with center (z,τ) and radius r*; namely,

$$B((z,\tau), r) := \{(x,t) \in \mathbb{R}^n \times (0,\infty) : \max(|x-z|, |t-\tau|) < r\}.$$

Lemma 3.9. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$, $\alpha \in (0,1)$, ϵ, $R \in (0, d_\Omega)$ and $\epsilon < R$. Then there exists a positive constant C, depending only on α, Ω and n, such that for all $f \in L^2(\Omega)$ and $x \in \Omega$,*

$$(3.11) \qquad \widetilde{S}_h^{\epsilon, R, \alpha}(f)(x) \leq C\,[1 + \ln(R/\epsilon)]^{1/2}\,\mathcal{N}_h(f)(x).$$

Proof. Fix $\alpha \in (0,1)$, $0 < \epsilon < R < d_\Omega$ and $x \in \Omega$. Let $f \in L^2(\Omega)$ and for all $t \in (0, d_\Omega)$, $u_t := e^{-t^2 L}(f)$. For all $(z, \tau) \in \Gamma_\alpha^{\epsilon, R}(x)$, let

$$E_{(z,\tau)} := B((z,\tau), \gamma\tau) \cap (\Omega \times (0, d_\Omega)),$$

where $\gamma \in (0,1)$ is a positive constant which is determined later. By the Besicovitch covering lemma, there exists a subcollection $\{E_{(z_j, \tau_j)}\}_j$ of $\{E_{(z,\tau)}\}_{(z,\tau) \in \Gamma_\alpha^{\epsilon, R}(x)}$ such that

$$(3.12) \qquad \Gamma_\alpha^{\epsilon, R}(x) \subset \bigcup_j E_{(z_j, \tau_j)} \quad \text{and} \quad \sum_j \chi_{E_{(z_j, \tau_j)}} \leq M,$$

where M is a positive integer depending only on n. For each j, we *denote $E_{(z_j, \tau_j)}$ simply by E_j*. Then we have the following two facts for E_j:

(i) *For each j, if $(y,t) \in E_j$, then $t \sim \tau_j \sim d_j$, where d_j denotes the distance from E_j to the bottom boundary $\Omega \times \{0\}$.*

Indeed, if $(y,t) \in E_j$, we then have $(1-\gamma)\tau_j < t < (1+\gamma)\tau_j$, which implies that $t \sim \tau_j$. By $d_j = (1-\gamma)\tau_j$, we see that

$$d_j < t < (1+\gamma)\tau_j = \frac{1+\gamma}{1-\gamma}d_j.$$

Thus, $t \sim d_j$.

(ii) *For each j, let*

$$\widetilde{E}_j := B((z_j, \tau_j), 9\gamma\tau_j) \cap (\Omega \times (0, d_\Omega)).$$

If $\gamma \in (0, \frac{1-\alpha}{18})$, then $\widetilde{E}_j \subset \Gamma^{\epsilon/2, 2R}(x)$.

Indeed, for all $(y,t) \in \widetilde{E}_j$, since $(z_j, \tau_j) \in \Gamma_\alpha^{\epsilon, R}(x)$, it follows that $|y - z_j| < 9\gamma\tau_j$ and $|x - z_j| < \alpha\tau_j$. By this, we find that

$$(3.13) \qquad |x-y| < |x-z_j| + |z_j - y| < (9\gamma + \alpha)\tau_j.$$

Moreover, by $|t - \tau_j| < 9\gamma\tau_j$, we know that

$$(1 - 9\gamma)\tau_j < t < (1 + 9\gamma)\tau_j,$$

which implies that $\tau_j < \frac{t}{1-9\gamma}$ if $\gamma \in (0, 1/9)$. From this and (3.13), it follows that $|x - y| < \frac{9\gamma+\alpha}{1-9\gamma}t$. Thus, to guarantee that $\widetilde{E}_j \subset \Gamma(x)$, it suffices to choose $\gamma \in (0, \frac{1-\alpha}{18}]$. Furthermore, by the facts that for any j and $(y,t) \in \widetilde{E}_j$,

$$(1 - 9\gamma)\tau_j < t < (1 + 9\gamma)\tau_j,$$

and $\epsilon < \tau_j < R$, to ensure that $t \in (\epsilon/2, 2R)$, it suffices to take $\gamma \in (0, \frac{1}{18})$. Thus, if we choose $\gamma \in (0, \frac{1-\alpha}{18}]$, we then conclude that for each j, $\widetilde{E}_j \subset \Gamma^{\epsilon/2, 2R}(x)$.

Now we show (3.11). By the fact that $R \in (0, d_\Omega)$ and Lemma 3.8, we know that for all $t \in (\epsilon, R)$,

$$|Q(x,t) \cap \Omega| \sim t^n.$$

From this, (3.12), the above two facts (i) and (ii), and Remark 3.7 (in which, if $\tau_j \in (\epsilon, \frac{d_\Omega}{1+\gamma}]$, we choose $t_0 = (1+\gamma)\tau_j$, $r = \gamma\tau_j$ and $c = \frac{4}{\gamma}$, and if $\tau_j \in (\frac{d_\Omega}{1+\gamma}, d_\Omega)$, we choose $t_0 = d_\Omega$, $r = \gamma\tau_j$ and $c = \frac{4}{\gamma(1+\gamma)^2}$; and in both cases, we need to choose $\gamma \in (0, \min\{\frac{2}{81}, \frac{1-\alpha}{18}\})$), it follows that

$$
\begin{aligned}
\left[\widetilde{S}_h^{\epsilon, R, \alpha}(f)(x)\right]^2 &\sim \int_{\Gamma_\alpha^{\epsilon, R}(x)} |t\nabla u_t(y)|^2 \frac{dy\, dt}{t^{n+1}} \lesssim \sum_j \int_{E_j} |t\nabla u_t(y)|^2 \frac{dy\, dt}{t^{n+1}} \\
&\lesssim \sum_j \int_{(1-\gamma)\tau_j}^{\min\{(1+\gamma)\tau_j, d_\Omega\}} \int_{Q(z_j, \gamma\tau_j)\cap\Omega} t|\nabla u_t(y)|^2 \frac{dy\, dt}{t^n} \\
&\lesssim \sum_j d_j^{-n} \frac{1}{(\gamma\tau_j)^2} \int_{\widetilde{E}_j} t|u_t(y)|^2 \, dy\, dt \\
&\lesssim \left\{ \sum_j d_j^{-n} (\gamma\tau_j)^{-2} |\widetilde{E}_j| (1+9\gamma)\tau_j \right\} [\mathcal{N}_h(f)(x)]^2 \\
&\sim \left\{ \sum_j \int_{E_j} \frac{dy\, dt}{t^{n+1}} \right\} [\mathcal{N}_h(f)(x)]^2 \lesssim \int_{\Gamma^{\epsilon/2, 2R}(x)} \frac{dy\, dt}{t^{n+1}} [\mathcal{N}_h(f)(x)]^2 \\
&\lesssim \int_{\epsilon/2}^{2R} \left\{ \int_{\mathbb{R}^n} \chi_{Q(0,1)}\left(\frac{x-y}{t}\right) dy \right\} t^{-(n+1)}\, dt\, [\mathcal{N}_h(f)(x)]^2 \\
&\sim [1 + \ln(R/\epsilon)] [\mathcal{N}_h(f)(x)]^2,
\end{aligned}
$$

which implies that

$$\widetilde{S}_h^{\epsilon, R, \alpha}(f)(x) \lesssim [1 + \ln(R/\epsilon)]^{1/2} \mathcal{N}_h(f)(x).$$

Thus, (3.11) holds, which completes the proof of Lemma 3.9. $\qquad\square$

Lemma 3.10. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$, let L be as in* (1.3), *and let $d_\Omega := 2\mathrm{diam}(\Omega)$. Then there exists a positive constant C such that for all $\gamma \in (0, 1]$, $\lambda \in (0, \infty)$, ϵ, $R \in (0, d_\Omega)$ with $\epsilon < R$ and $f \in H_{\Phi, \mathcal{N}_h}(\Omega) \cap L^2(\Omega)$,*

$$(3.14) \qquad \left| \left\{ x \in \Omega : \ \widetilde{S}_h^{\epsilon, R, 1/20}(f)(x) > 2\lambda, \ \mathcal{N}_h(f)(x) \leq \gamma\lambda \right\} \right|$$
$$\leq C\gamma^2 \left| \left\{ x \in \Omega : \ \widetilde{S}_h^{\epsilon, R, 1/2}(f)(x) > \lambda \right\} \right|.$$

We point out that Lemma 3.10 plays a key role in the proof of Proposition 3.5. The inequality (3.14) is usually called the "*good-λ inequality*" concerning the maximal function $\mathcal{N}_h(f)$ and the truncated area functions $\widetilde{S}_h^{\epsilon, R, 1/20}(f)$ and $\widetilde{S}_h^{\epsilon, R, 1/2}(f)$.

Proof of Lemma 3.10. To prove this lemma, we borrow some ideas from [3] and [4]. Fix $0 < \epsilon < R < d_\Omega$, $\gamma \in (0, 1]$ and $\lambda \in (0, \infty)$. Let $f \in H_{\Phi, \mathcal{N}_h}(\Omega) \cap L^2(\Omega)$ and

$$O := \left\{ x \in \Omega : \ \widetilde{S}_h^{\epsilon, R, 1/2}(f)(x) > \lambda \right\}.$$

It is easy to show that O is an open subset of Ω.

Now we show (3.14) by considering the following two cases for O:

Case 1) $O \neq \Omega$. In this case, let

$$(3.15) \qquad\qquad O = \bigcup_k (Q_k \cap \Omega)$$

be the Whitney decomposition of O, where $\{Q_k\}_k$ are dyadic cubes of $\mathbb{R}^n$ with disjoint interiors and $(2Q_k) \cap \Omega \subset O \subset \Omega$, but

$$((4Q_k) \cap \Omega) \cap (\Omega \setminus O) \neq \emptyset.$$

To show (3.14), by (3.15) and the disjoint property of $\{Q_k\}_k$, it suffices to show that for all k,

$$(3.16) \quad \left| \left\{ x \in Q_k \cap \Omega : \ \widetilde{S}_h^{\epsilon, R, 1/20}(f)(x) > 2\lambda, \ \mathcal{N}_h(f)(x) \leq \gamma\lambda \right\} \right| \lesssim \gamma^2 |Q_k \cap \Omega|.$$

From now on, we fix k and denote by l_k the sidelength of Q_k.

If $x \in Q_k \cap \Omega$, then

$$(3.17) \qquad\qquad \widetilde{S}_h^{\max\{10l_k, \epsilon\}, R, 1/20}(f)(x) \leq \lambda.$$

Indeed, pick $x_k \in (4Q_k) \cap \Omega$ with $x_k \notin O$. For any $(y, t) \in \Omega \times (0, d_\Omega)$, if $|x - y| < \frac{t}{20}$ and $t \geq \max\{10l_k, \epsilon\}$, then

$$|x_k - y| \leq |x_k - x| + |x - y| < 4l_k + \frac{t}{20} < \frac{t}{2},$$

which implies that

$$\Gamma_{1/20}^{\max\{10l_k, \epsilon\}, R}(x) \subset \Gamma_{1/2}^{\max\{10l_k, \epsilon\}, R}(x_k).$$

By this, we know that

$$\widetilde{S}_h^{\max\{10l_k, \epsilon\}, R, 1/20}(f)(x) \leq \widetilde{S}_h^{\max\{10l_k, \epsilon\}, R, 1/2}(f)(x_k) \leq \lambda.$$

Thus, the claim (3.17) holds.

If $\epsilon \geq 10l_k$, by (3.17), we see that (3.16) holds. If $\epsilon < 10l_k$, to show (3.16), by the fact that

$$\widetilde{S}_h^{\epsilon, R, 1/20}(f) \leq \widetilde{S}_h^{\epsilon, 10l_k, 1/20}(f) + \widetilde{S}_h^{10l_k, R, 1/20}(f)$$

and (3.17), it remains to show that

(3.18) $$|\{x \in Q_k \cap F : g(x) > \lambda\}| \lesssim \gamma^2 |Q_k \cap \Omega|,$$

where $g := \widetilde{S}_h^{\epsilon, 10l_k, 1/20}(f)$ and

$$F := \{x \in \Omega : \mathcal{N}_h(f)(x) \leq \gamma\lambda\}.$$

By Chebyshev's inequality, we see that (3.18) follows from

(3.19) $$\int_{Q_k \cap F} [g(x)]^2 \, dx \lesssim (\gamma\lambda)^2 |Q_k \cap \Omega|.$$

Now we prove (3.19). It is easy to see that F is a closed subset of Ω.

If $\epsilon \geq 5l_k$, then by the definitions of g and F, and Lemma 3.9, we have

$$\int_{Q_k \cap F} [g(x)]^2 \, dx \lesssim \int_{Q_k \cap F} [\mathcal{N}_h(f)(x)]^2 \, dx \lesssim (\gamma\lambda)^2 |Q_k \cap F| \lesssim (\gamma\lambda)^2 |Q_k \cap \Omega|,$$

which shows (3.19) in this case.

Assume from now on that $\epsilon < 5l_k$. Let

(3.20) $$G := \big\{(y, t) \in \Omega \times (\epsilon, \min\{10l_k, d_\Omega\}) : \ \psi(y) < \tfrac{t}{20}\big\},$$

where

(3.21) $$\psi(y) := \operatorname{dist}(y, Q_k \cap F).$$

By the geometric properties of Ω, we have

(3.22) $$\int_{Q_k \cap F} [g(x)]^2 \, dx \lesssim \int_G t |\nabla u_t(y)|^2 \, dy \, dt.$$

Indeed, if Ω is unbounded, by Lemma 3.8, we know that for all $x \in \Omega$ and $t \in (0, \infty)$,

$$|Q(x, t) \cap \Omega| \sim |Q(x, t)|.$$

Thus, in this case, we have

$$\int_{Q_k \cap F} [g(x)]^2 \, dx = \int_{Q_k \cap F} \Big\{ \int_{\Gamma_{1/20}^{\epsilon, 10l_k}(x)} |t\nabla u_t(y)|^2 \, \frac{dy \, dt}{t |Q(x, t) \cap \Omega|} \Big\} \, dx$$

$$\lesssim \int_G \Big\{ \int_\Omega t^{1-n} \chi_{Q(0,1)} \Big(\frac{20[x-y]}{t} \Big) \, dx \Big\} |\nabla u_t(y)|^2 \, dy \, dt$$

$$\lesssim \int_G t |\nabla u_t(y)|^2 \, dy \, dt.$$

That is, (3.22) holds in this case. If Ω is bounded, we first assume that $\mathrm{diam}(\Omega)$ $\leq 10l_k$. Then

$$\int_{Q_k \cap F} [g(x)]^2 \, dx = \int_{Q_k \cap F} \Big\{ \int_{\Gamma_{1/20}^{\epsilon, \, \mathrm{diam}(\Omega)}(x)} |t\nabla u_t(y)|^2 \, \frac{dy \, dt}{t|Q(x,t) \cap \Omega|} \Big\} \, dx$$

$$+ \int_{Q_k \cap F} \Big\{ \int_{\Gamma_{1/20}^{\mathrm{diam}(\Omega), \, 10l_k}} \cdots \Big\} \, dx$$

$$\lesssim \int_G \Big\{ \int_\Omega t^{1-n} \chi_{Q(0,1)} \Big(\frac{20[x-y]}{t} \Big) \, dx \Big\} |\nabla u_t(y)|^2 \, dy \, dt$$

$$+ \int_G \Big\{ \frac{1}{|\Omega|} \int_\Omega t \chi_{Q(0,1)} \Big(\frac{20[x-y]}{t} \Big) \, dx \Big\} |\nabla u_t(y)|^2 \, dy \, dt$$

$$\lesssim \int_G t \, |\nabla u_t(y)|^2 \, dy \, dt,$$

which is the desired relation. If Ω is bounded and $\mathrm{diam}(\Omega) > 10l_k$, then

$$g \leq \widetilde{S}_h^{\epsilon, \, \mathrm{diam}(\Omega), \, 1/20}(f),$$

which, together with an argument similar to the above, shows that (3.22) also holds in this case. Thus, (3.22) is always true.

Let

$$E := \big\{ y \in \Omega : \text{ there exists } t \in (\epsilon, \min\{10l_k, \, d_\Omega\}) \text{ such that } \psi(y) < \tfrac{t}{20} \big\}.$$

Then $E \subset 2Q_k \cap \Omega$. Indeed, if $y \in E$, then there exist $t \in (\epsilon, \min\{10l_k, \, d_\Omega\})$ such that $(y,t) \in G$ and $x \in Q_k \cap F$ such that $|x - y| < \tfrac{t}{20}$. By $t < 10l_k$, we have $|x - y| < \tfrac{10l_k}{20} = \tfrac{l_k}{2}$, which implies that $E \subset 2Q_k \cap \Omega$.

Let

$$\widetilde{G} := \big\{ (y,t) \in \Omega \times \big(\tfrac{\epsilon}{5}, \min\{40l_k, \, d_\Omega\} \big) : \ \psi(y) < t \big\}.$$

Then for all $(y,t) \in \widetilde{G}$,

$$(3.23) \qquad\qquad\qquad |u_t(y)| \leq \gamma\lambda.$$

Indeed, for any $(y,t) \in \widetilde{G}$, there exists $x \in Q_k \cap F$ such that $|x - y| < t$ with $t \in (\tfrac{\epsilon}{5}, \min\{40l_k, \, d_\Omega\})$, which implies that $(y,t) \in \Gamma(x)$. Thus, by the definitions of F and $\mathcal{N}_h(f)$, we have

$$|u_t(y)| \leq \mathcal{N}_h(f)(x) \leq \gamma\lambda.$$

To finish the proof of Lemma 3.10, we need the following conclusion, which is just Lemma 3.5 in [54]:

Lemma 3.11. *Let*

$$D := \big\{ (y,t) \in \Omega \times (\epsilon, 10l_k) : \ \psi(y) < \tfrac{t}{20} \big\}$$

and

$$D_1 := \big\{ (y,t) \in \Omega \times \big(\tfrac{\epsilon}{2}, 20l_k \big) : \ \psi(y) < \tfrac{t}{10} \big\},$$

where ψ is as in (3.21).

Then there exists $\widetilde{\zeta} \in C^\infty(D_1) \cap C(\overline{D}_1)$ satisfying that

$$0 \leq \widetilde{\zeta} \leq 1, \ \widetilde{\zeta} \equiv 1 \ on \ D, \quad |\nabla\widetilde{\zeta}(y,t)| \lesssim \frac{1}{t} \ for \ all \ (y,t) \in D_1,$$

and

$$\mathrm{supp}\,\widetilde{\zeta} \subset D_1 \cup \left\{\partial\Omega \times \left(\tfrac{\epsilon}{2}, 20l_k\right)\right\},$$

where above and in what follows, $\overline{D}_1$ denotes the closure of D_1 in $\mathbb{R}^{n+1}$.

Now we continue proving Lemma 3.10 by using Lemma 3.11. Let

$$G_1 := \left\{(y,t) \in \Omega \times \left(\tfrac{\epsilon}{2}, \min\{20l_k, d_\Omega\}\right) : \ \psi(y) < \tfrac{t}{10}\right\}$$

and let $\widetilde{\zeta}$ be as in Lemma 3.11. Let $\zeta := \widetilde{\zeta}\chi_{\overline{\Omega}\times(0,d_\Omega]}$. Then

$$\zeta \in C^\infty(G_1) \cap C(\overline{G}_1), \ 0 \leq \zeta \leq 1, \ \zeta \equiv 1 \ \text{on} \ G, |\nabla\zeta(y,t)| \lesssim \frac{1}{t} \ \text{for all} \ (y,t) \in G_1,$$

and

$$\mathrm{supp}\,\zeta \subset G_1 \cup \left\{\partial\Omega \times \left(\tfrac{\epsilon}{2}, \min\{20l_k, d_\Omega\}\right)\right\}.$$

Recall that $u_t := e^{-t^2 L}(f)$ for all $t \in (0, d_\Omega)$. By $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on G, and (1.1), we have

$$(3.24) \quad \int_G t|\nabla u_t(y)|^2 \, dy \, dt \leq \int_{G_1} t|\nabla u_t(y)|^2 \zeta(y,t) \, dy \, dt$$

$$\leq \delta^{-1}\Re \int_{G_1} tA(y)\nabla u_t(y) \cdot \nabla\overline{u_t(y)}\zeta(y,t) \, dy \, dt =: \delta^{-1}\Re\mathrm{I},$$

where $A(y)$ and δ are as in (1.1). Let

$$\mathrm{J} := \int_{G_1} tA(y)\nabla u_t(y) \cdot \nabla\zeta(y,t)\overline{u_t(y)} \, dy \, dt.$$

For all $t \in (\epsilon/2, \min\{20l_k, d_\Omega\})$ and all $y \in \Omega$, let $\zeta_t(y) := \zeta(y,t)$. Then $\zeta_t \in C^\infty(\Omega)$. By (1.19) in page 23 of [42], we know that for all $t \in (0, d_\Omega)$,

$$u_t \in D(L) \subset W_0^{1,2}(\Omega),$$

which, together with $\zeta_t \in C^\infty(\Omega)$, implies that for all $t \in (0, d_\Omega)$, $u_t\zeta_t \in W_0^{1,2}(\Omega)$. From this, (1.2) and the fact that

$$\partial_t u_t + 2tL u_t = 0$$

in $L^2(\Omega)$, it follows that

$$(3.25) \quad \mathrm{I} = \int_{G_1} tA(y)\nabla u_t(y) \cdot \nabla\overline{u_t(y)}\zeta(y,t) \, dydt$$

$$= \int_{G_1} tA(y)\nabla u_t(y) \cdot \nabla(\overline{u}_t\zeta_t)(y) \, dydt - \int_{G_1} tA(y)\nabla u_t(y) \cdot \nabla\zeta_t(y)\overline{u_t(y)} \, dydt$$

$$= \int_{G_1} tL u_t(y)(\overline{u}_t\zeta_t)(y) \, dy \, dt - \mathrm{J}$$

$$= -\frac{1}{2}\int_{G_1} \partial_t u_t(y)(\overline{u}_t\zeta_t)(y) \, dy \, dt - \mathrm{J} =: -\frac{1}{2}\mathrm{I}_1 - \mathrm{J}.$$

For I_1, by the fact that $2\Re((\partial_t u_t)\overline{u}_t) = \partial_t |u_t|^2$ and integration by parts, we conclude that

$$\Re I_1 = \frac{1}{2} \int_{G_1} \partial_t |u_t(y)|^2 \zeta(y,t)\, dy\, dt$$
$$= \frac{1}{2} \Big\{ \int_{\partial G_1} |u_t(y)|^2 \zeta(y,t) N(y,t) \cdot (0,0,\dots,1)\, d\sigma(y,t) - \int_{G_1} |u_t(y)|^2 \partial_t \zeta(y,t) dy\, dt \Big\},$$

where ∂G_1 denotes the boundary of G_1, $N(y,t)$ is the unit outward normal vector to G_1, and $d\sigma$ is the surface measure over ∂G_1. This, combined with (3.25), implies that

$$(3.26) \quad \Re I = -\frac{1}{2}\Re I_1 - \Re J = \frac{1}{4}\Big\{ \int_{G_1} |u_t(y)|^2 \partial_t \zeta(y,t)\, dy\, dt$$
$$- \int_{\partial G_1} |u_t(y)|^2 \zeta(y,t) N(y,t) \cdot (0,0,\dots,1)\, d\sigma(y,t) \Big\}$$
$$- \Re\Big\{ \int_{G_1} t A(y) \nabla u_t(y) \cdot \nabla \zeta_t(y) \overline{u_t(y)}\, dy\, dt \Big\}.$$

By $\operatorname{supp}\zeta \subset G_1 \cup \{\partial\Omega \times (\frac{\epsilon}{2}, \min\{20 l_k,\, d_\Omega\})\}$ and the fact that

$$N(y,t) \cdot (0,\dots,0,1) = 0$$

on $\partial\Omega \times (\frac{\epsilon}{2}, \min\{20 l_k,\, d_\Omega\})$, we obtain

$$(3.27) \qquad \int_{\partial G_1} |u_t(y)|^2 \zeta(y,t) N(y,t) \cdot (0,\dots,0,1)\, d\sigma(y,t) = 0.$$

From $\zeta \equiv 1$ on G, we deduce that $\nabla\zeta \equiv 0$ on G. Thus, by this, (3.26) and (3.27), we have

$$\Re I = \frac{1}{4} \int_{G_1 \setminus G} |u_t(y)|^2 \partial_t \zeta(y,t)\, dy dt - \Re\Big\{ \int_{G_1 \setminus G} t A(y) \nabla u_t(y) \cdot \nabla \zeta_t(y) \overline{u_t(y)}\, dy dt \Big\}$$
$$(3.28) \quad =: I_2 + I_3.$$

First, we estimate I_2. By $G_1 \subset \widetilde{G}$ and (3.23), we find that for all $(y,t) \in G_1 \setminus G$, $|u_t(y)| \le \gamma\lambda$. Moreover,

$$G_1 \setminus G = \Big\{(y,t) \in \Omega \times \big(\tfrac{\epsilon}{2}, \min\{20 l_k,\, d_\Omega\}\big):\ \tfrac{t}{20} \le \psi(y) < \tfrac{t}{10}\Big\}$$
$$\cup \Big\{(y,t) \in \Omega \times \big(\tfrac{\epsilon}{2}, \min\{20 l_k,\, d_\Omega\}\big):\ \psi(y) < \tfrac{t}{10},\ \tfrac{\epsilon}{2} \le t < \epsilon\Big\}$$
$$\cup \Big\{(y,t) \in \Omega \times \big(\tfrac{\epsilon}{2}, \min\{20 l_k,\, d_\Omega\}\big):\ \psi(y) < \tfrac{t}{10},\ 10 l_k \le t < 20 l_k\Big\}.$$

From these observations and the fact that for all $(y,t) \in G_1$,

$$|\nabla\zeta(y,t)| \lesssim \frac{1}{t},$$

we deduce that

$$(3.29) \quad \int_{G_1 \setminus G} |u_t(y)|^2 |\partial_t \zeta(y,t)| \, dy \, dt$$

$$\lesssim (\gamma\lambda)^2 \int_{G_1 \setminus G} \frac{dy \, dt}{t}$$

$$\lesssim (\gamma\lambda)^2 \int_{H_1} \left\{ \int_{\epsilon/2}^{\epsilon} \frac{dt}{t} + \int_{10l_k}^{20l_k} \frac{dt}{t} + \int_{10\psi(y)}^{20\psi(y)} \frac{dt}{t} \right\} dy$$

$$\lesssim (\gamma\lambda)^2 |H_1|,$$

where

$$H_1 := \left\{ y \in G_1 : \text{ there exists } t \in \left(\tfrac{\epsilon}{2}, \min\{20l_k, d_\Omega\}\right) \text{ such that } (y,t) \in G_1 \right\}.$$

For all $y \in H_1$, we know that there exists $t \in \left(\frac{\epsilon}{2}, \min\{20l_k, d_\Omega\}\right)$ such that $(y,t) \in G_1$. From this and the definition of G_1, it follows that there exists $x \in Q_k \cap F$ such that $|x - y| < \frac{t}{10}$ with $t \in \left(\frac{\epsilon}{2}, \min\{20l_k, d_\Omega\}\right)$. Thus, $|x - y| < 2l_k$, which implies that $y \in (5Q_k) \cap \Omega$. By this, we know that $H_1 \subset (5Q_k) \cap \Omega$, which, together with (3.19), implies that

$$(3.30) \quad |\mathrm{I}_2| \lesssim \int_{G_1 \setminus G} |u_t(y)|^2 |\partial_t \zeta(y,t)| \, dy \, dt \lesssim (\gamma\lambda)^2 |H_1| \lesssim (\gamma\lambda)^2 |Q_k \cap \Omega|.$$

To estimate I_3, by the facts that $|\nabla \zeta(y,t)| \lesssim \frac{1}{t}$ for all $(y,t) \in G_1$ and that $|u_t(y)| \leq \gamma\lambda$ for all $(y,t) \in G_1$, we have

$$(3.31) \quad |\mathrm{I}_3| \lesssim \int_{G_1 \setminus G} |\nabla u_t(y)| \, |u_t(y)| \, dy \, dt \lesssim \gamma\lambda \int_{G_1 \setminus G} |\nabla u_t(y)| \, dy \, dt.$$

Now, we need to show

$$(3.32) \quad \int_{G_1 \setminus G} |\nabla u_t(y)| \, dy \, dt \lesssim \gamma\lambda |Q_k \cap \Omega|.$$

For all $(y,t) \in (G_1 \setminus G)$ and $\delta_1 \in (0,1)$, let

$$E_{(y,t)} := B((y,t), \delta_1 t) \cap (\Omega \times (0, d_\Omega)), \quad \text{and}$$

$$\widetilde{E}_{(y,t)} := B((y,t), 9\delta_1 t) \cap (\Omega \times (0, d_\Omega)).$$

Take δ_1 small enough such that for all $(y,t) \in (G_1 \setminus G)$,

$$\widetilde{E}_{(y,t)} \subset \left\{ (y,t) \in \Omega \times \left(\tfrac{\epsilon}{5}, \min\{30l_k, d_\Omega\}\right) : \tfrac{t}{40} < \psi(y) < \tfrac{t}{2} \right\}$$
$$\cup \left\{ (y,t) \in \Omega \times \left(\tfrac{\epsilon}{5}, \min\{30l_k, d_\Omega\}\right) : \psi(y) < \tfrac{t}{2}, \tfrac{\epsilon}{5} < t < 2\epsilon \right\}$$
$$\cup \left\{ (y,t) \in \Omega \times \left(\tfrac{\epsilon}{5}, \min\{30l_k, d_\Omega\}\right) : \psi(y) < \tfrac{t}{2}, 5l_k < t < 30l_k \right\} := G_2.$$

By the Besicovitch covering lemma, there exists a sequence $\{E_{(y_j,t_j)}\}_j$ of sets which are a bounded covering of $G_1 \setminus G$. Let $E_j := E_{(y_j,t_j)}$ and $\widetilde{E}_j := \widetilde{E}_{(y_j,t_j)}$. Notice that for all $(y,t) \in E_j$, $t \sim t_j \sim r(E_j)$, where $r(E_j)$ denotes the radius of E_j. From this, Hölder's inequality, Remark 3.7 (in which, if $\tau_j \in (\epsilon, \frac{d_\Omega}{1+\delta_1}]$, we choose $t_0 = (1+\delta_1)\tau_j$, $r = \delta_1\tau_j$ and $c = \frac{4}{\delta_1}$, and if $\tau_j \in (\frac{d_\Omega}{1+\delta_1}, d_\Omega)$, we then choose $t_0 = d_\Omega$,

$r = \delta_1 \tau_j$ and $c = \frac{4}{\delta_1(1+\delta_1)^2}$; and in both cases, we need $\delta_1 \in (0, 2/81))$, and the fact that for all j and $(y, t) \in \widetilde{E}_j$, $|u_t(y)| \leq \gamma\lambda$, it follows that

$$
(3.33) \quad \int_{G_1 \setminus G} |\nabla u_t(y)| \, dy \, dt
$$

$$
\lesssim \sum_j \int_{E_j} |\nabla u_t(y)| \, dy \, dt \lesssim \sum_j |E_j|^{1/2} \Big\{ \int_{E_j} |\nabla u_t(y)|^2 \, dy \, dt \Big\}^{1/2}
$$

$$
\lesssim \sum_j |E_j|^{1/2} [r(E_j)]^{-1} \Big\{ \int_{\widetilde{E}_j} |u_t(y)|^2 \, dy \, dt \Big\}^{1/2}
$$

$$
\lesssim \gamma\lambda \sum_j |E_j| [r(E_j)]^{-1} \lesssim \gamma\lambda \int_{G_2} \frac{dy \, dt}{t}
$$

$$
\lesssim \gamma\lambda \int_{H_2} \Big\{ \int_{\epsilon/5}^{\epsilon} \frac{dt}{t} + \int_{5l_k}^{30l_k} \frac{dt}{t} + \int_{2\psi(y)}^{40\psi(y)} \frac{dt}{t} \Big\} \, dy \lesssim \gamma\lambda |H_2|,
$$

where

$$
H_2 := \big\{ y \in \Omega : \text{ there exists } t \in \big(\tfrac{\epsilon}{5}, 30l_k \big) \text{ such that } (y, t) \in G_2 \big\}.
$$

Similar to the estimate of H_1, we also have $|H_2| \lesssim |Q_k \cap \Omega|$, which, together with (3.33), implies that (3.32) holds. Thus, by (3.31) and (3.32), we see that

$$
|\mathrm{I}_3| \lesssim (\gamma\lambda)^2 |Q_k \cap \Omega|,
$$

which, together with (3.22), (3.28) and (3.30), implies that (3.19) holds. This finishes the proof of Lemma 3.10 in Case 1).

Case 2) $O = \Omega$. In this case, we claim that Ω is bounded. Otherwise, $|\Omega| = \infty$. Indeed, if Ω is unbounded, then $\mathrm{diam}(\Omega) = \infty$. By this and Lemma 3.8, we know that for any cube Q with its center x_Q in Ω,

$$
|\Omega| \geq |Q \cap \Omega| \gtrsim |Q|,
$$

which, together with the arbitrariness of Q, implies that $|\Omega| = \infty$. Moreover, from $f \in H_{\Phi, \mathcal{N}_h}(\Omega)$, we deduce that $\mathcal{N}_h(f) \in L^\Phi(\Omega)$, which, together with Lemma 3.9, implies that $\widetilde{S}_h^{\epsilon, R, 1/2}(f) \in L^\Phi(\Omega)$. By this and the definition of O, we have $|O| < \infty$, which conflicts with $|O| = |\Omega| = \infty$. Thus, the claim holds.

By Lemma 3.9, we know that there exists a positive constant C_1 such that for all $R \in (\mathrm{diam}(\Omega), d_\Omega)$ and $x \in \Omega$,

$$
(3.34) \qquad \widetilde{S}_h^{\mathrm{diam}(\Omega), R, 1/20}(f)(x) \leq C_1 \mathcal{N}_h(f)(x).
$$

Now we continue the proof of Lemma 3.10 by using (3.34). Without loss of generality, we may assume that $R \geq \mathrm{diam}(\Omega)$. Otherwise, we replace R by $\mathrm{diam}(\Omega)$ in (3.14). If $\gamma \geq \frac{1}{C_1}$, then

$$
\big| \big\{ x \in \Omega : \widetilde{S}_h^{\epsilon, R, 1/20}(f)(x) > 2\lambda, \, \mathcal{N}_h(f)(x) \leq \gamma\lambda \big\} \big| \leq |\Omega| \leq C_1^2 \gamma^2 |O| \lesssim \gamma^2 |O|,
$$

which shows Lemma 3.10 in the case that $O = \Omega$ and $\gamma \geq \frac{1}{C_1}$.

If $\gamma < \frac{1}{C_1}$, by the fact that $\mathcal{N}_h(f)(x) \leq \gamma\lambda$ for all $x \in F$ and (3.34), we have that for any $R \geq \mathrm{diam}(\Omega)$ and $x \in F$,

$$\widetilde{S}_h^{\mathrm{diam}(\Omega),\, R,\, 1/20}(f)(x) \leq C_1 \mathcal{N}_h(f)(x) \leq \frac{1}{\gamma}\gamma\lambda = \lambda,$$

which implies that

$$\big\{x \in \Omega: \ \widetilde{S}_h^{\epsilon,\, R,\, 1/20}(f)(x) > 2\lambda,\, \mathcal{N}_h(f)(x) \leq \gamma\lambda\big\}$$
$$\subset \big\{x \in \Omega: \ \widetilde{S}_h^{\epsilon,\, \mathrm{diam}(\Omega),\, 1/20}(f)(x) > \lambda,\, \mathcal{N}_h(f)(x) \leq \gamma\lambda\big\}.$$

Thus, to finish the proof of Lemma 3.10 in this case, it suffices to show that

$$\big|\big\{x \in \Omega: \ \widetilde{S}_h^{\epsilon,\, \mathrm{diam}(\Omega),\, 1/20}(f)(x) > \lambda,\, \mathcal{N}_h(f)(x) \leq \gamma\lambda\big\}\big| \lesssim \gamma^2 |O|.$$

The proof of this is similar to that of (3.18) with $10l_k$ and $Q_k \cap F$ respectively replaced by $\mathrm{diam}(\Omega)$ and Ω. We omit the details. This completes the proof of Lemma 3.10. $\qquad\square$

Lemma 3.12. *Let Φ, Ω and L be as in Proposition 3.1. For all α, $\beta \in (0,\infty)$, $0 \leq \epsilon < R < d_\Omega$ and all $f \in L^2(\Omega)$,*

$$\int_\Omega \Phi\big(\widetilde{S}_h^{\epsilon,\, R,\, \alpha}(f)(x)\big)\, dx \sim \int_\Omega \Phi\big(\widetilde{S}_h^{\epsilon,\, R,\, \beta}(f)(x)\big)\, dx,$$

where the implicit constants are independent of ϵ, R and f.

The proof of Lemma 3.12 is similar to that of Proposition 4 in [13]. We omit the details.

Now we show Proposition 3.5 by using Lemmas 3.10 and 3.12.

Proof of Proposition 3.5. Let $f \in H_{\Phi,\, \mathcal{N}_h}(\Omega) \cap L^2(\Omega)$. By the upper type 1 and the lower type p_Φ properties of Φ, we know that

$$\Phi(t) \sim \int_0^t \frac{\Phi(s)}{s}\, ds$$

for all $t \in (0,\infty)$. From this, Fubini's theorem and Lemma 3.10, it follows that for all ϵ, $R \in (0, d_\Omega)$ with $\epsilon < R$ and $\gamma \in (0,1]$,

$$(3.35) \quad \int_\Omega \Phi\big(\widetilde{S}_h^{\epsilon,\, R,\, 1/20}(f)(x)\big)\, dx$$

$$\sim \int_\Omega \int_0^{\widetilde{S}_h^{\epsilon,\, R,\, 1/20}(f)(x)} \frac{\Phi(t)}{t}\, dt\, dx \sim \int_0^\infty \frac{\Phi(t)}{t} \sigma_{\widetilde{S}_h^{\epsilon,\, R,\, 1/20}(f)}(t)\, dt$$

$$\lesssim \int_0^\infty \frac{\Phi(t)}{t} \sigma_{\mathcal{N}_h(f)}(\gamma t)\, dt + \gamma^2 \int_0^\infty \frac{\Phi(t)}{t} \sigma_{\widetilde{S}_h^{\epsilon,\, R,\, 1/2}(f)}(t/2)\, dt$$

$$\lesssim \frac{1}{\gamma} \int_0^\infty \frac{\Phi(t)}{t} \sigma_{\mathcal{N}_h(f)}(t)\, dt + \gamma^2 \int_0^\infty \frac{\Phi(t)}{t} \sigma_{\widetilde{S}_h^{\epsilon,\, R,\, 1/2}(f)}(t)\, dt$$

$$\sim \frac{1}{\gamma} \int_\Omega \Phi\big(\mathcal{N}_h(f)(x)\big)\, dx + \gamma^2 \int_\Omega \Phi\big(\widetilde{S}_h^{\epsilon,\, R,\, 1/2}(f)(x)\big)\, dx,$$

where

$$\sigma_{\widetilde{S}_h^{\epsilon,\,R,\,1/20}(f)}(t) := \big|\big\{x \in \Omega : \ \widetilde{S}_h^{\epsilon,\,R,\,1/20}(f)(x) > t\big\}\big|.$$

Furthermore, by Lemma 3.12, (3.35) and $\widetilde{S}_h^{\epsilon,\,R,\,1/2}(f) \leq \widetilde{S}_h^{\epsilon,\,R}(f)$, we conclude that for all ϵ, $R \in (0, d_\Omega)$ with $\epsilon < R$ and $\gamma \in (0, 1]$,

$$\int_\Omega \Phi\big(\widetilde{S}_h^{\epsilon,\,R}(f)(x)\big)\,dx \sim \int_\Omega \Phi\big(\widetilde{S}_h^{\epsilon,\,R,\,1/20}(f)(x)\big)\,dx$$
$$\lesssim \frac{1}{\gamma} \int_\Omega \Phi\left(\mathcal{N}_h(f)(x)\right)\,dx + \gamma^2 \int_\Omega \Phi\big(\widetilde{S}_h^{\epsilon,\,R}(f)(x)\big)\,dx,$$

which, together with the facts that for all $\lambda \in (0, \infty)$,

$$\widetilde{S}_h^{\epsilon,\,R}(f/\lambda) = \widetilde{S}_h^{\epsilon,\,R}(f)/\lambda \quad \text{and} \quad \mathcal{N}_h(f/\lambda) = \mathcal{N}_h(f)/\lambda,$$

implies that there exists a positive constant C_2 such that

$$(3.36) \qquad \int_\Omega \Phi\left(\frac{\widetilde{S}_h^{\epsilon,\,R}(f)(x)}{\lambda}\right)\,dx$$
$$\leq C_2\left\{\frac{1}{\gamma} \int_\Omega \Phi\left(\frac{\mathcal{N}_h(f)(x)}{\lambda}\right)\,dx + \gamma^2 \int_\Omega \Phi\left(\frac{\widetilde{S}_h^{\epsilon,\,R}(f)(x)}{\lambda}\right)\,dx\right\}.$$

Take $\gamma \in (0, 1]$ such that $C_2\gamma^2 = 1/2$. Then by (3.36), we see that for all $\lambda \in (0, \infty)$,

$$\int_\Omega \Phi\left(\frac{\widetilde{S}_h^{\epsilon,\,R}(f)(x)}{\lambda}\right)\,dx \lesssim \int_\Omega \Phi\left(\frac{\mathcal{N}_h(f)(x)}{\lambda}\right)\,dx.$$

By the Fatou lemma, upon letting $\epsilon \to 0$ and $R \to d_\Omega$, we find that for any $\lambda \in (0, \infty)$,

$$\int_\Omega \Phi\left(\frac{\widetilde{S}_h(f)(x)}{\lambda}\right)\,dx \lesssim \int_\Omega \Phi\left(\frac{\mathcal{N}_h(f)(x)}{\lambda}\right)\,dx,$$

which implies that

$$\|\widetilde{S}_h(f)\|_{L^\Phi(\Omega)} \lesssim \|\mathcal{N}_h(f)\|_{L^\Phi(\Omega)}.$$

This finishes the proof of Proposition 3.5. □

Proposition 3.13. *Let Φ, Ω and L be as in Proposition 3.1. Then under* DBC, *there exists a positive constant C such that for all $f \in L^2(\Omega)$,*

$$\|S_h(f)\|_{L^\Phi(\Omega)} \leq C\big\|\widetilde{S}_h(f)\big\|_{L^\Phi(\Omega)}.$$

Proof. To show this proposition, we borrow some ideas from [22]. Fix ϵ, $R \in (0, d_\Omega)$ with $\epsilon < R$ and $x \in \Omega$. Let $f \in L^2(\Omega)$ and, for $\alpha \in (0, \infty)$,

$$\widetilde{\widetilde{\Gamma}}_\alpha^{\epsilon,\,R}(x) := \{(y, t) \in \mathbb{R}^n \times (\epsilon, R) : \ |x - y| < \alpha t\}.$$

Take $\eta \in C_c^\infty(\mathbb{R}^n \times (0, \infty))$ such that $\eta \equiv 1$ on $\widetilde{\widetilde{\Gamma}}_1^{\epsilon, R}(x)$, $0 \le \eta \le 1$,

$$\operatorname{supp} \eta \subset \widetilde{\widetilde{\Gamma}}_{3/2}^{\epsilon/2, R/2}(x)$$

and for all $(y, t) \in \widetilde{\widetilde{\Gamma}}_{3/2}^{\epsilon/2, 2R}(x)$, $|\nabla \eta(y, t)| \lesssim \frac{1}{t}$. By the choice of η, we see that

$$\text{for all } t \in (\epsilon/2, 2R), \ \eta_t(\cdot) := \eta(\cdot, t) \in C^\infty(\Omega).$$

In the rest part of this proof, we denote $e^{-t^2 L}(f)$ by u_t for all $t \in (0, d_\Omega)$. Then by (1.19) in page 23 of [42], we know that

$$\text{for any given } t \in (0, d_\Omega), \ u_t \in D(L) \subset W_0^{1, 2}(\Omega).$$

Moreover, by the fact that for all $t \in (0, d_\Omega)$,

$$L u_t = e^{-\frac{t^2}{2} L}\big(L e^{-\frac{t^2}{2} L}(f)\big),$$

and (1.19) in page 23 of [42] again, we know that

$$L u_t \in D(L) \subset W_0^{1, 2}(\Omega),$$

which, together with $\eta_t \in C^\infty(\Omega)$, implies that

$$\text{for all } t \in (0, d_\Omega), \ (L u_t)\eta_t \in W_0^{1, 2}(\Omega).$$

From this, (1.2), the facts that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $\widetilde{\widetilde{\Gamma}}_1^{\epsilon, R}(x)$, and Hölder's inequality, we deduce that

$$S_h^{\epsilon, R}(f)(x) = \Big\{ \int_{\Gamma_1^{\epsilon, R}(x)} \Big| t^2 L e^{-t^2 L}(f)(y) \Big|^2 \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/2}$$

$$\le \Big\{ \int_{\Gamma_{3/2}^{\epsilon/2, 2R}(x)} t^2 L e^{-t^2 L}(f)(y) \overline{t^2 L e^{-t^2 L}(f)(y)} \eta(y, t) \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/2}$$

$$\le \Big\{ \int_{\Gamma_{3/2}^{\epsilon/2, 2R}(x)} t \Big| A(y) \nabla u_t(y) \cdot t \nabla \overline{(t^2 L u_t)(y)} \Big| \eta(y, t) \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/2}$$

$$+ \Big\{ \int_{\Gamma_{3/2}^{\epsilon/2, 2R}(x)} t \Big| A(y) \nabla u_t(y) \cdot \nabla \eta(y, t) \overline{t^3 L u_t(y)} \Big| \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/2}$$

(3.37) $\qquad \lesssim \Big\{ \int_{\Gamma_{3/2}^{\epsilon/2, 2R}(x)} |t \nabla u_t(y)|^2 \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/4}$

$$\times \Big\{ \int_{\Gamma_{3/2}^{\epsilon/2, 2R}(x)} \big| t \nabla (t^2 L u_t)(y) \big|^2 \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/4}$$

$$+ \Big\{ \int_{\Gamma_{3/2}^{\epsilon/2, 2R}(x)} |t \nabla u_t(y)|^2 \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/4}$$

$$\times \Big\{ \int_{\Gamma_{3/2}^{\epsilon/2, 2R}(x)} |t^2 L u_t(y)|^2 \frac{dy\, dt}{t |Q(x, t) \cap \Omega|} \Big\}^{1/4}.$$

For all $(z, \tau) \in \Gamma_{3/2}^{\epsilon/2, \, 2R}(x)$, let

$$E_{(z, \tau)} := B((z, \tau), \gamma\tau) \cap (\Omega \times (0, d_\Omega)),$$

where γ is a positive constant which is determined later. From the Besicovitch covering lemma, it follows that there exists a subcollection $\{E_{(z_j, \tau_j)}\}_j$ of $\{E_{(z, \tau)}\}_{(z, \tau) \in \Gamma_{3/2}^{\epsilon/2, \, 2R}(x)}$ such that (3.12) holds in this case. For each j, we denote $E_{(z_j, \tau_j)}$ simply by E_j. Similar to the facts (i) and (ii) appearing in the proof of Lemma 3.9, we have the following two facts for E_j:

(i) For each j, if $(y, t) \in E_j$, then $t \sim d_j \sim r(E_j)$, where d_j and $r(E_j)$ denote, respectively, the distance from E_j to the bottom boundary $\Omega \times \{0\}$ and the radius of E_j.

(ii) For each j, let

$$\widetilde{E}_j := B((z_j, \tau_j), 9\gamma\tau_j) \cap (\Omega \times (0, d_\Omega)).$$

If $\gamma \in (0, 1/54)$, then $\widetilde{E}_j \subset \Gamma_2^{\epsilon/4, \, 4R}(x)$.

For all $t \in (0, d_\Omega)$, let $v_t := Le^{-tL}(f)$. Then we see that

$$\partial_t v_t + Lv_t = 0.$$

Thus, from Remark 3.7 (in which, if $\tau_j \in (\epsilon, \frac{d_\Omega}{1+\gamma}]$, we choose $t_0 = (1+\gamma)\tau_j$, $r = \gamma\tau_j$ and $c = \frac{4}{\gamma}$, and if $\tau_j \in (\frac{d_\Omega}{1+\gamma}, d_\Omega)$, we choose $t_0 = d_\Omega$, $r = \gamma\tau_j$ and $c = \frac{4}{\gamma(1+\gamma)^2}$; and in both cases, we need to choose $\gamma \in (0, 1/54)$), we deduce that for each j,

$$\int_{E_j} t |\nabla(Lu_t)(y)|^2 \, dy \, dt \lesssim \frac{1}{[r(E_j)]^2} \int_{\widetilde{E}_j} t |Lu_t(y)|^2 \, dy \, dt.$$

By this, the above facts (i) and (ii), and (3.12), we conclude that

$$\int_{\Gamma_{3/2}^{\epsilon/2, \, 2R}(x)} \left| t\nabla(t^2 Lu_t)(y) \right|^2 \frac{dy \, dt}{t|Q(x, t) \cap \Omega|} \leq \sum_j \int_{E_j} \left| t\nabla(t^2 Lu_t)(y) \right|^2 \frac{dy \, dt}{t|Q(x, t) \cap \Omega|}$$

$$\sim \sum_j \frac{[r(E_j)]^4}{|Q(x, r(E_j)) \cap \Omega|} \int_{E_j} t \left| \nabla(Lu_t)(y) \right|^2 \, dy \, dt$$

$$\lesssim \sum_j \frac{[r(E_j)]^2}{|Q(x, r(E_j)) \cap \Omega|} \int_{\widetilde{E}_j} t \left| Lu_t(y) \right|^2 \, dy \, dt$$

$$\sim \sum_j \frac{1}{[r(E_j)]^2} \int_{\widetilde{E}_j} \left| t^3 Lu_t(y) \right|^2 \frac{dy \, dt}{t|Q(x, t) \cap \Omega|}$$

$$\lesssim \int_{\Gamma_2^{\epsilon/4, \, 4R}(x)} \left| t^2 Lu_t(y) \right|^2 \frac{dy \, dt}{t|Q(x, t) \cap \Omega|},$$

which, together with (3.37), implies that

$$S_h^{\epsilon, \, R}(f)(x) \lesssim \left[\widetilde{S}_h^{\epsilon/2, \, 2R, \, 3/2}(f)(x) \right]^{1/2} \left[S_h^{\epsilon/4, \, 4R, \, 2}(f)(x) \right]^{1/2}.$$

By the Fatou lemma, upon letting $\epsilon \to 0$ and $R \to d_\Omega$, we know that

$$S_h(f)(x) \lesssim [\widetilde{S}_h^{3/2}(f)(x)]^{1/2} \, [S_h^2(f)(x)]^{1/2},$$

which, together with Cauchy's inequality, implies that there exists a positive constant C_2 such that for all $\varepsilon \in (0,1)$,

$$(3.38) \qquad S_h(f)(x) \le \frac{C_2}{\varepsilon} \widetilde{S}_h^{3/2}(f)(x) + \varepsilon \, S_h^2(f)(x).$$

Similar to the proof of Lemma 3.12, we find that there exists a positive constant C_3 such that for all $g \in L^2(\Omega)$,

$$\int_\Omega \Phi \left(S_h^2(g)(y) \right) \, dy \le C_3 \int_\Omega \Phi \left(S_h(g)(y) \right) \, dy.$$

From this, (3.38), the strictly lower type p_Φ and the upper type 1 properties of Φ, it follows that there exists a positive constant C such that for all $x \in \Omega$,

$$\int_\Omega \Phi \left(S_h(f)(x) \right) \, dx \le \int_\Omega \Phi \left(\frac{C_2}{\varepsilon} \widetilde{S}_h^{3/2}(f)(x) \right) dx + \int_\Omega \Phi \left(S_h^2(f)(x) \right) \, dx$$

$$(3.39) \qquad \le \frac{CC_2}{\varepsilon} \int_\Omega \Phi \left(\widetilde{S}_h^{3/2}(f)(x) \right) \, dx + C_3 \varepsilon^{p_\Phi} \int_\Omega \Phi \left(S_h(f)(x) \right) \, dx.$$

Take $\varepsilon \in (0,1)$ small enough such that $C_3 \, \varepsilon^{p_\Phi} \le 1/2$. By this, (3.39) and Lemma 3.12, we know that

$$\int_\Omega \Phi(S_h(f)(x)) \, dx \lesssim \int_\Omega \Phi \big(\widetilde{S}_h(f)(x) \big) \, dx,$$

which, together with the facts that for all $\lambda \in (0, \infty)$,

$$S_h(f/\lambda) = S_h(f)/\lambda \quad \text{and} \quad \widetilde{S}_h(f/\lambda) = \widetilde{S}_h(f)/\lambda,$$

implies that

$$\int_\Omega \Phi \left(\frac{S_h(f)(x)}{\lambda} \right) dx \lesssim \int_\Omega \Phi \left(\frac{\widetilde{S}_h(f)(x)}{\lambda} \right) dx.$$

This completes the proof of Proposition 3.13. $\qquad \qquad \square$

To complete the proof of Theorem 1.9, we need the following key proposition:

Proposition 3.14. *Let* Φ, Ω *and* L *be as in Theorem* 1.9. *Assume that* L *satisfies* DBC *and the semigroup generated by* L *has the Gaussian property* $(G_{\mathrm{diam}(\Omega)})$.

(i) *If* Ω *is unbounded, then*

$$(H_{\Phi, \, S_h}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi, \, r}(\Omega) \cap L^2(\Omega))$$

and there is a positive constant C *such that for all* $f \in H_{\Phi, \, S_h}(\Omega) \cap L^2(\Omega)$,

$$\|f\|_{H_{\Phi, \, r}(\Omega)} \le C \|f\|_{H_{\Phi, \, S_h}(\Omega)}.$$

(ii) *If Ω is bounded, then*

$$(H_{\Phi,\, S_h\, d_\Omega}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi,\, r}(\Omega) \cap L^2(\Omega))$$

and there is a positive constant C such that for all $f \in H_{\Phi,\, S_h,\, d_\Omega}(\Omega) \cap L^2(\Omega)$,

$$\|f\|_{H_{\Phi,\, r}(\Omega)} \leq C \|f\|_{H_{\Phi,\, S_h,\, d_\Omega}(\Omega)}.$$

Moreover, if, in addition, $n \geq 3$ and (G_∞) holds, then

$$\begin{aligned}
(H_{\Phi,\, \widetilde{S}_h,\, d_\Omega}(\Omega) \cap L^2(\Omega)) &= (H_{\Phi,\, \widetilde{S}_h}(\Omega) \cap L^2(\Omega)) \\
&= (H_{\Phi,\, S_h,\, d_\Omega}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\, S_h}(\Omega) \cap L^2(\Omega))
\end{aligned}$$

with equivalent norms.

To show Proposition 3.14, we need the atomic decomposition of the tent space on Ω. Now we recall some definitions and notions about the tent space, which was initially introduced by Coifman, Meyer and Stein [13] on $\mathbb{R}^n$, and then generalized by Russ [45] to spaces of homogeneous type in the sense of Coifman and Weiss [14] and [15]. Recall that it is well known that the strongly Lipschitz domain Ω is a space of homogeneous type. For all measurable functions g on $\Omega \times (0, \infty)$ and $x \in \Omega$, define

$$\mathcal{A}(g)(x) := \left\{ \int_{\widetilde{\Gamma}(x)} |g(x,t)|^2 \frac{dy}{|Q(x,t) \cap \Omega|} \frac{dt}{t} \right\}^{1/2},$$

where

$$\widetilde{\Gamma}(x) := \{(y,t) \in \Omega \times (0,\infty) : \ |y - x| < t\}.$$

In what follows, we denote by $T_\Phi(\Omega)$ the *space of all measurable functions g on $\Omega \times (0,\infty)$ such that $\mathcal{A}(g) \in L^\Phi(\Omega)$* and for any $g \in T_\Phi(\Omega)$, define its *quasi-norm* by

$$\|g\|_{T_\Phi(\Omega)} := \|\mathcal{A}(g)\|_{L^\Phi(\Omega)} := \inf \left\{ \lambda \in (0,\infty) : \ \int_\Omega \Phi\left(\frac{\mathcal{A}(g)(x)}{\lambda}\right) dx \leq 1 \right\}.$$

When $\Phi(t) := t$ for all $t \in (0,\infty)$, we denote $T_\Phi(\Omega)$ simply by $T_1(\Omega)$.

A function a on $\Omega \times (0, \infty)$ is called a *$T_\Phi(\Omega)$-atom* if

(i) there exists a cube

$$Q := Q(x_Q, l(Q)) \subset \mathbb{R}^n$$

with $x_Q \in \Omega$ and $l(Q) \in (0,\infty) \cap (0, d_\Omega]$ such that $\operatorname{supp} a \subset \widehat{Q \cap \Omega}$, where here and in what follows,

$$\widehat{Q \cap \Omega} := \left\{ (y,t) \in \Omega \times (0,\infty) : \ |y - x_Q| < \tfrac{l(Q)}{2} - t \right\};$$

(ii)

$$\int_{\widehat{Q \cap \Omega}} |a(y,t)|^2 \frac{dy\, dt}{t} \leq |Q \cap \Omega|^{-1} [\rho(|Q \cap \Omega|)]^{-2}.$$

Since Φ is of upper type 1, it is easy to see that there exists a positive constant C such that for all $T_\Phi(\Omega)$-atoms a, we have $\|a\|_{T_\Phi(\Omega)} \leq C$ (see [28]). By a slight modification of the proof of Theorem 3.1 in [28], we have the following atomic decomposition for functions in $T_\Phi(\Omega)$. We omit the details.

Lemma 3.15. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$ and let Φ satisfy Assumption (A). Then, for any $f \in T_\Phi(\Omega)$, there exist a sequence $\{a_j\}_j$ of $T_\Phi(\Omega)$-atoms and a sequence $\{\lambda_j\}_j$ of numbers such that*

$$(3.40) \qquad f(x,t) = \sum_j \lambda_j a_j(x,t) \quad \text{for almost every } (x,t) \in \Omega \times (0,\infty).$$

Moreover, there exists a positive constant C such that for all $f \in T_\Phi(\Omega)$,

$$\Lambda(\{\lambda_j a_j\}_j) := \inf\left\{ \lambda \in (0,\infty) : \sum_j |Q_j \cap \Omega| \Phi\left(\frac{|\lambda_j| \|a_j\|_{T_2^2(\Omega \times (0,\infty))}}{\lambda |Q_j \cap \Omega|^{1/2}}\right) \leq 1 \right\}$$

$$(3.41) \qquad \leq C \|f\|_{T_\Phi(\Omega)},$$

where $Q_j \cap \Omega$ appears in the support of a_j and

$$\|a_j\|_{T_2^2(\Omega \times (0,\infty))} := \left\{ \int_{\widehat{Q_j \cap \Omega}} |a_j(y,t)|^2 \frac{dy\,dt}{t} \right\}^{1/2}.$$

In [4] (see page 183), Auscher and Russ showed the following property of strongly Lipschitz domains, which plays an important role in the proof of Proposition 3.14:

Lemma 3.16. *Let Ω be a strongly Lipschitz domain of $\mathbb{R}^n$. Then there exists $\rho(\Omega) \in (0,\infty)$ such that for any cube Q satisfying $l(Q) < \rho(\Omega)$ and $2Q \subset \Omega$ but $4Q \cap \partial\Omega \neq \emptyset$, where $\partial\Omega$ denotes the boundary of Ω, there exists a cube $\widetilde{Q} \subset \Omega^c$ such that $l(\widetilde{Q}) = l(Q)$ and the distance from $\widetilde{Q}$ to Q is comparable to $l(Q)$. Furthermore, $\rho(\Omega) = \infty$ if Ω^c is unbounded.*

Now we show Proposition 3.14 by applying Lemmas 3.4, 3.15 and 3.16.

Proof of Proposition 3.14. We first prove (i) of Proposition 3.14 by borrowing some ideas from the proof of Theorem C in page 594 of [15] (see also [23] and [31]). Recall that in this case, since Ω is unbounded, we have $\mathrm{diam}(\Omega) = \infty$. Let $f \in H_{\Phi, S_h}(\Omega) \cap L^2(\Omega)$. Then by the H^∞-functional calculus for L, we know that

$$(3.42) \qquad f = 8 \int_0^\infty (t^2 L e^{-t^2 L})(t^2 L e^{-t^2 L})(f) \frac{dt}{t}$$

in $L^2(\Omega)$ (see also the equation (9) in [25]). Since $f \in H_{\Phi, S_h}(\Omega)$, we find that $S_h(f) \in L^\Phi(\Omega)$, which implies that $t^2 L e^{-t^2 L}(f) \in T_\Phi(\Omega)$ and

$$\|f\|_{H_{\Phi, S_h}(\Omega)} = \left\| t^2 L e^{-t^2 L}(f) \right\|_{T_\Phi(\Omega)}.$$

Then from Lemma 3.15, we deduce that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $T_\Phi(\Omega)$-atoms such that for almost every $(x, t) \in \Omega \times (0, \infty)$,

$$(3.43) \qquad t^2 L e^{-t^2 L}(f)(x) = \sum_j \lambda_j a_j(x, t).$$

For each j, let

$$\alpha_j := 8 \int_0^\infty t^2 L e^{-t^2 L}(a_j) \frac{dt}{t}.$$

Then by (3.42) and (3.43), similar to the proof of Proposition 4.2 in [29], we see that

$$(3.44) \qquad f = \sum_j \lambda_j \alpha_j$$

in $L^2(\Omega)$. For any $T_\Phi(\Omega)$-atom a supported in $\widehat{Q \cap \Omega}$, let

$$(3.45) \qquad \alpha := 8 \int_0^\infty t^2 L e^{-t^2 L}(a) \frac{dt}{t}.$$

To show Proposition 3.14, it suffices to show that there exist a function $\widetilde{\alpha}$ on $\mathbb{R}^n$ such that

$$(3.46) \qquad \widetilde{\alpha}|_\Omega = \alpha$$

and a sequence $\{b_i\}_i$ of harmless constant multiples of $(\rho, 2, 0)$-atoms, with the constant depending on i, such that $\widetilde{\alpha} = \sum_i b_i$ in $L^2(\mathbb{R}^n)$ and

$$(3.47) \qquad \sum_i |Q_i| \, \Phi\Big(\frac{\|b_i\|_{L^2(\mathbb{R}^n)}}{|Q_i|^{1/2}}\Big) \lesssim |Q \cap \Omega| \, \Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q \cap \Omega|^{1/2}}\Big),$$

where for each i, $\operatorname{supp} b_i \subset Q_i$ and $Q \cap \Omega$ appears in the support of a. Indeed, if (3.46) and (3.47) hold, then by (3.46), we know that for each j, there exists a function $\widetilde{\alpha}_j$ on $\mathbb{R}^n$ such that $\widetilde{\alpha}_j|_\Omega = \alpha_j$. Let

$$\widetilde{f} := \sum_j \lambda_j \widetilde{\alpha}_j.$$

Then $\widetilde{f}|_\Omega = f$. Furthermore, from (3.47), we deduce that there exists a sequence $\{b_{j,i}\}_{j,i}$ of harmless constant multiples of $(\rho, 2, 0)$-atoms, with the constant depending on j and i, such that

$$\widetilde{f} = \sum_j \sum_i \lambda_j b_{j,i}$$

and

$$\sum_{j,i} |Q_{j,i}| \Phi\Big(\frac{|\lambda_j| \|b_{j,i}\|_{L^2(\mathbb{R}^n)}}{|Q_{j,i}|^{1/2}}\Big) \lesssim \sum_j |Q_j \cap \Omega| \Phi\Big(\frac{|\lambda_j| \|a_j\|_{T_2^2(\Omega \times (0,\infty))}}{|Q_j \cap \Omega|^{1/2}}\Big),$$

where for each j and i, $\operatorname{supp} b_{j,i} \subset Q_{j,i}$ and $Q_j \cap \Omega$ appears in the support of a_j, which, together with the facts that for all $\lambda \in (0, \infty)$,

$$\|b_{i,j}/\lambda\|_{L^2(\mathbb{R}^n)} = \|b_{i,j}\|_{L^2(\mathbb{R}^n)}/\lambda \quad \text{and} \quad \|a_j/\lambda\|_{T_2^2(\Omega \times (0,\infty))} = \|a_j\|_{T_2^2(\Omega \times (0,\infty))}/\lambda,$$

implies that for all $\lambda \in (0, \infty)$,

$$\sum_{j,i} |Q_{j,i}| \Phi\Big(\frac{|\lambda_j| \|b_{j,i}\|_{L^2(\mathbb{R}^n)}}{\lambda |Q_{j,i}|^{1/2}}\Big) \lesssim \sum_j |Q_j \cap \Omega| \Phi\Big(\frac{|\lambda_j| \|a_j\|_{T_2^2(\Omega \times (0,\infty))}}{\lambda |Q_j \cap \Omega|}\Big).$$

From this and Lemmas 3.4 and 3.15, it follows that $\widetilde{f} \in H_\Phi(\mathbb{R}^n)$ and

(3.48) $$\|\widetilde{f}\|_{H_\Phi(\mathbb{R}^n)} \sim \|\widetilde{f}\|_{H^{\rho,2,0}(\mathbb{R}^n)} \lesssim \|f\|_{H_{\Phi, S_h}(\Omega)}.$$

Thus, $f \in H_{\Phi, r}(\Omega)$ and

$$\|f\|_{H_{\Phi, r}(\Omega)} \lesssim \|f\|_{H_{\Phi, S_h}(\Omega)},$$

which, together with the arbitrariness of $f \in H_{\Phi, S_h}(\Omega) \cap L^2(\Omega)$, implies the desired conclusion, Proposition 3.14 (i).

Let $Q := Q(x_0, r_0)$. Now we show (3.46) and (3.47) by considering the following two cases for Q appearing in the support of a:

Case 1) $8Q \cap \Omega^c \neq \emptyset$. In this case, let

$$R_k(Q) := (2^{k+1}Q \setminus 2^k Q) \cap \Omega$$

if $k \geq 3$ and $R_0(Q) := 8Q \cap \Omega$. Let

$$J_\Omega := \{k \in \mathbb{N} : k \geq 3, |R_k(Q)| > 0\}.$$

For $k \in J_\Omega \cup \{0\}$, let

$$\chi_k := \chi_{R_k(Q)}, \quad \widetilde{\chi}_k := |R_k(Q)|^{-1}\chi_k \quad \text{and} \quad m_k := \int_{R_k(Q)} \alpha(x)\,dx.$$

Then we have

(3.49) $$\alpha = \alpha\chi_0 + \sum_{k \in J_\Omega} \alpha\chi_k$$

almost everywhere and also in $L^2(\Omega)$. Take the cube $\widetilde{Q} \subset \mathbb{R}^n$ such that the center $x_{\widetilde{Q}}$ of $\widetilde{Q}$ satisfies that

$$x_{\widetilde{Q}} \in \Omega^c, \; l(\widetilde{Q}) = l(Q) \text{ and } \operatorname{dist}(Q, \widetilde{Q}) \sim l(Q).$$

Then there exists a cube Q_0^* such that $(Q \cup \widetilde{Q}) \subset Q_0^*$ and

(3.50) $$l(Q_0^*) \sim l(Q).$$

Let
$$b_0 := \alpha\chi_0 - \frac{1}{|\widetilde{Q}\cap\Omega^{\mathrm c}|}\Big\{\int_{R_0(Q)}\alpha(x)\,dx\Big\}\,\chi_{\widetilde{Q}\cap\Omega^{\mathrm c}}.$$

Then
$$\int_{\mathbb{R}^n}b_0(x)\,dx = 0 \quad\text{and}\quad \operatorname{supp}b_0\subset Q_0^*.$$

Similar to the proof of (3.36) in [54], we have

(3.51)
$$\|\alpha\|_{L^2(\Omega)}\lesssim\|a\|_{T_2^2(\Omega\times(0,\infty))}.$$

By the facts that $\Omega^{\mathrm c}$ is an unbounded strongly Lipschitz domain and Lemma 3.8, we know that $|\widetilde{Q}\cap\Omega^{\mathrm c}|\sim|\widetilde{Q}|$. From this, Hölder's inequality, (3.50) and (3.51), we deduce that

$$\begin{aligned}
\|b_0\|_{L^2(\mathbb{R}^n)} &\le \|\alpha\|_{L^2(\Omega)} + \frac{1}{|\widetilde{Q}\cap\Omega^{\mathrm c}|^{1/2}}\Big\{\int_{R_0(\Omega)}|\alpha(x)|^2\,dx\Big\}^{1/2}|Q\cap\Omega|^{1/2}\\
&\lesssim \|\alpha\|_{L^2(\Omega)} + \|\alpha\|_{L^2(\Omega)}\frac{|Q|^{1/2}}{|\widetilde{Q}|^{1/2}}\lesssim\|a\|_{T_2^2(\Omega\times(0,\infty))}\\
&\lesssim \frac{1}{|Q\cap\Omega|^{1/2}\rho(|Q\cap\Omega|)}\sim\frac{1}{|Q|^{1/2}\rho(|Q|)}\sim\frac{1}{|Q_0^*|^{1/2}\rho(|Q_0^*|)}.
\end{aligned}$$

Thus, we know that b_0 is a harmless constant multiple of a $(\rho, 2, 0)$-atom and, by the upper 1 property of Φ,

(3.52)
$$\begin{aligned}
|Q_0^*|\Phi\Big(\frac{\|b_0\|_{L^2(\mathbb{R}^n)}}{|Q_0^*|^{1/2}}\Big) &\lesssim |Q|\Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{|Q|^{1/2}}\Big)\\
&\lesssim |Q\cap\Omega|\Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{|Q\cap\Omega|^{1/2}}\Big).
\end{aligned}$$

To finish the proof in this case, we need the following Fact 1, whose proof is similar to the usual Whitney decomposition of an open set in $\mathbb{R}^n$ (see, for example, [47]). We omit the details.

Fact 1. *For all $k\in J_\Omega$, there exists the Whitney decomposition $\{Q_{k,i}\}_i$ of $R_k(Q)$ about $\partial\Omega$, where $\{Q_{k,i}\}_i$ are dyadic cubes of $\mathbb{R}^n$ with disjoint interiors, and for each i, $2Q_{k,i}\subset\Omega$ but $4Q_{k,i}\cap\partial\Omega\neq\emptyset$.*

Notice that Fact 1 was also used in pages 304–305 of [10] and in page 167 of [4]. Let $\{Q_{k,i}\}_{k\in J_\Omega,i}$ be as in Fact 1. Then for each $k\in J_\Omega$,

$$\alpha\chi_{R_k(Q)} = \sum_i\alpha\chi_{Q_{k,i}}$$

almost everywhere. In what follows, for all $t\in(0,\infty)$, let

$$D_t := s\partial_s K_s|_{s=t^2}.$$

Then for all $x \in R_k(Q)$, by (3.45), Lemmas 2.3 and 3.8, and Hölder's inequality, we see that

$$(3.53) \quad |\alpha(x)| \lesssim \int_0^{r_0} \int_{Q \cap \Omega} |D_t(x,y)| |a(y,t)| \frac{dy\,dt}{t} \lesssim \int_0^{r_0} \int_{Q \cap \Omega} \frac{e^{-\alpha \frac{|x-y|^2}{t^2}}}{t^n} |a(y,t)| \frac{dy\,dt}{t}$$

$$\lesssim \|a\|_{T_2^2(\Omega \times (0,\infty))} \Big\{ \int_0^{r_0} \int_{Q \cap \Omega} \frac{t^2}{|x-y|^{2(n+1)}} \frac{dy\,dt}{t} \Big\}^{1/2}$$

$$\lesssim |x - x_0|^{-(n+1)} r_0 |Q \cap \Omega|^{1/2} \|a\|_{T_2^2(\Omega \times (0,\infty))}$$

$$\lesssim 2^{-k(n+1)} |Q \cap \Omega|^{-1/2} \|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

Moreover, by Lemma 3.16, we know that for each k and i, there exists a cube $\widetilde{Q}_{k,i} \subset \Omega^c$ such that

$$l(\widetilde{Q}_{k,i}) = l(Q_{k,i}) \quad \text{and} \quad \text{dist}\,(\widetilde{Q}_{k,i}, Q_{k,i}) \sim l(Q_{k,i}).$$

Then for each k and i, there exists a cube $Q_{k,i}^*$ such that

$$(Q_{k,i} \cup \widetilde{Q}_{k,i}) \subset Q_{k,i}^* \quad \text{and} \quad l(Q_{k,i}^*) \sim l(Q_{k,i}).$$

For each k and i, let

$$b_{k,i} := \alpha \chi_{Q_{k,i}} - \frac{1}{|\widetilde{Q}_{k,i}|} \Big\{ \int_{Q_{k,i}} \alpha(x)\,dx \Big\} \chi_{\widetilde{Q}_{k,i}}.$$

Then

$$\int_{\mathbb{R}^n} b_{k,i}(x)\,dx = 0 \quad \text{and} \quad \text{supp}\, b_{k,i} \subset Q_{k,i}^*.$$

Furthermore, by (3.53) and Hölder's inequality, we conclude that

$$(3.54) \qquad \|b_{k,i}\|_{L^2(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^2(Q_{k,i})}$$

$$\lesssim 2^{-k(n+1)} |Q \cap \Omega|^{-1/2} |Q_{k,i}^*|^{1/2} \|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

Thus, for each k and i, $b_{k,i}$ is a constant multiple of some $(\rho, 2, 0)$-atom with a constant depending on k and i. Let

$$\widetilde{\alpha} := b_0 + \sum_{k \in J_\Omega} \sum_i b_{k,i}.$$

Then, by the constructions of b_0 and $\{b_{k,i}\}_{k \in J_\Omega, i}$, we know that $\widetilde{\alpha}|_\Omega = \alpha$. Moreover, we claim that $\sum_{k \in J_\Omega} \sum_i b_{k,i}$ converges in $L^2(\mathbb{R}^n)$. Indeed, let M denote the usual Hardy–Littlewood maximal operator. Then by (3.53), the boundedness of the vector-valued Hardy–Littlewood maximal operator established by Fefferman and Stein in Theorem 1(1) of [19], and the disjoint property of $\{Q_{k,i}\}_i$, we find

that, for each $k \in J_\Omega$,

$$\int_{\mathbb{R}^n} \Big[\sum_i b_{k,i}(x) \Big]^2 dx$$

$$\leq \int_{\mathbb{R}^n} \Big[\sum_i 2^{-k(n+1)} |Q \cap \Omega|^{-1/2} \|a\|_{T_2^2(\Omega \times (0,\infty))} \chi_{Q_{k,i}^*}(x) \Big]^2 dx$$

$$\lesssim 2^{-2k(n+1)} |Q \cap \Omega|^{-1} \|a\|_{T_2^2(\Omega \times (0,\infty))}^2 \int_{\mathbb{R}^n} \Big\{ \sum_i \big[M(\chi_{Q_{k,i}})(x) \big]^2 \Big\}^2 dx$$

$$\lesssim 2^{-2k(n+1)} |Q \cap \Omega|^{-1} \|a\|_{T_2^2(\Omega \times (0,\infty))}^2 \int_{\mathbb{R}^n} \Big\{ \sum_i \big[\chi_{Q_{k,i}}(x) \big]^2 \Big\}^2 dx$$

$$\sim 2^{-2k(n+1)} |Q \cap \Omega|^{-1} \|a\|_{T_2^2(\Omega \times (0,\infty))}^2 |R_k(Q)| \lesssim 2^{-k(n+2)} \|a\|_{T_2^2(\Omega \times (0,\infty))}^2,$$

which, together with Minkowski's inequality, implies that

$$\Big\| \sum_{k \in J_\Omega} \sum_i b_{k,i} \Big\|_{L^2(\mathbb{R}^n)} \leq \sum_{k \in J_\Omega} \Big\| \sum_i b_{k,i} \Big\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{k \in J_\Omega} 2^{-k(n/2+1)} \|a\|_{T_2^2(\Omega \times (0,\infty))}$$

$$(3.55) \qquad\qquad\qquad\qquad \lesssim \|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

Thus, the claim holds and hence

$$\widetilde{\alpha} = b_0 + \sum_{k \in J_\Omega} \sum_i b_{k,i}$$

in $L^2(\mathbb{R}^n)$. Furthermore, by (3.52), (3.54), the lower type p_Φ property and $p_\Phi \in (n/(n+1), 1]$, we see that

$$(3.56) \quad |Q_0^*| \Phi\Big(\frac{\|b_0\|_{L^2(\mathbb{R}^n)}}{|Q_0^*|^{1/2}} \Big) + \sum_{k \in J_\Omega} \sum_i |Q_{k,i}^*| \Phi\Big(\frac{\|b_{k,i}\|_{L^2(\mathbb{R}^n)}}{|Q_{k,i}^*|^{1/2}} \Big)$$

$$\lesssim |Q \cap \Omega| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q \cap \Omega|^{1/2}} \Big)$$

$$\qquad + \sum_{k \in J_\Omega} \sum_i |Q_{k,i}^*| \Phi\Big(\frac{2^{-k(n+1)} |Q \cap \Omega|^{-1/2} |Q_{k,i}^*|^{1/2} \|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q_{k,i}^*|^{1/2}} \Big)$$

$$\lesssim |Q \cap \Omega| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q \cap \Omega|^{1/2}} \Big)$$

$$\qquad + \sum_{k=3}^{\infty} |(2^{(k+1)n} Q) \cap \Omega| \Phi\Big(\frac{2^{-k(n+1)} \|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q \cap \Omega|^{1/2}} \Big)$$

$$\lesssim |Q \cap \Omega| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q \cap \Omega|^{1/2}} \Big) \Big\{ 1 + \sum_{k=3}^{\infty} 2^{-[k(n+1)p_\Phi - kn]} \Big\}$$

$$\lesssim |Q \cap \Omega| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q \cap \Omega|^{1/2}} \Big),$$

which implies that $\widetilde{\alpha} \in H_\Phi(\mathbb{R}^n)$ and (3.47) in Case 1.

Case 2) $8Q \subset \Omega$. In this case, let $k_0 \in \mathbb{N}$ be such that $2^{k_0}Q \subset \Omega$ but $(2^{k_0+1}Q) \cap \partial\Omega \neq \emptyset$. Then $k_0 \geq 3$. Let

$$R_k(Q) := (2^{k+1}Q \setminus 2^k Q) \cap \Omega \text{ for } k \in \mathbb{N},$$

and $R_0(Q) := 2Q$. Let

$$J_{\Omega, k_0} := \{k \in \mathbb{N}: \ k \geq k_0 + 1, |R_k(Q)| > 0\}.$$

For $k \in \mathbb{Z}_+$, let

$$\chi_k := \chi_{R_k(Q)}, \ \widetilde{\chi}_k := |R_k(Q)|^{-1}\chi_k, \ m_k := \int_{R_k(Q)} \alpha(x)\, dx,$$

$M_k := \alpha\chi_k - m_k\widetilde{\chi}_k$ and $\widetilde{M}_k := \alpha\chi_k$. Then

$$\alpha = \sum_{k=0}^{k_0} M_k + \sum_{k \in J_{\Omega, k_0}} \widetilde{M}_k + \sum_{k=0}^{k_0} m_k\widetilde{\chi}_k.$$

For $k \in \{0, \ldots, k_0\}$, by the definition of M_k, we know that

$$\int_{\mathbb{R}^n} M_k(x)\, dx = 0 \quad \text{and} \quad \operatorname{supp} M_k \subset 2^{k+1}Q.$$

Moreover, if $k = 0$, by Hölder's inequality and (3.51), we have

$$(3.57) \quad \|M_0\|_{L^2(\mathbb{R}^n)} \lesssim \|a\|_{T_2^2(\Omega \times (0,\infty))} \lesssim |Q|^{-1/2}[\rho(Q)]^{-1} \lesssim |2Q|^{-1/2}[\rho(2Q)]^{-1},$$

and, if $k \in \{1, \ldots, k_0\}$, similar to the proof of (3.54), we have

$$(3.58) \qquad \|M_k\|_{L^2(\mathbb{R}^n)} \lesssim \|\alpha\|_{L^2(R_k(Q))} \lesssim 2^{-k(n/2+1)}\|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

Thus, for each $i \in \{0, \ldots, k_0\}$, M_k is a constant multiple of a $(\rho, 2, 0)$-atom with a constant depending on k. Furthermore, from (3.57), we deduce that

$$(3.59) \qquad |2Q|\Phi\Big(\frac{\|M_0\|_{L^2(\mathbb{R}^n)}}{|2Q|^{1/2}}\Big) \lesssim |Q|\,\Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q|^{1/2}}\Big).$$

By (3.58), the lower type p_Φ property and $p_\Phi \in (n/(n+1), 1]$, we then obtain

$$(3.60) \qquad \sum_{k=1}^{k_0} |2^{k+1}Q|\,\Phi\Big(\frac{\|M_k\|_{L^2(\mathbb{R}^n)}}{|2^{k+1}Q|^{1/2}}\Big) \lesssim \sum_{k=1}^{k_0} |2^k Q|\,\Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{2^{-k(n+1)}|Q|^{1/2}}\Big)$$

$$\lesssim |Q|\,\Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q|^{1/2}}\Big).$$

For each $k \in J_{\Omega, k_0}$, by Fact 1, there exists the Whitney decomposition $\{Q_{k,i}\}_i$ of $R_k(Q)$ about $\partial\Omega$ such that $\cup_i Q_{k,i} = R_k(Q)$ and for each i, $Q_{k,i}$ satisfies that

$$2Q_{k,i} \subset \Omega \quad \text{and} \quad 4Q_{k,i} \cap \partial\Omega \neq \emptyset.$$

Then $\widetilde{M}_k = \sum_i \alpha \chi_{Q_{k,i}}$ almost everywhere. Moreover, by Lemma 3.16, for each k and i, there exists a cube $\widetilde{Q}_{k,i} \subset \Omega^{\complement}$ such that

$$l(\widetilde{Q}_{k,i}) = l(Q_{k,i}) \quad \text{and} \quad \operatorname{dist}(\widetilde{Q}_{k,i}, Q_{k,i}) \sim l(Q_{k,i}).$$

Then for each k and i, there exists a cube $Q_{k,i}^*$ such that

$$(Q_{k,i} \cup \widetilde{Q}_{k,i}) \subset Q_{k,i}^* \quad \text{and} \quad l(Q_{k,i}^*) \sim l(Q_{k,i}).$$

For each k and i, let

$$b_{k,i} := \alpha \chi_{Q_{k,i}} - \frac{1}{|\widetilde{Q}_{k,i}|}\Big\{ \int_{Q_{k,i}} \alpha(x)\,dx \Big\} \chi_{\widetilde{Q}_{k,i}}.$$

Then

$$\int_{\mathbb{R}^n} b_{k,i}(x)\,dx = 0 \quad \text{and} \quad \operatorname{supp} b_{k,i} \subset Q_{k,i}^*.$$

Furthermore, similar to the proofs of (3.56) and (3.55), we see that for each $k \in J_{\Omega, k_0}$ and i, $b_{k,i}$ is a constant multiple of a $(\rho, 2, 0)$-atom with a constant depending on k and i, and

$$(3.61) \qquad \sum_{k \in J_{\Omega, k_0}} \sum_i |Q_{k,i}^*| \Phi\Big(\frac{\|b_{k,i}\|_{L^2(\mathbb{R}^n)}}{|Q_{k,i}^*|^{1/2}} \Big) \lesssim |Q| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega \times (0,\infty))}}{|Q|^{1/2}} \Big).$$

For $j \in \{0, \ldots, k_0\}$, let $N_j := \sum_{k=j}^{k_0} m_k$. It is easy to see that

$$(3.62) \qquad \sum_{k=0}^{k_0} m_k \widetilde{\chi}_k = \sum_{k=1}^{k_0} (\widetilde{\chi}_k - \widetilde{\chi}_{k-1}) N_k + N_0 \widetilde{\chi}_0.$$

For any $k \in \{1, \ldots, k_0\}$, by (3.53) and $|\widetilde{\chi}_k - \widetilde{\chi}_{k-1}| \lesssim |2^k Q|^{-1}$, we know that

$$(3.63) \qquad \|(\widetilde{\chi}_k - \widetilde{\chi}_{k-1}) N_k\|_{L^2(\mathbb{R}^n)} \lesssim |2^k Q|^{-1/2} |N_k|$$

$$\lesssim |2^k Q|^{-1/2} \Big(\sum_{j=k}^{\infty} 2^{-j} \Big) |Q|^{1/2} \|a\|_{T_2^2(\Omega \times (0,\infty))}$$

$$\lesssim 2^{-k(n/2+1)} \|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

This, together with

$$\int_{\mathbb{R}^n} [\widetilde{\chi}_k(x) - \widetilde{\chi}_{k-1}(x)]\,dx = 0 \quad \text{and} \quad \operatorname{supp}(\widetilde{\chi}_k - \widetilde{\chi}_{k-1}) \subset 2^k Q,$$

yields that for each $k \in \{1, \ldots, k_0\}$, $(\widetilde{\chi}_k - \widetilde{\chi}_{k-1}) N_k$ is a constant multiple of a $(\rho, 2, 0)$-atom with a constant depending on k. Furthermore, by (3.63), the lower

type p_Φ property of Φ and $p_\Phi \in (n/(n+1), 1]$, we have

$$(3.64) \qquad \sum_{k=1}^{k_0} |2^k Q| \Phi\Big(\frac{\|(\widetilde{\chi}_k - \widetilde{\chi}_{k-1})N_k\|_{L^2(\mathbb{R}^n)}}{|2^k Q|^{1/2}}\Big)$$

$$\lesssim \sum_{k=1}^{k_0} |2^k Q| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{2^{-k(n+1)}|Q|^{1/2}}\Big) \lesssim |Q| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{|Q|^{1/2}}\Big).$$

Finally we deal with $N_0\widetilde{\chi}_0$. By

$$2^{k_0-1} r_0 < \operatorname{dist}(x_0, \partial\Omega) \leq 2^{k_0} r_0,$$

we know that there exist a positive integer M and a sequence $\{Q_{0,i}\}_{i=1}^M$ of cubes such that

(i) $M \sim 2^{k_0}$;

(ii) for all $i \in \{1, \ldots, M\}$, $l(Q_{0,i}) = 2r_0$ and $Q_{0,i} \subset \Omega$;

(iii) for all $i \in \{1, \ldots, M-1\}$, $Q_{0,i} \cap Q_{0,i+1} \neq \emptyset$ and

$$\operatorname{dist}(Q_{0,i}, \partial\Omega) \geq \operatorname{dist}(Q_{0,i+1}, \partial\Omega);$$

(iv) $2Q_{0,M} \cap \partial\Omega \neq \emptyset$.

Then by Lemma 3.16, there exists a cube $Q_{0,M+1} \subset \Omega^c$ such that

$$l(Q_{0,M+1}) = r_0 \quad \text{and} \quad \operatorname{dist}(Q_{0,M}, Q_{0,M+1}) \sim r_0.$$

Let

$$b_{0,1} := N_0\widetilde{\chi}_0 - \frac{N_0}{|Q_{0,1}|}\chi_{Q_{0,1}} \quad \text{and} \quad b_{0,i} := \frac{N_0}{|Q_{0,i-1}|}\chi_{Q_{0,i-1}} - \frac{N_0}{|Q_{0,i}|}\chi_{Q_{0,i}}$$

with $i \in \{2, \ldots, M+1\}$. Obviously, for all $i \in \{1, \ldots, M+1\}$, by the definition of $b_{0,i}$, we see that

$$\int_{\mathbb{R}^n} b_{0,i}(x)\, dx = 0$$

and there exists a cube $Q_{0,i}^* \subset \mathbb{R}^n$ such that $\operatorname{supp} b_{0,i} \subset Q_{0,i}^*$ and

$$(3.65) \qquad l(Q_{0,i}^*) \sim l(Q).$$

To finish the proof of Proposition 3.14 (i) in this case, we need another fact as follows.

Fact 2. *Let L be as in* (1.3) *and satisfy* DBC. *Let Ω, Q and k_0 be as above. Assume that* (G_∞) *holds. For all $x \in \Omega$, let*

$$\delta(x) := \operatorname{dist}(x, \partial\Omega).$$

Then there exist positive constants C and β, independent of k_0 and Q, such that for all $x \in Q$,

$$\Big|\int_{2^{k_0}Q} \partial_t K_t(y, x)\, dy\Big| \leq \frac{C}{t} e^{-\frac{\beta[\delta(x)]^2}{t}}.$$

Now we continue the proof of Proposition 3.14 (i) by using Fact 2. By Fact 2, equation (3.45) and Hölder's inequality, we conclude that

$$(3.66) \quad |N_0| = \Big| \int_{2^{k_0}Q} \alpha(x)\,dx \Big| = 8 \Big| \int_{2^{k_0}Q} \Big\{ \int_0^\infty \int_\Omega D_t(x,y) a(y,t) \frac{dy\,dt}{t} \Big\} dx \Big|$$

$$\leq 8 \int_0^\infty \int_\Omega \Big| \int_{2^{k_0}Q} D_t(x,y)\,dx \Big| |a(y,t)| \frac{dy\,dt}{t}$$

$$\lesssim \|a\|_{T_2^2(\Omega\times(0,\infty))} \Big\{ \int_0^{r_0} \int_Q e^{-\frac{2\beta[\delta(y)]^2}{t^2}} \frac{dy\,dt}{t} \Big\}^{1/2}$$

$$\lesssim \|a\|_{T_2^2(\Omega\times(0,\infty))} \Big\{ \int_0^{r_0} \int_Q \Big(\frac{t}{2^{k_0}r_0} \Big)^{2(n+1)/n} \frac{dy\,dt}{t} \Big\}^{1/2}$$

$$\lesssim 2^{-k_0(n+1)/n}|Q|^{1/2}\|a\|_{T_2^2(\Omega\times(0,\infty))}.$$

For each $i \in \{1, \ldots, M+1\}$, from the definition of $b_{0,i}$, (3.65) and (3.66), it follows that

$$(3.67) \quad \|b_{0,i}\|_{L^2(\mathbb{R}^n)} \lesssim |N_0||Q|^{-1/2} \lesssim 2^{-k_0(n+1)/n}\|a\|_{T_2^2(\Omega\times(0,\infty))}$$

$$\lesssim 2^{-k_0(n+1)/n}|Q|^{-1/2}[\rho(|Q|)]^{-1}$$

$$\sim 2^{-k_0(n+1)/n}|Q_{0,i}|^{-1/2}[\rho(|Q_{0,i}|)]^{-1},$$

which, together with the facts that

$$\int_{\mathbb{R}^n} b_{0,i}(x)\,dx = 0 \quad \text{and} \quad \operatorname{supp} b_{0,i} \subset Q_{0,i}^*,$$

implies that $b_{0,i}$ is a constant multiple of a $(\rho, 2, 0)$-atom with a constant depending on i. Furthermore, by (3.67), the fact that $M \sim 2^{k_0}$ and (3.65), we have

$$(3.68) \quad \sum_{i=1}^{M+1} |Q_{0,i}^*| \Phi\Big(\frac{\|b_{0,i}\|_{L^2(\mathbb{R}^n)}}{|Q_{0,i}|^{1/2}} \Big) \lesssim \sum_{i=1}^{M+1} |Q| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{2^{k_0(n+1)/n}|Q|^{1/2}} \Big)$$

$$\lesssim M 2^{-\frac{k_0(n+1)p_\Phi}{n}} |Q| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{|Q|^{1/2}} \Big)$$

$$\lesssim 2^{k_0[1-(n+1)p_\Phi/n]} |Q| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{|Q|^{1/2}} \Big)$$

$$\lesssim |Q| \Phi\Big(\frac{\|a\|_{T_2^2(\Omega\times(0,\infty))}}{|Q|^{1/2}} \Big).$$

Let

$$\widetilde{\alpha} := \sum_{i=1}^{k_0} M_k + \sum_{k\in J_{\Omega,k_0}} \sum_i b_{k,i} + \sum_{k=1}^{k_0} (\widetilde{\chi}_k - \widetilde{\chi}_{k-1})N_k + \sum_{i=1}^{M+1} b_{0,i}.$$

Similar to the proof of (3.55), we know that the above equality holds in $L^2(\mathbb{R}^n)$. It is easy to see that $\widetilde{\alpha}|_\Omega = \alpha$. Furthermore, from (3.59), (3.60), (3.61), (3.63) and (3.70), it follows that $\widetilde{\alpha} \in H_\Phi(\mathbb{R}^n)$ and (3.47) holds.

To finish the proof of Proposition 3.14 (i), we need to show Fact 2.

Fix $x \in Q$. Choose $\psi_1 \in C_c^\infty(\Omega)$ such that $0 \le \psi_1 \le 1$, $\psi_1 \equiv 1$ on $Q(x, \frac{\delta(x)}{8})$,

$$\operatorname{supp} \psi_1 \subset Q\left(x, \frac{\delta(x)}{4}\right), \quad \text{and} \quad |\nabla\psi_1(z)| \lesssim \frac{1}{\delta(x)} \text{ for all } z \in \Omega.$$

Then we see that

$$(3.69) \quad \left| \int_{2^{k_0}Q} \partial_t K_t(y, x) \, dy \right|$$

$$\le \left| \int_{2^{k_0}Q} \partial_t K_t(y, x)\psi_1(y) \, dy \right| + \left| \int_{2^{k_0}Q} \partial_t K_t(y, x) \left[1 - \psi_1(y) \right] dy \right|$$

$$\le \left| \int_{2^{k_0}Q} \partial_t K_t(y, x)\psi_1(y) \, dy \right| + \int_{2^{k_0}Q \setminus Q(x, \delta(x)/8)} |\partial_t K(y, x)| \, dy$$

$$=: \mathrm{I}_1 + \mathrm{I}_2.$$

We first estimate I_1. It was proved by Auscher and Russ in Proposition A.4 of [4] that for all $y \in \Omega$, $t \in (0, \infty)$ and all $r \in (0, \infty)$,

$$(3.70) \quad \left\{ \int_{\{z \in \Omega: \, r \le |y-z| \le 2r\}} |\nabla_z K_t(z, y)|^2 \, dz \right\}^{1/2} \lesssim t^{-\frac{1}{2}-\frac{n}{4}} \left(\frac{r}{\sqrt{t}} \right)^{\frac{n-2}{2}} e^{-\gamma\frac{r^2}{t}},$$

where γ is a positive constant independent of y, t and r. Notice that

$$\partial_t K_t(\cdot, x) + L K_t(\cdot, x) = 0$$

and $Q(x, \frac{\delta(x)}{4}) \subset 2^{k_0}Q$. From this, the facts that $\psi_1 \equiv 1$ on $Q(x, \frac{\delta(x)}{8})$,

$$\operatorname{supp} \psi_1 \subset Q\left(x, \frac{\delta(x)}{4}\right), \quad |\nabla\psi_1(y)| \lesssim \frac{1}{\delta(x)} \quad \text{for all } y \in \Omega,$$

Hölder's inequality and (3.70), it follows that

$$(3.71) \quad \mathrm{I}_1 = \left| \int_\Omega L_y K_t(y, x)\psi_1(y) \, dy \right| = \left| \int_\Omega A(y)\nabla_y K_t(y, x) \cdot \nabla_y\psi_1(y) \, dy \right|$$

$$\lesssim \int_{\{y \in \Omega: \, \frac{\delta(x)}{8} \le |x-y| \le \frac{\delta(x)}{4}\}} |\nabla_y K_t(y, x)||\nabla_y\psi_1(y)| \, dy$$

$$\lesssim \left\{ \int_{\{y \in \Omega: \, \frac{\delta(x)}{8} \le |x-y| \le \frac{\delta(x)}{4}\}} |\nabla_y K_t(y, x)|^2 \, dy \right\}^{1/2}$$

$$\times \left\{ \int_{\{y \in \Omega: \, \frac{\delta(x)}{8} \le |x-y| \le \frac{\delta(x)}{4}\}} |\nabla_y\psi_1(y)|^2 \, dy \right\}^{1/2}$$

$$\lesssim t^{-\frac{1}{2}-\frac{n}{4}} \left[\frac{\delta(x)}{\sqrt{t}} \right]^{\frac{n-2}{2}} e^{-\frac{\gamma[\delta(x)]^2}{16t}} [\delta(x)]^{\frac{n-2}{2}}$$

$$\sim \frac{1}{t} \left[\frac{\delta(x)}{\sqrt{t}} \right]^{n-2} e^{-\frac{\gamma[\delta(x)]^2}{16t}} \lesssim \frac{1}{t} e^{-\frac{\gamma[\delta(x)]^2}{32t}}.$$

For I_2, by Lemma 2.3, we conclude that

$$I_2 \lesssim \int_{\Omega \setminus Q(x, \delta(x)/8)} \frac{1}{t^{n/2+1}} e^{-\frac{\alpha|x-y|^2}{t}} \, dy \lesssim \frac{1}{t} e^{-\frac{\alpha[\delta(x)]^2}{2^9 t}} \int_{\mathbb{R}^n} \frac{e^{-\frac{\alpha|y|^2}{t}}}{t^{n/2}} \, dy \lesssim \frac{1}{t} e^{-\frac{\alpha[\delta(x)]^2}{2^9 t}}.$$

From this, (3.69) and (3.71), it follows that Fact 2 holds, which completes the proof of Proposition 3.14 (i).

Now we prove (ii) of Proposition 3.14. To this end, let $f \in H_{\Phi, S_h, d_\Omega}(\Omega) \cap L^2(\Omega)$. Recall that $d_\Omega := 2\mathrm{diam}(\Omega)$. We write $(d_\Omega)^2$ simply as d_Ω^2. It is easy to see that for all $z \in \mathbb{C}$ satisfying $z \neq 0$ and $|\arg z| \in (0, \pi/2)$,

$$8 \int_0^{d_\Omega} (t^2 z e^{-t^2 z})(t^2 z e^{-t^2 z}) \frac{dt}{t} + (2d_\Omega^2 z + 1) e^{-2d_\Omega^2 z} = 1;$$

this, together with the H^∞-functional calculus for L, implies that for all $f \in L^2(\Omega)$,

$$(3.72) \quad f = 8 \int_0^{d_\Omega} (t^2 L e^{-t^2 L})(t^2 L e^{-t^2 L})(f) \frac{dt}{t} + \left[2d_\Omega^2 L e^{-2d_\Omega^2 L}(f) + e^{-2d_\Omega^2 L}(f) \right]$$
$$=: f_1 + f_2.$$

We first deal with f_1. By the fact that $f \in H_{\Phi, \mathcal{N}_h}(\Omega) \cap L^2(\Omega)$, and Propositions 3.5 and 3.13, we know that $S_h(f) \in L^\Phi(\Omega)$. From this and the definition of the space $T_\Phi(\Omega)$, it follows that

$$t^2 L e^{-t^2 L}(f) \chi_{\Omega \times (0, d_\Omega)} \in T_\Phi(\Omega)$$

and

$$\|f\|_{H_{\Phi, S_h}(\Omega)} = \left\| t^2 L e^{-t^2 L}(f) \chi_{\Omega \times (0, d_\Omega)} \right\|_{T_\Phi(\Omega)}.$$

Then by Lemma 3.15, there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $T_\Phi(\Omega)$-atoms such that for almost every $(x, t) \in \Omega \times (0, \infty)$,

$$(3.73) \qquad t^2 L e^{-t^2 L}(f)(x) \chi_{\Omega \times (0, d_\Omega)}(x, t) = \sum_j \lambda_j a_j(x, t).$$

For each j, let

$$\alpha_j := 8 \int_0^\infty t^2 L e^{-t^2 L}(a_j) \frac{dt}{t}.$$

Then by the fact that

$$f_1 = 8 \int_0^\infty \left(t^2 L e^{-t^2 L} \right) \left(t^2 L e^{-t^2 L}(f) \chi_{\Omega \times (0, d_\Omega)} \right) \frac{dt}{t}$$

and (3.73), similar to the proof of Proposition 4.2 in [29], we know that $f_1 = \sum_j \lambda_j \alpha_j$ in $L^2(\Omega)$. Also, similar to the proof of (3.48), there exists $\widetilde{f}_1 \in H_\Phi(\mathbb{R}^n)$ such that

$$\widetilde{f}_1|_\Omega = f_1 \quad \text{and} \quad \|\widetilde{f}_1\|_{H_\Phi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\Phi, S_h}(\Omega)},$$

which implies that $f_1 \in H_{\Phi, r}(\Omega)$ and

$$(3.74) \qquad \|f_1\|_{H_{\Phi, r}(\Omega)} \lesssim \|f\|_{H_{\Phi, S_h}(\Omega)}.$$

Now we deal with f_2. Since Ω is bounded, there exists a closed cube $\widetilde{Q}_0 \subset \mathbb{R}^n$ such that $x_{\widetilde{Q}_0} \in \Omega$, $l(\widetilde{Q}_0) \sim d_\Omega$ and $\Omega \subset \widetilde{Q}_0$. Take cubes $\widetilde{Q}_1$, $\widetilde{Q}_2$ such that

$$\widetilde{Q}_1 \subset \Omega^c, \ l(\widetilde{Q}_1) \sim d_\Omega, \ (\widetilde{Q}_0 \cup \widetilde{Q}_1) \subset \widetilde{Q}_2 \text{ and } l(\widetilde{Q}_2) \sim d_\Omega.$$

Let

$$\widetilde{f}_2 := f_2 \chi_{\widetilde{Q}_0} - \frac{1}{|\widetilde{Q}_1|} \Big[\int_\Omega f_2(y) \, dy \Big] \chi_{\widetilde{Q}_1}.$$

Then $\widetilde{f}_2|_\Omega = f_2$. It is easy to see that $\operatorname{supp} \widetilde{f}_2 \subset \widetilde{Q}_2$,

$$\int_{\mathbb{R}^n} \widetilde{f}_2(y) \, dy = 0 \quad \text{and} \quad \|\widetilde{f}_2\|_{L^2(\mathbb{R}^n)} \lesssim \|f_2\|_{L^2(\Omega)}.$$

Thus, we see that $\widetilde{f}_2$ is a harmless constant multiple of some $(\rho, 2, 0)$-atom. Denote by $\widetilde{K}$ the kernel of $2d_\Omega^2 L e^{-d_\Omega^2 L} + e^{-d_\Omega^2 L}$. Then by Lemma 2.3, we know that for all $x, y \in \Omega$,

$$|\widetilde{K}(x, y)| \lesssim \frac{1}{d_\Omega^n} e^{-\frac{\alpha|x-y|^2}{d_\Omega^2}},$$

where α is as in (2.1), which, together with the fact that Ω is bounded, implies that

$$\sup_{z \in \Omega} |f_2(z)| = \sup_{z \in \Omega} \Big| \int_\Omega \widetilde{K}(z, y) e^{-d_\Omega^2 L}(f)(y) \, dy \Big| \lesssim \|e^{-d_\Omega^2 L}(f)\|_{L^1(\Omega)}.$$

From this, the upper type 1 property of Φ, and the facts that $\Omega \subset \widetilde{Q}_2$ and $l(\widetilde{Q}_2) \sim d_\Omega$, we deduce that for all $\lambda \in (0, \infty)$,

$$|\widetilde{Q}_2| \, \Phi \Big(\frac{\|\widetilde{f}_2\|_{L^2(\mathbb{R}^n)}}{\lambda |\widetilde{Q}_2|^{1/2}} \Big) \lesssim |\widetilde{Q}_2| \, \Phi \Big(\frac{\|f_2\|_{L^2(\Omega)}}{\lambda |\widetilde{Q}_2|^{1/2}} \Big) \lesssim |\widetilde{Q}_2| \, \Phi \Big(\frac{\sup_{z \in \Omega} |f_2(z)|}{\lambda} \Big)$$

$$\lesssim |\widetilde{Q}_2| \, \Phi \Big(\frac{\|e^{-d_\Omega^2 L}(f)\|_{L^1(\Omega)}}{\lambda} \Big) \sim \Phi \Big(\frac{\|e^{-d_\Omega^2 L}(f)\|_{L^1(\Omega)}}{\lambda} \Big).$$

By this, Lemma 3.4 and the definition of $H_{\Phi, r}(\Omega)$, we know that $f_2 \in H_{\Phi, r}(\Omega)$ and

$$(3.75) \quad \|f_2\|_{H_{\Phi, r}(\Omega)} \leq \|\widetilde{f}_2\|_{H_\Phi(\mathbb{R}^n)} \lesssim \inf \Big\{ \lambda \in (0, \infty) \colon \Phi \Big(\frac{\|e^{-d_\Omega^2 L}(f)\|_{L^1(\Omega)}}{\lambda} \Big) \leq 1 \Big\}.$$

Thus, from (3.72), (3.74) and (3.75), it follows that $f \in H_{\Phi, r}(\Omega)$ and

$$\|f\|_{H_{\Phi, r}(\Omega)} \lesssim \|f\|_{H_{\Phi, S_h, d_\Omega}(\Omega)},$$

which, together with the arbitrariness of $f \in H_{\Phi, S_h, d_\Omega}(\Omega) \cap L^2(\Omega)$, implies that the first part of Proposition 3.14 (ii) holds.

We now show the second part of Proposition 3.13 (ii). We first prove that

$$(3.76) \qquad (H_{\Phi, S_h, d_\Omega}(\Omega) \cap L^2(\Omega)) = (H_{\Phi, S_h}(\Omega) \cap L^2(\Omega))$$

with equivalent norms. Obviously, we have

$$(H_{\Phi, S_h, d_\Omega}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi, S_h}(\Omega) \cap L^2(\Omega))$$

by their definitions.

To show the converse, let $f \in H_{\Phi, S_h}(\Omega) \cap L^2(\Omega)$. By Lemma 3.6, the contraction property of $\{e^{-tL}\}_{t \geq 0}$ on $L^2(\Omega)$ and Hölder's inequality, we know that for all $x \in \Omega$,

$$[S_h(f)(x)]^2 \gtrsim \int_{d_\Omega/2}^{d_\Omega} \int_\Omega \left| t^2 L e^{-t^2 L}(f)(y) \right|^2 \frac{dy\, dt}{t^{n+1}} \gtrsim d_\Omega^{-n} \| L e^{-d_\Omega^2 L}(f) \|_{L^2(\Omega)}^2,$$

which implies that

$$(3.77) \qquad \inf_{x \in \Omega} S_h(f)(x) \gtrsim d_\Omega^{-n/2} \| L e^{-d_\Omega^2 L}(f) \|_{L^2(\Omega)}.$$

To continue the proof, we need the following fact, whose proof is similar to the proof of Proposition 5.3 in [1]. We omit the details.

Fact 3. *Let $1 < p < q < \infty$ and $\alpha := \frac{1}{2}(\frac{n}{p} - \frac{n}{q})$. Assume that (G_∞) holds. Then $L^{-\alpha}$ is bounded from $L^p(\Omega)$ to $L^q(\Omega)$.*

By $n \geq 3$, we know that there exist $p_0 \in (1, 2]$ and $q_0 \in (1, \infty)$ such that $1/p_0 = 2/n + 1/q_0$. Then $(n/p_0 - n/q_0)/2 = 1$. By this, Fact 3, (3.77) and Hölder's inequality, we see that

$$\left\| e^{-d_\Omega^2 L}(f) \right\|_{L^1(\Omega)} \left\| e^{-d_\Omega^2 L}(f) \right\|_{L^{q_0}(\Omega)} \sim \left\| L^{-1} L e^{-d_\Omega^2 L}(f) \right\|_{L^{q_0}(\Omega)}$$
$$\lesssim \left\| L e^{-d_\Omega^2 L}(f) \right\|_{L^{p_0}(\Omega)} \lesssim d_\Omega^{n/p_0} \inf_{x \in \Omega} S_h(f)(x),$$

which, together with the upper type 1 property of Φ, (3.77) and (3.65), implies that for all $\lambda \in (0, \infty)$,

$$\Phi\Big(\frac{\| e^{-d_\Omega^2 L}(f) \|_{L^1(\Omega)}}{\lambda} \Big) \lesssim \Phi\Big(\frac{\inf_{x \in \Omega} S_h(f)(x)}{\lambda} \Big) \lesssim \int_\Omega \Phi\Big(\frac{S_h(f)(x)}{\lambda} \Big)\, dx.$$

From this, it follows that $f \in H_{\Phi, S_h, d_\Omega}(\Omega)$ and

$$\|f\|_{H_{\Phi, S_h, d_\Omega}(\Omega)} \lesssim \|f\|_{H_{\Phi, S_h}(\Omega)},$$

which, together with the arbitrariness of $f \in H_{\Phi, S_h}(\Omega) \cap L^2(\Omega)$, implies that

$$(H_{\Phi, S_h}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi, S_h, d_\Omega}(\Omega) \cap L^2(\Omega)).$$

Thus, (3.76) holds.

By Propositions 3.1, 3.5, 3.13 and 3.14, we have

$$(3.78) \qquad (H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,S_h,\,d_\Omega}(\Omega) \cap L^2(\Omega))$$

with equivalent norms. To finish the proof of the second part of Proposition 3.14 (ii), let $f \in H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)$. By

$$e^{-d_\Omega^2 L}(f) = e^{-\frac{d_\Omega^2}{2}L}\big(e^{-\frac{d_\Omega^2}{2}L}(f)\big),$$

estimate (2.1) and the fact that $|\Omega| < \infty$, we know that for all $x \in \Omega$,

$$\big\|e^{-d_\Omega^2 L}(f)\big\|_{L^1(\Omega)} \lesssim \int_\Omega \sup_{y \in \Omega} \big|e^{-\frac{d_\Omega^2}{2}L}(f)(y)\big|\, dz$$

$$\lesssim d_\Omega^n \sup_{y \in \Omega,\, t \in (0, d_\Omega),\, |x-y| < t} \big|e^{-t^2 L}(f)(y)\big| \sim \mathcal{N}_h(f)(x).$$

From this, it follows that

$$\|e^{-d_\Omega^2 L}(f)\|_{L^1(\Omega)} \lesssim \inf_{x \in \Omega} \mathcal{N}_h(f)(x),$$

which implies that for all $\lambda \in (0, \infty)$,

$$(3.79) \quad \Phi\Big(\frac{\|e^{-d_\Omega^2 L}(f)\|_{L^1(\Omega)}}{\lambda}\Big) \lesssim \Phi\Big(\frac{\inf_{x \in \Omega} \mathcal{N}_h(f)(x)}{\lambda}\Big)$$

$$\lesssim \frac{1}{|\Omega|}\int_\Omega \Phi\Big(\frac{\mathcal{N}_h(f)(x)}{\lambda}\Big)\, dx \sim \int_\Omega \Phi\Big(\frac{\mathcal{N}_h(f)(x)}{\lambda}\Big)\, dx.$$

By Proposition 3.1 and (3.79), we conclude that

$$(H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega)),$$

which, together with Proposition 3.13, (3.76), (3.78) and the obvious fact that

$$(H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi,\,\widetilde{S}_h}(\Omega) \cap L^2(\Omega)),$$

implies that

$$(H_{\Phi,\,\widetilde{S}_h}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi,\,S_h}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,S_h,\,d_\Omega}(\Omega) \cap L^2(\Omega))$$

$$= (H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega))$$

$$\subset (H_{\Phi,\,\widetilde{S}_h}(\Omega) \cap L^2(\Omega)).$$

From this, we deduce that

$$(H_{\Phi,\,\widetilde{S}_h}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,S_h}(\Omega) \cap L^2(\Omega))$$

$$= (H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega))$$

with equivalent norms. This finishes the proof of Proposition 3.14 (ii) and hence Proposition 3.14. □

Combining Propositions 3.1, 3.5, 3.13 with 3.14, we then obtain Theorem 1.9.

Proof of Theorem 1.9. We first prove Theorem 1.9 (i). By Propositions 3.1, 3.5, 3.13 and 3.14 (i), we know that

$$(H_{\Phi,\,r}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,\widetilde{S}_h}(\Omega) \cap L^2(\Omega))$$
$$= (H_{\Phi,\,S_h}(\Omega) \cap L^2(\Omega))$$

with equivalent norms, which, together with the fact that

$$H_{\Phi,\,r}(\Omega) \cap L^2(\Omega),\ H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega),\ H_{\Phi,\,\widetilde{S}_h}(\Omega) \cap L^2(\Omega) \text{ and } H_{\Phi,\,S_h}(\Omega) \cap L^2(\Omega)$$

are, respectively, dense in

$$H_{\Phi,\,r}(\Omega),\ H_{\Phi,\,\mathcal{N}_h}(\Omega),\ H_{\Phi,\,\widetilde{S}_h}(\Omega) \text{ and } H_{\Phi,\,S_h}(\Omega),$$

and a density argument, implies that the spaces

$$H_{\Phi,\,r}(\Omega),\ H_{\Phi,\,\mathcal{N}_h}(\Omega),\ H_{\Phi,\,\widetilde{S}_h}(\Omega) \text{ and } H_{\Phi,\,S_h}(\Omega)$$

coincide with equivalent norms, which completes the proof of Theorem 1.9 (i).

Now we prove Theorem 1.9 (ii). By Proposition 3.5, we know that for all $f \in H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)$,

$$\|f\|_{H_{\Phi,\,\widetilde{S}_h}(\Omega)} \lesssim \|f\|_{H_{\Phi,\,\mathcal{N}_h}(\Omega)},$$

which, together with (3.79), implies that

$$\|f\|_{H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega)} \lesssim \|f\|_{H_{\Phi,\,\mathcal{N}_h}(\Omega)}$$

for all $f \in H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)$. By the arbitrariness of $f \in H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)$, we know that

$$(H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)) \subset (H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega)).$$

From this, Propositions 3.1, 3.13, and 3.14 (ii), it follows that

$$(H_{\Phi,\,r}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega)) = (H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega))$$
$$= (H_{\Phi,\,S_h,\,d_\Omega}(\Omega) \cap L^2(\Omega))$$

with equivalent norms. This, together with the fact that the spaces $H_{\Phi,\,r}(\Omega) \cap L^2(\Omega),\ H_{\Phi,\,\mathcal{N}_h}(\Omega) \cap L^2(\Omega),\ (H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \cap L^2(\Omega))$, and $H_{\Phi,\,S_h,\,d_\Omega}(\Omega) \cap L^2(\Omega)$ are, respectively, dense in

$$H_{\Phi,\,r}(\Omega),\ H_{\Phi,\,\mathcal{N}_h}(\Omega),\ H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \text{ and } H_{\Phi,\,S_h,\,d_\Omega}(\Omega),$$

and a density argument, implies that the spaces

$$H_{\Phi,\,r}(\Omega),\ H_{\Phi,\,\mathcal{N}_h}(\Omega),\ H_{\Phi,\,\widetilde{S}_h,\,d_\Omega}(\Omega) \text{ and } H_{\Phi,\,S_h,\,d_\Omega}(\Omega)$$

coincide with equivalent norms, which is the desired conclusion.

Moreover, if $n \geq 3$ and (G_∞) holds, by the second part of Proposition 3.14 (ii) and the fact that $H_{\Phi, \widetilde{S}_h, \, d_\Omega}(\Omega) \cap L^2(\Omega)$, $H_{\Phi, \widetilde{S}_h}(\Omega) \cap L^2(\Omega)$, $H_{\Phi, S_h, \, d_\Omega}(\Omega) \cap L^2(\Omega)$, and $H_{\Phi, \, S_h}(\Omega) \cap L^2(\Omega)$ are, respectively, dense in

$$H_{\Phi, \widetilde{S}_h, \, d_\Omega}(\Omega), \ \ H_{\Phi, \widetilde{S}_h}(\Omega), \ \ H_{\Phi, S_h, \, d_\Omega}(\Omega) \text{ and } H_{\Phi, \, S_h}(\Omega),$$

together with a density argument, we find that the spaces

$$H_{\Phi, \widetilde{S}_h, \, d_\Omega}(\Omega), \ \ H_{\Phi, \widetilde{S}_h}(\Omega), \ \ H_{\Phi, S_h, \, d_\Omega}(\Omega) \text{ and } H_{\Phi, \, S_h}(\Omega)$$

coincide with equivalent norms, which completes the proof of Theorem 1.9 (ii) and hence that of Theorem 1.9.   □

Assume that (G_∞) holds. For all $t \in (0, \infty)$, let $P_t := e^{-t\sqrt{L}}$. For all $f \in L^2(\Omega)$ and $x \in \Omega$, let

$$S_P(f)(x) := \Big\{ \int_{\widetilde{\Gamma}(x)} |t\partial_t P_t(f)(y)|^2 \frac{dy \, dt}{t|Q(x,t) \cap \Omega|} \Big\}^{1/2}$$

and

$$\widetilde{H}_{S_P}^1(\Omega) := \big\{ f \in L^2(\Omega) : \ \|f\|_{H_{S_P}^1(\Omega)} < \infty \big\},$$

where

$$\|f\|_{H_{S_P}^1(\Omega)} := \|S_P(f)\|_{L^1(\Omega)}.$$

The *Hardy space* $H_{S_P}^1(\Omega)$ is then defined to be the completion of $\widetilde{H}_{S_P}^1(\Omega)$ in the norm $\| \cdot \|_{H_{S_P}^1(\Omega)}$.

Proposition 3.17. *Let Ω and L be as in Theorem 1.9. Assume that (G_∞) holds. Then $H_r^1(\Omega) = H_{S_P}^1(\Omega)$ with equivalent norms.*

Proof. Similar to the proof of Proposition 3.1, we see that

$$(3.80) \qquad \big(H_r^1(\Omega) \cap L^2(\Omega) \big) \subset \big(H_{S_P}^1(\Omega) \cap L^2(\Omega) \big).$$

To finish the proof of Proposition 3.17, it suffices to show

$$(3.81) \qquad \big(H_{S_P}^1(\Omega) \cap L^2(\Omega) \big) \subset \big(H_r^1(\Omega) \cap L^2(\Omega) \big).$$

Indeed, if (3.81) holds, by (3.80), we have

$$(H_r^1(\Omega) \cap L^2(\Omega)) = (H_{S_P}^1(\Omega) \cap L^2(\Omega))$$

with equivalent norms, which, together with the fact that $H_r^1(\Omega) \cap L^2(\Omega)$ and $H_{S_P}^1(\Omega) \cap L^2(\Omega)$ are respectively dense in $H_r^1(\Omega)$ and $H_{S_P}^1(\Omega)$, and a density argument, implies that the spaces $H_r^1(\Omega)$ and $H_{S_P}^1(\Omega)$ coincide with equivalent norms, which completes the proof of Proposition 3.17.

To show (3.81), let $f \in H^1_{S_P}(\Omega) \cap L^2(\Omega)$. Then by the H^∞-functional calculus for L and the equation (13) in page 164 of [4], we conclude that

$$(3.82) \qquad f = 4 \int_0^\infty t\sqrt{L}P_t(t\sqrt{L}P_t)(f)\frac{dt}{t}$$

in $L^2(\Omega)$. By $f \in H^1_{S_P}(\Omega)$, we see that $S_P(f) \in L^1(\Omega)$, which, together with $t\sqrt{L}P_t(f) = t\partial_t P_t(f)$, implies that

$$t\sqrt{L}P_t(f) \in T_1(\Omega) \quad \text{and} \quad \|f\|_{H^1_{S_P}(\Omega)} = \|t\sqrt{L}P_t(f)\|_{T_1(\Omega)}.$$

Then from Lemma 3.15, it follows that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $T_1(\Omega)$-atoms such that

$$(3.83) \qquad t\sqrt{L}P_t(f) = \sum_j \lambda_j a_j.$$

For each j, let

$$\alpha_j := 4 \int_0^\infty t\sqrt{L}P_t(a_j)\frac{dt}{t}.$$

Then by (3.82) and (3.83), similar to the proof of Proposition 4.2 in [29], we have $f = \sum_j \lambda_j \alpha_j$ in $L^2(\Omega)$. For any $T_1(\Omega)$-atom a supported in $\widehat{Q \cap \Omega}$, let

$$\alpha := 4 \int_0^\infty t\sqrt{L}P_t(a)\frac{dt}{t}.$$

To show (3.81), similar to the proof of Proposition 3.14 (i), it suffices to show that for α as above, there exist a function $\widetilde{\alpha}$ on $\mathbb{R}^n$ such that

$$(3.84) \qquad \widetilde{\alpha}|_\Omega = \alpha$$

and a sequence $\{b_i\}_i$ of harmless constant multiples of $(1, 2, 0)$-atoms, with the constant depending on i, such that $\widetilde{\alpha} = \sum_i b_i$ in $L^2(\mathbb{R}^n)$ and

$$(3.85) \qquad \sum_i |Q_i|^{1/2}\|b_i\|_{L^2(\mathbb{R}^n)} \lesssim |Q \cap \Omega|^{1/2}\|a\|_{T_2^2(\Omega\times(0,\infty))},$$

where for each i, $\operatorname{supp} b_i \subset Q_i$.

Let $Q := Q(x_0, r_0)$. Now we show (3.84) and (3.85) by considering the following two cases for Q:

Case 1) $8Q \cap \Omega^c \neq \emptyset$. In this case, the proofs of (3.84) and (3.85) are similar to Case 1) of the proof of Proposition 3.14. We omit the details.

Case 2) $8Q \subset \Omega$. In this case, let k_0, J_{Ω, k_0}, $R_0(Q)$, $R_k(Q)$, χ_k, $\widetilde{\chi}_k$, m_k, M_k and $\widetilde{M}_k$ be as in Case 2) of the proof of Proposition 3.14. Then

$$\alpha = \sum_{k=0}^{k_0} M_k + \sum_{k \in J_{\Omega, k_0}} \widetilde{M}_k + \sum_{k=0}^{k_0} m_k\widetilde{\chi}_k.$$

Similar to the proof of (3.59) and (3.60), we conclude that

$$(3.86) \quad |2Q|^{1/2}\|M_0\|_{L^2(\mathbb{R}^n)} + \sum_{k=0}^{k_0} |2^k Q|^{1/2}\|M_k\|_{L^2(\mathbb{R}^n)} \lesssim |Q|^{1/2}\|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

For each $k \in J_{\Omega, k_0}$, by Fact 1, there exists the Whitney decomposition $\{Q_{k,i}\}_i$ of $R_k(Q)$ about $\partial\Omega$ such that $\cup_i Q_{k,i} = R_k(Q)$ and for each i, $Q_{k,i}$ satisfies that

$$2Q_{k,i} \subset \Omega \quad \text{and} \quad 4Q_{k,i} \cap \partial\Omega \neq \emptyset.$$

Then $\widetilde{M}_k = \sum_i \alpha \chi_{Q_{k,i}}$ almost everywhere. Moreover, by Lemma 3.16, for each k and i, there exists a cube $\widetilde{Q}_{k,i} \subset \Omega^c$ such that

$$l(\widetilde{Q}_{k,i}) = l(Q_{k,i}) \quad \text{and} \quad \text{dist}\,(\widetilde{Q}_{k,i}, Q_{k,i}) \sim l(Q_{k,i}).$$

Then for each k and i, there exists a cube $Q_{k,i}^*$ such that

$$(Q_{k,i} \cup \widetilde{Q}_{k,i}) \subset Q_{k,i}^* \quad \text{and} \quad l(Q_{k,i}^*) \sim l(Q_{k,i}).$$

For any k and i, let

$$b_{k,i} := \alpha \chi_{Q_{k,i}} - \frac{1}{|\widetilde{Q}_{k,i}|}\Big\{\int_{Q_{k,i}} \alpha(x)\,dx\Big\}\chi_{\widetilde{Q}_{k,i}}.$$

Then

$$\int_{\mathbb{R}^n} b_{k,i}(x)\,dx = 0 \quad \text{and} \quad \text{supp}\, b_{k,i} \subset Q_{k,i}^*.$$

Furthermore, similar to (3.59) and (3.56), we know that for each $k \in J_{\Omega, k_0}$ and i, $b_{k,i}$ is a constant multiple of some $(1, 2, 0)$-atom, with the constant depending on k and i, and

$$(3.87) \quad \sum_{k \in J_{\Omega, k_0}} \sum_i |Q_{k,i}^*|^{1/2}\|b_{k,i}\|_{L^2(\mathbb{R}^n)} \lesssim |Q|^{1/2}\|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

For $j \in \{0, \ldots, k_0\}$, let $N_j := \sum_{k=j}^{k_0} m_k$. It is easy to see that

$$(3.88) \quad \sum_{k=0}^{k_0} m_k \widetilde{\chi}_k = \sum_{k=1}^{k_0} (\widetilde{\chi}_k - \widetilde{\chi}_{k-1})N_k + N_0\widetilde{\chi}_0.$$

Similar to the proofs of (3.63) and (3.66), we see that for each $k \in \{1, \ldots, k_0\}$, $(\widetilde{\chi}_k - \widetilde{\chi}_{k-1})N_k$ is a constant multiple of a $(1, 2, 0)$-atom, with the constant depending on k, and

$$(3.89) \quad \sum_{k=1}^{k_0} |2^k Q|^{1/2}\|(\widetilde{\chi}_k - \widetilde{\chi}_{k-1})N_k\|_{L^2(\mathbb{R}^n)} \lesssim |Q|^{1/2}\|a\|_{T_2^2(\Omega \times (0,\infty))}.$$

Finally we deal with $N_0\widetilde{\chi}_0$. Let M, $\{Q_{0,i}\}_{i=0}^{M+1}$ and $\{b_{0,i}\}_{i=0}^{M+1}$ be as in Case 2) of the proof of Proposition 3.14 (i). For all $t \in (0,\infty)$, we denote the kernel of P_t by p_t. Then by Fact 2 and the subordination formula associated with L that

$$e^{-t\sqrt{L}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4u}L} e^{-u} u^{-1/2}\, du$$

(see (A.1) in page 180 of [4]), we conclude that for all $x \in Q$,

$$\Big| \int_{2^{k_0}Q} \partial_t p_t(y,x)\, dy \Big| \lesssim \frac{1}{t}\Big\{ 1 + \frac{\delta(x)}{t} \Big\}^{-1},$$

where $\delta(x)$ for $x \in Q$ is as in Fact 2 of the proof of Proposition 3.14. From this and Hölder's inequality, it follows that

$$
\begin{aligned}
|N_0| &= \Big| \int_{2^{k_0}Q} \alpha(x)\, dx \Big| \sim \Big| \int_{2^{k_0}Q} \Big\{ \int_0^\infty \int_\Omega t\partial_t p_t(x,y) a(y,t) \frac{dy\,dt}{t} \Big\} dx \Big| \\
&\lesssim \int_0^\infty \int_\Omega \Big| \int_{2^{k_0}Q} t\partial_t p_t(x,y)\, dx \Big| |a(y,t)| \frac{dy\,dt}{t} \\
&\lesssim \|a\|_{T_2^2(\Omega\times(0,\infty))} \Big\{ \int_0^{r_0} \int_Q \Big[1 + \frac{\delta(y)}{t} \Big]^{-2} \frac{dy\,dt}{t} \Big\}^{1/2} \\
&\lesssim \|a\|_{T_2^2(\Omega\times(0,\infty))} \Big\{ \int_0^{r_0} \int_Q \Big(\frac{t}{2^{k_0}r_0} \Big)^2 \frac{dy\,dt}{t} \Big\}^{1/2} \lesssim 2^{-k_0} |Q|^{1/2} \|a\|_{T_2^2(\Omega\times(0,\infty))}.
\end{aligned}
$$
(3.90)

For each $i \in \{1, \ldots, M+1\}$, by the definition of $b_{0,i}$, (3.90) and the fact that $l(Q_{0,i}) \sim l(Q)$, we have

$$
\begin{aligned}
\text{(3.91)} \qquad \|b_{0,i}\|_{L^2(\mathbb{R}^n)} &\lesssim |N_0| |Q|^{-1/2} \lesssim 2^{-k_0} \|a\|_{T_2^2(\Omega\times(0,\infty))} \\
&\lesssim 2^{-k_0} |Q|^{-1/2} [\rho(|Q|)]^{-1} \sim 2^{-k_0} |Q_{0,i}|^{-1/2} [\rho(|Q_{0,i}|)]^{-1},
\end{aligned}
$$

which, together with the facts that

$$\int_{\mathbb{R}^n} b_{0,i}(x)\, dx = 0 \quad \text{and} \quad \operatorname{supp} b_{0,i} \subset Q_{0,i}^*,$$

implies that $b_{0,i}$ is a constant multiple of some $(1,2,0)$-atom with the constant depending on i. Furthermore, by (3.91) and the fact that $M \sim 2^{k_0}$, we obtain

$$
\begin{aligned}
\text{(3.92)} \qquad \sum_{i=1}^{M+1} |Q_{0,i}^*|^{1/2} \|b_{0,i}\|_{L^2(\mathbb{R}^n)} &\sim \sum_{i=1}^{M+1} 2^{-k_0} |Q|^{1/2} \|a\|_{T_2^2(\Omega\times(0,\infty))} \\
&\sim |Q|^{1/2} \|a\|_{T_2^2(\Omega\times(0,\infty))}.
\end{aligned}
$$

Let

$$\widetilde{\alpha} := \sum_{i=1}^{k_0} M_k + \sum_{k\in J_{\Omega,k_0}} \sum_i b_{k,i} + \sum_{k=1}^{k_0} (\widetilde{\chi}_k - \widetilde{\chi}_{k-1}) N_k + \sum_{i=1}^{M+1} b_{0,i}.$$

By an argument similar to that used to obtain the estimate (3.55), we know that the series in the definition of $\widetilde{\alpha}$ converges in $L^2(\mathbb{R}^n)$. It is easy to see that $\widetilde{\alpha}|_\Omega = \alpha$. Furthermore, by (3.86), (3.87), (3.89) and (3.92), we conclude that $\widetilde{\alpha} \in H_\Phi(\mathbb{R}^n)$ and there holds (3.85), which completes the proof of Proposition 3.17.   □

From Proposition 3.17, we deduce that for any given $f \in H^1_{S_P}(\Omega)$, there exists an atomic decomposition, which gives a positive answer to the question asked by Duong and Yan in Remarks (iii) of page 485 in [17] in the case that $p = 1$. However, it is still unknown whether this method also works for $p < 1$ but near to 1. This case seems to need a stronger estimate than (3.90).

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Dachun Yang: School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China.
E-mail: dcyang@bnu.edu.cn

Sibei Yang: School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China.
E-mail: yangsibei@mail.bnu.edu.cn