



Wavelet approach to operator-valued Hardy spaces

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Abstract. This paper is devoted to the study of operator-valued Hardy spaces via the wavelet method. This approach is parallel to that in the noncommutative martingale case. We show that our Hardy spaces defined by wavelets coincide with those introduced by Tao Mei via the usual Lusin and Littlewood–Paley square functions. As a consequence, we give an explicit complete unconditional basis of the Hardy space $H_1(\mathbb{R})$ when $H_1(\mathbb{R})$ is equipped with an appropriate operator space structure.

1. Introduction

In this paper, we exploit Meyer’s wavelet methods to study operator-valued Hardy spaces. We are motivated by two rapidly developing fields. The first one is the theory of noncommutative martingale inequalities. This theory was initiated already in the 1970’s. Its modern period of development began with Pisier and Xu’s seminal paper [20], in which the authors established the noncommutative Burkholder–Gundy inequalities and the Fefferman duality theorem between H_1 and BMO. Since then many classical results have been successfully transferred to the noncommutative world (see [11], [14], [15], [1]). In particular, motivated by [9], Mei [15] developed the theory of Hardy spaces on \mathbb{R}^n for operator-valued functions.

Our second motivation is the theory of wavelets founded by Meyer. It is now well known that this theory is important for many domains, in particular in harmonic analysis. For instance, it provides powerful tools for the theory of Calderón–Zygmund singular integral operators. More recently, Meyer’s wavelet methods were extended to study more sophisticated subjects in harmonic analysis. For example, the authors of [5] exploited the properties of Meyer’s wavelets to give a characterization of product BMO by commutators; [17] deals with the estimates of bi-parameter paraproducts.

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It is in this spirit that we wish to understand how useful wavelet methods are for noncommutative analysis. The most natural and possible way would be first to do this in the semicommutative case. This is exactly the purpose of the present paper which could be viewed as the first attempt at developing wavelet techniques for noncommutative analysis.

A wavelet basis of $L_2(\mathbb{R})$ is a complete orthonormal system $(w_I)_{I \in \mathcal{D}}$, where \mathcal{D} denotes the collection of all dyadic intervals in \mathbb{R} , w is a Schwartz function satisfying the properties needed for Meyer's construction in [16], and

$$w_I(x) := \frac{1}{|I|^{\frac{1}{2}}} w\left(\frac{x - c_I}{|I|}\right),$$

where c_I is the center of I . The central facts that we will need about wavelet bases are the orthogonality between different w_I 's, that $\|w\|_{L_2(\mathbb{R})} = 1$, and the regularity of w ,

$$\max(|w(x)|, |w'(x)|) \lesssim (1 + |x|)^{-m}, \quad \forall m \geq 2.$$

The analogy between wavelets and dyadic martingales is well known. The key observation is the following parallelism:

$$\sum_{|I|=2^{-n+1}} \langle f, w_I \rangle w_I \sim df_n,$$

where df_n denotes the n -th dyadic martingale difference of f . This parallelism indicates that martingale methods may be used to deal with wavelets. With this in mind, we develop the theory of operator-valued Hardy spaces, which are defined through wavelets, in the way that is well known in the noncommutative martingale case. Then we show that our Hardy and BMO spaces coincide with Mei's.

This paper is organized as follows. In Section 1, we will give some preliminaries on noncommutative analysis, and the definitions in our setting of $\mathcal{H}_p(\mathbb{R}, \mathcal{M})$ with $1 \leq p < \infty$ and $L_q \mathcal{MO}(\mathbb{R}, \mathcal{M})$ with $2 < q \leq \infty$. In Section 2, we are concerned with three duality results. The most important one is the noncommutative analogue of the famous Fefferman duality theorem between $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$ and $\mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$. The second is the duality between $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ and $L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$ with $1 < p < 2$, where we need a noncommutative version of Doob's inequality. This is why we consider the case $1 < p < 2$ independently. The last one is the duality between $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ and $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ when $1 < p < \infty$. As a corollary of the last two results, we identify $\mathcal{H}_q^c(\mathbb{R}, \mathcal{M})$ and $L_q^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$ with $2 < q < \infty$. Section 3 deals with interpolation for our Hardy spaces. In the last section, we show that our Hardy spaces coincide with those of [15]. Consequently, we can give an explicit completely unconditional basis for the space $H_1(\mathbb{R})$, when $H_1(\mathbb{R})$ is equipped with an appropriate operator space structure.

We end this introduction by introducing the convention that, throughout the paper, c will denote an absolute positive constant, which may vary from line to line, and c_p will denote a positive constant depending only on p .

2. Preliminaries

2.1. Operator-valued noncommutative L_p -spaces

Let \mathcal{M} be a von Neumann algebra equipped with a normal semifinite faithful trace τ and let $S_{\mathcal{M}}^+$ be the set of all positive elements x in \mathcal{M} with $\tau(s(x)) < \infty$, where $s(x)$ is the smallest projection e such that $exe = x$. Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^+$. Then any $x \in S_{\mathcal{M}}$ has finite trace, and $S_{\mathcal{M}}$ is a w^* -dense $*$ -subalgebra of \mathcal{M} .

Let $1 \leq p < \infty$. For any $x \in S_{\mathcal{M}}$, the operator $|x|^p$ belongs to $S_{\mathcal{M}}^+$ ($|x| = (x^*x)^{\frac{1}{2}}$). We define

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad \forall x \in S_{\mathcal{M}}.$$

One can check that $\|\cdot\|_p$ is well defined and is a norm on $S_{\mathcal{M}}$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M})$ which is the usual noncommutative L_p -space associated with (\mathcal{M}, τ) . For convenience, we usually set $L_{\infty}(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm $\|\cdot\|_{\mathcal{M}}$. The elements of $L_p(\mathcal{M}, \tau)$ can be described as closed densely defined operators on H (H being the Hilbert space on which \mathcal{M} acts). We refer the reader to [21] for more information on noncommutative L_p -spaces.

In this paper, we are concerned with three operator-valued noncommutative L_p -spaces. The first one is the noncommutative space $L_p(\mathcal{M}; \ell_2^c)$ (resp. $L_p(\mathcal{M}; \ell_2^r)$), which is studied at length in [9]. For this space, we need the following properties. In the sequel, p' will always denote the conjugate index of p .

Lemma 2.1. *Let $1 \leq p < \infty$. Then*

$$(2.1) \quad (L_p(\mathcal{M}; \ell_2^c))^* = L_{p'}(\mathcal{M}; \ell_2^c).$$

Thus, for $f = (f_k)_k \in L_p(\mathcal{M}; \ell_2^c)$ and $g = (g_k)_k \in L_{p'}(\mathcal{M}; \ell_2^c)$, we have

$$|\tau(\langle f, g \rangle)| \leq \|f\|_{L_p(\mathcal{M}; \ell_2^c)} \|g\|_{L_{p'}(\mathcal{M}; \ell_2^c)},$$

where

$$\langle f, g \rangle = \sum_k f_k g_k^*.$$

Lemma 2.2. *Let $1 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$(2.2) \quad [L_{p_0}(\mathcal{M}; \ell_2^c), L_{p_1}(\mathcal{M}; \ell_2^c)]_{\theta} = L_p(\mathcal{M}; \ell_2^c).$$

A similar equality holds for row spaces.

The second space is the ℓ_{∞} -valued noncommutative space $L_p(\mathcal{M}; \ell_{\infty})$, which is studied by Pisier [19] for an injective \mathcal{M} , and by Junge [8] for a general \mathcal{M} (see also [11] and [13] for more properties). For this space, we need the following property:

Lemma 2.3. *Let $1 \leq p < \infty$. Then*

$$(L_p(\mathcal{M}; \ell_1))^* = L_{p'}(\mathcal{M}; \ell_\infty).$$

Thus, for $x = (x_n)_n \in L_p(\mathcal{M}; \ell_1)$ and $y = (y_n)_n \in L_{p'}(\mathcal{M}; \ell_\infty)$, we have

$$(2.3) \quad \left| \sum_{n \geq 1} \tau(x_n y_n) \right| \leq \|x\|_{L_p(\mathcal{M}; \ell_1)} \|y\|_{L_{p'}(\mathcal{M}; \ell_\infty)},$$

where $L_p(\mathcal{M}; \ell_1)$ is the space of all sequences $x = (x_n)_n$ such that

$$\|(x_k)_k\|_{L_p(\mathcal{N}; \ell_1)} = \sup_{x_n = \sum_k a_{n,k}^* b_{n,k}} \left\| \sum_{n,k} |a_{n,k}|^2 \right\|_p^{1/2} \left\| \sum_{n,k} |b_{n,k}|^2 \right\|_p^{1/2}.$$

The third space is $L_p(\mathcal{M}; \ell_\infty^c)$ for $2 \leq p \leq \infty$, which was introduced in [4] and is related with the second space by

$$\|(x_n)_n\|_{L_p(\mathcal{M}; \ell_\infty^c)} = \|(|x_n|^2)_n\|_{L_{\frac{p}{2}}(\mathcal{M}; \ell_\infty)}.$$

These are normed spaces for the following characterization of the norm:

$$\|(x_n)_n\|_{L_p(\mathcal{M}; \ell_\infty^c)} = \inf_{x_n = y_n a} \| (y_n) \|_{\ell_\infty(L_\infty(\mathcal{M}))} \|a\|_{L_p(\mathcal{M})}.$$

We need the following interpolation results for these spaces (see [18]):

Lemma 2.4. *Let $2 \leq p_0 < p < p_1 \leq \infty$, $0 < \theta < 1$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then*

$$(2.4) \quad [L_{p_0}(\mathcal{M}; \ell_\infty^c), L_{p_1}(\mathcal{M}; \ell_\infty^c)]_\theta = L_p(\mathcal{M}; \ell_\infty^c).$$

2.2. Operator-valued Hardy spaces

In this paper, for simplicity, we denote $L_\infty(\mathbb{R}) \bar{\otimes} \mathcal{M}$ by \mathcal{N} . As indicated in the introduction, one can observe that we have the operator-valued Calderón identity

$$(2.5) \quad f(x) = \sum_{I \in \mathcal{D}} \langle f, w_I \rangle w_I(x) = \sum_{I \in \mathcal{D}} \int_{\mathbb{R}} f(y) \overline{w_I(y)} dy w_I(x),$$

which holds when $f \in L_2(\mathcal{N})$. As in the classical case, for $f \in S_{\mathcal{N}}$, we define the two Littlewood–Paley square functions as

$$(2.6) \quad S_c(f)(x) = \left(\sum_{I \in \mathcal{D}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbb{1}_I(x) \right)^{\frac{1}{2}},$$

$$(2.7) \quad S_r(f)(x) = \left(\sum_{I \in \mathcal{D}} \frac{|\langle f^*, w_I \rangle|^2}{|I|} \mathbb{1}_I(x) \right)^{\frac{1}{2}}.$$

For $1 \leq p < \infty$, define

$$\|f\|_{\mathcal{H}_p^c} = \|S_c(f)\|_{L_p(\mathcal{N})} \quad \text{and} \quad \|f\|_{\mathcal{H}_p^r} = \|S_r(f)\|_{L_p(\mathcal{N})}.$$

These are norms, which can be seen easily from the space $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$. So we define the space $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ (resp., $\mathcal{H}_p^r(\mathbb{R}, \mathcal{M})$) as the completion of $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^c})$ (resp., $(S_{\mathcal{N}}, \|\cdot\|_{\mathcal{H}_p^r})$).

Now, we define the operator-valued Hardy spaces as follows: for $1 \leq p < 2$,

$$(2.8) \quad \mathcal{H}_p(\mathbb{R}, \mathcal{M}) = \mathcal{H}_p^c(\mathbb{R}, \mathcal{M}) + \mathcal{H}_p^r(\mathbb{R}, \mathcal{M})$$

with the norm

$$\|f\|_{\mathcal{H}_p} = \inf\{\|g\|_{\mathcal{H}_p^c} + \|h\|_{\mathcal{H}_p^r} : f = g + h, g \in \mathcal{H}_p^c, h \in \mathcal{H}_p^r\},$$

and for $2 \leq p < \infty$,

$$(2.9) \quad \mathcal{H}_p(\mathbb{R}, \mathcal{M}) = \mathcal{H}_p^c(\mathbb{R}, \mathcal{M}) \cap \mathcal{H}_p^r(\mathbb{R}, \mathcal{M})$$

with the norm

$$\|f\|_{\mathcal{H}_p} = \max\{\|f\|_{\mathcal{H}_p^c}, \|f\|_{\mathcal{H}_p^r}\}.$$

We can identify $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ as a subspace of $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$, which is related with the two maps below:

Definition 2.5. (i) The map Φ is defined from $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ to $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ by

$$(2.10) \quad \Phi(f) = \left(\frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right)_{I \in \mathcal{D}}.$$

(ii) The projection map Ψ is defined from $L_2(\mathcal{N}; \ell_2^c(\mathcal{D}))$ to $\mathcal{H}_2^c(\mathbb{R}, \mathcal{M})$ by

$$(2.11) \quad \Psi((g_I)) = \sum_{I \in \mathcal{D}} \int \frac{g_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I dy \cdot w_I.$$

2.3. Operator-valued \mathcal{BMO} spaces

For $\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$(2.12) \quad \begin{aligned} \|\varphi\|_{\mathcal{BMO}^c} &= \sup_{J \in \mathcal{D}} \left\| \left(\frac{1}{|J|} \sum_{I \subset J} |\langle \varphi, w_I \rangle|^2 \right)^{\frac{1}{2}} \right\|_{\mathcal{M}} \quad \text{and} \\ \|\varphi\|_{\mathcal{BMO}^r} &= \|\varphi^*\|_{\mathcal{BMO}^c(\mathbb{R}, \mathcal{M})}. \end{aligned}$$

These are again norms modulo constant functions. Define

$$\begin{aligned} \mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) &= \{\varphi \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^c} < \infty\}, \quad \text{and} \\ \mathcal{BMO}^r(\mathbb{R}, \mathcal{M}) &= \{\varphi \in L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{\mathcal{BMO}^r} < \infty\}. \end{aligned}$$

Now we define

$$\mathcal{BMO}(\mathbb{R}, \mathcal{M}) = \mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) \cap \mathcal{BMO}^r(\mathbb{R}, \mathcal{M}).$$

As in the martingale case [11], we can also define $L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$ for all $2 < p \leq \infty$. For $\varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2}))$, set

$$(2.13) \quad \|\varphi\|_{L_p^c \mathcal{MO}} = \left\| \left(\frac{1}{|I_k^x|} \sum_{I \subset I_k^x} |\langle \varphi, w_I \rangle|^2 \right)_k \right\|_{L_{\frac{p}{2}}(\mathcal{N}; \ell_\infty)}^{\frac{1}{2}}$$

and

$$\|\varphi\|_{L_p^r \mathcal{MO}} = \|\varphi^*\|_{L_p^c \mathcal{MO}},$$

where I_k^x denotes the unique dyadic interval with length 2^{-k+1} that contains x . We will use the convention adopted in [13] for the norm in $L_{\frac{p}{2}}(\mathcal{N}; \ell_\infty)$. Thus

$$\left\| \left(\frac{1}{|I_k^x|} \sum_{I \subset I_k^x} |\langle \varphi, w_I \rangle|^2 \right)_k \right\|_{L_{\frac{p}{2}}(\mathcal{N}; \ell_\infty)}^{\frac{1}{2}} = \left\| \sup_k^+ \frac{1}{|I_k^x|} \sum_{I \subset I_k^x} |\langle \varphi, w_I \rangle|^2 \right\|_{L_{\frac{p}{2}}(\mathcal{N})}^{\frac{1}{2}}$$

These are norms, which follow from the Banach spaces $L_p(\mathcal{N} \otimes B(\ell_2(\mathcal{D})); \ell_\infty^c)$. Again, we can define

$$\begin{aligned} L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M}) &= \{ \varphi \in L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_p^c \mathcal{MO}} < \infty \}, \quad \text{and} \\ L_p^r \mathcal{MO}(\mathbb{R}, \mathcal{M}) &= \{ \varphi \in L_p(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dx}{1+x^2})) : \|\varphi\|_{L_p^r \mathcal{MO}} < \infty \}. \end{aligned}$$

Define

$$L_p \mathcal{MO}(\mathbb{R}, \mathcal{M}) = L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M}) \cap L_p^r \mathcal{MO}(\mathbb{R}, \mathcal{M}).$$

Note that $L_\infty^c \mathcal{MO}(\mathbb{R}, \mathcal{M}) = \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$. It is easy to check that all the spaces we have defined here are Banach spaces with respect to the relevant norms.

3. Duality

To prove the first two duality results in this section, we need the following non-commutative version of Doob’s inequality from [8].

Let $(\mathcal{E}_n)_n$ be the conditional expectation with respect to a filtration $(\mathcal{N}_n)_n$ of \mathcal{N} .

Lemma 3.1. *Let $1 < p \leq \infty$ and $f \in L_p(\mathcal{N})$. Then*

$$(3.1) \quad \left\| \sup_n^+ \mathcal{E}_n(f) \right\|_{L_p(\mathcal{N})} \leq c_p \|f\|_{L_p(\mathcal{N})}.$$

Theorem 3.2. *We have*

$$(3.2) \quad (\mathcal{H}_1^c(\mathbb{R}, \mathcal{M}))^* = \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$$

with equivalent norms. That is, every $\varphi \in \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$ induces a continuous linear functional l_φ on $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$ by

$$(3.3) \quad l_\varphi(f) = \tau \int \varphi^* f, \quad \forall f \in S_{\mathcal{N}}.$$

Conversely, for every $l \in (\mathcal{H}_1^c(\mathbb{R}, \mathcal{M}))^*$, there exists a $\varphi \in \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$ such that $l = l_\varphi$. Moreover,

$$c^{-1} \|\varphi\|_{\mathcal{BMO}^c} \leq \|l_\varphi\|_{(\mathcal{H}_1^c)^*} \leq c \|\varphi\|_{\mathcal{BMO}^c}$$

where $c > 0$ is a universal constant.

Similarly, duality holds between \mathcal{H}_1^r and \mathcal{BMO}^r , and between \mathcal{H}_1 and \mathcal{BMO} , with equivalent norms.

In order to adapt the arguments in the martingale case, we need to define the truncated square functions for $n \in \mathbb{Z}$,

$$S_{c,n}(f)(x) = \left(\sum_{k=-\infty}^n \sum_{|I|=2^{-k+1}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbb{1}_I(x) \right)^{\frac{1}{2}}.$$

Proof. Since $S_{\mathcal{N}}$ is dense in $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$, by an approximation argument we only need to prove the inequality

$$|l_\varphi(f)| \leq c \|\varphi\|_{\mathcal{BMO}^c} \|f\|_{\mathcal{H}_1^c}$$

for $f \in S_{\mathcal{N}}$. By approximation we may assume that $S_{c,n}(f)(x)$ is invertible in \mathcal{M} for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then we have

$$\begin{aligned} |l_\varphi(f)| &= \left| \tau \int \varphi^* f dx \right| = \left| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \langle \varphi, w_I \rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'} \rangle w_{I'} dx \right| \\ &= \left| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I \rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \sum_{|I'|=2^{-n+1}} \frac{\langle f, w_{I'} \rangle}{|I'|^{\frac{1}{2}}} \mathbb{1}_{I'} dx \right| \\ &\leq \sum_n \left(\tau \int \left| \sum_{|I|=2^{-n+1}} \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right|^2 S_{c,n}^{-1}(f) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\tau \int \left| \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right|^2 S_{c,n}(f) \right)^{\frac{1}{2}} \\ &\leq \left(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{-1}(f) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I \rangle|^2}{|I|} \mathbb{1}_I S_{c,n}(f) \right)^{\frac{1}{2}} \\ &= A \cdot B. \end{aligned}$$

In the above estimates, the second equality uses the orthogonality of the w_I 's on different levels, and the third one the orthogonality of the w_I 's on the same level and the disjointness of different dyadic I 's on the same level; the first inequality uses the Hölder inequality of Lemma 2.1, and the second one uses the Cauchy-Schwarz inequality and the disjointness of different I 's on the same level.

Now, let us estimate A :

$$\begin{aligned}
 A^2 &= \sum_n \tau \int (S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{-1}(f) \\
 &= \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f))(1 + S_{c,n-1}(f) S_{c,n}^{-1}(f)) \\
 &\leq \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f)) \|1 + S_{c,n-1}(f) S_{c,n}^{-1}(f)\|_\infty \\
 &\leq 2 \sum_n \tau \int (S_{c,n}(f) - S_{c,n-1}(f)) = 2 \|f\|_{\mathcal{H}_1^c}.
 \end{aligned}$$

For the first inequality, we have used the Hölder inequality and the positivity of $S_{c,n}(f) - S_{c,n-1}(f)$.

The second term is estimated as follows:

$$\begin{aligned}
 B^2 &= \sum_k \tau \int (S_{c,k}(f) - S_{c,k-1}(f)) \sum_{n \geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I \rangle|^2}{|I|} \mathbf{1}_I \\
 &= \sum_k \tau \sum_j (S_{c,k}(f) - S_{c,k-1}(f)) \int_{I_k^j} \sum_{n \geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I \rangle|^2}{|I|} \mathbf{1}_I \\
 &= \sum_k \tau \sum_j \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \frac{1}{|I_k^j|} \sum_{I \subset I_k^j} |\langle \varphi, w_I \rangle|^2 \\
 &\leq \sum_k \sum_j \tau \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \left\| \frac{1}{|I_k^j|} \sum_{I \subset I_k^j} |\langle \varphi, w_I \rangle|^2 \right\|_\infty \\
 &\leq \|\varphi\|_{\mathcal{BMO}^c}^2 \sum_k \sum_j \tau \int_{I_k^j} (S_{c,k}(f) - S_{c,k-1}(f)) \\
 &= \|\varphi\|_{\mathcal{BMO}^c}^2 \|f\|_{\mathcal{H}_1^c}.
 \end{aligned}$$

We use, in the first equality, the Fubini theorem; and in the second equality, the fact that $S_{c,k-1}(f)$ and $S_{c,k}(f)$ are constant on the dyadic interval $I_k^j = [j2^{-k+1}, (j+1)2^{-k+1})$. For the first inequality the Hölder inequality and the positivity of $S_{c,n}(f) - S_{c,n-1}(f)$ are used.

Now, let us begin to deal with the other direction, i.e., supposing that l is a bounded linear functional on $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$, we want to find an operator-valued function φ in $\mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$ such that $l = l_\varphi$ and $l_\varphi(f) = \tau \int \varphi^* f$ for $f \in S_{\mathcal{N}}$. By the embedding operator Φ in (2.10) and by the Hahn–Banach theorem, l extends to a bounded continuous functional on $L_1(\mathcal{N}; \ell_2^c(\mathcal{D}))$ of the same norm. Then, by Lemma 2.1, there exists $g = (g_I)_{I \in \mathcal{D}}$ such that $\|g\|_{L_\infty(\mathcal{N}; \ell_2^c(\mathcal{D}))} = \|l\|$, and

$$l(f) = \tau \int \sum_{I \in \mathcal{D}} g_I^* \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbf{1}_I, \quad \forall f \in S_{\mathcal{N}}.$$

Now let $\varphi = \Psi(g)$, where Ψ is defined as (2.11). The orthogonality of the w_I 's yields

$$\begin{aligned} \left\| \sum_{I \subset J} |\langle \varphi, w_I \rangle|^2 \right\|_{\mathcal{M}} &= \left\| \sum_{I \subset J} \left| \int \frac{g_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right|^2 \right\|_{\mathcal{M}} \leq \left\| \sum_{I \subset J} \int_J |g_I|^2 \right\|_{\mathcal{M}} \\ &\leq |J| \left\| \sum_{I \subset J} |g_I|^2 \right\|_{L^\infty(\mathcal{N})} \leq |J| \|(g_I)_I\|_{L^\infty(\mathcal{N}; \ell_2^2(\mathcal{D}))}, \end{aligned}$$

where in the first inequality the Kadison–Schwarz inequality has been used. Also thanks to the orthogonality of the w_I 's, we get

$$l(f) = \tau \int \sum_{I \in \mathcal{D}} g_I^* \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau \int \varphi^* f$$

for all $f \in S_{\mathcal{N}}$. Therefore, we have completed the proof for the column spaces. Passing to adjoints, we have the conclusion concerning \mathcal{H}_1^r and \mathcal{BMO}^r . Finally, by the classical fact that the dual of a sum space is the intersection space, we obtain the duality between \mathcal{H}_1 and \mathcal{BMO} . \square

Theorem 3.3. *Let $1 < p < 2$. We have*

$$(3.4) \quad (\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^* = L_{p'}^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$$

with equivalent norms. That is, every $\varphi \in L_{p'}^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$ induces a continuous linear functional l_φ on $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ by

$$(3.5) \quad l_\varphi(f) = \tau \int \varphi^* f, \quad \forall f \in S_{\mathcal{N}}.$$

Conversely, for every $l \in (\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^*$, there exists an operator-valued function $\varphi \in L_{p'}^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$ such that $l = l_\varphi$ and

$$c_p^{-1} \|\varphi\|_{L_{p'}^c \mathcal{MO}} \leq \|l_\varphi\|_{(\mathcal{H}_p^c)^*} \leq \sqrt{2} \|\varphi\|_{L_{p'}^c \mathcal{MO}}.$$

Similarly, duality holds between \mathcal{H}_p^r and L_p^r , and between \mathcal{H}_p and $L_{p'} \mathcal{MO}$, with equivalent norms.

We need the following lemma from [11]. We recall it for the reader's convenience, but without proof.

Lemma 3.4. *Let s and t be two real numbers such that $s < t$ and $0 \leq s \leq 1 \leq t \leq 2$. Let x and y be two positive operators such that $x \leq y$ and $x^{t-s}, y^{t-s} \in L_1(\mathcal{N})$. Then*

$$\tau \int y^{-s/2} (y^t - x^t) y^{-s/2} \leq 2\tau \int y^{-(s+1-t)/2} (y - x) y^{-(s+1-t)/2}.$$

Proof of Theorem 3.3. We only need to prove the first assertion about \mathcal{H}_p^c . Since $S_{\mathcal{N}}$ is dense in $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$, by an approximation argument, we only need to prove the inequality

$$|l_\varphi(f)| \leq c \|\varphi\|_{L_p^c, \mathcal{MO}} \|f\|_{\mathcal{H}_p^c}$$

for $f \in S_{\mathcal{N}}$. By approximation we may assume that $S_{c,n}(f)(x)$ is invertible in \mathcal{M} for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. By the same principle as in the noncommutative martingale case as in [11], we have

$$\begin{aligned} |l_\varphi(f)| &= \left| \tau \int \varphi^* f dx \right| = \left| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \langle \varphi, w_I \rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'} \rangle w_{I'} dx \right| \\ &= \left| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I \rangle^*}{|I|^{\frac{1}{2}}} \mathbb{1}_I \sum_{|I'|=2^{-n+1}} \frac{\langle f, w_{I'} \rangle}{|I'|^{\frac{1}{2}}} \mathbb{1}_{I'} dx \right| \\ &\leq \sum_n \left(\tau \int \left| \sum_{|I|=2^{-n+1}} \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right|^2 S_{c,n}^{p-2}(f) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\tau \int \left| \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right|^2 S_{c,n}^{2-p}(f) \right)^{\frac{1}{2}} \\ &\leq \left(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{p-2}(f) \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I \rangle|^2}{|I|} \mathbb{1}_I S_{c,n}^{2-p}(f) \right)^{\frac{1}{2}} \\ &= A \cdot B. \end{aligned}$$

Now we need Lemma 3.4 to estimate the first term. Take $s = 2 - p$ and $t = 2$. The lemma yields

$$\begin{aligned} A^2 &= \sum_n \tau \int (S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{p-2}(f) \\ &= \sum_n \tau \int S_{c,n}^{-(2-p)/2}(f) (S_{c,n}^2(f) - S_{c,n-1}^2(f)) S_{c,n}^{-(2-p)/2}(f) \\ &\leq 2 \sum_n \tau \int S_{c,n}^{-(1-p)/2}(f) (S_{c,n}(f) - S_{c,n-1}(f)) S_{c,n}^{-(1-p)/2}(f) \\ &= 2 \sum_n \tau \int S_{c,n}(f) - S_{c,n-1}(f) S_{c,n}^{p-1}(f) \\ &\leq 2 \sum_n \tau \int S_{c,n}^p(f) - S_{c,n-1}^p(f) \\ &= 2 \|f\|_{\mathcal{H}_p^c}^p. \end{aligned}$$

The last inequality has used two elementary inequalities: $0 \leq S_{c,n-1}(f) \leq S_{c,n}(f)$ implies $S_{c,n-1}^{p-1}(f) \leq S_{c,n}^{p-1}(f)$ for $0 < p - 1 < 1$; and

$$\tau(S_{c,n-1}^p(f)) \leq \tau(S_{c,n-1}^{\frac{1}{2}}(f) S_{c,n}^{p-1}(f) S_{c,n-1}^{\frac{1}{2}}(f)).$$

The second term can be deduced from the nontrivial duality results in Lemma 2.3 for $1 < p < \infty$ as follows:

$$\begin{aligned}
 B^2 &= \sum_k \tau \int S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \sum_{n \geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I \rangle|^2}{|I|} \mathbb{1}_I \\
 &= \sum_k \tau \sum_j S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \int_{I_k^j} \sum_{n \geq k} \sum_{|I|=2^{-n+1}} \frac{|\langle \varphi, w_I \rangle|^2}{|I|} \mathbb{1}_I \\
 &= \sum_k \tau \sum_j \int \mathbb{1}_{I_k^j}(x) S_{c,k}^{2-p}(f)(x) - S_{c,k-1}^{2-p}(f)(x) \frac{1}{|I_k^j|} \sum_{I \subset I_k^j} |\langle \varphi, w_I \rangle|^2 dx \\
 &= \sum_k \tau \int S_{c,k}^{2-p}(f)(x) - S_{c,k-1}^{2-p}(f)(x) \frac{1}{|I_k^x|} \sum_{I \subset I_k^x} |\langle \varphi, w_I \rangle|^2 dx \\
 &\leq \left\| \sum_k S_{c,k}^{2-p}(f) - S_{c,k-1}^{2-p}(f) \right\|_{L_{(p'/2)'}} \left\| \sup_k \frac{1}{|I_k^x|} \sum_{I \subset I_k^x} |\langle \varphi, w_I \rangle|^2 \right\|_{L_{p'/2}} \\
 &= \|\varphi\|_{L_{p',\mathcal{M}}^c}^2 \|f\|_{\mathcal{H}_p^c}^{2-p}.
 \end{aligned}$$

The first equality uses the Fubini theorem, and the second one uses the fact that $S_{c,k-1}(f)$ and $S_{c,k}(f)$ are constant on the dyadic intervals with length 2^{-k+1} .

For the other direction, we can carry out the proof as in the case $p = 1$. Suppose that l is a bounded linear functional on $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$. By the embedding operator Φ , by the Hahn–Banach theorem, and by the results in Lemma 2.1, we can find $g = (g_I)_{I \in \mathcal{D}}$ such that $\|g\|_{L_{p',(\mathcal{N}; \ell_2^c(\mathcal{D}))}} = \|l\|$ and

$$l(f) = \tau \int \sum_{I \in \mathcal{D}} g_I^* \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I, \forall f \in S_{\mathcal{N}}.$$

Now let $\varphi = \Psi(g)$ be as defined in (2.11). The orthogonality of the w_I 's yields

$$\begin{aligned}
 \left\| \sup_n \frac{1}{|I_n^x|} \sum_{I \subset I_n^x} |\langle \varphi, w_I \rangle|^2 \right\|_{L_{p'/2}(\mathcal{N})} &= \left\| \sup_n \frac{1}{|I_n^x|} \sum_{I \subset I_n^x} \left| \int \frac{g_I}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right|^2 \right\|_{L_{p'/2}(\mathcal{N})} \\
 &\leq \left\| \sup_n \frac{1}{|I_n^x|} \sum_{I \subset I_n^x} \int_{I_n^x} |g_I|^2 \right\|_{L_{p'/2}(\mathcal{N})} \leq \left\| \sup_n \frac{1}{|I_n^x|} \int_{I_n^x} \sum_{I \subset I_n^x} |g_I|^2 \right\|_{L_{p'/2}(\mathcal{N})} \\
 &\leq \left\| \sup_n \frac{1}{|I_n^x|} \int_{I_n^x} \sum_{I \in \mathcal{D}} |g_I|^2 \right\|_{L_{p'/2}(\mathcal{N})} \leq c \left\| \sum_{I \in \mathcal{D}} |g_I|^2 \right\|_{L_{p'/2}(\mathcal{N})} \\
 &= c \|(g_I)_I\|_{L_{p',(\mathcal{N}; \ell_2^c(\mathcal{D}))}},
 \end{aligned}$$

where for the first inequality we have used the Kadison–Schwarz inequality, and the last inequality is (3.1). Also due to the orthogonality of the w_I 's, we get

$$l(f) = \tau \int \sum_{I \in \mathcal{D}} g_I^* \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I = \tau \int \varphi^* f,$$

for all $f \in S_{\mathcal{N}}$. Therefore, we have completed the proof. □

Instead of using the noncommutative Doob’s inequality, we will use the following noncommutative Stein inequality from [20] to prove the duality between the spaces \mathcal{H}_p^c , $1 < p < \infty$.

Let $(\mathcal{E}_n)_n$ be the conditional expectation with respect to a filtration $(\mathcal{N}_n)_n$ of \mathcal{N} .

Lemma 3.5. *Let $1 < p < \infty$ and $a = (a_n)_n \in L_p(\mathcal{N}; \ell_2^c)$. Then there exists a constant depending only on p such that*

$$(3.6) \quad \left\| \left(\sum_n |\mathcal{E}_n a_n|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{N})} \leq c_p \left\| \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{N})}.$$

Theorem 3.6. *For any $1 < p < \infty$, we have*

$$(3.7) \quad (\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^* = \mathcal{H}_{p'}^c(\mathbb{R}, \mathcal{M}).$$

Proof. By a reasoning similar to that in the proof of Theorem 3.2, we calculate

$$\begin{aligned} |l_\varphi(f)| &= \left| \tau \int \varphi^* f dx \right| = \left| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \langle \varphi, w_I \rangle^* w_I \sum_{|I'|=2^{-n+1}} \langle f, w_{I'} \rangle w_{I'} dx \right| \\ &= \left| \sum_n \tau \int \sum_{|I|=2^{-n+1}} \frac{\langle \varphi, w_I \rangle^*}{|I|^{\frac{1}{2}}} \mathbf{1}_I \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbf{1}_I dx \right| \\ &\leq \left\| \left(\sum_{I \in \mathcal{D}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbf{1}_I \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}, \mathcal{M})} \cdot \left\| \left(\sum_{I \in \mathcal{D}} \frac{|\langle \varphi, w_I \rangle|^2}{|I|} \mathbf{1}_I \right)^{\frac{1}{2}} \right\|_{L_{p'}(\mathbb{R}, \mathcal{M})}. \end{aligned}$$

Now, we turn to the proof of the inverse direction. Take a bounded linear functional $l \in (\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}))^*$. By the embedding operator Φ and the Hahn–Banach extension theorem, l extends to a bounded linear functional on $L_p(\mathcal{N}; \ell_2^c)$ with the same norm. Thus by (2.1), there exists a sequence $g = (g_I)_I$ such that

$$\|g\|_{L_q(\mathcal{N}; \ell_2^c(\mathcal{D}))} = \|l\|$$

and

$$l(f) = \tau \int \sum_{I \in \mathcal{D}} g_I^* \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbf{1}_I, \forall f \in S_{\mathcal{N}}.$$

Now let $\varphi = \Psi(g)$ where Ψ is defined in (2.11). Then, applying the Stein inequality (3.5) to the conditional expectation

$$\mathcal{E}_I(h) = \sum_J \frac{1}{|J|} \int_J h(y) dy \cdot \mathbf{1}_J,$$

where J is a dyadic interval with the same length as I , we get

$$\begin{aligned} \|\varphi\|_{\mathcal{H}_{p'}^c(\mathbb{R}, \mathcal{M})} &= \left\| \left(\sum_{I \in \mathcal{D}} \left| \frac{1}{|I|} \int_I g_I dy \cdot \mathbf{1}_I \right|^2 \right)^{\frac{1}{2}} \right\|_{L_{p'}(\mathcal{N})} \\ &\leq \left\| \left(\sum_{I \in \mathcal{D}} |\mathcal{E}_I(g_I)|^2 \right)^{\frac{1}{2}} \right\|_{L_{p'}(\mathcal{N})} \leq c_{p'} \left\| \left(\sum_{I \in \mathcal{D}} |g_I|^2 \right)^{\frac{1}{2}} \right\|_{L_{p'}(\mathcal{N})}. \end{aligned}$$

By the orthogonality of the w_I 's, we have

$$l(f) = \tau \int \sum_{I \in \mathcal{D}} g_I^* \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbf{1}_I = \tau \int \varphi^* f,$$

for all $f \in S_{\mathcal{N}}$. □

From the proof of the second part of Theorem 3.2, Theorem 3.3 and Theorem 3.6, we obtain the boundedness of Ψ , which we state as a corollary:

Corollary 3.7. (i) *Let $1 < p < \infty$. Then Ψ is a projection map from $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ onto $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ if we identify the latter as a subspace of the former.*

(ii) *For $2 < p \leq \infty$, Ψ is a bounded map from $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ to $L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$.*

Theorem 3.3 and Theorem 3.6 immediately imply the following corollary:

Corollary 3.8. *Let $2 < p < \infty$. Then*

$$\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}) = L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M}), \quad \forall 2 < p < \infty,$$

with equivalent norms.

4. Interpolation

This section is devoted to interpolation for our wavelet Hardy spaces. The interpolation results below will be needed in the next section to compare our Hardy spaces with those of Mei.

Lemma 4.1. *Let $1 < p_0 < p < p_1 < \infty$. We have*

$$(4.1) \quad [\mathcal{H}_{p_0}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}_{p_1}^c(\mathbb{R}, \mathcal{M})]_{\theta} = \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$$

with equivalent norms, where θ satisfies $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Proof. The embedding map Φ yields

$$[\mathcal{H}_{p_0}^c, \mathcal{H}_{p_1}^c]_{\theta} \subset \mathcal{H}_p^c.$$

On the other hand, it is the projection map Ψ from $L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ onto $\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$ stated in Corollary 3.7 that yields the inverse direction. □

Theorem 4.2. *Let $1 \leq q < p < \infty$. We have*

$$(4.2) \quad [\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}_q^c(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}} = \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})$$

with equivalent norms.

Proof. We will prove the theorem using the same general strategy as in [18].

Step 1. We prove the conclusion for $2 < q < p < \infty$. The identity can be seen easily from the following two inclusions. On the one hand, the operator Φ from (2.10), together with (2.2), yields

$$[\mathcal{H}_1^c(\mathbb{R}, \mathcal{M}), \mathcal{H}_q^c(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}} \subset \mathcal{H}_{p'}^c(\mathbb{R}, \mathcal{M}).$$

Then by duality and Corollary 3.8, we have

$$(4.3) \quad L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M}) \subset [\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), L_q^c \mathcal{MO}(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}}.$$

On the other hand, the operator \mathcal{T} identifying $L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M})$ as a subspace of $L_p(L_\infty(\mathcal{N} \otimes B(\ell_2(\mathcal{D})); \ell_\infty^c)$ and defined by

$$(4.4) \quad \mathcal{T}(\varphi) = \langle f, w_I \rangle |I_k^t|^{-\frac{1}{2}} \mathbb{1}_{I \subset I_k^t}(I) \otimes e_{I,1},$$

together with Lemma 2.4, yields

$$(4.5) \quad [\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), L_q^c \mathcal{MO}(\mathbb{R}, \mathcal{M})]_{\frac{q}{p}} \subset L_p^c \mathcal{MO}(\mathbb{R}, \mathcal{M}).$$

Step 2. We prove the conclusion for $1 < q < p < \infty$. This step can be divided into two substeps.

Substep 2.1: $p > 2$. Let $p < s < \infty$. By Step 1, we have

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})]_{\frac{p}{s}} = \mathcal{H}_s^c(\mathbb{R}, \mathcal{M}).$$

On the other hand, by Theorem 4.1, we have

$$[\mathcal{H}_q^c, \mathcal{H}_s^c]_\theta = \mathcal{H}_p^c,$$

where here (and in the rest of the paper) θ denotes the interpolation parameter. Then Wolff's interpolation theorem yields the result.

Substep 2.2: $p \leq 2$. Let $s > 2$, then by Substep 2.1, we have

$$[\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}), \mathcal{H}_p^c(\mathbb{R}, \mathcal{M})]_{\frac{p}{s}} = \mathcal{H}_s^c(\mathbb{R}, \mathcal{M}).$$

Together with Lemma 4.1, Wolff's interpolation theorem yields the result.

Step 3. We prove the conclusion for $1 = q < p < \infty$. Take $s > \max(p, 2)$. By Step 2 and duality (see Theorem 4.3.1 in [2]), we get

$$[\mathcal{H}_1^c, \mathcal{H}_s^c]_\theta = \mathcal{H}_p^c.$$

Then together with Step 2, Wolff's interpolation yields the conclusion. □

Theorem 4.3. *For $1 < p < \infty$, we have*

$$\mathcal{H}_p(\mathbb{R}, \mathcal{M}) = L_p(\mathcal{N})$$

with equivalent norms.

Proof. There are several ways to prove this result. One can prove it by the strategy used in [20] together with Stein’s inequality (3.5). Here, we just use that $L_p(\mathcal{M})$ with $1 < p < \infty$ is a UMD space and our $(w_I)_I$ is a complete orthonormal basis.

So, by Theorem 3.8 in [7], we have

$$\|f\|_{L_p(\mathcal{N})} \simeq \left(\mathbb{E} \left\| \sum_{I \in \mathcal{D}} \varepsilon_I \frac{\langle f, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right\|_{L_p(\mathcal{N})}^p \right)^{\frac{1}{p}}.$$

We complete the proof for $2 \leq p < \infty$ by Khintchine’s inequalities. Now, let us prove the case $1 < p < 2$. Let $f \in \mathcal{H}_p(\mathbb{R}, \mathcal{M})$, then for any $\epsilon > 0$, by the definition of $\mathcal{H}_p(\mathbb{R}, \mathcal{M})$, there exists a decomposition $f = f_c + f_r$ such that

$$\|f_c\|_{\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})} + \|f_r\|_{\mathcal{H}_p^r(\mathbb{R}, \mathcal{M})} \leq \|f\|_{\mathcal{H}_p(\mathbb{R}, \mathcal{M})} + \epsilon.$$

Take any $g \in L_{p'}(\mathcal{N})$, by the results for $p' > 2$, the operator-valued Calderón identity (2.5) yields

$$\begin{aligned} |\tau \int g f^*| &= \left| \sum_{I \in \mathcal{D}} \tau \int \frac{\langle g, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f^*, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right| \\ &\leq \left| \sum_{I \in \mathcal{D}} \tau \int \frac{\langle g, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f_c^*, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right| + \left| \sum_{I \in \mathcal{D}} \tau \int \frac{\langle g, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \cdot \frac{\langle f_r^*, w_I \rangle}{|I|^{\frac{1}{2}}} \mathbb{1}_I \right| \\ &\leq \|S_c(g)\|_{L_{p'}(\mathcal{N})} \|S_c(f_c)\|_{L_p(\mathcal{N})} + \|S_r(g)\|_{L_{p'}(\mathcal{N})} \|S_r(f_r)\|_{L_p(\mathcal{N})} \\ &\leq c_{p'} \|g\|_{L_{p'}(\mathcal{N})} (\|f\|_{\mathcal{H}_p(\mathbb{R}, \mathcal{M})} + \epsilon). \end{aligned}$$

Taking the sup and letting $\epsilon \rightarrow 0$, we get the desired result.

Finally, we prove the inverse inequality. Let $f \in L_p(\mathcal{N})$. By duality, we can find two sequences of functions $(F_{c,I})_I \in L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))$ and $(F_{r,I})_I \in L_p(\mathcal{N}; \ell_2^r(\mathcal{D}))$ such that $F_{c,I} + F_{r,I} = \langle f, w_I \rangle |I|^{-\frac{1}{2}} \mathbb{1}_I$ and

$$\|(F_{c,I})_I\|_{L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))} + \|(F_{r,I})_I\|_{L_p(\mathcal{N}; \ell_2^r(\mathcal{D}))} \leq \|f\|_{L_p(\mathcal{N})}.$$

Let $f_c = \Psi((F_{c,I})_I)$ and $f_r = \Psi((F_{r,I})_I)$. By identity (2.5), we have $f = f_c + f_r$. On the other hand, by the Stein inequality (3.5), we have that $\|f_c\|_{\mathcal{H}_p^c(\mathbb{R}, \mathcal{M})} \leq \|(F_{c,I})_I\|_{L_p(\mathcal{N}; \ell_2^c(\mathcal{D}))}$ and $\|f_r\|_{\mathcal{H}_p^r(\mathbb{R}, \mathcal{M})} \leq \|(F_{r,I})_I\|_{L_p(\mathcal{N}; \ell_2^r(\mathcal{D}))}$. So we have found the desired decomposition of f . \square

Theorem 4.4. *The following results hold with equivalent norms:*

(i) *Let $1 \leq q < p < \infty$. We have*

$$(4.6) \quad [\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} = L_p(\mathcal{N}).$$

(ii) *Let $1 < q < p \leq \infty$. We have*

$$(4.7) \quad [\mathcal{H}_1(\mathbb{R}, \mathcal{M}), L_p(\mathcal{N})]_{\frac{p'}{q}} = L_q(\mathcal{N}).$$

(iii) *Let $1 < p < \infty$. We have*

$$(4.8) \quad [\mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_{\frac{1}{p}} = L_p(\mathcal{N}).$$

In order to prove this theorem, we need the following result from the theory of interpolation. We formulate it here without proof.

Lemma 4.5. *Let $A_0, B_0, A_1,$ and B_1 be four Banach spaces satisfying the property needed for interpolation. Then*

$$[A_0 + B_0, A_1 + B_1]_\theta \supset [A_0, A_1]_\theta + [B_0, B_1]_\theta$$

and

$$[A_0 \cap B_0, A_1 \cap B_1]_\theta \subset [A_0, A_1]_\theta \cap [B_0, B_1]_\theta.$$

Proof of Theorem 4.4. (i) We use a strategy, similar to, but different from, that used to prove Theorem 4.2.

Step 1. We prove the results for $2 \leq q < p < \infty$. By Theorem 4.3, Theorem 4.2 and Lemma 4.5, we have

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}} \subset L_p(\mathcal{N}).$$

The inverse direction follows from $L_\infty(\mathcal{N}) \subset \mathcal{BMO}(\mathbb{R}, \mathcal{M})$:

$$L_p(\mathcal{N}) = [L_\infty(\mathcal{N}), L_q(\mathcal{N})]_{\frac{q}{p}} \subset [\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_q(\mathcal{N})]_{\frac{q}{p}}.$$

Step 2. We prove the results for $1 \leq q < 2 \leq p < \infty$. By Step 1, we have

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_2(\mathcal{N})]_{\frac{2}{p}} = L_p(\mathcal{N}).$$

Together with

$$L_2(\mathcal{N}) = [L_p(\mathcal{N}), L_q(\mathcal{N})]_\theta,$$

Wolff’s interpolation yields the conclusion.

Step 3. We prove the results for $1 \leq q < p < 2$. By Step 2, we have

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_p(\mathcal{N})]_{\frac{p}{2}} = L_2(\mathcal{N}).$$

Together with

$$L_p(\mathcal{N}) = [L_2(\mathcal{N}), L_q(\mathcal{N})]_\theta,$$

Wolff’s interpolation yields the conclusion.

(ii) The results for $1 < q < p < \infty$ can be immediately proved by duality and the partial results in (i). For $p = \infty$, take $q < s < \infty$. Then, by Wolff’s argument, we get the conclusion.

(iii) First, we prove the conclusion for $p < 2$. By (i) and (ii), we have

$$[\mathcal{BMO}(\mathbb{R}, \mathcal{M}), L_p(\mathcal{N})]_{\frac{p}{2}} = L_{p'}(\mathcal{N}) \quad \text{and} \quad [\mathcal{H}_1(\mathbb{R}, \mathcal{M}), L_{p'}(\mathcal{N})]_{\frac{p}{2}} = L_p(\mathcal{N}).$$

We end with Wolff’s argument. The proof for $p > 2$ is the same. Finally, when $p = 2$, we can take $s > 2$. By the results for $p \neq 2$ and the reiteration theorem (see Theorem 4.6.1 in [2]), we get

$$\begin{aligned} L_2 &= [L_s, L_{s'}]_\theta = [\mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_{\frac{1}{s}}, \mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_{\frac{1}{s'}}]_\theta \\ &= [\mathcal{BMO}(\mathbb{R}, \mathcal{M}), \mathcal{H}_1(\mathbb{R}, \mathcal{M})]_\theta. \end{aligned}$$

□

5. Comparison with Mei’s results

We denote by $H_p^c(\mathbb{R}, \mathcal{M})$ the column Hardy space defined in [15] through operator-valued Lusin square function, and by $BMO^c(\mathbb{R}, \mathcal{M})$ the column bounded mean oscillation space appearing in matrix-valued harmonic analysis (see, e.g., [15]). We have the following result:

Theorem 5.1. *We have*

$$\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) = BMO^c(\mathbb{R}, \mathcal{M})$$

with equivalent norms. Similar results holds for the row spaces. Consequently, $\mathcal{BMO}(\mathbb{R}, \mathcal{M}) = BMO(\mathbb{R}, \mathcal{M})$ with equivalent norms.

The theorem can be easily seen from the corresponding $BMO(\mathbb{R}, H)$ -spaces. However, we also use the idea in the proof of Theorem 1.2 in [7] to prove the theorem.

Proof. $\mathcal{BMO}^c(\mathbb{R}, \mathcal{M}) \subset BMO^c(\mathbb{R}, \mathcal{M})$. Let $\varphi \in \mathcal{BMO}_c(\mathbb{R}, \mathcal{M})$. As in the beginning of the proof of Theorem 1.2 in [7], fix a finite interval $I \subset \mathbb{R}$, and consider the collections of dyadic intervals:

- (1) $\mathcal{D}_1 := \{J \in \mathcal{D}; 2|J| > |I|\}$,
- (2) $\mathcal{D}_2 := \{J \in \mathcal{D}; 2|J| \leq |I|, 2J \cap 2I = \emptyset\}$,
- (3) $\mathcal{D}_3 := \{J \in \mathcal{D}; 2|J| \leq |I|, 2J \cap 2I \neq \emptyset\}$.

Let $a_J = \langle \varphi, \omega_J \rangle$. Then we have an *a priori* formal series

$$\varphi_1(x) = \sum_{J \in \mathcal{D}_1} a_J [\omega_J(x) - \omega_J(c_I)], \quad \varphi_i(x) = \sum_{J \in \mathcal{D}_i} a_J \omega_J(x), \quad i = 2, 3,$$

where c_I is the center of the interval I . Write $\varphi_I = \varphi_1 + \varphi_2 + \varphi_3$. By an argument similar to that in [7], we only need to prove

$$\left\| \frac{1}{|I|} \int_I |\varphi_I(x)|^2 dx \right\|_{\mathcal{M}} < \infty.$$

By scaling we can assume:

$$\sup_I \frac{1}{|I|} \left\| \sum_{J \subset I} |a_J|^2 \right\| = 1.$$

Then we have the obvious bound $\|a_J\| \leq |J|^{\frac{1}{2}}$ for the individual terms.

We have the following estimates for φ_1 :

$$\begin{aligned} \left\| \frac{1}{|I|} \int_I |\varphi_1(x)|^2 dx \right\| &\leq \frac{1}{|I|} \left(\sum_{J \in \mathcal{D}_1} \|a_J\| |\omega_J(x) - \omega_J(c_I)| \right)^2 dx \\ &\leq c \frac{1}{|I|} \int_I \left[\sum_{J \in \mathcal{D}_1} |J|^{\frac{1}{2}} |I| |J|^{-\frac{3}{2}} \left(1 + \frac{\text{dist}(I, J)}{|J|} \right)^{-2} \right]^2 dx \\ &= c \left[\sum_{j=0}^{\infty} \sum_{|J| \in (2^{j-1}, 2^j] |I|} |I| |J|^{-1} \left(1 + \frac{\text{dist}(I, J)}{|J|} \right)^{-2} \right]^2 < \infty. \end{aligned}$$

For φ_2 ,

$$\begin{aligned} \left\| \frac{1}{|I|} \int_I |\varphi_2(x)|^2 dx \right\| &\leq \frac{1}{|I|} \int_I \left\| \sum_{\mathcal{D}_2} a_J \omega_J(x) \right\|^2 dx \leq \frac{1}{|I|} \int_I \left(\sum_{\mathcal{D}_2} \|a_J\| |\omega_J(x)| \right)^2 dx \\ &\leq c \frac{1}{|I|} \int_I \left[\sum_{\mathcal{D}_2} |J|^{\frac{1}{2}} |J|^{-\frac{1}{2}} \left(\frac{\text{dist}(I, J)}{|J|} \right)^{-2} \right]^2 dx \\ &= c \left[\sum_{j=1}^{\infty} \sum_{|J| \in (2^{-j-1}, 2^{-j}) |I|, \text{dist}(I, J) > 2^{-1}|I|} \left(\frac{\text{dist}(I, J)}{|J|} \right)^{-2} \right]^2 < \infty. \end{aligned}$$

And for φ_3 ,

$$\left\| \frac{1}{|I|} \int_I |\varphi_3(x)|^2 dx \right\| \leq \frac{1}{|I|} \left\| \sum_{J \in \mathcal{D}_3} |a_J|^2 \right\| \leq \frac{1}{|I|} \left\| \sum_{J \subset 4I} |a_J|^2 \right\| < \infty.$$

Hence, we deduce that:

$$\left\| \int_I |\varphi_I(x)|^2 dx \right\|_{\mathcal{M}} \leq c \sum_{i=1}^3 \left\| \int_I |\varphi_i(x)|^2 dx \right\|_{\mathcal{M}} \leq c|I|.$$

Now we turn to the proof of the inverse direction, $\text{BMO}^c(\mathbb{R}, \mathcal{M}) \subset \mathcal{BMO}^c(\mathbb{R}, \mathcal{M})$. Let $\varphi \in \text{BMO}^c(\mathbb{R}, \mathcal{M})$. The proof is very similar to that of Lemma 4.1 in Mei's work [15]. For any dyadic interval $I \subset \mathbb{R}$, write $\varphi = \varphi_1 + \varphi_2 + \varphi_3$, where $\varphi_1 = (\varphi - \varphi_{2I})\chi_{2I}$, $\varphi_2 = (\varphi - \varphi_{2I})\chi_{2I^c}$, $\varphi_3 = \varphi_{2I}$.

Thus

$$\sum_{J \subset I} |\langle \varphi, \omega_J \rangle|^2 \leq 2 \left(\sum_{J \subset I} |\langle \varphi_1, \omega_J \rangle|^2 + \sum_{J \subset I} |\langle \varphi_2, \omega_J \rangle|^2 \right).$$

We have the following estimates for φ_1 :

$$\left\| \sum_{J \subset I} |\langle \varphi_1, \omega_J \rangle|^2 \right\| \leq \left\| \int |\varphi_1(x)|^2 dx \right\| \leq c \left\| \int_{2I} |\varphi - \varphi_{2I}|^2 \right\| \leq c|I|.$$

And for φ_2 ,

$$\begin{aligned} \left\| \sum_{J \subset I} |\langle \varphi_2, \omega_J \rangle|^2 \right\| &= \left\| \sum_{J \subset I} \left| \sum_{k=1}^{\infty} \int_{2^{k+1}I/2^k I} \varphi_2 \omega_J dx \right|^2 \right\| \\ &\leq \left\| \sum_{J \subset I} \left(\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \int_{2^{k+1}I/2^k I} |\varphi_2|^2 \right) \left(\sum_{k=1}^{\infty} 2^{2k} \int_{2^{k+1}I/2^k I} |\omega_J|^2 \right) \right\| \\ &\leq c \left(\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \left\| \int_{2^{k+1}I} |\varphi - \varphi_{2I}|^2 \right\| \right) \left(\sum_{J \subset I} \sum_{k=1}^{\infty} 2^{2k} \int_{2^{k+1}I/2^k I} |\omega_J|^2 \right) \\ &\leq c|I| \|\varphi\|_{\mathcal{BMO}^c}^2 \sum_{j=0}^{\infty} 2^j \sum_{k=1}^{\infty} \int_{2^{k+1}I/2^k I} 2^{2k} \frac{|2^{-j}I|^3}{|2^k I|^4} \leq c|I|. \end{aligned}$$

Therefore, $\left\| \sum_{J \subset I} |\langle \varphi, \omega_J \rangle|^2 \right\| \leq c|I|$, which completes our proof. \square

Combined with Theorem 3.3 and Theorem 4.2, we have the following corollary:

Corollary 5.2. *For $1 \leq p < \infty$, we have*

$$\mathcal{H}_p^c(\mathbb{R}, \mathcal{M}) = H_p^c(\mathbb{R}, \mathcal{M}).$$

Similar results hold for \mathcal{H}_p^r and H_p^r , and for \mathcal{H}_p and H_p .

If $\mathcal{M} = \mathbb{C}$, $\mathcal{H}_1(\mathbb{R}, \mathbb{C})$ is just the usual Hardy space $H_1(\mathbb{R})$ on \mathbb{R} . $H_1(\mathbb{R})$ also has the following characterization:

$$H_1(\mathbb{R}) = \{f \in L_1(\mathbb{R}) : H(f) \in L_1(\mathbb{R})\},$$

where H is the Hilbert transform. For any $f \in H_1(\mathbb{R})$,

$$\|f\|_{H_1(\mathbb{R})} \approx \|f\|_{L_1(\mathbb{R})} + \|H(f)\|_{L_1(\mathbb{R})}.$$

Thus $H_1(\mathbb{R})$ can be viewed as a subspace of $L_1(\mathbb{R}) \oplus_1 L_1(\mathbb{R})$. The latter direct sum has its natural operator structure as an L_1 space. This induces an operator space structure on $H_1(\mathbb{R})$. Although $(w_I)_{I \in \mathcal{D}}$ is an unconditional basis of $H_1(\mathbb{R})$, Ricard [22] (see also [23]) proved that $H_1(\mathbb{R})$ does not have a complete unconditional basis. However, in noncommutative analysis, one can introduce another natural operator space structure on $H_1(\mathbb{R})$ as follows: $S_1(H_1(\mathbb{R})) = \mathcal{H}_1(\mathbb{R}, B(\ell_2))$, where S_1 is the trace class on ℓ_2 . Then we have the following result. Note that Ricard [23] obtained a similar result using Hilbert space techniques.

Corollary 5.3. *The complete orthogonal system $(w_I)_{I \in \mathcal{D}}$ of $L_2(\mathbb{R})$ is a completely unconditional basis for $H_1(\mathbb{R})$ if we define the operator space structure imposed on $H_1(\mathbb{R})$ by $S_1(H_1(\mathbb{R})) = \mathcal{H}_1(\mathbb{R}, B(\ell_2))$.*

Proof. Fix a finite subset $\mathcal{I} \subset \mathcal{D}$. Let $T_\epsilon f := \sum_{I \in \mathcal{I}} \epsilon_I \langle f, w_I \rangle w_I$, where $\epsilon_I = \pm 1$. By the definition of $\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})$, the orthogonality of $(w_I)_{I \in \mathcal{D}}$ yields immediately that

$$\begin{aligned} \|T_\epsilon f\|_{\mathcal{H}_1^c} &= \left\| \left(\sum_{I \in \mathcal{I}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbf{1}_I(x) \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{N})} \\ &\leq \left\| \left(\sum_{I \in \mathcal{D}} \frac{|\langle f, w_I \rangle|^2}{|I|} \mathbf{1}_I(x) \right)^{\frac{1}{2}} \right\|_{L_1(\mathcal{N})} = \|f\|_{\mathcal{H}_1^c}. \end{aligned}$$

Similarly, the above inequality holds for $\mathcal{H}_1^r(\mathbb{R}, \mathcal{M})$. Now, let $f \in \mathcal{H}_1(\mathbb{R}, \mathcal{M})$. Then, for any $\epsilon > 0$, there exists a decomposition $f = g + h$ such that

$$\|g\|_{\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})} + \|h\|_{\mathcal{H}_1^r(\mathbb{R}, \mathcal{M})} \leq \|f\|_{\mathcal{H}_1(\mathbb{R}, \mathcal{M})} + \epsilon.$$

Therefore

$$\begin{aligned} \|T_\epsilon f\|_{\mathcal{H}_1(\mathbb{R}, \mathcal{M})} &\leq \|T_\epsilon g\|_{\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})} + \|T_\epsilon h\|_{\mathcal{H}_1^r(\mathbb{R}, \mathcal{M})} \\ &\leq \|g\|_{\mathcal{H}_1^c(\mathbb{R}, \mathcal{M})} + \|h\|_{\mathcal{H}_1^r(\mathbb{R}, \mathcal{M})} \leq \|f\|_{\mathcal{H}_1(\mathbb{R}, \mathcal{M})} + \epsilon. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get the result. □

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