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Revisiting the multifractal analysis of measures

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Abstract. New proofs of theorems on the multifractal formalism are given. They yield results even at points q for which Olsen's functions $b(q)$ and $B(q)$ differ. Indeed, we provide an example of a measure for which the functions b and B differ and for which the Hausdorff dimensions of the sets X_α (the level sets of the local Hölder exponent) are given by the Legendre transform of b and their packing dimensions by the Legendre transform of B.

1. Introduction

The multifractal formalism aims at expressing the dimension of the level sets of the local Hölder exponent of some set function μ in terms of the Legendre transform of some "free energy" function (see [7], [5], and [6] for early works). If such a formula holds, one says that μ satisfies the multifractal formalism. At first, the formalism used "boxes", or in other terms took place in a totally disconnected metric space. In this context, the closeness to large deviation theory is patent. To get rid of these boxes and have a formalism meaningful in geometric measure theory, Olsen [8] introduced a formalism which is now commonly used. See also Pesin's monograph [9] on multifractality and dynamical systems. At this stage of the theory, whether it dealt with boxes or not, the formalism was proven to hold when there exists an auxiliary measure, a so-called *Gibbs measure*. Later, it was shown that this formalism holds under the condition that Olsen's Hausdorff-like multifractal measure be positive (see [2] in the totally disconnected case, [3] in general). So, the situation when $b(q) = B(q)$ (in Olsen's notation) is fairly well understood.

Here, we elaborate on the previous proofs. There is a vector version of Olsen's constructions [10], and, in particular, of the functions b and B. However, in this setting b and B are functions of several variables. In this work, we show that the restriction of these functions to a suitable affine subspace can be used to estimate the Hausdorff and Tricot dimensions of some level sets. In particular, this gives

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some results even in the case when $b \neq B$. Although notation is inherently complicated, we provide a simple proof of already known results, and we obtain some new estimates. In particular, we provide an example of a measure on the interval $[0,1]$ for which the functions b and B differ and for which the Hausdorff dimensions of the sets X_α (the level sets of the local Hölder exponent) are given by the Legendre transform of b, and their packing dimensions by the Legendre transform of B.

2. Notations and definitions

We deal with a metric space $(\mathbb{X}, \mathsf{d})$ having the *Besicovitch property:*

There exists an integer constant C_B such that one can extract C_B countable families $\big\{\{\mathsf{B}_{j,k}\}_k\big\}_{1\le j\le C_B}$ from any collection $\mathcal{B}$ of balls so that

1. $\bigcup_{j,k} \mathsf{B}_{j,k}$ contains the centers of the elements of $\mathcal{B}$,
2. for any j and $k \neq k'$, $\mathsf{B}_{j,k} \cap \mathsf{B}_{j,k'} = \emptyset$.

Notations

$\mathsf{B}(x,r)$ stands for the open ball $\mathsf{B}(x,r) = \{y \in \mathbb{X}\ ;\ \mathsf{d}(x,y) < r\}$. The letter B with or without a subscript will implicitly stand for such a ball. When dealing with a collection of balls $\{\mathsf{B}_i\}_{i\in I}$, the notation $\mathsf{B}_i = \mathsf{B}(x_i, r_i)$ will implicitly be assumed.

By a δ-cover of $E \subset \mathbb{X}$, we mean a collection of *balls* of radii not exceeding δ whose union contains E. A *centered cover* of E is a cover of E consisting of balls whose centers belong to E.

By a δ-packing of $E \subset \mathbb{X}$, we mean a collection of disjoint balls of radii not exceeding δ centered in E.

By a Besicovitch δ-cover of $E \subset \mathbb{X}$, we mean a centered δ-cover of E which can be decomposed into C_B packings.

If E is a subset of $\mathbb{X}$, $\dim_H E$ stands for its Hausdorff dimension and $\dim_P E$ for its packing dimension (introduced by Tricot [12]).

Let $\mathscr{B}$ stand for the set of balls of $\mathbb{X}$ and $\mathscr{F}$ for the set of maps from $\mathscr{B}$ to $[0,+\infty)$.

The set of $\mu \in \mathscr{F}$ such that $\mu(\mathsf{B}) = 0$ implies $\mu(\mathsf{B}') = 0$ for all $\mathsf{B}' \subset \mathsf{B}$ will be denoted by $\mathscr{F}^*$. For such a μ, one defines its support S_μ to be the complement of the set

$$\bigcup \{\mathsf{B} \in \mathscr{B}\ ;\ \mu(\mathsf{B}) = 0\}.$$

Multifractal measures and separator functions

For $\mu = (\mu_1, \ldots, \mu_m) \in \mathscr{F}^m$, $E \subset \mathbb{X}$, $q = (q_1, \ldots, q_m) \in \mathbb{R}^m$, $t \in \mathbb{R}$, and $\delta > 0$, one sets

$$\overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E) = \sup\Big\{ \sum\nolimits^* r_j^t \prod_{k=1}^m \mu_k(\mathsf{B}_j)^{q_k}\ ;\ \{\mathsf{B}_j\} \text{ a } \delta\text{-packing of } E\Big\},$$

where $*$ means that one only sums the terms for which $\prod_k \mu_k(\mathsf{B}_j) \neq 0$,

$$\overline{\mathscr{P}}^{q,t}_{\mu}(E) = \lim_{\delta \searrow 0} \overline{\mathscr{P}}^{q,t}_{\mu,\delta}(E),$$

$$\mathscr{P}^{q,t}_{\mu}(E) = \inf\left\{ \sum \overline{\mathscr{P}}^{q,t}_{\mu}(E_j) \ ; \ E \subset \bigcup E_j \right\},$$

and

$$\begin{aligned}
\overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E) &= \inf\left\{ \sum^{*} r_j^t \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} \ ; \{\mathsf{B}_j\} \text{ a centered } \delta\text{-cover of } E \right\}, \\
\overline{\mathscr{H}}^{q,t}_{\mu}(E) &= \lim_{\delta \searrow 0} \overline{\mathscr{H}}^{q,t}_{\mu,\delta}(E), \\
\mathscr{H}^{q,t}_{\mu}(E) &= \sup\left\{ \overline{\mathscr{H}}^{q,t}_{\mu}(F) \ ; \ F \subset E \right\},
\end{aligned}$$

It is known that $\overline{\mathscr{H}}^{q,t}_{\mu}$ is σ-subadditive, and that $\mathscr{P}^{q,t}_{\mu}$ and $\mathscr{H}^{q,t}_{\mu}$ are outer measures. When d is an ultrametric, then $\mathscr{H}^{q,t}_{\mu} = \overline{\mathscr{H}}^{q,t}_{\mu}$.

When $m = 1$, these measures have been defined by Olsen [8]. When μ is identically 1 these quantities do not depend on q. They will be simply denoted by $\overline{\mathscr{P}}^{t}_{\delta}(E)$, $\overline{\mathscr{P}}^{t}(E)$, $\mathscr{P}^{t}(E)$, $\overline{\mathscr{H}}^{t}_{\delta}(E)$, $\overline{\mathscr{H}}^{t}(E)$, and $\mathscr{H}^{t}(E)$, respectively. They are the classical packing pre-measures and measures introduced by Tricot [12], and the Hausdorff centered pre-measures and measures [11]. The centered Hausdorff measures also define the Hausdorff dimension.

It will prove convenient to use the following notations, when $m = 1$:

$$\overline{\mu}_\delta = \overline{\mathscr{H}}^{1,0}_{\mu,\delta}, \quad \overline{\mu} = \overline{\mathscr{H}}^{1,0}_{\mu}, \quad \text{and} \quad \mu^\sharp = \mathscr{H}^{1,0}_{\mu}.$$

Also, as usual, one considers the following functions:

$$\begin{aligned}
\tau_{\mu,E}(q) &= \inf\{t \in \mathbb{R} \ ; \ \overline{\mathscr{P}}^{q,t}_{\mu}(E) = 0\} = \sup\{t \in \mathbb{R} \ ; \ \overline{\mathscr{P}}^{q,t}_{\mu}(E) = \infty\} \\
B_{\mu,E}(q) &= \inf\{t \in \mathbb{R} \ ; \ \mathscr{P}^{q,t}_{\mu}(E) = 0\} = \sup\{t \in \mathbb{R} \ ; \ \mathscr{P}^{q,t}_{\mu}(E) = \infty\}, \\
b_{\mu,E}(q) &= \inf\{t \in \mathbb{R} \ ; \ \mathscr{H}^{q,t}_{\mu}(E) = 0\} = \sup\{t \in \mathbb{R} \ ; \ \mathscr{H}^{q,t}_{\mu}(E) = \infty\}.
\end{aligned}$$

It is well known [8], [10] that τ and B are convex and that $b \leq B \leq \tau$. Let J_τ, J_B, and J_b stand for the interiors of the sets where respectively τ, B, and b are finite.

When μ is identically 1 we will denote these quantities by $\overline{\dim}_B E$, $\dim_P E$, and $\dim_H E$. The first one is the Minkowski–Bouligand dimension (or upper box-dimension), the second is the Tricot (packing) dimension [12], and the last the Hausdorff dimension.

Here is an alternate definition of $\tau_{\mu,E}$. Fix $\lambda < 1$ and define

$$\widetilde{\mathscr{P}}^{q,t}_{\mu,\delta}(E) = \sup\left\{ \sum^{*} r_j^t \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} \ ; \{\mathsf{B}_j\} \text{ a packing of } E \text{ with } \lambda\delta < r_j \leq \delta \right\},$$

$$\widetilde{\mathscr{P}}^{q,t}_{\mu}(E) = \overline{\lim_{\delta \searrow 0}} \widetilde{\mathscr{P}}^{q,t}_{\mu,\delta}(E),$$

$$\widetilde{\tau}_{\mu,E}(q) = \sup\left\{ t \in \mathbb{R} \ ; \widetilde{\mathscr{P}}^{q,t}_{\mu}(E) = +\infty \right\}.$$

Lemma 2.1. *One has* $\widetilde{\tau}_{\mu,E} = \tau_{\mu,E}$.

Proof. Obviously $\widetilde{\mathscr{P}}_{\mu}^{q,t}(E) \leq \overline{\mathscr{P}}_{\mu}^{q,t}(E)$, so $\widetilde{\tau}_{\mu,E} \leq \tau_{\mu,E}$. To prove the converse inequality, one only has to consider the case $\tau_{\mu,E}(q) > -\infty$.

Choose $\gamma < \tau_{\mu,E}(q)$ and $\varepsilon > 0$ such that $\gamma + \varepsilon < \tau_{\mu,E}(q)$. There exists n_0 such that, for all $n > n_0$, there exists a λ^n-packing $\{\mathsf{B}_j\}$ of E such that

$$\sum r_j^{\gamma+\varepsilon} \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} > 1.$$

As

$$\sum r_j^{\gamma+\varepsilon} \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} = \sum_{i\geq 0} \sum_{\lambda < r_j \lambda^{-(n+i)} \leq 1} r_j^{\gamma+\varepsilon} \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k},$$

there exists $i \geq 0$ such that

$$\sum_{\lambda < r_j \lambda^{-(n+i)} \leq 1} r_j^{\gamma+\varepsilon} \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} > \lambda^{i\varepsilon}(1-\lambda^{\varepsilon}),$$

from which it follows

$$\sum_{\lambda < r_j \lambda^{-(n+i)} \leq 1} r_j^{\gamma} \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} > \lambda^{-(n+i)\varepsilon} \lambda^{i\varepsilon}(1-\lambda^{\varepsilon}) = \lambda^{-n}(1-\lambda^{\varepsilon}),$$

and $\widetilde{\mathscr{P}}_{\mu}^{q,\gamma}(E) = +\infty$. □

Corollary 2.2. *For any* $\lambda < 1$, *one has*

$$\tau_{\mu,E}(q) = \overline{\lim_{\delta \searrow 0}} \frac{-1}{\log \delta} \log \sup \Big\{ \sum^{*} \prod_{k=1}^{m} \mu_k(\mathsf{B}_j)^{q_k} \ ; \ \{\mathsf{B}_j\} \text{ a packing of } E \text{ with } \lambda\delta < r_j \leq \delta \Big\}.$$

Level sets of local Hölder exponents

Let μ be an element of $\mathscr{F}^*$. For α, $\beta \in \mathbb{R}$, one sets

$$\overline{X}_{\mu}(\alpha) = \Big\{ x \in \mathsf{S}_{\mu} \ ; \ \overline{\lim_{r \searrow 0}} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} \leq \alpha \Big\},$$

$$\underline{X}_{\mu}(\alpha) = \Big\{ x \in \mathsf{S}_{\mu} \ ; \ \underline{\lim_{r \searrow 0}} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} \geq \alpha \Big\},$$

$$X_{\mu}(\alpha, \beta) = \underline{X}_{\mu}(\alpha) \cap \overline{X}_{\mu}(\beta),$$

and

$$X_{\mu}(\alpha) = \underline{X}_{\mu}(\alpha) \cap \overline{X}_{\mu}(\alpha).$$

3. Results

First, we revisit the Billingsley and Tricot lemmas [4], [12].

Lemma 3.1. *Let E be a subset of $\mathbb{X}$ and ν an element of $\mathscr{F}$.*

a) *If $B_{\nu,E}(1)\le 0$, then*

$$\dim_H E \le \sup_{x\in E}\underline{\lim}_{r\searrow 0}\frac{\log\nu\big(\mathsf{B}(x,r)\big)}{\log r}, \tag{3.1}$$

$$\dim_P E \le \sup_{x\in E}\overline{\lim}_{r\searrow 0}\frac{\log\nu\big(\mathsf{B}(x,r)\big)}{\log r}. \tag{3.2}$$

b) *If $\nu^\sharp(E)>0$, then*

$$\dim_H E \ge \operatorname*{ess\,sup}_{x\in E,\,\nu^\sharp}\underline{\lim}_{r\searrow 0}\frac{\log\nu\big(\mathsf{B}(x,r)\big)}{\log r}, \tag{3.3}$$

$$\dim_P E \ge \operatorname*{ess\,sup}_{x\in E,\,\nu^\sharp}\overline{\lim}_{r\searrow 0}\frac{\log\nu\big(\mathsf{B}(x,r)\big)}{\log r}, \tag{3.4}$$

where

$$\operatorname*{ess\,sup}_{x\in E,\,\nu^\sharp}\chi(x)=\inf\Big\{t\in\mathbb{R};\ \nu^\sharp\big(E\cap\{\chi>t\}\big)=0\Big\}.$$

Proof. Take

$$\gamma>\sup_{x\in E}\underline{\lim}_{r\searrow 0}\frac{\log\nu\big(\mathsf{B}(x,r)\big)}{\log r}$$

and $\eta>0$. Since $B_{\nu,E}(1)\le 0$ there exists a partition $E=\bigcup E_j$ such that $\sum\overline{\mathscr{P}}_\nu^{1,\eta/2}(E_j)<1$. Therefore we have that $\sum\overline{\mathscr{P}}_\nu^{1,\eta}(E_j)=0$.

Let F be a subset of E_k and let δ be a positive number. For all $x\in F$, there exists $r\le\delta$ such that $\nu\big(\mathsf{B}(x,r)\big)\ge r^\gamma$. By the Besicovitch property, there exists a centered δ-cover $\{\mathsf{B}_j\}$ of F, which can be decomposed into C_B packings, such that $\nu(\mathsf{B}_j)\ge r_j^\gamma$. We then have

$$\sum r_j^{\gamma+\eta}\le\sum r_j^\eta\nu(\mathsf{B}_j)\le C_B\overline{\mathscr{P}}_{\nu,\delta}^{1,\eta}(E_k).$$

Therefore we have $\overline{\mathscr{H}}^{\gamma+\eta}(F)=0$, $\mathscr{H}^{\gamma+\eta}(E_k)=0$, and finally $\mathscr{H}^{\gamma+\eta}(E)=0$. Then (3.1) easily follows.

To prove (3.2), take

$$\gamma>\sup_{x\in E}\overline{\lim}_{r\searrow 0}\frac{\log\nu\big(\mathsf{B}(x,r)\big)}{\log r}$$

and $\eta>0$. As previously, there exists a partition $E=\bigcup E_j$ such that $\sum\overline{\mathscr{P}}_\nu^{1,\eta}(E_j)=0$.

For all $x\in E$, there exists $\delta>0$ such that, for all $r\le\delta$, one has $\nu\big(\mathsf{B}(x,r)\big)\ge r^\gamma$. Consider the set

$$E(n)=\big\{x\in E\ ;\ \forall r\le 1/n,\ \nu\big(\mathsf{B}(x,r)\big)\ge r^\gamma\big\}.$$

Let $\{\mathsf{B}_j\}$ be a δ-packing of $E_k \cap E(n)$, with $\delta \le 1/n$. One has

$$\sum r_j^{\gamma+\eta} \le \sum_j r_j^{\eta} \nu(\mathsf{B}_j) \le \overline{\mathscr{P}}^{1,\eta}_{\nu,\delta}(E_k),$$

from which $\overline{\mathscr{P}}^{\gamma+\eta}\big(E_k \cap E(n)\big) = 0$ follows.

So we have $\mathscr{P}^{\gamma+\eta}\big(E(n)\big) = 0$. Since $E = \bigcup_{n\ge 1} E(n)$, one has $\dim_P E \le \gamma + \eta$, and hence (3.2).

To prove (3.3), take

$$\gamma < \operatorname*{ess\,sup}_{x\in E,\, \nu^\sharp} \varliminf_{r\searrow 0} \frac{\log \nu\big(\mathsf{B}(x,r)\big)}{\log r}$$

and consider the set $F = \Big\{x \in E \ ;\ \varliminf_{r\searrow 0} \frac{\log \nu(\mathsf{B}(x,r))}{\log r} > \gamma\Big\}$. We have $\nu^\sharp(F) > 0$. For all $x \in F$, there exists $\delta > 0$ such that, for all $r \le \delta$, one has $\nu\big(\mathsf{B}(x,r)\big) \le r^\gamma$. Consider the set

$$F(n) = \big\{x \in F \ ;\ \forall r \le 1/n,\ \nu\big(\mathsf{B}(x,r)\big) \le r^\gamma\big\}\,.$$

We have $F = \bigcup_{n\ge 1} F(n)$. Since $\nu^\sharp(F) > 0$, there exists n such that $\nu^\sharp\big(F(n)\big) > 0$, and therefore there is a subset G of $F(n)$ such that $\overline{\nu}(G) > 0$. Then for any centered δ-cover $\{\mathsf{B}_j\}$ of G, with $\delta \le 1/n$, one has

$$\overline{\nu}_\delta(G) \le \sum \nu(\mathsf{B}_j) \le \sum r_j^\gamma.$$

Therefore,

$$\overline{\nu}_\delta(G) \le \overline{\mathscr{H}}^{\gamma}_{\delta}(G), \quad \text{and} \quad 0 < \overline{\nu}(G) \le \overline{\mathscr{H}}^{\gamma}(G) \le \mathscr{H}^\gamma(G),$$

which implies $\dim_H E \ge \dim_H G \ge \gamma$.

To prove (3.4), take

$$\gamma < \operatorname*{ess\,sup}_{x\in E,\, \nu^\sharp} \varlimsup_{r\searrow 0} \frac{\log \nu\big(\mathsf{B}(x,r)\big)}{\log r}$$

and consider the set $F = \Big\{x \in E \ ;\ \varlimsup_{r\searrow 0} \frac{\log \nu(\mathsf{B}(x,r))}{\log r} > \gamma\Big\}$. We have $\nu^\sharp(F) > 0$, so there exists a subset F' of F such that $\overline{\nu}(F') > 0$. Let G be a subset of F'. Then, for all $x \in G$, for all $\delta > 0$, there exists $r \le \delta$ such that $\nu\big(\mathsf{B}(x,r)\big) \le r^\gamma$. Then for all δ, by using the Besicovitch property, there exists a collection $\{\{\mathsf{B}_{j,k}\}_j\}_{1\le k\le C_B}$ of δ-packings of G which together cover G and such that $\nu(\mathsf{B}_{j,k}) \le r_{j,k}^\gamma$. Then one has

$$\overline{\nu}_\delta(G) \le \sum_{j,k} \nu(\mathsf{B}_{j,k}) \le \sum r_{j,k}^\gamma.$$

This implies that there exists k such that $\sum_j r_{j,k}^\gamma \geq \frac{1}{C_B}\overline{\nu}_\delta(G)$. So we have $\overline{\mathscr{P}}_\delta^\gamma(G) \geq \frac{1}{C_B}\overline{\nu}_\delta(G)$. This implies $\overline{\mathscr{P}}^\gamma(G) \geq \frac{1}{C_B}\overline{\nu}(G)$. Hence, if $F' = \bigcup G_j$, one has

$$\sum \overline{\mathscr{P}}^\gamma(G_j) \geq \frac{1}{C_B}\sum \overline{\nu}(G_j) \geq \frac{1}{C_B}\overline{\nu}(F') > 0,$$

so $\mathscr{P}^\gamma(F') > 0$. Therefore, $\dim_P F \geq \gamma$. Then (3.4) easily follows. □

Lemma 3.2. *Let μ and ν be elements of $\mathscr{F}^*$ and $\mathscr{F}$ respectively. Set $\varphi(t) = B_{(\mu,\nu),\mathsf{S}_\mu}(t,1)$ and assume that $\varphi(0) = 0$ and $\nu^\sharp(\mathsf{S}_\mu) > 0$. Then one has*

$$\nu^\sharp\big({}^{\complement}X_\mu\big(-\varphi_r'(0), -\varphi_l'(0)\big)\big) = 0,$$

where φ_l' and φ_r' are the left-hand and right-hand derivatives of φ.

The same result holds with $\varphi(t) = \tau_{(\mu,\nu),\mathsf{S}_\mu}(t,1)$.

Proof. Take $\gamma > -\varphi_l'(0)$, and choose γ' and $t > 0$ such that $\gamma > \gamma' > -\varphi_l'(0)$ and $\varphi(-t) < \gamma' t$. Then $\mathscr{P}_{(\mu,\nu)}^{(-t,1),\gamma' t}(\mathsf{S}_\mu) = 0$, so there exists a countable partition $\mathsf{S}_\mu = \bigcup E_j$ of S_μ such that

$$\sum_j \overline{\mathscr{P}}_{(\mu,\nu)}^{(-t,1),\gamma' t}(E_j) \leq 1\,,$$

and therefore $\overline{\mathscr{P}}_{(\mu,\nu)}^{(-t,1),\gamma t}(E_j) = 0$ for all j.

Consider the set

$$E(\gamma) = \Big\{x \in \mathsf{S}_\mu \ ; \ \overline{\lim_{r\searrow 0}} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} > \gamma\Big\}.$$

If $x \in E(\gamma)$, for all $\delta > 0$, there exists $r \leq \delta$ such that $\mu\big(\mathsf{B}(x,r)\big) \leq r^\gamma$. Let F be a subset of $E(\gamma)$. Set $F_j = F \cap E_j$.

For $\delta > 0$, for all j, one can find a Besicovitch δ-cover $\{\mathsf{B}_{j,k}\}$ of F_j such that $\mu(\mathsf{B}_{j,k}) \leq r_{j,k}^\gamma$.

We have,

$$\overline{\nu}_\delta(F_j) \leq \sum_k \nu(\mathsf{B}_{j,k}) = \sum_k \mu(\mathsf{B}_{j,k})^{-t}\mu(\mathsf{B}_{j,k})^t\nu(\mathsf{B}_{j,k}) \leq \sum_k \mu(\mathsf{B}_{j,k})^{-t} r_{j,k}^{\gamma t}\nu(\mathsf{B}_{j,k}),$$

which, together with the Besicovitch property, implies

$$\overline{\nu}_\delta(F_j) \leq C_B \overline{\mathscr{P}}_{(\mu,\nu),\delta}^{(-t,1),\gamma t}(E_j).$$

so

$$\overline{\nu}(F_j) \leq C_B \overline{\mathscr{P}}_{(\mu,\nu)}^{(-t,1),\gamma t}(E_j) = 0.$$

This implies $\overline{\nu}(F) = 0$, and $\nu^\sharp(E(\gamma)) = 0$.

We conclude that

$$\nu^\sharp\Big(\Big\{x \in \mathsf{S}_\mu \ ; \ \overline{\lim_{r\searrow 0}} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} > -\varphi_l'(0)\Big\}\Big) = 0.$$

In the same way, one proves that

$$\nu^\sharp\Big(\Big\{x \in \mathsf{S}_\mu \ ; \ \underline{\lim_{r\searrow 0}} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} < -\varphi_r'(0)\Big\}\Big) = 0.$$

□

Corollary 3.3. *With the same notations and hypotheses as in Lemma 3.2, one has*

$$\dim_H X_\mu\big(-\varphi_r'(0), -\varphi_l'(0)\big) \geq \inf\Big\{ \underline{\lim_{r\searrow 0}} \frac{\log \nu\big(\mathsf{B}(x,r)\big)}{\log r} \ ; \ x \in X_\mu\big(-\varphi_r'(0), -\varphi_l'(0)\big)\Big\}$$

and

$$\dim_P X_\mu\big(-\varphi_r'(0), -\varphi_l'(0)\big) \geq \inf\Big\{ \overline{\lim_{r\searrow 0}} \frac{\log \nu\big(\mathsf{B}(x,r)\big)}{\log r} \ ; \ x \in X_\mu\big(-\varphi_r'(0), -\varphi_l'(0)\big)\Big\}.$$

Note that statements in Corollary 3.3 are weaker than what can be deduced from Lemma 3.2 and Lemma 3.1-b.

The previous lemmas contain the now classical results on multifractal analysis [8], [3], [10]. Indeed, let μ be a element of $\mathscr{F}^*$. Until the end of this section, we will write b, τ, and B instead of b_{μ,S_μ}, $\tau_{\mu,\mathsf{S}_\mu}$, and B_{μ,S_μ}. For $q \geq 0$, take $\nu(\mathsf{B}) = \mu(\mathsf{B})^q r^{B(q)}$. Then the corresponding φ of Lemma 3.2 is $B_{(\mu,\nu),\mathsf{S}_\mu}(t,1) = B(q+t) - B(q)$ and, for $x \in \overline{X}_\mu(\alpha)$, one has

$$\overline{\lim_{r\searrow 0}} \frac{\log \nu\big(\mathsf{B}(x,r)\big)}{\log r} = q \overline{\lim_{r\searrow 0}} \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} + B(q) \leq q\alpha + B(q).$$

So, by (3.2) of Lemma 3.1, one gets

$$\dim_P \overline{X}_\mu(\alpha) \leq \inf_{q\geq 0} q\alpha + B(q).$$

In the same way, we get

$$\dim_P \underline{X}_\mu(\alpha) \leq \inf_{q\leq 0} q\alpha + B(q).$$

If moreover we assume that $\mathscr{H}_\mu^{q,B(q)}(\mathsf{S}_\mu) > 0$, we have $\nu^\sharp(\mathsf{S}_\mu) > 0$, and therefore, by Lemma 3.2,

$$\nu^\sharp\left(\big\{X_\mu\big(-B_r'(q), -B_l'(q)\big)\big\}\right) > 0.$$

Therefore, by (3.3) of Lemma 3.1, we have

$$\dim_H \big\{X_\mu\big(-B_r'(q), -B_l'(q)\big)\big\} \geq \begin{cases} -q\, B_r'(q) + B(q) & \text{if } q \geq 0, \\ -q\, B_l'(q) + B(q) & \text{if } q \leq 0. \end{cases}$$

Recall that the Legendre transform of a function χ is defined to be $\chi^*(\alpha) = \inf_{q\in\mathbb{R}} q\alpha + \chi(q)$.

All this gives a new proof of the following theorem (see [2] in the totally disconnected case, [3] in general).

Theorem 3.4. *If B has a derivative at some point $q \in J_B$ and if $\mathscr{H}_\mu^{q,B(q)}(\mathsf{S}_\mu) > 0$, then*

$$\dim_H X_\mu\big(-B'(q)\big) = B^*\big(-B'(q)\big).$$

The same statement holds with τ instead of B.

In [3] it is shown that if $B'(q)$ exists and if $\dim_H X_\mu\big(-B'(q)\big) = B^*\big(-B'(q)\big)$, then $b(q) = B(q)$.

We now deal with the case when $b(q) \neq B(q)$. The following notation will prove convenient: for a real function ψ, we set

$$\psi_l^\flat(q) = \overline{\lim_{t\searrow 0}}\, \frac{\psi(q-t)-\psi(q)}{-t} \quad \text{and} \quad \psi_r^\flat(q) = \overline{\lim_{t\searrow 0}}\, \frac{\psi(q+t)-\psi(q)}{t}.$$

Lemma 3.5. *Let μ and ν be elements of $\mathscr{F}^*$ and $\mathscr{F}$ respectively. Set $\varphi(t) = b_{(\mu,\nu),\mathsf{S}_\mu}(t,1)$ and assume that $\varphi(0) = 0$ and $\nu^\sharp(\mathsf{S}_\mu) > 0$. Then one has*

$$\nu^\sharp\left(\left\{x \in \mathsf{S}_\mu\ ;\ \underline{\lim_{r\searrow 0}}\, \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} > -\varphi_l^\flat(0)\right\}\right) = 0$$

and

$$\nu^\sharp\left(\left\{x \in \mathsf{S}_\mu\ ;\ \overline{\lim_{r\searrow 0}}\, \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} < -\varphi_r^\flat(0)\right\}\right) = 0.$$

Proof. Take $\gamma > -\varphi_l^\flat(0) = \underline{\lim}_{t\searrow 0} \frac{\varphi(-t)}{t}$ and choose $t > 0$ such that $\gamma t > \varphi(-t)$. We have $\mathscr{H}_{(\mu,\nu)}^{(-t,1),\gamma t}(\mathsf{S}_\mu) = 0$.

Consider the set

$$E = \left\{x \in \mathsf{S}_\mu\ ;\ \underline{\lim_{r\searrow 0}}\, \frac{\log \mu\big(\mathsf{B}(x,r)\big)}{\log r} > \gamma\right\}.$$

For all $x \in E$, there exists $\delta > 0$ such that, for all $r < \delta$, one has $\mu\big(\mathsf{B}(x,r)\big) < r^\gamma$.

Set $E_n = \big\{x \in \mathsf{S}_\mu\ ;\ \forall r \le 1/n,\ \mu\big(\mathsf{B}(x,r)\big) < r^\gamma\big\}$ and consider a subset F of E_n. If $\{\mathsf{B}_j\}_j$ is any centered δ-cover of F with $\delta < 1/n$, one has

$$\overline{\nu}_\delta(F) \le \sum \nu(\mathsf{B}_j) = \sum \mu(\mathsf{B}_j)^{-t}\mu(\mathsf{B}_j)^t\nu(\mathsf{B}_j) \le \sum \mu(\mathsf{B}_j)^{-t} r_j^{\gamma t}\nu(\mathsf{B}_j).$$

Therefore

$$\overline{\nu}_\delta(F) \le \overline{\mathscr{H}}_{(\mu,\nu),\delta}^{(-t,1),\gamma t}(F).$$

Then we have

$$\overline{\nu}(F) \le \overline{\mathscr{H}}_{(\mu,\nu)}^{(-t,1),\gamma t}(F) \le \mathscr{H}_{(\mu,\nu)}^{(-t,1),\gamma t}(\mathsf{S}_\mu) = 0.$$

This implies $\nu^\sharp(E_n) = 0$ and $\nu^\sharp(E) = 0$. This proves the first assertion. The second one is proved in the same way. □

Proposition 3.6. *Let μ be an element of $\mathcal{F}$. Suppose that, for some $q \in J_b$, $\mathscr{H}_\mu^{q,b(q)}(\mathsf{S}_\mu) > 0$, and consider the set*

$$E = \Big\{ x \in \mathsf{S}_\mu \ ; \ \underline{\lim}_{r \searrow 0} \frac{\log \mu\left(\mathsf{B}(x,r)\right)}{\log r} \le -b_l^\flat(q) \text{ and } \overline{\lim}_{r \searrow 0} \frac{\log \mu\left(\mathsf{B}(x,r)\right)}{\log r} \ge -b_r^\flat(q) \Big\}.$$

Then we have

$$\dim_P E \ge \begin{cases} b(q) - q\, b_r^\flat(q), & \text{if } q \ge 0, \\ b(q) - q\, b_l^\flat(q), & \text{if } q \le 0. \end{cases}$$

In particular, if $b'(q)$ exists one has

$$\dim_P \Big\{ x \in \mathsf{S}_\mu \ ; \ \underline{\lim}_{r \searrow 0} \frac{\log \mu\left(\mathsf{B}(x,r)\right)}{\log r} \le -b'(q) \le \overline{\lim}_{r \searrow 0} \frac{\log \mu\left(\mathsf{B}(x,r)\right)}{\log r} \Big\} \ge b(q) - q\, b'(q).$$

Proof. This results from Lemma 3.5 and (3.4) of Lemma 3.1. □

4. An example

Now, we can deal with the example given in [3] (Theorem 2.6). We take for $\mathbb{X}$ the space $\{0,1\}^{\mathbb{N}^*}$ endowed with the ultrametric which assigns diameter 2^{-n} to cylinders of order n.

We are given two numbers p and $\tilde{p}$ such that $0 < p < \tilde{p} \le 1/2$, and a sequence of integers $1 = t_0 < t_1 < \cdots < t_n < \cdots$ such that $\lim_{n\to\infty} t_n/t_{n+1} = 0$.

We define a probability measure μ on $\{0,1\}^{\mathbb{N}^*}$: the measure assigned to the cylinder $[\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n]$ is

$$\mu\big([\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n]\big) = \prod_{j=1}^{n} \varpi_j,$$

where

- if $t_{2k-1} \le j < t_{2k}$ for some k, then $\varpi_j = p$ if $\varepsilon_j = 0$, and $\varpi_j = 1 - p$ otherwise,
- if $t_{2k} \le j < t_{2k+1}$ for some k, then $\varpi_j = \tilde{p}$ if $\varepsilon_j = 0$, and $\varpi_j = 1 - \tilde{p}$ otherwise.

In fact, the measure considered in [3] is obtained by taking the image of μ under the natural binary coding of numbers in $[0,1]$ composed with the Gray code. The purpose of using the Gray code was to get a doubling measure on $[0,1]$.

For $q \in \mathbb{R}$, define

$$\theta(q) = \log_2\big(p^q + (1-p)^q\big) \quad \text{and} \quad \tilde{\theta}(q) = \log_2\big(\tilde{p}^q + (1-\tilde{p})^q\big).$$

Then it follows from [3] that for $0 < q < 1$ we have

$$b(q) = \theta(q) < \tilde{\theta}(q) = B(q),$$

and, for $q < 0$ or $q > 1$,

$$b(q) = \tilde{\theta}(q) < \theta(q) = B(q).$$

We wish to prove the following result:

Proposition 4.1. 1) *For* $\alpha \in \big(-\log_2(1-\tilde{p}), -\log_2\tilde{p}\big)$, *we have*

$$\dim_H X_\mu(\alpha) = \inf_{q\in\mathbb{R}} b(q) + \alpha q.$$

2) *For* $\alpha \in \big(-\log_2(1-\tilde{p}), -\log_2\tilde{p}\big) \setminus \big([-B_r'(0), -B_l'(0)] \cup [-B_r'(1), -B_l'(1)]\big)$, *we have*

$$\dim_P X_\mu(\alpha) = \inf_{q\in\mathbb{R}} B(q) + \alpha q.$$

Proof. We consider the measure ν constructed as μ with parameters r and $\tilde{r}$ instead of p and $\tilde{p}$. We impose the condition

$$r\log p + (1-r)\log(1-p) = \tilde{r}\log\tilde{p} + (1-\tilde{r})\log(1-\tilde{p}). \tag{4.1}$$

As both r and $\tilde{r}$ should belong to the interval $(0,1)$, we must have

$$\log\frac{1-p}{1-\tilde{p}} < r\log\frac{1-p}{p} < \log\frac{1-p}{\tilde{p}}. \tag{4.2}$$

From Corollary 2.2, it is easy to compute $\varphi(x) = \tau_{(\mu,\nu),\mathsf{S}_\mu}$. We have

$$\varphi(x) = \log_2\max\left\{\big(p^x r + (1-p)^x(1-r)\big), \big(\tilde{p}^x\tilde{r} + (1-\tilde{p})^x(1-\tilde{r})\big)\right\}.$$

Condition (4.1) implies that $\varphi'(0)$ exists. We set

$$\alpha = -\varphi'(0) = -r\log_2 p - (1-r)\log_2(1-p) = r\log_2\frac{1-p}{p} - \log_2(1-p). \tag{4.3}$$

It results from (4.2) that α can take any value in the interval $\big(-\log_2(1-\tilde{p}), -\log_2\tilde{p}\big)$.

Moreover, the strong law of large numbers shows that we have

$$\underline{\lim}_{n\to\infty}\frac{\log_2\nu\big(\mathsf{B}(x,2^{-n})\big)}{-n} = \min\{\mathsf{h}(r),\mathsf{h}(\tilde{r})\}$$

and

$$\overline{\lim}_{n\to\infty}\frac{\log_2\nu\big(\mathsf{B}(x,2^{-n})\big)}{-n} = \max\{\mathsf{h}(r),\mathsf{h}(\tilde{r})\}$$

for ν-almost every x, where we set $\mathsf{h}(r) = -r\log_2 r - (1-r)\log_2(1-r)$.

Then it results from Lemmas 3.2 and 3.1-b that

$$\dim_H X_\mu(\alpha) \geq \min\{\mathsf{h}(r),\mathsf{h}(\tilde{r})\} \tag{4.4}$$

and

$$\dim_P X_\mu(\alpha) \geq \max\{\mathsf{h}(r),\mathsf{h}(\tilde{r})\}, \tag{4.5}$$

where r, $\tilde{r}$, and α are linked by (4.1) and (4.3).

If α is defined by (4.3), we have

$$\alpha = -\theta'(q) \quad \text{if} \quad q = \frac{\log \frac{1-r}{r}}{\log \frac{1-p}{p}} \quad \text{and} \quad \alpha = -\tilde{\theta}'(\tilde{q}) \quad \text{if} \quad \tilde{q} = \frac{\log \frac{1-\tilde{r}}{\tilde{r}}}{\log \frac{1-\tilde{p}}{\tilde{p}}}. \tag{4.6}$$

Now fix q and $\tilde{q}$ as above in (4.6). One can check that, for these values of q and $\tilde{q}$, one has

$$\theta(q) - q\,\theta'(q) = \mathsf{h}(r) \quad \text{and} \quad \tilde{\theta}(\tilde{q}) - \tilde{q}\,\tilde{\theta}'(\tilde{q}) = \mathsf{h}(\tilde{r}). \tag{4.7}$$

In order to have $\theta(q) = b(q)$, we must have $0 < q < 1$, which means

$$\log_2 \frac{1}{p^p(1-p)^{1-p}} < \alpha < \log_2 \frac{1}{\sqrt{p(1-p)}}. \tag{4.8}$$

In order to have $\tilde{\theta}(\tilde{q}) = b(\tilde{q})$, we must have $\tilde{q} < 0$ or $\tilde{q} > 1$, which means

$$\alpha > \log_2 \frac{1}{\sqrt{\tilde{p}(1-\tilde{p})}} \tag{4.9}$$

or

$$\alpha < \log_2 \frac{1}{\tilde{p}^{\tilde{p}}(1-\tilde{p})^{1-\tilde{p}}}. \tag{4.10}$$

One can check that at least one of the conditions (4.8), (4.9) and (4.10) is fulfilled.

But for any q such that $b'(q)$ exists, we have (see [8] or [1]) that

$$\dim_H X_\mu\big(-b'(q)\big) \le b(q) - q\,b'(q). \tag{4.11}$$

The first assertion then results from (4.4), (4.7), and (4.11).

In order to have $\theta(q) = B(q)$, we must have $q < 0$ or $q > 1$, which means

$$\alpha > \log_2 \frac{1}{\sqrt{p(1-p)}} = -B_l'(0) \quad \text{or} \quad \alpha < \log_2 \frac{1}{p^p(1-p)^{1-p}} = -B_r'(1).$$

In order to have $\tilde{\theta}(\tilde{q}) = B(\tilde{q})$, we must have $0 < \tilde{q} < 1$, which means

$$-B_l'(1) = \log_2 \frac{1}{\tilde{p}^{\tilde{p}}(1-\tilde{p})^{1-\tilde{p}}} < \alpha < \log_2 \frac{1}{\sqrt{\tilde{p}(1-\tilde{p})}} = -B_r'(0).$$

Then assertion (2) follows as before. □

Remark 4.2. Proposition 4.1 also holds for the measure considered in [3]. Indeed, using the Gray code before projecting on $[0, 1]$ yields doubling measures.

5. The vector case

As in [10] one may consider expressions of the form $\exp -\langle q, \varkappa(\mathsf{B})\rangle$ instead of $\mu(\mathsf{B})^q$, where $\varkappa$ takes its values in the dual $\mathbb{E}'$ of a separable Banach space $\mathbb{E}$ and $q \in \mathbb{E}$.

Let ν be an element of $\mathscr{F}$. For $E \subset \mathbb{X}$, $q \in \mathbb{E}$, $t \in \mathbb{R}$, and $\delta > 0$, one sets

$$\begin{aligned}
\overline{\mathscr{P}}_\delta^{q,t}(E) &= \sup\left\{\sum r_j^t \mathrm{e}^{-\langle q,\varkappa(\mathsf{B}_j)\rangle}\nu(\mathsf{B}_j)\ ;\ \{\mathsf{B}_j\} \text{ a } \delta\text{-packing of } E\right\},\\
\overline{\mathscr{P}}^{q,t}(E) &= \lim_{\delta\searrow 0} \overline{\mathscr{P}}_\delta^{q,t}(E),\\
\mathscr{P}^{q,t}(E) &= \inf\left\{\sum \overline{\mathscr{P}}^{q,t}(E_j)\ ;\ E\subset \bigcup E_j\right\},
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mathscr{H}}_\delta^{q,t}(E) &= \inf\left\{\sum r_j^t \mathrm{e}^{-\langle q,\varkappa(\mathsf{B}_j)\rangle}\nu(\mathsf{B}_j)\ ;\ \{\mathsf{B}_j\} \text{ a centered } \delta\text{-cover of } E\right\},\\
\overline{\mathscr{H}}^{q,t}(E) &= \lim_{\delta\searrow 0} \overline{\mathscr{H}}_\delta^{q,t}(E),\\
\mathscr{H}^{q,t}(E) &= \sup\left\{\overline{\mathscr{H}}^{q,t}(F)\ ;\ F\subset E\right\},
\end{aligned}$$

For a function χ from $\mathbb{E}$ to $\mathbb{R}$, and for $v \in \mathbb{E}$ of norm 1, one defines

$$\partial_v\chi(0) = \lim_{t\searrow 0}\frac{\chi(tv)-\chi(0)}{t} \quad\text{and}\quad \partial_v^*\chi(0) = \overline{\lim_{t\searrow 0}} -\frac{\chi(tv)-\chi(0)}{t}.$$

With these notations we have the following analogues of Lemmas 3.2 and 3.5:

Lemma 5.1. *Let $\varphi(q)$ be one of the following functions:*

$$\inf\left\{t\ ;\ \overline{\mathscr{P}}^{q,t}(\mathbb{X}) = 0\right\} \quad \textit{or} \quad \inf\left\{t\ ;\ \mathscr{P}^{q,t}(\mathbb{X}) = 0\right\}.$$

Assume that $\varphi(0) = 0$ and that $\partial_v\varphi(0)$ at 0 is a lower semi-continuous function of v. Then one has

$$\nu^\sharp\left\{x\ ;\ \underline{\lim}_{r\searrow 0}\frac{\langle v,\varkappa\big(\mathsf{B}(x,r)\big)}{-\ln r} < -\partial_v\varphi(0) \textit{ for some } v\in\mathbb{E}\right\} = 0.$$

Lemma 5.2. *Set $\varphi(q) = \inf\{t\ ;\ \mathscr{H}^{q,t}(\mathbb{X}) = 0\}$ and assume that $\varphi(0) = 0$ and that $\partial_v^*\chi(0)$ is a lower semi-continuous function of v. Then one has*

$$\nu^\sharp\left\{x\ ;\ \overline{\lim}_{r\searrow 0}\frac{\langle v,\varkappa\big(\mathsf{B}(x,r)\big)}{-\ln r} < -\partial_v^*\varphi(0) \textit{ for some } v\in\mathbb{E}\right\} = 0.$$

The proofs follow the same lines as those above and as the proofs in [10]. As a corollary we get the following result (with the notations of [10]):

Theorem 5.3. *Let $B(q)=\inf\{t\in\mathbb{R}\,;\mathscr{H}_\varkappa^{q,t}(\mathbb{X}) = 0\}$. Assume that, at some point q, the function B is differentiable with derivative $B'(q)$ and that $\mathscr{H}_\varkappa^{q,B(q)}(\mathbb{X}) > 0$. Then one has*

$$\dim_H\left\{x\ ;\ \forall v\in\mathbb{E}, \lim_{r\searrow 0}\frac{\big\langle v,\varkappa\big(\mathsf{B}(x,r)\big)\big\rangle}{\log r} = -B'(q)v\right\} = B(q) - B'(q)q.$$

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