

On genuine infinite algebraic tensor products

Chi-Keung Ng

Abstract. In this paper, we study genuine infinite tensor products of some algebraic structures. By a genuine infinite tensor product of vector spaces, we mean a vector space $\bigotimes_{i \in I} X_i$ whose linear maps coincide with multilinear maps on an infinite family $\{X_i\}_{i\in I}$ of vector spaces. After establishing its existence, we give a direct sum decomposition of $\bigotimes_{i\in I} X_i$ over a set $\Omega_{I:X}$, through which we obtain a more concrete description and some properties of $\bigotimes_{i\in I} X_i$. If $\{A_i\}_{i\in I}$ is a family of unital ^{*}-algebras, we define, through a subgroup $\Omega_{I;A}^{\text{ut}} \subseteq \Omega_{I;A}$, an interesting subalgebra $\bigotimes_{i \in I}^{\text{ut}} A_i$. When all A_i are C^* -algebras or group algebras, it is the linear span of $\bigotimes_{i\in I}^{\text{ut}}C$ is the group algebra of $\Omega_{I;\mathbb{C}}^{\text{ut}}$. In general, $\bigotimes_{i\in I}^{\text{ut}}A_i$ can be identified the tensor products of unitary elements of A_i . Moreover, it is shown that with the algebraic crossed product of a cocycle twisted action of $\Omega_{I;A}^{\text{ut}}$. On the other hand, if $\{H_i\}_{i\in I}$ is a family of inner product spaces, we define a Hilbert $C^*(\Omega_{I;\mathbb{C}}^{\text{ut}})$ -module $\bar{\bigotimes}_{i\in I}^{\text{mod}} H_i$, which is the completion of a subspace $\bigotimes_{i\in I}^{\text{unit}} H_i$ of $\bigotimes_{i\in I} H_i$. If $\chi_{\Omega_{I;\mathbb{C}}^{\text{ut}}}$ is the canonical tracial state on $C^*(\Omega_{I;\mathbb{C}}^{\text{ut}})$, then $\bar{\bigotimes}^{\text{mod}}_{i\in I}H_i \otimes_{\chi_{\Omega_{I,\mathbb{C}}^{\text{ut}}}} \mathbb{C}$ coincides with the Hilbert space $\bar{\bigotimes}^{\phi_1}_{i\in I}H_i$ given by a very elementary algebraic construction and is a natural dilation of the infinite direct product $\prod \otimes_{i\in I} H_i$ as defined by J. von Neumann. We will show that the canonical representation of $\bigotimes_{i\in I}^{\text{ut}}\mathcal{L}(H_i)$ on $\overline{\bigotimes}_{i\in I}^{\phi_1}H_i$ is injective (note that the canonical representation of $\mathcal{Q}_{i\in I}^{\mathrm{ut}}\mathcal{L}(H_i)$ on $\prod \otimes_{i\in I} H_i$ is not injective). We will also show that if $\{A_i\}_{i\in I}$ is a family of unital Hilbert algebras, then so is $\bigotimes_{i\in I}^{\text{ut}} A_i$.

1. Introduction

In this paper, we study infinite tensor products of some algebraic structures. In the literature, infinite tensor products are often defined as inductive limit of finite tensor products (see, e.g., [\[4\]](#page-26-0), [\[21\]](#page-27-1) [\[9\]](#page-26-1), [\[14\]](#page-26-2), and [\[15\]](#page-27-2)). As far as we know, the only

Mathematics Subject Classification (2010): Primary 15A69, 46M05; Secondary 16G99, 16S35, 20C07, 46C05, 46L99, 47A80.

Keywords: Infinite tensor products, unital [∗]-algebras, twisted crossed products, inner product spaces, representations.

alternative approach so far is the one by J. von Neumann, concerning *infinite direct products of Hilbert spaces* (see [\[20\]](#page-27-3)). Some authors used this approach to define infinite tensor products of other functional analytic structures (see, e.g., [\[3\]](#page-26-3), [\[11\]](#page-26-4) and [\[13\]](#page-26-5)). The work of von Neumann attracted the attention of many physicists who are interested in "quantum mechanics with infinite degrees of freedom", as well as mathematicians whose interest is in the field of operator algebras (see, e.g., [\[1\]](#page-26-6), [\[2\]](#page-26-7), [\[3\]](#page-26-3), [\[8\]](#page-26-8), [\[12\]](#page-26-9), [\[17\]](#page-27-4), and [\[19\]](#page-27-5)).

However, von Neumann's approach is not appropriate for purely algebraic objects. The aim of this article is to study "genuine infinite algebraic tensor products" (i.e. ones that are defined in terms of multilinear maps instead of through inductive limits) of some algebraic structures. There are several motivations behind this study.

1. Conceptually speaking, it is natural to define "infinite tensor products" as the object that produces a unique linear map from a multilinear map on a given infinite family of objects (see Definition [2.1\)](#page-3-0). As infinite direct products of Hilbert spaces are important in both physics and mathematics, it is believed that such infinite tensor products of algebraic structures are also important.

2. We want to construct an infinite tensor product of Hilbert spaces that is easier for non-analyst to grasp (compare with the infinite direct product as defined by J. von Neumann; see Lemma [4.2](#page-13-0) and Remark [4.7](#page-16-0) (d)) and is more natural (see Theorem [4.8,](#page-17-0) Example [4.10](#page-19-0) and Example [5.6\)](#page-23-0). Veumani
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n a fami
 $G_i := \{$

3. Given a family of groups $\{G_i\}_{i\in I}$, it is well known that the group algebra of group. the group

$$
\bigoplus_{i \in I} G_i := \left\{ [g_i]_{i \in I} \in \Pi_{i \in I} G_i : g_i = e \text{ except for finite number of } i \in I \right\}
$$

is an inductive limit of finite tensor products. However, if one wants to consider the group algebra $\mathbb{C}[\Pi_{i\in I}G_i]$, one is forced to consider a "bigger version of tensor products" (see Example [3.1\)](#page-8-0).

In this article, the algebraic structures that we consider are vector spaces, unital [∗]-algebras, inner product spaces, and [∗]-representations of unital [∗]-algebras on Hilbert spaces. In our study, we discovered some interesting phenomena about infinite tensor products that do not have counterparts in the case of finite tensor products. Most of these phenomena relate to a certain object, $\Omega_{I:X}$, defined as in Remark $2.4(d)$ $2.4(d)$, which "encodes the asymptotic information" of a given family $\{X_i\}_{i\in I}$.

In Section [2,](#page-3-1) we will begin our study by defining the infinite tensor product $(\bigotimes_{i\in I} X_i, \Theta_X)$ of a family $\{X_i\}_{i\in I}$ of vector spaces. Two particular concerns are
bases of \bigotimes X_i and the relationship between \bigotimes X_i and inductive limits of finite broatics. Most of these phenomena relate to a certain object, $\Omega_{I;X}$, denned as
in Remark 2.4(d), which "encodes the asymptotic information" of a given fam-
ily $\{X_i\}_{i\in I}$.
In Section 2, we will begin our study by d tensor products of $\{X_i\}_{i\in I}$ (which depend on choices of fixed elements in $\Pi_{i\in I}X_i$). In Section 2, we will begin our study by defining the infinit $(\bigotimes_{i \in I} X_i, \Theta_X)$ of a family $\{X_i\}_{i \in I}$ of vector spaces. Two partic bases of $\bigotimes_{i \in I} X_i$ and the relationship between $\bigotimes_{i \in I} X_i$ and induct tenso is, we obtain a direct sum decomposition of $\bigotimes_{i \in I} X_i$ indexed by
eorem 2.5) with all the direct summands being inductive limits
ducts (see Proposition 2.6 (b)). From this, we also obtain that
 $\Psi : \bigotimes L(X_i; Y_i) \to L(\bigotimes X_i$ a set $\Omega_{I:X}$ (see Theorem [2.5\)](#page-5-1) with all the direct summands being inductive limits of finite tensor products (see Proposition $2.6(b)$ $2.6(b)$). From this, we also obtain that the canonical map

$$
\Psi : \bigotimes_{i \in I} L(X_i; Y_i) \to L(\bigotimes_{i \in I} X_i; \bigotimes_{i \in I} Y_i)
$$

ON GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS 331

is injective (but not surjective). As a consequence, $\bigotimes_{i \in I} X_i$ is automatically a

faithful module over the big unital commutative algebra $\bigotimes_{i \in I} C$ (see Corolla ON GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS 331

is injective (but not surjective). As a consequence, $\bigotimes_{i \in I} X_i$ is automatically a

faithful module over the big unital commutative algebra $\bigotimes_{i \in I} \mathbb{C}$ (see Co and Example [2.10\)](#page-8-1). Moreover, one may regard the canonical map (e). As a consequential commutative α , one may regard Θ ^c : Π_{*i*∈I}^C → ⊘

$$
\Theta_{\mathbb C}:\Pi_{i\in I}{\mathbb C}\to\bigotimes\nolimits_{i\in I}{\mathbb C}
$$

as a generalized multiplication (see Example [2.10](#page-8-1) (a)). In this sense, one can make sense of infinite products like $(-1)^{I}$. $\Theta_{\mathbb{C}} : \Pi_{i \in I} \mathbb{C} \to \bigotimes_{i \in I} \mathbb{C}$

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se of infinite products like $(-1)^I$.

Clearly, $\bigotimes_{i \in I} A_i$ is a unital [∗]-algebra if all

 $\Theta_{\mathbb{C}} : \Pi_i \in I^{\mathbb{C}} \to \bigotimes_{i \in I} \mathbb{C}$
as a generalized multiplication (see Example 2.10(a)). In this sense, one can make
sense of infinite products like $(-1)^I$.
Clearly, $\bigotimes_{i \in I} A_i$ is a unital [∗]-algebra if al sum over a subgroup $\Omega_{I;A}^{\text{ut}}$ of the semigroup $\Omega_{I;A}$. The reasons for considering this subgroup after Proposition subgroup of the discussion after Propositions (see the discussion after Proposition subalgebra are that it has good representations (see the discussion after Proposi-tion [5.1\)](#page-20-0), and it is big enough to contain $\mathbb{C}[\Pi_{i\in I}G_i]$ when $A_i = \mathbb{C}[G_i]$ for all $i \in I$ (see Example [3.1](#page-8-0)(a)). Moreover, if all A_i are generated by their unitary elements study in Section 3, a natural *-subalgebra $\bigotimes_{i \in I}^{\infty} A_i$ of $\bigotimes_{i \in I} A_i$ with
sum over a subgroup $\Omega_{I;A}^{ut}$ of the semigroup $\Omega_{I;A}$. The reasons for
subalgebra are that it has good representations (see the d $\sum_{i\in I}^{u_i} A_i$ is the linear sum over a subgroup $\Omega_{I;A}^{\infty}$ or the semigroup $\Omega_{I;A}$. The reasons for considering
subalgebra are that it has good representations (see the discussion after Prop
tion 5.1), and it is big enough to contain $\mathbb{C}[\$ $\sum_{i\in I}$ and be identified with the crossed products of some twisted actions in the sense of Busby and Smith (i.e., a cocycle action with a 2-cocycle) of $\Omega_{I:A}^{\text{ut}}$ on $\bigotimes_{i\in I}^e A_i$ H_i for all sample in the set of A_i is the limit of $\bigotimes_{i=1}^n H_i$ on $\bigotimes_{i=1}^n H_i$ on $\bigotimes_{i=1}^n H_i$ or $\bigotimes_{i=1}^n H_i$ (the unital ^{*}-algebra inductive limit of finite tensor products of A_i). Moreover, it is
shown that \otimes^{ut} C can be identified with the group algebra of Ω^{ut} . Corollary 3.4) span of the tensor products of unitary elements in A_i . We will show that $\bigotimes_{i\in I}^{\text{ut}} A_i$ span or the tensor products or unitary elements in A_i . We will show that $\bigotimes_{i \in I} A_i$ can be identified with the crossed products of some twisted actions in the sense of Busby and Smith (i.e., a cocycle action with a We will also study the center of $\bigotimes_{i\in I}^{\text{ut}} A_i$ in the case when A_i is generated by its unitary elements (for all $i \in I$).
Le Costian 4 are will consider

In Section [4,](#page-12-0) we will consider tensor products of inner product spaces. If ${H_i}_{i\in I}$ is a family of inner product spaces, we define a natural inner product on a sub-Shown that $\mathbf{Q}_{i\in I}^{\text{out}}$ is a family the center of $\mathbf{Q}_{i\in I}^{\text{out}}A_i$ in the case when A_i is generated by its unitary elements (for all $i \in I$).
In Section 4, we will consider tensor products of inner product s unitary elements

In Section 4,

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space $\bigotimes_{i\in I}^{\text{unit}} H_i$ (

completion $\overline{\bigotimes}_{i\in I}^{\phi_i}$ in Section 4, we will consider tensor products of inner product spaces. If $\{H_i\}_{i\in I}$
is a family of inner product spaces, we define a natural inner product on a subspace $\bigotimes_{i\in I}^{\text{unit}} H_i$ of $\bigotimes_{i\in I} H_i$ (see Lem is a family of inner product spaces, we define a natural inner product on a subspace $\bigotimes_{i\in I}^{\text{unit}} H_i$ of $\bigotimes_{i\in I} H_i$ (see Lemma 4.2 (b)). In the case of Hilbert spaces, the completion $\overline{\bigotimes}_{i\in I}^{\phi_1} H_i$ of \big the construction for $\bar{\bigotimes}_{i\in I}^{\phi_1} H_i$ is totally algebraical and is more natural (see Example 4.10 and Example 5.6). Note also that one can construct $\prod \otimes_{i\in I} H_i$ in a similar space $\bigotimes_{i\in I}^{\text{unit}} H_i$ of $\bigotimes_{i\in I}^{\infty} H_i$ (see Lemma 4.2(b)). In the case of Hilbert spaces, the completion $\overline{\bigotimes}_{i\in I}^{\phi_1} H_i$ of $\bigotimes_{i\in I}^{\text{unit}} H_i$ is a "natural dilation" of the infinite direct product $\$ way as $\bar{\mathcal{Q}}_{i\in I}^{\phi_1}H_i$ (see Remark [4.7](#page-16-0) (d)). On the other hand, there is an inner product $\prod \otimes_{i \in I} H_i$ as defined by J. von Neumann in [20] (see Remark 4.7(b)). Note that
the construction for $\overline{\otimes}_{i \in I}^{\phi_1} H_i$ is totally algebraical and is more natural (see Exam-
ple 4.10 and Example 5.6). Note also th r pre-inner products on $\bigotimes_{i\in I}^{\text{unif}} H_i$ (see Remark [4.9](#page-19-1) (a)).

Section [5](#page-20-1) will be devoted to the study of [∗]-representations of unital [∗]-algebras. More precisely, if $\Psi_i : A_i \to \mathcal{L}(H_i)$ is a unital *-representation $(i \in I)$, we define a canonical [∗]-representation s on $\bigotimes_{i \in I} H_i$ (
y of *-represent
mital *-represent
 $A_i \rightarrow \mathcal{L}(\bar{\bigotimes}_{i \in I}^{\phi_1})$

$$
\bigotimes\nolimits_{i \in I}^{\phi_1} \Psi_i \; : \; \bigotimes\nolimits_{i \in I}^{\mathrm{ut}} A_i \; \to \; \mathcal{L}(\bar{\bigotimes}\nolimits_{i \in I}^{\phi_1} H_i).
$$

More precisely, if $\Psi_i : A_i \to \mathcal{L}(H_i)$ is a unital *-representation $(i \in I)$, we define a
canonical *-representation
 $\bigotimes_{i \in I}^{\phi_1} \Psi_i : \bigotimes_{i \in I}^{\text{ut}} A_i \to \mathcal{L}(\overline{\bigotimes}_{i \in I}^{\phi_1} H_i).$
We will show in Theorem [5.3](#page-21-0) (c) th injective. This is equivalent to the canonical *-representations of $\mathcal{Q}_{i\in I}^{\text{uf}}\mathcal{L}(H_i)$ on $\bar{\mathbf{Q}}_{i\in I}^{\phi_1}H_i$ being injective, and is related to the "strong faithfulness" of the canonical
ortion of \mathbf{Q}^{ut} on \mathbf{Q}^{unit} (see Bemark 5.4(b)). Note however, that the correaction of $\Omega_{I;\mathcal{L}(H)}^{\text{ut}}$ on $\Omega_{I;H}^{\text{unit}}$ (see Remark [5.4](#page-22-0)(b)). Note however, that the corre-We will show in Theorem 5.3 (c) that if all the Ψ_i are injective, then $\bigotimes_{i\in I}^{\phi_1} \Psi_i$ is also
injective. This is equivalent to the canonical *-representations of $\bigotimes_{i\in I}^{\psi_1} \mathcal{L}(H_i)$ on
 $\overline{\bigotimes}_{i\in I}^{\phi_$ Consequently, if (H_i, π_i) is a unitary representation of a group G_i that induces an injective ^{*}-representation of $\mathbb{C}[G_i]$ on H_i $(i \in I)$, then we obtain an injective C. K. NG

Consequently, if (H_i, π_i) is a unitary representation of a group G_i that induces

an injective *-representation of $\mathbb{C}[G_i]$ on H_i $(i \in I)$, then we obtain an injective

"tensor type" *-representation of Consequently, if (H_i, π_i) is a unitary
an injective *-representation of $\mathbb{C}[G_i]$
"tensor type" *-representation of $\mathbb{C}[\Pi]$
the other hand, we will show that \bigoplus $\bigoplus_{\rho \in \Pi_i \in I} S(A_i)$ $\bar{\mathcal{S}}_{i\in I}^{\phi_1} H_{\rho_i}, \mathcal{S}_{i\in I}^{\phi_1} \pi_{\rho_i}$ is an injec $i \in I^{11} \rho_i$, $\bigotimes_i \in I^{n} \rho_i$
 \bigcirc^* 1 1 (C) Consequently, if (H_i, π_i) is
an injective *-representation
"tensor type" *-representation
the other hand, we will show
tive *-representation of $\bigotimes_{i \in \mathbb{Z}}^{\text{ut}}$ $\sum_{i\in I}^{u_{\text{t}}} A_i$ when all the A_i are C^* -algebras (Corollary [5.9\)](#page-24-0). an injective *-representation of $\mathbb{C}[G_i]$ on H_i $(i \in I)$, then we obtain an inje
"tensor type" *-representation of $\mathbb{C}[\Pi_{i \in I} G_i]$ on $\bar{\bigotimes}_{i \in I}^{\phi_1} H_i$ (see Corollary 5.7).
the other hand, we will show that Finally, we show that if all the A_i are unital Hilbert algebras, then so is $\bigotimes_{i\in I}^{\text{ut}} A_i$.

Notation 1.1. i) In this article, all the vector spaces, algebras as well as inner product spaces are over the complex field \mathbb{C} , although some results remain valid if one considers the real field instead.

ii) Throughout this article, I is an infinite set, and \mathfrak{F} is the set of all non-empty finite subsets of I.

iii) For any vector space X, we write $X^* := X \setminus \{0\}$ and define X^* to be the set of linear functionals on X. If Y is another vector space, we denote by $X \otimes Y$ and $L(X; Y)$ respectively, the algebraic tensor product of X and Y, and the set of linear maps from X to Y. We also write $L(X) := L(X; X)$.

iv) If $\{X_i\}_{i\in I}$ is a family of vector spaces and $x \in \Pi_{i\in I}X_i$, we denote by x_i the "ith-coordinate" of x (i.e. $x = [x_i]_{i \in I}$). If $x, y \in \Pi_{i \in I} X_i$ are such that $x_i = y_i$
except for a finite number of $i \in I$ we write except for a finite number of $i \in I$, we write

$$
x_i = y_i
$$
 e.f.

v) If V is a normed space, we denote by $\mathcal{L}(V)$ and V' the set of bounded linear operators and the set of bounded linear functionals, respectively, on V . Moreover, we set $\mathfrak{S}_1(V) := \{x \in V : ||x|| = 1\}$ and $B_1(V) := \{x \in V : ||x|| \leq 1\}.$

vi) If A is a unital ^{*}-algebra, we denote by e_A the identity of A and write $U_A := \{a \in A : a^*a = e_A = aa^*\}.$

2. Tensor products of vector spaces

In this section, $\{X_i\}_{i\in I}$ *and* $\{Y_i\}_{i\in I}$ *are families of non-zero vector spaces.*

Definition 2.1. Let Y be a vector space. A map $\Phi : \Pi_{i \in I} X_i \to Y$ is said to **2. Tensor products of vector spaces**
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 Definition 2.1. Let *Y* be a vector space. A map $\Phi : \Pi_{i \in I} X$

be *multilinear* if Φ is linear be *multilinear* if Φ is linear on each variable. Suppose that $\bigotimes_{i\in I} X_i$ is a vector In this section, $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ are families of non-zero vector spaces.
 Definition 2.1. Let Y be a vector space. A map $\Phi : \Pi_{i\in I}X_i \to Y$ is said to be *multilinear* if Φ is linear on each variable. space Y and any multilinear map $\Phi: \Pi_{i\in I} X_i \to Y$, there exists a unique linear **Definition 2.1.** Let Y be a vector space. A map $\Phi : \Pi_{i \in I} X_i \to Y$ is said to be *multilinear* if Φ is linear on each variable. Suppose that $\bigotimes_{i \in I} X_i$ is a vector space and $\Theta_X : \Pi_{i \in I} X_i \to \bigotimes_{i \in I} X_i$ is a mul *tensor product* of $\{X_i\}_{i\in I}$. We will denote $\otimes_{i\in I} x_i := \Theta_X(x)$ ($x \in \Pi_{i\in I} X_i$) and set be *multilin*
space Y an
space Y an
map $\tilde{\Phi}$: ξ
tensor proc
 $X^{\otimes I} := \bigotimes$ $X^{\otimes I} := \bigotimes_{i \in I} X_i$ if all X_i are equal to the same vector space X.

Let us first give the following simple example showing that non trivial multilinear maps with an infinite number of variables do exist. They are also crucial for some constructions later on.

Example 2.2. (a) Let $\Pi_{i\in I}^1 \mathbb{C} := \{\beta \in \Pi_{i\in I} \mathbb{C} : \beta_i = 1 \text{ e.f.}\}\$ and set

LGEBRAIC TENSOR PRODUCTS
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$$
\Pi_{i\in I}^1 \mathbb{C} := \{ \beta \in \Pi_{i\in I} \mathbb{C} : \beta_i = 1 \text{ e.f.} \}
$$

\n $\varphi_1(\beta) := \begin{cases} \Pi_{i\in I} \beta_i & \text{if } \beta \in \Pi_{i\in I}^1 \mathbb{C}, \\ 0 & \text{otherwise.} \end{cases}$

It is not hard to check that φ_1 is a non-zero multilinear map from $\Pi_{i\in I}\mathbb{C}$ to \mathbb{C} . If $\phi_1 \cdot \mathbb{C} \to \mathbb{C}$ is the linear functional induced by ϕ_1 (the existence of \mathbb{C}). $\phi_1 : \bigotimes_{i \in I} \mathbb{C} \to \mathbb{C}$ is the linear functional induced by φ_1 (the existence of $\bigotimes_{i \in I} \mathbb{C}$ will be established in Proposition 2.3(a)) then ϕ_1 is an involutive unital map $\varphi_1(\beta) := \begin{cases} \Pi_{i \in I} \beta_i & \text{if } \beta \in \Pi_{i \in I}^1 \mathbb{C}, \\ 0 & \text{otherwise.} \end{cases}$
t hard to check that φ_1 is a non-zero multilinear map from $\Pi_{i \in I} \mathbb{C}$ to $\varphi_i \in \mathbb{C} \to \mathbb{C}$ is the linear functional induced by φ_1 (will be established in Proposition [2.3](#page-4-0)(a)), then ϕ_1 is an involutive unital map.

(b) Let $\Pi_{e}^0 \Pi_{e}^0 \Gamma := \{ \beta \in \Pi_{i \in I} \mathbb{C} : \sum_{i \in I} |\beta_i - 1| < \infty \}.$ For each $\beta \in \Pi_{i \in I}^0 \mathbb{C}$, net $\{ \Pi_{i \in I} \beta_i \}_{i \in \mathbb{Z}}$ converges to a complex number denoted by $\Pi_{i \in I} \beta_i$ (see the net ${\Pi_{i \in F} \beta_i}_{F \in \mathfrak{F}}$ converges to a complex number, denoted by ${\Pi_{i \in I} \beta_i}_{F \in \mathfrak{F}}$ (see, e.g., 2.4.1 in [\[20\]](#page-27-3)). We define $\varphi_0(\beta) := \Pi_{i \in I} \beta_i$ whenever $\beta \in \Pi_{i \in I}^0 \mathbb{C}$ and set $\varphi_0(\alpha) = \alpha \operatorname{tr} \alpha = 0$. As in part (a) φ_0 induces an involutive unital linear functional $\varphi_0|_{\Pi_i \in I\mathbb{C}\setminus \Pi_{i \in I}^0 \mathbb{C}} \equiv 0$. As in part (a), φ_0 induces an involutive unital linear functional (b) L
the net
e.g., 2.4.
 $\varphi_0|_{\Pi_{i\in I}\mathbb{C}}$
 ϕ_0 on \bigotimes ϕ_0 on $\bigotimes_{i\in I}\mathbb{C}$.

Clearly, infinite tensor products are unique (up to linear bijections) if they exist. The existence of infinite tensor products follows from a similar argument as that for finite tensor products, but we give an outline here for future reference. Clearly, infinite tensor products are unique (up to linear biexist. The existence of infinite tensor products follows from a sime that for finite tensor products, but we give an outline here for further proposition 2.3. (Clearly, infinite tensor products are unique (up to linear bijections) in
it. The existence of infinite tensor products follows from a similar argum
t for finite tensor products, but we give an outline here for future ref

is an algebra (*respectively, a* [∗]*-algebra*) *with* $(\otimes_{i\in I} a_i)(\otimes_{i\in I} b_i) := \otimes_{i\in I} a_ib_i$ (*and*
 $(\otimes_{i\in I} a_i)^* := (\otimes_{i\in I} a_i^*)$) for $a \in \Pi_{i\in I} A$. $(\otimes_{i \in I} a_i)^* := (\otimes_{i \in I} a_i^*)$ *for* $a, b \in \Pi_{i \in I} A_i$. (b) If $\{A_i\}_{i\in I}$ is a family of algebras (respectively, *-alg
 is an algebra (respectively, a *-algebra) with $(\otimes_{i\in I} a_i)(\otimes_{i\in I}$
 $(\otimes_{i\in I} a_i)^* := (\otimes_{i\in I} a_i^*)$ for $a, b \in \Pi_{i\in I} A_i$.

(c) If $\Psi_i : A_i \to L(X_i)$ is a mily of algebras (respection), a^* -algebra) with $(\otimes_{i\in I} a_i \circ a_i) \in \Pi_{i\in I} A_i$.

is a homomorphism for $\bigotimes_{i\in I} A_i \to L(\bigotimes_{i\in I} X_i)$

(c) *If* $\Psi_i : A_i \to L(X_i)$ *is a homomorphism for each* $i \in I$ *, there is a canonical* 71

˜ i∈I Ψ(⊗i∈I ^ai)⊗i∈I ^xi ⁼ [⊗]i∈I ^Ψi(ai)xi (^a [∈] ^Πi∈IAi; ^x [∈] ^Πi∈IXi). (d) *If* A = n=0 ^Aⁿ *is a graded algebra and*

∞ $\sum_{n=0}^{\infty} M_n$ *is a graded left* A-
 *n*ⁱth</sub> $(\tilde{\bigotimes}_{i \in I} \Psi_i)(\otimes_{i \in I} a_i) \otimes_{i \in I} x_i = \otimes_{i \in I} \Psi_i(a_i) x_i \quad (a \in I$

(d) If $A = \bigoplus_{n=0}^{\infty} A_n$ *is a graded algebra and* $\bigoplus_{n=0}^{\infty} \mathbb{R}$
 module, then $\bigoplus_{n=0}^{\infty} \bigotimes_{k \geq n} M_k$ *is a graded A-module with* $\begin{aligned} \mathcal{A}_{i\in I} &\longrightarrow (\Im(\varepsilon_{1}^{k}A_{n})\otimes \varepsilon_{1}^{k}A_{n})\ \mathcal{A}_{n} &\longrightarrow \mathcal{A}_{n}^{k} &\longrightarrow \math$

$$
a_m(\otimes_{k\geq n}x_k)=\otimes_{k\geq n}a_mx_k\in \bigotimes_{k\geq m+n}M_k(a_m\in A_m;x\in \Pi_{k\geq n}M_k).
$$

Proof. Parts (b), (c) and (d) follow from the universal property of tensor products, and we will only give a brief account for part (a). Let V be the free vector space
generated by elements in $\Pi_{\{x\}} X$ and let $\Theta_0 : \Pi_{\{x\}} X \to V$ be the canonical man generated by elements in $\Pi_{i\in I}X_i$ and let $\Theta_0 : \Pi_{i\in I}X_i \to V$ be the canonical map.
Suppose that $W := \text{span } W_0$ where Suppose that $W := \text{span } W_0$, where roof. Parts

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uppose the
 $W_0 := \{$

Suppose that
$$
W := \text{span } W_0
$$
, where
\n
$$
W_0 := \{ \lambda \Theta_0(u) + \Theta_0(v) - \Theta_0(w) : \lambda \in \mathbb{C}; u, v, w \in \Pi_{i \in I} X_i; \exists i_0 \in I \text{ with}
$$
\n
$$
(2.1) \qquad \lambda u_{i_0} + v_{i_0} = w_{i_0} \text{ and } u_j = v_j = w_j, \forall j \in I \setminus \{i_0\} \}.
$$
\nIf we put $\bigotimes_{i \in I} X_i := V/W$, and set Θ_X to be the composition of Θ_0 with the quotient map from V to V/W , then they will satisfy the requirement in Defini-

quotient map from *V* to V/W , then they will satisfy the requirement in Definition 2.1. tion [2.1.](#page-3-0) \Box

In the following remark, we list some observations that will be used implicitly throughout this article. 334

In the following remark, we list some observations that will be used

throughout this article.
 Remark 2.4. (a) As Θ_X is multilinear, $\bigotimes_{i \in I} X_i = \text{span} \Theta_X \left(\prod_{i \in I} X_i^{\times} \right)$.

(b) If I and I are non-number l

(b) If I_1 and I_2 are non-empty disjoint subsets of I with $I = I_1 \cup I_2$, it follows,
in the universal property that $\bigotimes X \cong (\bigotimes X) \otimes (\bigotimes X)$ canoni-In the following remark, we list some observations that will be
throughout this article.
 Remark 2.4. (a) As Θ_X is multilinear, $\bigotimes_{i \in I} X_i = \text{span } \Theta_X \big(\Pi_{i \in \Theta} \big)$

(b) If I_1 and I_2 are non-empty disjoint su $_{i\in I_1} X_i) \otimes (\bigotimes_{j\in I_2} X_j)$ canonically. mark 2.4. (a) As Θ_X is multilinear,

(b) If I_1 and I_2 are non-empty disjoin

in the universal property, that $\bigotimes_{i \in I} I_i$

y.

(c) $\bigotimes_{i \in I} (X_i \otimes Y_i) \cong (\bigotimes_{i \in I} X_i) \otimes (\bigotimes_{i \in I} I_i)$

- $i \in I(X_i \otimes Y_i) \cong (\bigotimes_{i \in I} X_i) \otimes (\bigotimes_{i \in I} Y_i)$ canonically.
- (d) For any $x, y \in \Pi_{i \in I} X_i^{\times}$, we write

$$
x \sim y \quad \text{if} \quad x_i = y_i \quad e.f.
$$

Obviously, \sim is an equivalence relation on $\Pi_{i \in I} X_i^{\times}$, and we set $[x]_{\sim}$ to be the invalence class of $x \in \Pi_{i \in I} X^{\times}$. Let $\Omega_{i} x$ be the collection of such equivalence equivalence class of $x \in \Pi_{i \in I} X_i^{\times}$. Let $\Omega_{I,X}$ be the collection of such equivalence
classes. It is not hard to see that $\Omega_{I,X}$ is a quotient group of $\Pi_{I,X}(\mathbb{C}^{\times})$ and that it classes. It is not hard to see that $\Omega_{I;\mathbb{C}}$ is a quotient group of $\Pi_{i\in I}\mathbb{C}^{\times}$, and that it acts freely on $\Omega_{I:X}$.

(e) The element $\otimes_{i\in I} 1 \in \mathbb{C}^{\otimes I}$ is non-zero. In fact, if $\otimes_{i\in I} 1=0$, then $\mathbb{C}^{\otimes I}=(0)$ (by Proposition [2.3](#page-4-0) (b)), and this implies the only multilinear map from $\Pi_{i\in I}\mathbb{C}$ to \mathbb{C} is zero, which contradicts Example [2.2.](#page-3-2)

The "asymptotic object" $\Omega_{I:X}$ defined in (d) above is crucial in the study of genuine infinite tensor products, as can be seen from our next result. Let us first give some more notation here. For every $u \in \prod_{i \in I} X_i^{\times}$, we set ⁱ e "asymptotic object" $\Omega_{I;X}$ defined in (d) are infinite tensor products, as can be seen from more notation here. For every $u \in \Pi_{i \in I}$
 $\frac{u}{i \in I} X_i := \{x \in \Pi_{i \in I} X_i : x \sim u\}$ and $\bigotimes_{i \in I}^u X_i$

$$
\Pi_{i\in I}^u X_i := \{ x \in \Pi_{i\in I} X_i : x \sim u \} \quad \text{and} \quad \bigotimes_{i\in I}^u X_i := \text{span}\,\Theta_X(\Pi_{i\in I}^u X_i).
$$

If $u \sim v$, then $\prod_{i\in I}^u X_i = \prod_{i\in I}^v X_i$, and we will also write $\prod_{i\in I}^{\lfloor u\rfloor_\sim} X_i := \prod_{i\in I}^u X_i$ give some more notation h
 $\Pi_{i\in I}^u X_i := \{x \in \Pi_{i\in I}.$

If $u \sim v$, then $\Pi_{i\in I}^u X_i$

and $\bigotimes_{i\in I}^{[u]_i} X_i := \bigotimes_{i\in I}^u X_i$. me more notar
 $\sum_{i\in I}^u X_i := \{x \in$
 $\sum_{i\in I} w_i$, then $\prod_{i\in I}^{|u|_{\sim}} X_i := \bigotimes_{i\in I}^u$ $\text{If } u \sim v, \text{ then}$

and $\bigotimes_{i \in I}^{[u]_{\sim}} X_i := \emptyset$

Theorem 2.5. \otimes $\prod_{i\in I}^{u} X_i =$
 $\prod_{i\in I}^{u} X_i =$
 $\sum_{i\in I}^{u} X_i$.
 $\prod_{i\in I} X_i = \bigoplus$

 $\bigoplus_{\omega \in \Omega_{I;X}} \bigotimes_{i \in I}^{\omega} X_i.$

Proof. Suppose that $x^{(1)},...,x^{(n)} \in \Pi_{i\in I} X_i^{\times}$ and that $0 = n_0 < \cdots < n_N = n$
is a sequence satisfying $x^{(n_k+1)} \sim ... \sim x^{(n_k+1)}$ for $k \in \{0, N-1\}$ but is a sequence satisfying $x^{(n_k+1)} \sim \cdots \sim x^{(n_{k+1})}$ for $k \in \{0,\ldots,N-1\}$, but $x^{(n_k)} \propto x^{(n_l)}$ whenever $1 \leq k \neq l \leq N$. We first show that if $\nu_1, \ldots, \nu_n \in \mathbb{C}$ with $\sum_{l}^{n} \nu_l \Theta_{\mathcal{X}}(x^{(l)}) = 0$ then **Theorer**
Proof. Si
is a sequ
 $x^{(n_k)} \nsim \text{with } \sum_{l=1}^{n}$ with $\sum_{l=1}^n \nu_l \Theta_X(x^{(l)})=0$, then hat

$$
\sum_{l=n_k+1}^{n_{k+1}} \nu_l \Theta_X(x^{(l)}) = 0 \quad (k = 0, \dots, N-1).
$$

In fact, by the proof of Proposition [2.3](#page-4-0) (a), there exist $m \in \mathbb{N}$, $\mu_1, \ldots, \mu_m \in \mathbb{C}$ and $\lambda_k \Theta_0(u^{(k)}) + \Theta_0(v^{(k)}) - \Theta_0(w^{(k)}) \in W_0$ $(k = 1, ..., m)$ such that ⁺¹_{i_k+1} $\nu_l \Theta_X$
 \vdots of Prope
 \vdots Θ_0
 \vdots Θ_0
 \vdots Θ_0

$$
\sum_{l=1}^{n} \nu_l \Theta_0(x^{(l)}) = \sum_{k=1}^{m} \mu_k (\lambda_k \Theta_0(u^{(k)}) + \Theta_0(v^{(k)}) - \Theta_0(w^{(k)})).
$$

Observe that if one of the elements in $\{u^{(k)}, v^{(k)}, w^{(k)}\}$ is equivalent to $x^{(1)}$ (un-
der ∞) then so are the other two (see (2.1)). After renaming one may assume der ∼), then so are the other two (see (2.1)). After renaming, one may assume that $u^{(k)} \sim v^{(k)} \sim w^{(k)} \sim x^{(1)}$ for $k = 1, \ldots, m_1$, but none of $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ is equivalent to $x^{(1)}$ when $k \in \{m_1 + 1, \ldots, m\}$.

Since the two sets

$$
\{x^{(n_1+1)}, \ldots, x^{(n)}\} \cup \{u^{(m_1+1)}, \ldots, u^{(m)}\} \cup \{v^{(m_1+1)}, \ldots, v^{(m)}\} \cup \{w^{(m_1+1)}, \ldots, w^{(m)}\}
$$

and

$$
\{x^{(1)}, \ldots, x^{(n_1)}\} \cup \{u^{(1)}, \ldots, u^{(m_1)}\} \cup \{v^{(1)}, \ldots, v^{(m_1)}\} \cup \{w^{(1)}, \ldots, w^{(m_1)}\}
$$

disjoint and elements in $\Theta_0 (\Pi_{i \in I} X_i)$ are linearly independent in V, we have

$$
\sum_{i=1}^{n_1} \nu_i \Theta_0(x^{(l)}) - \sum_{i=1}^{m_1} \mu_k (\lambda_k \Theta_0(u^{(k)}) + \Theta_0(v^{(k)}) - \Theta_0(w^{(k)})) = 0.
$$

are disjoint and elements in $\Theta_0 (\Pi_{i \in I} X_i)$ are linearly independent in V, we have

are disjoint and elements in
$$
\Theta_0 (\Pi_{i \in I} X_i)
$$
 are linearly independent in V , we have\n
$$
\sum_{l=1}^{n_1} \nu_l \Theta_0(x^{(l)}) - \sum_{k=1}^{m_1} \mu_k (\lambda_k \Theta_0(u^{(k)}) + \Theta_0(v^{(k)}) - \Theta_0(w^{(k)})) = 0.
$$
\nThis implies that\n
$$
\sum_{l=1}^{n_1} \nu_l \Theta_X(x^{(l)}) = 0.
$$
\nSimilarly,\n
$$
\sum_{l=n_k+1}^{n_{k+1}} \nu_l \Theta_X(x^{(l)}) = 0 \text{ for }
$$

 $k = 1, \ldots, N - 1.$

The above shows that

$$
\left(\bigotimes_{i\in I}^{\omega_M} X_i\right) \cap \left(\sum_{k=1}^{M-1} \bigotimes_{i\in I}^{\omega_k} X_i\right) = \{0\}
$$

when $\omega_1, \ldots, \omega_M$ are distinct elements in $\Omega_{I,X}$. On the other hand, for every $\left(\bigotimes_{i\in I}^{w_M} X_i\right) \cap \left(\sum_{k=1}^{M-1} \bigotimes_{i\in I}^{w_k} X_i\right) = \{0\}$
when $\omega_1, \dots, \omega_M$ are distinct elements in $\Omega_{I;X}$. On the other hand, for every $x \in \Pi_{i\in I} X_i^{\times}$, one has $\Theta_X(x) \in \bigotimes_{i\in I}^{[x]_{\sim}^{\times}} X_i$. These gi ts in $\Omega_{I;X}$. (
 X_i . These given

the has a linear
 $X_i \longrightarrow \bigotimes_{i}^u$

For any $F \in \mathfrak{F}$ and $u \in \Pi_{i \in I} X_i^{\times}$, one has a linear map

$$
J^u_F \,:\, \bigotimes\nolimits_{i \in F} X_i \,\longrightarrow\, \bigotimes\nolimits_{i \in I}^u X_i
$$

given by $J_F^u(\otimes_{i \in F} x_i) := \otimes_{j \in I} \tilde{x}_j \ (x_i \in X_i)$, where $\tilde{x}_j := x_j$ when $j \in F$, and $\tilde{x}_j := u_j$ when $j \in I \setminus F$ $\tilde{x}_j := u_j$ when $j \in I \setminus F$. given by $J_F^u(\otimes_{i \in F} x_i) := \otimes_{j \in I} \tilde{x}_j \ (x_i \in X_i)$, where $\tilde{x}_j := x_j$ when $j \in F$, and $\tilde{x}_j := u_j$ when $j \in I \setminus F$.
For any $F, G \in \mathfrak{F}$ with $F \subseteq G$, a similar construction gives a linear map $J_{G,F}^u$:

ven by $J_F^u(\otimes_{i\in F} x_i) := \otimes_{j\in I} \tilde{x}_j$ $(x_i \in X_i)$, where $\tilde{x}_j := x_j$ when $j \in F$, and
 $:= u_j$ when $j \in I \setminus F$.

For any $F, G \in \mathfrak{F}$ with $F \subseteq G$, a similar construction gives a linear map $J_{G;F}^u$:
 $_{i\in F} X_i \to \bigotimes_{i$ system in the category of vector spaces with linear maps as morphisms.

Proposition 2.6. (a) J_F^u *is injective for any* $u \in \Pi_{i \in I} X_i^{\times}$ *and* $F \in \mathfrak{F}$ *. Conse-*
guently $\Theta_{Y}(u) \neq 0$ *quently,* $\Theta_X(u) \neq 0$. $\epsilon_F X_i \to \bigotimes_{i \in G} X_i$. It is clear that $(\bigotimes_i$

tem in the category of vector spaces with li
 pposition 2.6. (a) J_F^u is injective for any

ntly, $\Theta_X(u) \neq 0$.

(b) The inductive limit of $(\bigotimes_{i \in F} X_i, J_{G;F}^u)$. $E_{F} X_i, J_{G;F}^* \Big|_{F \subseteq G \in \mathfrak{F}}$ is an induction
mear maps as morphisms.
 $u \in \Pi_{i \in I} X_i^{\times}$ and $F \in \mathfrak{F}$. Con
 $F \subseteq G \in \mathfrak{F}$ is $(\bigotimes_{i \in I}^u X_i, \{J_F^u\}_{F \in \mathfrak{F}})$. . र−

Proof. (a) Suppose that $a \in \text{ker } J_F^u$ and $\psi \in (\bigotimes_{i \in F} X_i)^*$. For each $j \in I \setminus F$, choose $f \in X^*$ with $f \cdot (u \cdot) = 1$. Bemark 2.4(b) and the universal property give a choose $f_j \in X_j^*$ with $f_j(u_j)=1$. Remark [2.4](#page-5-0) (b) and the universal property give a linear map $\check{\psi}$: $\bigotimes_{i \in I} X_i \to \mathbb{C}^{\otimes I}$ satisfying

$$
\check{\psi}(\otimes_{i\in I}x_i)=\psi(\otimes_{i\in F}x_i)\left(\otimes_{j\in I\setminus F}f_j(x_j)\right)\qquad(x\in\Pi_{i\in I}X_i).
$$

Thus, $\psi(a)(\otimes_{i\in I}1) = \check{\psi}(J_F^u(a)) = 0$, which implies that $a = 0$ (as ψ is arbitrary)
as required. On the other hand, if $i_0 \in I$, then $\Theta_X(u) = I^u$, $(u_0) \neq 0$ as required. On the other hand, if $i_0 \in I$, then $\Theta_X(u) = J_{\{i_0\}}^u(u_{i_0}) \neq 0$.

(b) This follows directly from part (a). \Box

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Part (b) of the above implies that $\Theta_X(C^{\omega})$ is a basis for $\bigotimes_{i\in I}^{\omega} X_i$, where C^{ω} is

lefined in the following result as defined in the following result.

Corollary 2.7. (a) Let $c : \Omega_{I;X} \to \Pi_{i \in I} X_i^{\times}$ be a cross section. For each $\omega \in \Omega_{I;X}$
and $i \in I$ we pick a basis R^{ω} of X; that contains $c(\omega)$; and set *and* $i \in I$, we pick a basis B_i^{ω} of X_i that contains $c(\omega)_i$ and set **Corollary 2.7.** (a) Let $c : \Omega_{I;X} \to \Pi_{i \in I} X_i^{\times}$ be a cond $i \in I$, we pick a basis B_i^{ω} of X_i that contains $C^{\omega} := \{x \in \Pi_{i \in I}^{\omega} X_i : x_i \in B_i^{\omega}$

If $C := \bigcup_{\omega \in \Omega_{I;X}} C^{\omega}$, then $\Theta_X(C)$ is a basis for \otimes

$$
C^{\omega} := \{ x \in \Pi_{i \in I}^{\omega} X_i : x_i \in B_i^{\omega}, \forall i \in I \}.
$$

 $\sum_{i\in I} X_i$. \overline{r}

(b) *If* $\Phi_i : X_i \to Y_i$ *is an injective linear map* $(i \in I)$ *, the induced linear map*
 $\bigotimes_{i=1}^{\infty} \Phi_i : \bigotimes_{i=1}^{\infty} X_i \to \bigotimes_{i=1}^{\infty} Y_i$ *is injective* $i \in I$ $\Phi_i : \bigotimes_{i \in I} X_i \to \bigotimes_{i \in I} Y_i$ *is injective.* C^{ω} , the
 $\Omega_{I;X}$ C^{ω} , the
 $\colon X_i \to Y_i$
 $i \in I$ $X_i \to \bigotimes$ *If* $C := \bigcup_{\omega \in \Omega_{I; X}} C^{\omega}$, then $\Theta_X(C)$ is a basis for $\bigotimes_{i \in I} X_i$.

(b) *If* $\Phi_i : X_i \to Y_i$ is an injective linear map $(i \in I)$, the induced linear map $\bigotimes_{i \in I} \Phi_i : \bigotimes_{i \in I} X_i \to \bigotimes_{i \in I} Y_i$ is injective.
 Propo

the universal property) *is injective.*

Proof. Suppose that $T^{(1)}, \ldots, T^{(n)} \in \Pi_{i \in I} L(X_i; Y_i)^\times$ are mutually inequivalent ele-**Proposition 2.8.** The map $\Psi : \bigotimes_{i \in I} L(X_i)$
the universal property) is injective.
Proof. Suppose that $T^{(1)}, \ldots, T^{(n)} \in \Pi_{i \in I} L(\text{ments (under } \sim), F \in \mathfrak{F}, R^{(1)}, \ldots, R^{(n)} \in \bigotimes$
 $(k = 1, \ldots, n)$ satisfying $i \in F$ $L(X_i; Y_i)$ with $S^{(k)} := J_F^{T^{(k)}}(R^{(k)})$ $(k = 1, \ldots, n)$ satisfying

$$
\Psi\bigl(\sum\nolimits_{k=1}^n S^{(k)}\bigr) = 0.
$$

Using an induction argument, it suffices to show that $S^{(1)} = 0$.

If $x = 1$, we take any $x \in \Pi$, Y^{\times} with $T^{(1)}x \neq 0$ ($i \in I$).

If $n = 1$, we take any $x \in \prod_{i \in I} X_i^{\times}$ with $T_i^{(1)} x_i \neq 0$ $(i \in I)$. If $n > 1$, we claim that there is $x \in \Pi_{i \in I} X_i^{\times}$ such that

$$
[T_i^{(1)}x_i]_{i\in I} \in \Pi_{i\in I}Y_i^{\times}
$$
 and $[T_i^{(k)}x_i]_{i\in I} \sim [T_i^{(1)}x_i]_{i\in I}$ $(k = 2,...,n)$.

In fact, let $I^k := \{i \in I : T_i^{(k)} \neq T_i^{(1)}\}$, which is an infinite set for any $k = 2, ..., n$.
For any $i \in I$ we put $N_i := \{k \in \{2, ..., n\} : i \in I^k\}$ and take any $x_i \in I$ For any $i \in I$, we put $N_i := \{k \in \{2,\ldots,n\} : i \in I^k\}$ and take any $x_i \in Y \setminus (1 + \log(T^{(k)} - T^{(1)}) \cup \log T^{(1)})$ (note that Y cannot be a finite union $X_i \setminus$ $[T_i^{(1)}x_i]_{i \in I}$ ∈ $\Pi_{i \in I}Y_i^{\times}$ and $[T_i^{(k)}x_i]_{i \in I} \sim [T_i^{(1)}x_i]_{i \in I}$ ($k = 2, ..., n$).
 x^* , let $I^k := \{i \in I : T_i^{(k)} \neq T_i^{(1)}\}$, which is an infinite set for any $k = 2, ..., n$,
 x^* i ∈ I , we put $N_i := \{k \in \{2, ..., n\} : i$ of proper subspaces). Thus, $T_i^{(1)}x_i \neq 0$ (for each $i \in I$) and $T_i^{(k)}x_i \neq T_i^{(1)}x_i$ (for $k \in I$) and $i \in I^k$) $k \in \{2, ..., n\}$ and $i \in I^k$).

Now, we have

$$
\Psi(S^{(1)})\big(\bigotimes\nolimits_{i \in I}^x X_i\big) \cap \Big(\sum\nolimits_{k=2}^n \Psi(S^{(k)})\big(\bigotimes\nolimits_{i \in I}^x X_i\big)\Big) = (0)
$$

by Theorem [2.5](#page-5-1) and the fact that $\Psi(S^{(l)})(\bigotimes_{i\in I}^x X_i)$ by Theorem 2.5 and the fact that $\mathcal{L}(S^{(1)}) \bigotimes_{i \in I} \mathcal{L}_i, \forall i \in I, \forall i, j \in \mathbb{Z}$, where $g_i = I_i \cup I_i$, $(i \in I, l = 1, ..., n)$. Consequently, $\Psi(S^{(1)}) \big|_{\bigotimes_{i \in I}^x X_i} = 0$. As $T_i^{(1)} x_i \neq 0 \ (i \in I)$, it $\left(\bigotimes_{i \in I}^{x} X_i \right) = (0)$
 $\in \bigotimes_{i \in I}^{y^{(l)}} Y_i$, where $y_i^{(l)} = T_i^{(l)}$ *i* $(i \in I; l = 1, ..., n)$. Consequently, $\Psi(S^{(1)})|_{\bigotimes_{i \in I}^{x} X_i} = 0$. As $T_i^{(1)} x_i \neq 0$ $(i \in I)$, it is easy to see that $R^{(1)} = 0$ as required.

Note that Ψ is not surjective even if $X_i = Y_i = \mathbb{C}$ $(i \in I)$ since in this c is easy to see that $R^{(1)} = 0$ as required.

Note that Ψ is not surjective even if $X_i = Y_i = \mathbb{C}$ $(i \in I)$ since in this case, Ψ $i\in I^{\mathbb{C}}$ is commutative while $L(\bigotimes_{i\in I} \mathbb{C})$ is not.

The following result follows from Propositions [2.3](#page-4-0) (c) and [2.8](#page-7-1) as well as Corollary [2.7](#page-7-2) (b). It says that an infinite tensor product of vector spaces is automatically a faithful module over a big commutative algebra.

CON GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS
 Corollary 2.9. *If* X_i *is a faithful* A_i -module ($i \in I$), then $\bigotimes_{i \in I} X_i$ *is a faithful*
 $\bigotimes_{i \in I} A_i$ -module *In narticular* $\bigotimes_{i \in I} X_i$ *is a faithful uni* $\bigotimes_{i\in I} A_i$ -module. In particular, $\bigotimes_{i\in I} Y_i$ is a faithful unital $\mathbb{C}^{\otimes I}$ -module. i GENUINE INFINITE ALGEBRAIC T
prollary 2.9. If X_i is a faithfu
 $i \in I$ ^Ai-module. In particular, \otimes Corollary 2.9. If X_i is a faithful A_i -module $(i \in I)$, then $\bigotimes_{i \in I} X_i$ is a faithful $\bigotimes_{i \in I} A_i$ -module. In particular, $\bigotimes_{i \in I} Y_i$ is a faithful unital $\mathbb{C}^{\otimes I}$ -module.
 Example 2.10. (a) If $\beta \in \Pi_{$

 $F \in \mathfrak{F}$ and $\mu_i \in \mathbb{C}$ $(i \in F)$, we have $J_F^{\beta}(\otimes_{i \in F} \mu_i) = (\prod_{i \in F} \mu_i/\beta_i) (\otimes_{i \in I} \beta_i)$. $\epsilon_I A_i$ -module. In particular, $\overline{\otimes}_{i \in I} Y_i$ is a faithful unital $\mathbb{C}^{\otimes I}$ -module.
 ample 2.10. (a) If $\beta \in \Pi_{i \in I} \mathbb{C}^{\times}$, then $\overline{\otimes}_{i \in I}^{\beta} \mathbb{C} = \mathbb{C} \cdot \otimes_{i \in I} \beta_i$. In fact, for $\overline{\mathfrak{F}}$ an

 $k=1$ $\frac{1}{\epsilon}$ **Example 2.10.** (a) If $\beta \in \Pi_{i \in I} \mathbb{C}^{\times}$, then $\bigotimes_{i \in I}^{\beta} \mathbb{C} = \mathbb{C} \cdot \bigotimes_{i \in I} \beta_i$. In fact, for any $F \in \mathfrak{F}$ and $\mu_i \in \mathbb{C}$ $(i \in F)$, we have $J_F^{\beta}(\bigotimes_{i \in F} \mu_i) = (\Pi_{i \in F} \mu_i/\beta_i) (\bigotimes_{i \in I} \beta_i)$.

(**EXAMPLE 2.10.** (a) If $\rho \in \Pi_{i \in I} \cup \dots$, then $\bigotimes_{i \in I} \cup \dots \otimes_{i \in I} \rho_i$. In fact,
 $F \in \mathfrak{F}$ and $\mu_i \in \mathbb{C}$ $(i \in F)$, we have $J^{\beta}_{F}(\otimes_{i \in F} \mu_i) = (\Pi_{i \in F} \mu_i/\beta_i) (\otimes_{i \in I} \beta_i)$.

(b) For $n \in \mathbb{N}$, let $I_1, \$ (b) For $n \in \mathbb{N}$, let I_1, \ldots, I_n be infinite disjoint subsets of I with $I = \bigcup_{k=1}^n I_k$ ι) ϵ

(c) Let G be a subgroup of $\mathbb{T}^n \subseteq (\mathbb{C}^\times)^n$ (where $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$). If $\overline{\beta^{(1)}}, \ldots, \overline{\beta^{(m)}}$ are distinct elements in G and $\overline{\beta^{(1)}}, \ldots, \overline{\beta^{(m)}} \in \Pi_{i \in I} \mathbb{C}^{\times}$ are as in $\frac{1}{2}$, $\frac{1}{2}$ part (b), then $\otimes_{i \in I} \widetilde{\beta_i^{(1)}}, \ldots, \otimes_{i \in I} \widetilde{\beta_i^{(m)}}$ are linearly independent in $\mathbb{C}^{\otimes I}$. Therefore, (c) Let G be a subgroup of $\mathbb{T}^n \subseteq (\mathbb{C}^\times)^n$ (where $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$). If $\overline{\beta^{(1)}, \ldots, \beta^{(m)}}$ are distinct elements in G and $\overline{\beta^{(1)}, \ldots, \beta^{(m)}} \in \Pi_{i \in I} \mathbb{C}^\times$ are as in part (b), then $\otimes_{i \in I} \overline$ group algebra $\mathbb{C}[G]$.

As $\otimes_{i\in I}\alpha_i = (\prod_{i\in I}\alpha_i)(\otimes_{i\in I}1)$ if $\alpha_i = 1$ e.f., one may regard $\otimes_{i\in I}\alpha_i$ as a generalization of the product. In this case, one can consider infinite products like $(-1)^{I}$.

3. Tensor products of unital *[∗]*-algebras

Throughout this section, A_i *is a unital* ^{*}-algebra with *identity* e_i ($i \in I$)*,* and we $\int \mathcal{L}^{\text{ut}}_{I;A} := \prod_{i \in I} U_{A_i} / \sim.$

Notice that in this case, $\Omega_{I,A}$ is a ^{*}-semigroup with identity and $\Omega_{I,A}^{\text{ut}}$ can
regarded as a subgroup of $\Omega_{I,A}$ with the inverse being the involution on $\Omega_{I,A}$. be regarded as a subgroup of $\Omega_{I;A}$ with the inverse being the involution on $\Omega_{I;A}$. Throughout this section, A_i is a unital *-algebra with identity e_i $(i \in I)$, and we
set $\Omega_{I;A}^{ut} := \Pi_{i \in I} U_{A_i} / \sim$.
Notice that in this case, $\Omega_{I;A}$ is a *-semigroup with identity and $\Omega_{I;A}^{ut}$ can
be regarded Notice tha
be regarded as
Moreover, $\bigotimes_{i \in \mathbb{N}}$
(3.1) $(\bigotimes_{i=1}^{\infty}$ this case, $\Omega_{I;A}$ is a *-semigroup with identity and Ω
ubgroup of $\Omega_{I;A}$ with the inverse being the involution c
is a $\Omega_{I;A}$ -graded *-algebra in the sense that for any ω, ω'
 $\left(\bigotimes_{i\in I}^{\omega'} A_i\right) \subseteq \bigotimes_{i\in$

Moreover,
$$
\bigotimes_{i \in I} A_i
$$
 is a $\Omega_{I;A}$ -graded *-algebra in the sense that for any $\omega, \omega' \in \Omega$
\n(3.1) $(\bigotimes_{i \in I}^{\omega} A_i) \cdot (\bigotimes_{i \in I}^{\omega'} A_i) \subseteq \bigotimes_{i \in I}^{\omega \omega'} A_i$ and $(\bigotimes_{i \in I}^{\omega} A_i)^* \subseteq \bigotimes_{i \in I}^{\omega^*} A_i$.
\nBy Proposition 2.6 (b), $\bigotimes_{i \in I}^e A_i$ can be identified with the unital *-alge

 $i\epsilon I A_i$ can be identified with the unital [∗]-algebra
products of 4. We will study the following ^{*}-subinductive limit of finite tensor products of A_i . We will study the following $*$ -sub-(3.1) $(\bigotimes_{i \in I}^{\omega} A_i) \cdot (\bigotimes_{i \in I} A_i)$
By Proposition 2.6 (b)
inductive limit of finite to
algebra that contains \bigotimes_i^{ω} algebra that contains $\bigotimes_{i\in I}^e A_i$: b), $\bigotimes_{i \in I}^{e} A_i$ can be identifiently
tensor products of A_i . We
 $\bigotimes_{i \in I}^{e} A_i$:
 $\bigotimes_{i \in I}^{u^{\text{t}}} A_i := \bigoplus_{\omega \in \Omega_{I;A}^{u^{\text{t}}}} \bigotimes$

$$
\bigotimes\nolimits_{i \in I}^{{\rm ut}} A_i \ := \ \bigoplus\nolimits_{\omega \in \Omega^{{\rm ut}}_{I; A}} \bigotimes\nolimits_{i \in I}^{\omega} A_i.
$$

The motivation for considering this subalgebra is partially Example [3.1](#page-8-0) (a) below, and partially because it has good representations (see the discussion after Proposition [5.1](#page-20-0) below). Moreover, if all the A_i are linear spans of U_{A_i} (in particu-The motivation for considering this subalgebra is partially Example 3.1 (a) be-
low, and partially because it has good representations (see the discussion after
Proposition 5.1 below). Moreover, if all the A_i are linear $\Theta_A(\Pi_{i\in I}U_{A_i})$. If $A_i = A$ for all $i \in I$, we write $A_{\mathrm{ut}}^{\otimes I} := \bigotimes_{i \in I}^{\mathrm{ut}} A_i$.

Example 3.1. (a) Let G_i be a group and $\mathbb{C}[G_i]$ be its group algebra $(i \in I)$. If $\Lambda : \Pi_{i \in I} G_i \to \Pi_{i \in I} U_{\mathbb{C}[G_i]}$ is the canonical map, then $\lambda := \Theta_{\mathbb{C}[G]} \circ \Lambda$ gives a *-isomorphism from $\mathbb{C}[\Pi_{i \in I} G_i]$ to the *-subalgebra Let G_i be a group and $\mathbb{C}[G_i]$ be its group
 $\mathbb{E}I^U\mathbb{C}[G_i]$ is the canonical map, then $\lambda := \mathbb{C}[\Pi_{i \in I} G_i]$ to the *-subalgebra
 $\mathbb{C}[G_i] := \sum_{t \in \Pi_{i \in I} G_i} \bigotimes_{i \in I}^{\Lambda(t)} \mathbb{C}[G_i] \subseteq \bigotimes_{i \in I}^{\mathfrak{u}^+}$

\nIn fact,
$$
\lambda
$$
 induces a *-homomorphism from $\mathbb{C}[\Pi_{i\in I}G_i]$ to the *-subalgebra\n

\n\n $\bigotimes_{i\in I}^{\Lambda(\Pi_{i\in I}G_i)}\mathbb{C}[G_i] := \sum_{t\in \Pi_{i\in I}G_i} \bigotimes_{i\in I}^{\Lambda(t)}\mathbb{C}[G_i] \subseteq \bigotimes_{i\in I}^{\text{ut}}\mathbb{C}[G_i].$ \n

\n\nIn fact, λ induces a *-homomorphism from $\mathbb{C}[\Pi_{i\in I}G_i]$ to $\bigotimes_{i\in I}^{\text{ut}}\mathbb{C}[G_i].$ Let q :\n

\n\n $\Pi_{i\in I}G_i \to \Pi_{i\in I}G_i/\oplus_{i\in I}G_i$ be the quotient map. For a fixed $s \in \Pi_{i\in I}G_i$, if we set\n

 $\Pi_{i\in I}G_i \to \Pi_{i\in I}G_i/\oplus_{i\in I}G_i$ be the quotient map. For a fixed $s \in \Pi_{i\in I}G_i$, if we set \sum_{t}
momorph
be the qu
 $G_i := \{$ In fact, λ induces a *-h
 $\Pi_{i \in I} G_i \to \Pi_{i \in I} G_i / \bigoplus_{i \in I} G_i$

then $s^{-1} (\bigoplus_{i \in I}^s G_i) = \bigoplus$

pendent elements in \otimes λ induces comomorphism from $\mathbb{C}[\Pi_{i\in \mathcal{I}}]$
 G_i be the quotient map. For
 $\prod_{i\in I} G_i$: $\prod_{i\in I} G_i$: $q(t)$
 $\prod_{i\in I} G_i$. Thus, $\{\lambda(t) : t \in \bigoplus_{i\in I} \mathbb{C}[G_i]$ (as $\lambda | \text{C}(\mathbb{C})|$, α , is a h

$$
\bigoplus_{i \in I}^{s} G_i := \{ t \in \Pi_{i \in I} G_i : q(t) = q(s) \},
$$

 $\Pi_{i\in I}G_i \to \Pi_{i\in I}G_i/\oplus_{i\in I}G_i$ be the quotient map. For a fixed $s \in \Pi_{i\in I}G_i$, if we set
 $\bigoplus_{i\in I}^s G_i := \{t \in \Pi_{i\in I}G_i : q(t) = q(s)\},$

then $s^{-1}(\bigoplus_{i\in I}^s G_i) = \bigoplus_{i\in I} G_i$. Thus, $\{\lambda(t) : t \in \bigoplus_{i\in I}^s G_i\}$ is a s the other hand, if $s^{(1)}, \ldots, s^{(n)} \in \Pi_{i \in I} G_i$ are such that $q(s^{(k)}) \neq q(s^{(l)})$ whenever
 $k \neq l$ then $\lambda(s^{(1)}) - \lambda(s^{(n)})$ are linearly independent in $\bigotimes_{\mathbb{C}} \mathbb{C}[G_i]$ (see Theo $k \neq l$, then $\lambda(s^{(1)}), \ldots, \lambda(s^{(n)})$ are linearly independent in $\bigotimes_{i \in I} \mathbb{C}[G_i]$ (see Theo-
 $\bigcirc_{i \in I} S_i$) $\bigcirc_{i \in I} S_i$ $\{i \in \Pi_i \in I \mid G_i : q(t) = q(s)\},\$
 $\{i_i$. Thus, $\{\lambda(t) : t \in \bigoplus_{i \in I}^s G_i\}$ is a $\{i_i\}$ (as $\lambda |_{\mathbb{C}[\bigoplus_{i \in I} G_i]}$ is a bijection or
 $\{i\} \in \Pi_{i \in I} G_i$ are such that $q(s^{(k)})$ are linearly independent in \bigotimes then $s^{-1}(\bigoplus_{i\in I}^s G_i) = \bigoplus_{i\in I} G_i$. Thus, $\{\lambda(t) : t \in \bigoplus_{i\in I}^s G_i\}$ is a set of line
pendent elements in $\bigotimes_{i\in I} \mathbb{C}[G_i]$ (as $\lambda|_{\mathbb{C}[\bigoplus_{i\in I} G_i]}$ is a bijection onto $\bigotimes_{i\in I}^e \mathbb{C}$
the other hand, $\bigcap_{i\in I}^{\Lambda(\prod_{i\in I}G_i)}\mathbb{C}[G_i].$ $i^{(k)} \neq q(s^{(l)})$ w
 $\bigotimes_{i \in I} \mathbb{C}[G_i]$ (se

ir $\bigotimes_{i \in I}^{\Lambda(\Pi_{i \in I} G_i)} \mathbb{C}$
 (α, u) , in the s
 $i \in I} G_i$ $\cong \bigotimes_i^e$

includes the set $\ddot{}$

(b) It is well known that there is a twisted action (α, u) , in the sense of Busby and Smith, of $\Omega_{I;G} := \Pi_{i \in I} G_i / \bigoplus_{i \in I} G_i$ on $\mathbb{C}[\bigoplus_{i \in I} G_i] \cong \bigotimes_{i \in I}^e \mathbb{C}[G_i]$
(see 2.1 in [5]) such that $\mathbb{C}[\Pi_{\{x \in I\}} G_i]$ is *-isomorphic to the algebraic crossed product (see 2.1 in [\[5\]](#page-26-10)) such that $\mathbb{C}[\Pi_{i\in I}G_i]$ is *-isomorphic to the algebraic crossed product \mathbb{R}^e $\mathbb{C}[G_i]$ \rtimes $\bigotimes_{i\in I}^{e}\mathbb{C}[G_i]\times_{\alpha,u}\Omega_{I;G}.$ (b) It is well known that there is a twisted action
by and Smith, of $\Omega_{I;G} := \Pi_{i \in I} G_i / \oplus_{i \in I} G_i$ on $\mathbb{C}[t]$
e.2.1 in [5]) such that $\mathbb{C}[\Pi_{i \in I} G_i]$ is *-isomorphic to the $\in_I \mathbb{C}[G_i] \rtimes_{\alpha,u} \Omega_{I;G}$.
There e

 $\prod_{i\in I}^{\mathfrak{u}} A_i$ given by inner automorphisms, i.e. of $\Pi_{i \in I}$
 $\binom{*}{i}$ (There exists a canonical action Ξ of $\Pi_{i\in I}U_{A_i}$ on $\bigotimes_{i\in I}^{\omega_i}A_i$ given by inner automorphisms, i.e.
 $\Xi_u(a) := (\otimes_{i\in I} u_i) \cdot a \cdot (\otimes_{i\in I} u_i^*) \qquad (u \in \Pi_{i\in I}U_{A_i}; a \in \bigotimes_{i\in I}^{\omega_i}A_i).$

This induces an action $\$

$$
\Xi_u(a) := (\otimes_{i \in I} u_i) \cdot a \cdot (\otimes_{i \in I} u_i^*) \qquad (u \in \Pi_{i \in I} U_{A_i}; a \in \bigotimes_{i \in I}^{\text{ut}} A_i)
$$

morphisms, i.e.
 $\Xi_u(a) := (\otimes_{i \in I} u_i) \cdot a \cdot (\otimes_{i \in I} u_i^*)$ $(u \in \Pi_{i \in I} U_{A_i}; a \in \bigotimes_{i \in I}^{ut} A_i)$.

This induces an action Ξ^e of $\Pi_{i \in I} U_{A_i}$ on the subalgebra $\bigotimes_{i \in I}^e A_i$. The following

result gives an identific page 166 of [\[16\]](#page-27-6)) of a cocycle twisted action (i.e., a twisted action in the sense of Busby and Smith) of $\Omega_{I;A}^{\text{ut}}$ on $\bigotimes_{i\in I}^e A_i$ induced by Ξ^e . $u_i) \cdot a \cdot (\otimes$
 Ξ^e of $\Pi_i \in \mathcal{A}$

ation of $\bigotimes_{\substack{\text{ocycle} \\ \text{out } \\ I; A}}$ on \bigotimes_i^e

Before we give this result, let us recall that an abelian group G is *divisible* if for any $g \in G$ and $n \in \mathbb{N}$, there is $h \in G$ with $g = h^n$.

page 166 of [16]) of a cocycle twisted action (i.e., a twisted action in the sense of Busby and Smith) of $\Omega_{I;A}^{\text{ut}}$ on $\bigotimes_{i\in I}^e A_i$ induced by Ξ^e .
Before we give this result, let us recall that an abelian gr $\text{such that } \bigotimes_{i\in I}^{\text{ut}} A_i \text{ is } \Omega_{I}^{\text{ut}}A \text{-}graded \text{ *-isomorphic to } (\bigotimes_{i\in I}^e A_i) \rtimes_{\Xi,m} \Omega_{I,A}^{\text{ut}}.$ before we give this result, let us recall that all about
any $g \in G$ and $n \in \mathbb{N}$, there is $h \in G$ with $g = h^n$.
eorem 3.2. (a) There is a cocycle twisted action
h that $\bigotimes_{i \in I}^{\text{ut}} A_i$ is $\Omega_{I;A}^{\text{ut}}$ -graded *-

(b) Suppose that all the A_i are commutative. If $\bigotimes_{i\in I}^e A_i$ is a unital ^{*}-subalgebra **Theorem 3.2.** (a) There is a cocycle twisted action $(\tilde{\Xi}, m)$

such that $\bigotimes_{i \in I}^{\text{ut}} A_i$ is $\Omega_{I,A}^{\text{ut}}$ -graded *-isomorphic to $(\bigotimes_{i \in I}^e A_i) \rtimes_{\Omega_{I,A}}^e$

(b) Suppose that all the A_i are commutative. If \big of a commutative *-algebra *B* with U_B being divisible, $\bigotimes_{i \in I}^{\text{ut}} A_i$ is $\Omega_{I;A}^{\text{ut}}$ -graded *of* a commutative *-algebra B with U_B being divisible, $\bigotimes_{i \in I}^{\text{u}} A_i$ is $\Omega_{I;A}^{\text{u}}$ -graded
*-isomorphic to a unital *-subalgebra of $B \otimes \mathbb{C}[\Omega_{I;A}^{\text{u}}]$. If $U_{\bigotimes_{i \in I}^e A_i}$ is itself divisible, $U_{i∈I}^{\text{ut}} A_i \cong (\bigotimes_{i∈I}^e A_i) \otimes \mathbb{C}[\Omega_{I;A}^{\text{ut}}]$ *as* $\Omega_{I;A}^{\text{ut}}$ -graded *-algebras.

Proof. Let $c : \Omega_{I;A}^{\text{ut}} \to \Pi_{i \in I} U_{A_i}$ be a cross section with $c([e]_{\sim}) = e$.

(a) For any $\mu, \nu \in \Omega_{I;A}^{\text{ut}},$ we set

$$
\tilde{\Xi}_{\mu} := \Xi_{c(\mu)}^e
$$
 and $m(\mu, \nu) := \otimes_{i \in I} c(\mu)_i c(\nu)_i c(\mu \nu)_i^{-1}$.

ON GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS

As $c(\mu)c(\nu) \sim c(\mu\nu)$, we have $m(\mu, \nu) \in \bigotimes_{i \in I}^e A_i$. It is easy to check that $(\check{\Xi}, m)$

is a twisted action in the sense of Busby and Smith. Furthermore, we define
 $\Psi :$ is a twisted action in the sense of Busby and Smith. Furthermore, we define ON GENI
As $c(\mu)$
is a twi
 $\Psi:({\bigotimes}_i^e)$ FINITE ALGEBRAIC TENSO:
 $c(\nu) \sim c(\mu\nu)$, we have $m(\mu, \nu)$

isted action in the sense of Bi
 $\sum_{i \in I}^e A_i$ $\forall \Sigma_{i,m}$ $\Omega_{I;A}^{\text{ut}} \rightarrow \bigotimes_{i \in I}^{\text{ut}} A_i$ by $\begin{align} &\text{Nc}(\nu) \sim c(\mu\nu),\\ &\text{isted action }i\in {}_{i\in I}A_{i})\rtimes_{\tilde{\Xi},m}\Omega,\\ &\Psi(f) \;:=\; \sum\limits_{i\in I} \end{align}$

$$
\Psi(f) := \sum_{\omega \in \Omega_{I;A}^{\text{ut}}} f(\omega)(\otimes_{i \in I} c(\omega)_i) \quad (f \in (\bigotimes_{i \in I}^e A_i) \rtimes_{\tilde{\Xi},m} \Omega_{I;A}^{\text{ut}}).
$$

It is not hard to verify that Ψ is a bijective $\Omega_{I;A}^{\text{ut}}$ -graded *-homomorphism.

(b) Let $\Pi_{i\in I}^e U_{A_i} := \Pi_{i\in I}^e I_{A_i} \cap \Pi_{i\in I} U_{A_i}$. By Baer's theorem, $\Theta_A|_{\Pi_{i\in I}^e U_{A_i}}$ can be ended to a group homomorphism $\varphi : \Pi_{i\in I} U_{A_i} \to U_P$. Since extended to a group homomorphism $\varphi : \Pi_{i \in I} U_{A_i} \to U_B$. Since (b) Let $\Pi_{i\in I}^e U_{A_i} := \Pi_{i\in I}^e A_i \cap \Pi_{i\in I} U_{A_i}$. By
extended to a group homomorphism $\varphi : \Pi_{i\in I}$
 $\varphi(c(\mu))\varphi(c(\nu))\varphi(c(\mu\nu))^{-1} = \otimes_{i\in I} c(\mu)$
the map $\Phi : \bigotimes_{i\in I}^{\text{ut}} A_i \to B \otimes \mathbb{C}[\Omega_{I;A}^{\text{ut}}]$ given by

$$
\varphi(c(\mu))\varphi(c(\nu))\varphi(c(\mu\nu))^{-1} = \otimes_{i \in I} c(\mu)_i c(\nu)_i c(\mu\nu)_i^{-1} \quad (\mu, \nu \in \Omega_{I;A}^{\text{ut}}),
$$

extended to a group homomorphism
$$
\varphi : \Pi_i \in I \cup A_i \to \cup B
$$
. Since
\n
$$
\varphi(c(\mu))\varphi(c(\nu))\varphi(c(\mu\nu))^{-1} = \otimes_{i \in I} c(\mu)_i c(\nu)_i c(\mu\nu)_i^{-1} \quad (\mu, \nu \in \Omega_{I;A}^{\text{ut}}),
$$
\nthe map $\Phi : \bigotimes_{i \in I}^{\text{ut}} A_i \to B \otimes \mathbb{C}[\Omega_{I;A}^{\text{ut}}]$ given by
\n(3.2) $\Phi(a) := (a \cdot \otimes_{i \in I} c(\omega)_i^{-1})\varphi(c(\omega)) \otimes \lambda(\omega) \quad (a \in \bigotimes_{i \in I}^{\omega} A_i; \omega \in \Omega_{I;A}^{\text{ut}})$
\nis a $\Omega_{I;A}^{\text{ut}}$ -graded *-homomorphism. If $\sum_{\omega \in \Omega_{I;A}^{\text{ut}}} a^{\omega} \in \ker \Phi$ (with $a^{\omega} \in \bigotimes_{i \in I}^{\omega} A_i$),

then for every $\omega \in \Omega_{I,\Lambda}^{\text{u}t}$, one has $(a^{\omega} \cdot \otimes_{i \in I} c(\omega)_i^{-1}) \varphi(c(\omega)) = 0$, which implies $a^{\omega} = 0$ and hence Φ is injective. The image of Φ is the linear span of $a^{\omega} = 0$, and hence Φ is injective. The image of Φ is the linear span of

$$
\{b\varphi(c(\omega))\otimes\lambda(\omega):b\in\bigotimes_{i\in I}^eA_i;\omega\in\Omega_{I;A}^{\mathrm{ut}}\},\
$$

and it is clear that Φ is surjective if $B = \bigotimes_{i \in I}^{e} A_i$. $\bigcup_{i\in I}A_i.$

Remark 3.3. (a) The cocycle twisted action (Ξ, m) depends on the choice of a cross section, and different cross sections may give different twisted actions (although their crossed products are all isomorphic). On the other hand, the map Φ in part (b) also depends on the choice of a cross section as well as the choice of an extension of Θ extension of $\Theta_A|_{\Pi_{i\in I}^eU_{A_i}}$.

(b) If S_i is a set and A_i is a ^{*}-subalgebra of $\ell^{\infty}(S_i)$ $(i \in I)$, then by Theothis fact use 18.4 in $[6]$ and 7.1 in $[7]$. term of $\Theta_A|_{\Pi_{i\in I}^e U_{A_i}}$.

(b) If S_i is a set and A_i is a *-subalgebra of $\ell^{\infty}(S_i)$ ($i \in I$), then by Theo-
 $\ell^{\infty}(S_i)$, $\mathcal{O}_i^{\text{ult}} A_i$ is a *-subalgebra of $\ell^{\infty}(\Pi_{i\in I} S_i) \otimes \mathbb{C}[\Omega_{I,A}^{\text{ut}}]$.

cross section, and dimerent cross sections may give dimerent twisted actions (ai-
though their crossed products are all isomorphic). On the other hand, the map Φ
in part (b) also depends on the choice of a cross sect graded *-algebras if and only if there is a group homomorphism $\pi : \Omega_{I;A}^{ut} \to U_{\bigotimes_{i \in I}^{ut}}$ ω
 ω _{*i*∈*I*}A_{*i*}
($\omega \in \Omega_{I,A}^{u}$). In fact, if such a π exists, one may replace

(*i*)) in (2,2) with $\sigma(\omega^{-1})$ and show that the concernenting Φ $(a \cdot \otimes_{i \in I} c(\omega)_i^{-1}) \varphi(c(\omega))$ in [\(3.2\)](#page-10-1) with $a\pi(\omega^{-1})$ and show that the corresponding Φ
is a *-isomorphism is a [∗]-isomorphism.

Clearly, the second statement of Theorem 3.2 (b) applies to the case when $A_i = \mathbb{C}^{n_i}$ for some $n_i \in \mathbb{N}$ ($i \in I$). In particular, Theorem [3.2](#page-9-0) (b) and its argument give the following corollary.

Corollary 3.4. *If* φ_1 *is as in Example* [2.2](#page-3-2)(a) *and* $\varphi : \Pi_{i \in I} \mathbb{T} \to \mathbb{T}$ *is a group homomorphism that extends* $\varphi_1|_{\Pi_{i\in I}^1\Gamma}$ (*its existence is guaranteed by Baer's theorem*)*,*
than $\Phi(\Omega_1,\Omega_2)$ (*i.e.*) $\chi(\Omega_1)$ (*i.e.*) *i.e.* π) *is a suall defined* * *isomorphism* from *then* $\Phi(\otimes_{i\in I}\alpha_i) := \varphi(\alpha)\lambda([\alpha]_{\sim})$ $(\alpha \in \Pi_{i\in I}\mathbb{T})$ *is a well-defined* *-*isomorphism from* $\mathbb{C}_{\rm ut}^{\otimes I}$ onto $\mathbb{C}[\Omega_{I;\mathbb{C}}^{\rm ut}].$

Conversely, it is clear that if $\varphi : \Pi_{i \in I} \mathbb{T} \to \mathbb{T}$ is any map such that Φ as defined in the above is a well-defined $*$ -isomorphism, then φ is a group homomorphism extending $\varphi_1|_{\Pi_{i\in I}^1\mathbb{T}}$. On the other hand, there is a simpler proof for Corollary [3.4.](#page-10-0) In fact, for $\alpha, \beta \in \Pi_{i \in I} \mathbb{T}$ with $\alpha \sim \beta$, one has $\varphi(\alpha)^{-1} \cdot \otimes_{i \in I} \alpha_i = \varphi(\beta)^{-1} \cdot \otimes_{i \in I} \beta_i$. Thus, $[\alpha] \sim \rightarrow \varphi(\alpha)^{-1} \cdot \otimes_{i \in I} \alpha_i$ is a well-defined group homomorphism from $\Omega_{I;C}^{\text{ut}}$

to $U_{\mathbb{C}_{\text{ut}}^{\otimes I}}$ such that $\{\varphi(\alpha)^{-1} \cdot \otimes_{i \in I} \alpha_i : [\alpha]_{\sim} \in \Omega_{I,\mathbb{C}}^{\text{ut}}\}$ is a basis for $\mathbb{C}_{\text{ut}}^{\otimes I}$.
 Example 3.5. For any subgroup $G \subseteq \mathbb{T}^n$, the algebra defined as ple 2.10(c) is a *-subalg Example 3.5. For any subgroup $G \subseteq \mathbb{T}^n$, the algebra defined as in Exam-ple [2.10](#page-8-1)(c) is a ^{*}-subalgebras of $\mathbb{C}_{ut}^{\otimes I}$.

In the remainder of this section, we will show that the center of $\bigotimes_{i\in I}^{ut} A_i$ is the tensor product of centers of the A_i when $A_i = \text{span } U_{A_i}$ for all $i \in I$.

If A is an algebra and G is a group, we denote by $Z(A)$ and $Z(G)$ the center of A and the center of G respectively. Clearly, the inclusion $\Pi_{i\in I}U_{Z(A_i)}\subseteq \Pi_{i\in I}U_{A_i}$ induces an injective group homomorphism from $\Omega_{I;Z(A)}^{\text{ut}}$ to $\Omega_{I;A}^{\text{ut}}$ and we regard the former as a subgroup of the latter former as a subgroup of the latter.

Theorem 3.6. *Suppose that there is* $F_0 \in \mathfrak{F}$ *with* $A_i = \text{span } U_{A_i}$ *for all* $i \in I_0 := I \setminus F_0$. $I \setminus F_0$.

(a) $Z(\Omega_{I;A}^{\text{ut}}) = \Omega_{I;Z(A)}^{\text{ut}}$. Moreover, $Z(\Omega_{I;A}^{\text{ut}}) = \Omega_{I;A}^{\text{ut}}$ *if and only if all but a finite*
nher of the A_i are commutative *number of the* A_i *are commutative.*
 $\begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} E & A_i \end{bmatrix}$ $\begin{aligned} \mathcal{L}_{I;Z(A)}^{\text{ut}} &\cdot \mathcal{L}_{I;Z(A)}^{\text{ut}} \cdot \mathcal{L}_{I;Z(A)}^{\text{ut}} \ &\cdot \mathcal{L}_{I;Z(A)}^{\text{ut}} \cdot \mathcal{L}_{I;Z(A)}^{\text{ut}} \end{aligned}$

(b) *Every element in* $\Omega_{I;A}^{\text{ut}} \setminus Z(\Omega_{I;A}^{\text{ut}})$ *has an infinite conjugacy class.*

(c)
$$
Z(\bigotimes_{i \in I}^{ut} A_i) = \bigotimes_{i \in I}^{ut} Z(A_i).
$$

Proof. (a) It is obvious that $\Omega_{I;\mathcal{Z}(A)}^{\text{ut}} \subseteq Z(\Omega_{I;A}^{\text{ut}})$. Suppose $u \in \Pi_{i\in I}U_{A_i}$ with $[u] \sim \notin \Omega^{ut}_{L^2(A)}$. There is an infinite subset $J \subseteq I_0$ such that $u_i \notin Z(A_i)$ $(i \in J)$.
For each $i \in I$ and see find $u \in U$, such that $u_i u \neq u_i u$. For any $i \in I$ I we put For each $i \in J$, one can find $v_i \in U_{A_i}$ such that $u_i v_i \neq v_i u_i$. For any $i \in I \setminus J$, we put $v_i = e_i$. Then $[v]_{\sim} \in \Omega_{I;A}^{\text{ut}}$ and $[u]_{\sim}[v]_{\sim} \neq [v]_{\sim}[u]_{\sim}$. Consequently, $[u]_{\sim} \notin Z(\Omega_{I;A}^{\text{ut}})$.
This argument also shows that if the set $I_i \in I : Z(A) \neq A$. is infinite then This argument also shows that if the set $\{i \in I : Z(A_i) \neq A_i\}$ is infinite, then $Z(\Omega_{I;A}^{\text{ut}}) \neq \Omega_{I;A}^{\text{ut}}$. Conversely, it is clear that $\Omega_{I;Z(A)}^{\text{ut}} = \Omega_{I;A}^{\text{ut}}$ if all but a finite numbers of the 4: are commutative numbers of the A_i are commutative.

(b) Suppose that $[u] \sim \text{C} \Omega_{I;A}^{\text{ut}} \setminus Z(\Omega_{I;A}^{\text{ut}})$ and $\{i_n\}_{n \in \mathbb{N}}$ is a sequence of distinct nearly I_{α} , I_{α} and I_{α} and $\{i_n\}_{n \in \mathbb{N}}$ is a sequence of distinct elements in I_0 such that $u_{i_n} \notin \mathbb{Z}(A_{i_n}) \ (n \in \mathbb{N})$. For each $n \in \mathbb{N}$, choose $v_{i_n} \in U_{A_{i_n}}$ with $v_{i_n}u_{i_n}v_{i_n}^* \neq u_{i_n}$. For any prime number p, we set $w_{i_n}^{(p)} := v_{i_n}$ $(n \in \mathbb{N}p)$, and $w_i^{(p)} := e_i$ if $i \in I \setminus \{i_n : n \in \mathbb{N}p\}$. If p and q are distinct prime numbers, then

$$
w_{i_n}^{(q)} u_{i_n} (w_{i_n}^{(q)})^* = u_{i_n} \neq w_{i_n}^{(p)} u_{i_n} (w_{i_n}^{(p)})^* \quad (n \in \mathbb{N}p \setminus \mathbb{N}q).
$$

Consequently, $w^{(q)}u(w^{(q)})^* \approx w^{(p)}u(w^{(p)})^*$, and the conjugacy class of $[u]_{\sim}$ is infinite infinite. $u_{i_n}(w_{i_n}^{(q)})^* = u_{i_n} \neq w_{i_n}^{(p)} u_{i_n}(w_{i_n}^{(p)})^*$ ($n \in \mathbb{N}p \setminus \mathbb{N}q$).
 $v)u(w^{(q)})^* \approx w^{(p)}u(w^{(p)})^*$, and the conjugacy class of $[u]_{\sim}$ is
 $\sum_{i \in I} A_i$) = $\bigotimes_{i \in F_0} Z(A_i) \otimes Z(\bigotimes_{i \in I_0}^{\mathsf{ut}} A_i)$, we may as $\overline{}$

(c) Since $Z(\bigotimes_{i=1}^{\text{ut}}$ span U_{A_i} for all $i \in I$. In this case, $Z(\bigotimes_{i=1}^{u} U_i)$
is the fixed point algebra of the action Ξ i∈IAi) = ut i∈IAⁱ Ξ acy class of $[u]_{\sim}$
y assume that A_i ,
where $(\bigotimes_{i \in I}^{\text{ut}} A_i)$
Moreover one b Ξ is the fixed point algebra of the action Ξ as defined above. Moreover, one has $\bigcup_{i\in I}^{\text{ut}}Z(A_i)\subseteq Z(\bigotimes_{i\in I}^{\text{ut}}Z_i)$ $I_A(i) = \bigotimes_{i \in F_0} Z(A_i) \otimes Z(\bigotimes_{i \in I_0}^{\text{ut}} A_i)$, we may assume th
 I. In this case, $Z(\bigotimes_{i \in I}^{\text{ut}} A_i) = (\bigotimes_{i \in I}^{\text{ut}} A_i)^{\Xi}$, where $(\bigotimes_{i \in I}^{\text{ut}} A_i)$ and it remains to show that $(\bigotimes_{i \in I}^{\text{ut}} A_i)^{\Xi} \subseteq \bigotimes_{i$ $\bigcup_{i\in I}^{\text{ut}} A_i \bigg)^\Xi \subseteq \bigotimes_{i\in I}^{\text{ut}} Z(A_i).$

Let $v^{(1)}, \ldots, v^{(n)} \in \Pi_{i \in I} U_{A_i}$ be mutually inequivalent elements, let $F \in \mathfrak{F}$, and ON GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS

Let $v^{(1)}, \ldots, v^{(n)} \in \Pi_{i \in I} U_{A_i}$ be mutually inequivalent elements, let

let $b_1, \ldots, b_n \in \bigotimes_{i \in F} A_i \setminus \{0\}$ be such that $a := \sum_{k=1}^n J_F^{v^{(k)}}(b_k) \in \bigotimes_{i \in F}^{u^{(k)}}$

f ici $o_1, \ldots, o_n \in \bigotimes_{i \in F^{\perp 1} i} [o]$ be such that $u := \sum_{k=1}^{\infty} o_F \quad (v_k) \in (\bigotimes_{i \in F^{\perp 1}} i)$. We first claim that $[v^{(k)}]_{\sim} \in \Omega_{I;Z(A)}^{\text{ut}}$ (k = 1, ..., n). Suppose, to the contrary, that $\setminus \{0\}$ be such that $a := \sum_{k=1}^{n} J_F^{v^{(k)}}(b_k) \in (\bigotimes_{i \in I}^{ut} A_i)^{\Xi}$. We

∈ $\Omega_{I;Z(A)}^{ut}$ ($k = 1,...,n$). Suppose, to the contrary, that
 $\downarrow_{I;A}^{ut}$). For every $u \in \Pi_{i \in I} U_{A_i}$, one has
 $(J_F^{v^{(1)}}(b_k)) \in (\bigotimes_{i \in I} [uv$ $[v^{(1)}]_{\sim} \notin \Omega^{\text{ut}}_{I;Z(A)} = Z(\Omega^{\text{ut}}_{I;A})$. For every $u \in \Pi_{i \in I} U_{A_i}$, one has

$$
\Xi_u(J_F^{v^{(1)}}(b_k)) \in \bigotimes_{i \in I}^{[uv^{(1)}u^*]_\sim} A_i \big) \setminus \{0\}.
$$

As $\Xi_u(a) = a$, we see that $[uv^{(1)}u^*]_{\sim} \in \{[v^{(1)}]_{\sim}, \ldots, [v^{(n)}]_{\sim}\}$, which contradicts the fact that $\{[uv^{(1)}v^*]_{\sim} \cdot [u] \in \Omega_{\epsilon}^{\text{ut}}\}$ is an infinite set (by part (b)) the fact that $\{[uv^{(1)}u^*]_{\sim} : [u]_{\sim} \in \Omega_{I,A}^{ut}\}$ is an infinite set (by part (b)).
By enlarging E we pay assume that $u^{(k)} \in \Pi_{I}$, U_{I} (b) 1

By enlarging F, we may assume that $v^{(k)} \in \Pi_{i \in I} U_{Z(A_i)}$ $(k = 1, ..., n)$. For each $u \in \Pi_{i \in I} U_{A_i}$ and $k \in \{1, ..., n\}$, one has $\Xi_u(J_F^{v^{(k)}}(b_k)) = J_F^{v^{(k)}}(b_k)$ and so,
 $b_i \in Z(\bigotimes A_i)$. Therefore, $a \in \bigotimes^{\text{ut}} Z(A_i)$ as expected $b_k \in Z$ (= *a*, we see that $[uv^{(1)}u^*]_{\sim} \in \{[v^{(1)}]_{\sim}, \ldots, [v^{(n)}]_{\sim}\}$, which contradicts
hat $\{[uv^{(1)}u^*]_{\sim} : [u]_{\sim} \in \Omega_{I,A}^{ut}\}$ is an infinite set (by part (b)).
arging *F*, we may assume that $v^{(k)} \in \Pi_{i \in I} U_{Z(A_i)}$ (The reader should notice that $\mathbb{Q}_{i\in I}^{\text{ut}}Z(A_i)$ equals $\bigoplus_{k\in I}U_{\mathbb{Z}}$

stead $u \in \Pi_{i\in I}U_{A_i}$ and $k \in \{1, ..., n\}$, one has $\Xi_u(J_F^{(k)})$
 $b_k \in Z(\bigotimes_{i\in F}A_i)$. Therefore, $a \in \bigotimes_{i\in I}^{\text{ut}}Z(A_i)$, as expected

 $\bigoplus_{\omega \in Z(\Omega_{I;A}^{\text{ut}})} \bigotimes_{i \in I}^{\omega} Z(A_i)$ ineader should notice that $\bigotimes_{i\in I} Z(A_i)$ equals $\bigoplus_{\omega\in Z(\Omega_{I,A}^{ut})}\bigotimes_{i\in I} Z(A_i)$ in-
 $\bigoplus_{\omega\in \Omega_{I,A}^{ut}}\bigotimes_{i\in I}^{\omega} Z(A_i)$ (strictly speaking, the latter object does not make sense).

Example 3.7. (a) If $n_i \in \mathbb{N}$ $(i \in I)$, then $Z(\bigotimes_{i \in I}^{ut} M_{n_i}(\mathbb{C})) \cong \mathbb{C}_{ut}^{\otimes I}$.

(b) If G_i are icc groups, then $Z(\bigotimes_{i\in I}^{\text{ut}} \mathbb{C}[G_i]) \cong \mathbb{C}_{\text{ut}}^{\otimes I}$ canonically.

We end this section with the following brief discussion on the non-unital case. Suppose that $\{A_i\}_{i\in I}$ is a family of ^{*}-algebras, not necessarily unital. If $M(A_i)$ **Example 3.7.** (a) If $n_i \in \mathbb{N}$ ($i \in I$), then $Z(\bigotimes_{i \in I}^{\text{ut}} M_{n_i}(\mathbb{C})) \cong \bigotimes_{\text{ut}}^{\text{ut}}$.

(b) If G_i are icc groups, then $Z(\bigotimes_{i \in I}^{\text{ut}} \mathbb{C}[G_i]) \cong \mathbb{C}_{\text{ut}}^{\otimes I}$ canonically.

We end this section wi $_{i\in I}^{\text{ut}}M(A_i)$ as follows: is section with
 $\{A_i\}_{i \in I}$ is a i

centraliser all

as follows:
 $A_i := \text{span}\left\{$ is the double c
 $\bigotimes_{i\in I}^{\text{ut}} M(A_i)$ as
 $\bigotimes_{i\in I}^{\text{ut}} A_i$

In general, $\bigotimes_{i\in I}^{\text{ut}}$ centraliser algebra of A_i ($i \in I$), we define an ideal, $\bigotimes_{i \in I}^{\infty} A_i$

s follows:
 $A_i := \text{span} \{J_F^u(a) : F \in \mathfrak{F} : a \in \bigotimes_{i \in F} A_i : u \in \Pi_{i \in I} U_{M(A_i)} \}.$
 $\bigcup_{i \in I} A_i$ is not a subset of $\bigotimes_{i \in I} A_i$. In a similar f $I^{\mu\nu}$

$$
\bigotimes_{i \in I}^{\text{ut}} A_i := \text{span} \{ J_F^u(a) : F \in \mathfrak{F}; a \in \bigotimes_{i \in F} A_i; u \in \Pi_{i \in I} U_{M(A_i)} \}
$$

eral, $\bigotimes_{i \in I}^{\text{ut}} A_i$ is not a subset of $\bigotimes_{i \in I} A_i$. In a similar fashion, we de

$$
\int_{i \in I}^e A_i := \text{span} \{ J_F^u(a) : F \in \mathfrak{F}; a \in \bigotimes_{i \in F} A_i; u \in \Pi_{i \in I} U_{M(A_i)}; u \sim
$$

In general, $\otimes_{i\in I}^{\text{ut}} A_i$ is not a

$$
\bigotimes_{i \in I}^{i} A_i := \text{span}\{J_F(u) : I \subset \mathfrak{h}, u \in \bigotimes_{i \in F}^{i} A_i, u \in H_i(E) \text{ and } H_i(E) \}
$$
\nIn general, $\bigotimes_{i \in I}^{i} A_i$ is not a subset of $\bigotimes_{i \in I} A_i$. In a similar fashion, we define\n
$$
\bigotimes_{i \in I}^{e} A_i := \text{span}\{J_F^u(a) : F \in \mathfrak{F}; a \in \bigotimes_{i \in F} A_i; u \in \Pi_{i \in I} U_{M(A_i)}; u \sim e\},
$$
\nwhich is an ideal of $\bigotimes_{i \in I}^{e} M(A_i)$. By the proof of Theorem 3.2 (a), one may identify

 $\sum_{i\in I} M(A_i)$. By the proof of Theorem [3.2](#page-9-0) (a), one may identify $\sum_{i\in I}^{\text{ut}} A_i$ as the ideal of $(\bigotimes_{i\in I}^e M(A_i)) \rtimes_{\breve{\Xi},m} \Omega_{I;M(A)}^{\text{ut}}$ consisting of functions from $\Omega_{I;M(A)}^{\text{ut}}$ to $\bigotimes_{i\in I}^e A_i$ having finite supports. $\bigotimes_{i \in I}^e A_i$:
hich is an idea
 $\bigvee_{i \in I}^{\mathsf{ut}} A_i$ as the
 $\bigcup_{I;M(A)}^{\mathsf{ut}}$ to \bigotimes_i^e

4. Tensor products of inner product spaces

Throughout this section, $(H_i, \langle \cdot, \cdot \rangle)$ *is a non-zero inner product space* $(i \in I)$ *. Moreover, we denote* $\Omega_{I;H}^{\text{unit}} := \Pi_{i \in I} \mathfrak{S}_1(H_i) / \sim$.

If B is a unital ^{*}-algebra and X is a unital left B-module, a map $\langle \cdot, \cdot \rangle_B : X \times$ $X \to B$ is called a *(left) Hermitian B*-form on X if $\langle ax+y, z \rangle_B = a \langle x, z \rangle_B + \langle y, z \rangle_B$ and $\langle x, y \rangle_B^* = \langle y, x \rangle_B$ $(x, y, z \in X; a \in B)$. It is easy to see that a Hermitian B_n -form on X can be regarded as a B_n -himodule man $A : X \otimes \tilde{X} \to B$ satisfying B-form on X can be regarded as a B-bimodule map $\theta : X \otimes X \to B$ satisfying $\theta(x\otimes \tilde{y})^* = \theta(y\otimes \tilde{x})$ (where X is the conjugate vector space of X regarded as a unital right B-module in the canonical way). Consequently, part (a) of the following result follows readily from the universal property of tensor products, while part (b) is easily verified. unital right *B*-module in the canonical way). Consequently, part (a) of the fol-
lowing result follows readily from the universal property of tensor products, while
part (b) is easily verified.
Proposition 4.1. (a) *Th*

 $\langle \otimes_{i \in I} x_i, \otimes_{i \in I} y_i \rangle_{\mathbb{C}^{\otimes I}} := \otimes_{i \in I} \langle x_i, y_i \rangle \langle x, y \in \Pi_{i \in I} H_i \rangle.$

(b) *For a fixed* $\mu \in \Omega_{I;H}^{\text{unit}}$, one has $\langle \Theta_H(x), \Theta_H(y) \rangle_{\mathbb{C}^{\otimes I}} = \Pi_{i \in I} \langle x_i, y_i \rangle (\otimes_{i \in I} 1)$
 $\mu \in \Pi^{\mu}$ *H*, *This induces an inner product on* \mathbb{S}^{μ} *H_r* that coincides with $(x, y \in \Pi_{i\in I}^{\mu} H_i)$. This induces an inner product on $\bigotimes_{i\in I}^{\mu} H_i$ that coincides with
the one given by the inductive limit of $(\bigotimes_{i\in I} H_i, I^{\mu})$ in the category of ion 4.1. (a) There is a Hermitian $\mathbb{C}^{\otimes I}$ -form
 $\mathbb{D}_{i\in I} y_i\rangle_{\mathbb{C}^{\otimes I}} := \mathbb{D}_{i\in I} \langle x_i, y_i \rangle \langle x, y \in \Pi_{i\in I} H_i \rangle.$
 Internal $\mu \in \Omega_{I;H}^{\text{unit}}$ *, one has* $\langle \Theta_H(x), \Theta_H(y) \rangle_{\mathbb{C}^{\otimes I}}$
 $\stackrel{\mu}{\underset{i\in I}{\text{even}}} H_i$ **Proposition 4.1.** (a) There is a Hermi
 $\langle \otimes_{i \in I} x_i, \otimes_{i \in I} y_i \rangle_{\mathbb{C}^{\otimes I}} := \otimes_{i \in I} \langle x_i, y_i \rangle$ $\langle x, y \in$

(b) For a fixed $\mu \in \Omega_{I;H}^{\text{unit}}$, one has $\langle \Theta_I$
 $(x, y \in \Pi_{i \in I}^{\mu} H_i)$. This induces an inner p

the $\sum_{i \in F} H_i, J_{G;F}^{\mu}$ _{$F \subseteq G \in \mathfrak{F}$ ^{, in the category of F}} *inner product spaces with isometries as morphisms.*

We want to construct a nice inner product space from the above Hermitian $\mathbb{C}^{\otimes I}$ -form. A naive idea is to appeal to a construction for Hilbert C^* -modules that produces a Hilbert space from a positive linear functional on $\mathbb{C}^{\otimes I}$. However, the difficulty is that there is no canonical order structure on $\mathbb{C}^{\otimes I}$. Nevertheless, we will make a similar construction using the functional ϕ_1 in Example [2.2](#page-3-2) (a). In this case one can only consider a subspace of $\bigotimes H$. (see Example 4.3 below) We want to construct a nice inner prod $\mathbb{C}^{\otimes I}$ -form. A naive idea is to appeal to a consproduces a Hilbert space from a positive lindifficulty is that there is no canonical order will make a similar construction us case, one can only consider a subspace of $\bigotimes_{i\in I} H_i$ (see Example [4.3](#page-14-0) below). produces a Hilbert space from a positive linear function
difficulty is that there is no canonical order structure will make a similar construction using the functional ϕ_1 is
case, one can only consider a subspace of

Lemma 4.2. *Define* $\langle \xi, \eta \rangle_{\phi_1} := \phi_1(\langle \xi, \eta \rangle_{\mathbb{C}^{\otimes I}})$ $(\xi, \eta \in \mathcal{Q}_{i \in I} H_i)$ *and set*

$$
\bigotimes_{i\in I}^{ct} H_i := \operatorname{span} \Theta_H(\Pi_{i\in I} B_1(H_i))
$$

Lemma 4.2. $\text{Define } \langle \xi, \eta \rangle_{\phi_1} := \phi_1(\langle \xi, \eta \rangle_{\mathbb{C}^{\otimes I}})$
 $\bigotimes_{i \in I}^{\text{ct}} H_i := \text{span } \Theta_H(\mathbb{C}^{\otimes I})$
 as well as $\bigotimes_{i \in I}^{\text{unit}} H_i := \text{span } \Theta_H(\Pi_{i \in I} \mathfrak{S}_1(H_i)).$

 $\bigotimes_{i \in I}^{\text{ct}} H_i := \text{span} \Theta_H(\Pi_{i \in I} B_1(H_i))$

well as $\bigotimes_{i \in I}^{\text{unit}} H_i := \text{span} \Theta_H(\Pi_{i \in I} \mathfrak{S}_1(H_i)).$

(a) For any $\mu \in \Omega_{I;H}^{\text{unit}}$, the restriction of $\langle \cdot, \cdot \rangle_{\phi_1}$ to $\bigotimes_{i \in I}^{\mu} H_i \times \bigotimes_{i \in I}^{\mu} H_i$ coincide *with the inner product in Proposition* [4.1](#page-13-1) (b)*.* well as $\bigotimes_{i \in I}^{\text{unit}} H_i := \text{span} \Theta_H(\Pi_{i \in I} \mathfrak{S}_1(H_i)).$

(a) For any $\mu \in \Omega_{I;H}^{\text{unit}}$, the restriction of $\langle \cdot, \cdot \rangle_{\phi_1}$ to the inner product in Proposition 4.1 (b).

(b) $\langle \cdot, \cdot \rangle_{\phi_1}$ is a positive sesquilinea ϵ ¹ ϵ ¹

(b) $\langle \cdot, \cdot \rangle_{\phi_1}$ *is a positive sesquilinear form on* $\bigotimes_{i \in I}^{\mathsf{ct}} H_i$ *and is an inner product on onell as* $\bigotimes_{i \in I}^{\text{unif}} H_i := \text{spa}$

(a) For any $\mu \in \Omega_{I;H}^{\text{unit}}$,
 with the inner product in

(b) $\langle \cdot, \cdot \rangle_{\phi_1}$ *is a positive*
 on $\bigotimes_{i \in I}^{\text{unif}} H_i$. Moreover, *if* $\begin{aligned} & \text{product} \ & \text{is a positive} \ & \text{Moreover} \ & \text{K} & := \left\{ \begin{aligned} \end{aligned} \right. \end{aligned}$

on
$$
\bigotimes_{i \in I}^{\text{unit}} H_i
$$
. Moreover, if
\n
$$
K := \left\{ y \in \bigotimes_{i \in I}^{\text{ct}} H_i : \langle x, y \rangle_{\phi_1} = 0, \forall x \in \bigotimes_{i \in I}^{\text{ct}} H_i \right\},
$$
\nthen $\bigotimes_{i \in I}^{\text{ct}} H_i = K \oplus \bigotimes_{i \in I}^{\text{unit}} H_i$ (as vector spaces).

 $K := \left\{ y \in \bigotimes_{i \in I}^{\text{ct}} H_i : \langle x, y \rangle_{\phi_1} = 0, \forall x \in \bigotimes_{i \in I}^{\text{ct}} H_i \right\},$
 $n \bigotimes_{i \in I}^{\text{ct}} H_i = K \oplus \bigotimes_{i \in I}^{\text{unit}} H_i \text{ (as vector spaces)}.$

(c) If $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$, then $\bigotimes_{i \in I}^{\text{unit}} H_i = (\bigotimes_{i \in I_1}^{\text{unit}} H_i) \otimes (\big$ *as inner product spaces.*

Proof. (a) This part is clear.

 $\prod_{i\in I}^{\text{ct}} H_i$. Let

\n- (a) This part is clear.
\n- (b) This part is clear.
\n- (c) It is obvious that
$$
\langle \cdot, \cdot \rangle_{\phi_1}
$$
 is a sesquilinear form on $\bigotimes_{i \in I}^{\mathcal{C}^t} H_i$. Let $E := \{ x \in \Pi_{i \in I} B_1(H_i) : ||x_i|| < 1 \text{ for an infinite number of } i \in I \}$
\n

(b) It is obvious that $\langle \cdot, \cdot \rangle_{\phi_1}$ is a sesquilinear form on $\bigotimes_{i \in I}^{\mathrm{ct}} H_i$. Let
 $E := \{ x \in \Pi_{i \in I} B_1(H_i) : ||x_i|| < 1 \text{ for an infinite number of } i \in I \}$

and $\tilde{K} := \mathrm{span} \Theta_H(E)$. Clearly, $\bigotimes_{i \in I}^{\mathrm{ct}} H_i = \tilde{K} \oplus \bigotimes_{i \in I}^{\mathrm{unit}}$ $\Pi_{i\in I}B_1(H_i)$ and $v \in E$, then $\langle u_i, v_i \rangle \neq 1$ for an infinite number of $i \in I$, which implies that $\langle \otimes_{i\in I} u_i, \otimes_{i\in I} v_i \rangle = 0$ Consequently $\tilde{K} \subset K$ implies that $\langle \otimes_{i \in I} u_i, \otimes_{i \in I} v_i \rangle_{\phi_1} = 0$. Consequently, $K \subseteq K$.

GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS

We claim that $\langle \xi, \xi \rangle_{\phi_1} \geq 0$ $(\xi \in \bigotimes_{i \in I}^{\text{ct}} H_i)$. Suppose that $\xi = \sum_{k=1}^n \lambda_k \otimes_{i \in I} u_i^{(k)}$ with $\lambda_1,\ldots,\lambda_n\in\mathbb{C}$ and $u^{(1)},\ldots,u^{(n)}\in\Pi_{i\in I}B_1(H_i)$. Then TE ALGEBRAIC TI
 $\langle \xi, \xi \rangle_{\phi_1} \geq 0 \ (\xi \in$

C and $u^{(1)}, \ldots, u$
 $\langle \xi, \xi \rangle_{\phi_1} = \sum_{k}^{n}$ $\phi_1 > \phi_2 > 0$

$$
\langle \xi, \xi \rangle_{\phi_1} = \sum_{k,l=1}^n \lambda_k \bar{\lambda}_l \phi_1 \big(\otimes_{i \in I} \langle u_i^{(k)}, u_i^{(l)} \rangle \big).
$$

As in the above, $\phi_1(\otimes_{i \in I} \langle u_i^{(k)}, u_i^{(l)} \rangle) = 0$ if either $u^{(k)}$ or $u^{(l)}$ is in E. Thus, by rescaling we may assume that rescaling, we may assume that

$$
u^{(1)},\ldots,u^{(n)}\in\Pi_{i\in I}\mathfrak{S}_1(H_i).
$$

Furthermore, we assume that there exist $0 = n_0 < \cdots < n_m = n$ such that $u^{(n_p+1)} \sim \cdots \sim u^{(n_{p+1})}$ for all $n \in \{0, \ldots, m-1\}$ but $u^{(n_p)} \approx u^{(n_q)}$ whenever $1 \leq$ $u^{(n_p+1)} \sim \cdots \sim u^{(n_{p+1})}$ for all $p \in \{0, \ldots, m-1\}$, but $u^{(n_p)} \nsim u^{(n_q)}$ whenever $1 \leq$ $p \neq q \leq m$. It is not hard to check that $u^{(k)} \sim u^{(l)}$ if and only if $\langle u_i^{(k)}, u_i^{(l)} \rangle = 1$ e.f. $(\text{as } \|u_i^{(k)}\|, \|u_i^{(l)}\| \le 1).$ Consequently, if $1 \le p \ne q \le m$,

(4.1)
$$
\phi_1\big(\otimes_{i\in I} \langle u_i^{(k)}, u_i^{(l)} \rangle\big) = 0 \quad \text{when } n_p < k \le n_{p+1} \text{ and } n_q < l \le n_{q+1}.
$$

Therefore, in order to show $\langle \xi, \xi \rangle_{\phi_1} \geq 0$, it suffices to consider the case when (as $||u_i^{\cdot}| \cdot ||, ||u_i^{\cdot}| \le 1$). Consequently, if $1 \le p \ne q \le m$,

(4.1) $\phi_1(\otimes_{i \in I} \langle u_i^{(k)}, u_i^{(l)} \rangle) = 0$ when $n_p < k \le n_{p+1}$ and $n_q < l \le n_{q+1}$.

Therefore, in order to show $\langle \xi, \xi \rangle_{\phi_1} \ge 0$, it suffices to consider $\langle \xi, \xi \rangle_{\phi_1} \geq 0$ by part (a). extra in order to show $\langle \xi, \xi \rangle_{\phi_1} \geq 0$, it suffices to consite $\langle \xi, \xi \rangle_{\phi_1} \geq 0$, it suffices to consitently $\langle \xi, \xi \rangle_{\phi_1} \geq 0$ by part (a) .

Next, we show that $\langle \cdot, \cdot \rangle_{\phi_1}$ is an inner product on $\$ Therefor
 $u^{(k)} \sim \langle \xi, \xi \rangle_{\phi_1}$
 Next
 $\xi = \sum_{k}^{n}$

that $\langle \xi \rangle$

 $\prod_{i\in I}^{univ} H_i$. Suppose that $\lambda_{k=1}^{n} \lambda_k \otimes_{i \in I} u_i^{(k)}$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $u^{(1)}, \ldots, u^{(n)} \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$ such $\beta \geq 0$. If n_0, \ldots, n_n are as above then that $\langle \xi, \xi \rangle_{\phi_1} = 0$. If n_0, \ldots, n_m are as above, then $\begin{array}{c} \n\text{chow } \text{thot } \langle u \rangle. \n\end{array}$

$$
\phi_1\left(\langle \sum\nolimits_{k=n_p+1}^{n_p+1} \lambda_k \otimes_{i \in I} u_i^{(k)}, \sum\nolimits_{l=n_q+1}^{n_q+1} \lambda_l \otimes_{i \in I} u_i^{(l)} \rangle_{\mathbb{C}^{\otimes I}}\right) = 0,
$$

because of [\(4.1\)](#page-14-1) and the positivity of $\langle \cdot, \cdot \rangle_{\phi_1}$. Hence, we may assume $u^{(k)} \sim u^{(l)}$
for all $k, l \in \{1, ..., n\}$ and apply part (a) to conclude that $\xi = 0$ for all $k, l \in \{1, ..., n\}$, and apply part (a) to conclude that $\xi = 0$.
Einelly, so ℓ , is an inner product on $\bigotimes^{\text{unit}} H$, and we have $\phi_1\left(\left\langle \sum_{k=n_p+1}^{n_{p+1}} \lambda_k \otimes_{i\in I} u_i^{(k)}, \sum_{l=n_q+1}^{n_{q+1}} \lambda_l \otimes_{i\in I} u_i^{(l)} \right\rangle_{\mathbb{C}^{\otimes I}}\right) = 0,$
ause of (4.1) and the positivity of $\langle \cdot, \cdot \rangle_{\phi_1}$. Hence, we may assume $u^{(k)} \sim u^{(l)}$
all $k, l \in \{1, ..., n\}$, and

because of (4.1) and the positivity of $\langle \cdot, \cdot \rangle_{\phi_1}$. Hence,
for all $k, l \in \{1, ..., n\}$, and apply part (a) to conclude
Finally, as $\langle \cdot, \cdot \rangle_{\phi_1}$ is an inner product on $\bigotimes_{i \in I}^{\text{unit}} H_i$ as
 $\tilde{K} \oplus \bigotimes_{i \in I}^{\text$ ause of (4.1) and the positivity of $\langle \cdot, \cdot \rangle_{\phi_1}$. Hence, we may assume $u^{(k)}$
all $k, l \in \{1, ..., n\}$, and apply part (a) to conclude that $\xi = 0$.
Finally, as $\langle \cdot, \cdot \rangle_{\phi_1}$ is an inner product on $\bigotimes_{i \in I}^{unit} H_i$

 $\sum_{i\in I_1} H_i) \otimes (0)$ (c) Observe that the linear bijection $\mathbf{F} \cdot (\mathbf{W}_{i \in I_1} H_i) \otimes (\mathbf{W}_{j \in I_2} H_j) \wedge \mathbf{W}_{i \in I} H_i$
as in Remark [2.4](#page-5-0) (b) restricts to a surjection from $(\mathbf{Q}_{i \in I_1}^{\text{unit}} H_i) \otimes (\mathbf{Q}_{j \in I_2}^{\text{unit}} H_j)$ to unit H_i . Moreover, for any $u, u' \in \Pi_{i \in I_1} \mathfrak{S}_1(H_i)$ and $v, v' \in \Pi_{j \in I_2} \mathfrak{S}_1(H_j)$,
the vector $(u, u') \propto (v, v')$ as elements in $\Pi_{i \in I} \mathfrak{S}_1(H_i)$ if and only if $u \propto u'$ and we have $(u, u') \sim (v, v')$ as elements in $\Pi_{i \in I} \mathfrak{S}_1(H_i)$ if and only if $u \sim u'$ and $v \sim u'$. $v \sim v'$. Thus, the argument in part (b) tells us that

$$
\langle (\otimes_{i \in I_1} u_i) \otimes (\otimes_{j \in I_2} v_j), (\otimes_{i \in I_1} u'_i) \otimes (\otimes_{j \in I_2} v'_j) \rangle_{\phi_1}
$$
\n
$$
= \langle \otimes_{i \in I_1} u_i, \otimes_{i \in I_1} u'_i \rangle_{\phi_1} \langle \otimes_{j \in I_2} v_j, \otimes_{j \in I_2} v'_j \rangle_{\phi_1}.
$$
\nis shows that $\Psi|_{(\bigotimes_{i \in I_1}^{\text{unit}} H_i) \otimes (\bigotimes_{j \in I_2}^{\text{unit}} H_j)}$ is inner product preserving.

\nWe denote by $\overline{\bigotimes}_{i \in I}^{\mu} H_i$ and $\overline{\bigotimes}_{i \in I}^{\phi_1} H_i$ the completions of $\bigotimes_{i \in I}^{\mu} H_i$ and $\bigotimes_{i \in I}^{\text{unit}} H_i$ and $\bigotimes_{i \in I}^{\text{unit}} H_i$.

This shows that $\Psi|_{(\bigotimes_{i \in I_1}^{\text{unit}} H_i) \otimes (\bigotimes_{j \in I_2}^{\text{unit}} H_j)})$ is inner product preserving. \Box

 $\begin{aligned} &\epsilon_{I_2} v_j, \otimes_{j \in I_2} v'_j \rangle_{\phi_1}. \\ &\text{serving.} \quad \Box \\ &\mu \atop i \in I} H_i \text{ and } \bigotimes_{i \in I}^{\text{unit}} H_i, \end{aligned}$ *respectively, under the norms induced by* $\langle \cdot, \cdot \rangle_{\phi_1}$.

Example 4.3. If $H_i = \mathbb{C}$ $(i \in I)$, then the sesquilinear form $\langle \cdot, \cdot \rangle_{\phi_1}$ is not positive 344
 Example 4.3. If $H_i = \mathbb{C}$ ($i \in I$), then the sesquilinear form $\langle \cdot, \cdot \rangle_{\phi_1}$ is not positive

on the whole space $\bigotimes_{i \in I} H_i$ since $\bigotimes_{i \in I} 1/2 - \bigotimes_{i \in I} 2\big)$, $\bigotimes_{i \in I} 1/2 - \bigotimes_{i \in I} 2\big)$, $\bigotimes_{$

Set $\Pi_{i\in I}^{eu}H_i := \{x \in \Pi_{i\in I}H_i : x_i \in \mathfrak{S}_1(H_i) \text{ except for a finite number of } i\}$ and K be an inner product space. A multilinear map $\Phi : \Pi^{eu}H_i \to K$ (i.e. Φ is let K be an inner product space. A multilinear map $\Phi : \Pi_{i \in I}^{\text{eu}} H_i \to K$ (i.e. Φ is
coordinatewise linear) is said to be *componentwise inner product preserving* if for coordinatewise linear) is said to be *componentwise inner product preserving* if for any $\mu, \nu \in \Omega_{I;H}^{\text{unit}},$

 $\langle \Phi(x), \Phi(y) \rangle = \delta_{\mu, \nu} \Pi_{i \in I} \langle x_i, y_i \rangle \quad (x \in \Pi_{i \in I}^{\mu} H_i; y \in \Pi_{i \in I}^{\nu} H_i),$

where $\delta_{\mu,\nu}$ is the Kronecker delta.

 $\langle \Phi(x), \Phi(y) \rangle = \delta_{\mu,\nu} \Pi_{i \in \mathbb{R}}$
where $\delta_{\mu,\nu}$ is the Kronecker delta.
Theorem 4.4. (a) $\bar{\bigotimes}^{\phi_1}_{i \in I} H_i \cong \bar{\bigoplus}$ $\bar{\bigoplus}^{\ell^2}_{\ldots}$ $\bigotimes_{\mu \in \Omega_{I;H}^{\text{unit}}}^{\ell^2} \bar{\mathfrak{O}}_{i \in I}^{\mu} H_i$ *canonically as Hilbert spaces.* necker delta.
 $\bar{\mathcal{S}}_{i\in I}^{\phi_1} H_i \cong \bar{\bigoplus}_{\mu \in I}^{\ell^2}$
 $i\in I$ $H_i \rightarrow \bigotimes_{i \in I}^{\text{unit}}$
 For any inner

(b) $\Theta_H|_{\Pi_{i\in I}^{eu}H_i}: \Pi_{i\in I}^{eu}H_i \to \bigotimes_{i\in I}^{unit}H_i$ is a componentwise inner product preserv-
multilinear man. For any inner product space K and any componentwise inner *ing multilinear map. For any inner product space* K *and any componentwise inner product preserving multilinear map* $\Phi : \Pi_{i \in I}^{\text{eu}} H_i \to K$, there is a unique isometry **Theoren**

(b) Θ
 ing multii
 product p
 $\tilde{\Phi} : \bigotimes_{i=1}^{m} I_i$ $\sum_{i\in I}^{\text{unit}} H_i \to K$ such that $\Phi = \tilde{\Phi} \circ \Theta_H|_{\Pi_{i\in I}^{\text{eu}} H_i}$. *Proof.* (a) CH $\prod_{i\in I} H_i \cdot \Pi_i \in \{1H_i \to \mathcal{O}_{i\in I} H$
 Product preserving multilinear map Φ
 $\tilde{\Phi}: \bigotimes_{i\in I}^{\text{unit}} H_i \to K$ such that $\Phi = \tilde{\Phi} \circ \Theta$
 Proof. (a) Clearly, $\bigotimes_{i\in I}^{\text{unit}} H_i = \sum_{\mu \in \Omega}$

Lemma [4.2](#page-13-0)

 $\mu \in \Omega_{i\in I}^{\text{unit}} \bigotimes_{i\in I}^{\mu} H_i$. Moreover, as in the proof of $\Phi: \Pi_{i\in I}^{\text{eu}} H_i \to K$, there is a unique isometry
 $\Theta_H|_{\Pi_{i\in I}^{\text{eu}} H_i}.$
 $\Omega_{I;H}^{\text{unit}} \bigotimes_{i\in I}^{\mu} H_i$. Moreover, as in the proof of $\mu_{i\in I}^{\mu} H_i$ and $\bigotimes_{i\in I}^{\mu} H_i$ are orthogonal if μ and ν
 $\Phi_{i\$ are distinct elements in $\Omega_{I;H}^{\text{unit}}$. The rest of the argument is standard. *Proof.* (a) Clearly, $\bigotimes_{i \in I}^{\text{unit}} H_i = \sum_{\mu \in \Omega_{I;H}} \bigotimes_{i \in I}^{\mu} H_i$. Moreover, as in the proof of Lemma 4.2 (b), the two subspaces $\bigotimes_{i \in I}^{\mu} H_i$ and $\bigotimes_{i \in I}^{\nu} H_i$ are orthogonal if μ and ν are disti

(b) It is easy to see that $\Theta_H|_{\Pi_{i\in I}^{\text{eu}}H_i}$ is componentwise inner product preserving. To show the existence of $\tilde{\Phi}$, we first define a multilinear map Φ_0 : $\Pi_{i \in I} H_i \to$ K by setting $\Phi_0 = \Phi$ on $\Pi_{i\in I}^{eu} H_i$ and $\Phi_0 = 0$ on $\Pi_{i\in I} H_i \setminus \Pi_{i\in I}^{eu}$ (b) It is easy to see that $\Theta_H|_{\Pi_{i\in I}^{\text{eu}}H_i}$ is componentwise inner product preserving.
The uniqueness of $\tilde{\Phi}$ follows from the fact that $\Theta_H(\Pi_{i\in I}^{\text{eu}}H_i)$ generates $\bigotimes_{i\in I}^{\text{unit}}H_i$.
To show the exist $\tilde{\Phi} := \tilde{\Phi}_0|_{\bigotimes_{i\in I}^{\text{unif}} H_i}$. Suppose that The uniqueness of $\tilde{\Phi}$ follows from the fact that

To show the existence of $\tilde{\Phi}$, we first define a

K by setting $\Phi_0 = \Phi$ on $\Pi_{i\in I}^{eu}H_i$ and $\Phi_0 = 0$
 $\bigotimes_{i\in I} H_i \to K$ be the induced linear map and so
 $u,$ $\sum_{i\in I}^{v} H_i$. If $u \propto v$, then $\langle \xi, \eta \rangle_{\phi_1} =$ To show the existence of $\tilde{\Phi}$, we first define a multilinear map

K by setting $\Phi_0 = \Phi$ on $\Pi_{i \in I}^{eu} H_i$ and $\Phi_0 = 0$ on $\Pi_{i \in I} H_i \setminus \Pi$
 $\bigotimes_{i \in I} H_i \to K$ be the induced linear map and set $\tilde{\Phi} := \tilde{\Phi}_0|_{\bigot$ and $\xi_0, \eta_0 \in \bigotimes_{i \in F} H_i$ such that
 F_{Lip} this same $\langle \tilde{\Phi}(\zeta), \tilde{\Phi}(\zeta) \rangle$ $\xi = J_F^u(\xi_0), \ \eta = J_F^v(\eta_0)$ and $u_i = v_i$ if $i \in I \setminus F$. In this case, $\langle \tilde{\Phi}(\xi), \tilde{\Phi}(\eta) \rangle =$
 $\langle \xi_0, \eta_0 \rangle = \langle \xi, \eta_0 \rangle$ $\langle \xi_0, \eta_0 \rangle = \langle \xi, \eta \rangle_{\phi_1}$. $\Phi = \langle \Psi(\xi), \Psi(\eta) \rangle$. Otherwise, there exist F
 $\xi = J_F^u(\xi_0), \eta = J_F^v(\eta_0)$ and $u_i = v_i$ if $i \langle \xi_0, \eta_0 \rangle = \langle \xi, \eta \rangle_{\phi_1}$.
 Example 4.5. Suppose that Φ and φ are

is the canonical orthonormal basis for ℓ^2

Example 4.5. Suppose that Φ and φ are as in Corollary [3.4,](#page-10-0) and $\{\delta_{\mu}\}_{{\mu \in \Omega_{I \cup C}^{\text{unit}}}}$ $\Omega_{I;\mathbb{C}}^{\text{unit}}$. Note that $\Omega_{I;\mathbb{C}}^{\text{ut}} = \Omega_{I;\mathbb{C}}^{\text{unit}}$ and $\Omega_{I;\mathbb{C}}^{\text{1}}$ counit given by $I(\lambda(\lceil_{\Omega}\rceil))$. consider the linear bijection $J : \mathbb{C}[\Omega_{I;\mathbb{C}}^{ut}] \to \mathbb{C}[\Omega_{I;\mathbb{C}}^{unt}]$ given by $J(\lambda([\alpha]_{\sim})) := \delta_{[\alpha]_{\sim}}$
 $(\alpha \in \Pi_{\mathbb{C}^*} \mathbb{T})$ By Example 2.10(a) and Theorem 4.4(a) the map $I \circ \Phi$ induces $(\alpha \in \Pi_{i \in I} \mathbb{T})$. By Example [2.10](#page-8-1)(a) and Theorem [4.4](#page-15-0)(a), the map $J \circ \Phi$ induces **Example 4.5.** Suppose that Φ and φ are as in
is the canonical orthonormal basis for $\ell^2(\Omega_{I;\mathbb{C}}^{\text{unit}})$
consider the linear bijection $J : \mathbb{C}[\Omega_{I;\mathbb{C}}^{\text{ut}}] \to \mathbb{C}[\Omega_{I;\mathbb{C}}^{\text{unit}}]$
 $(\alpha \in \Pi_{i\in I}\mathbb{T})$. By $\Omega_{I;\mathbb{C}}^{\text{unit}}$ such that $\hat{\Phi}(\otimes_{i\in I}\beta_i)$ = $\varphi(\beta)\delta_{\lbrack\beta\rbrack\sim}$ $(\beta\in\Pi_{i\in I}\mathbb{T}).$ $\in \Pi_{i \in I}(\mathbb{T})$. By Example 2.10 (a)
 δ illbert space isomorphism $\hat{\Phi}$:
 δ) $\delta_{[\beta]_{\sim}}$ ($\beta \in \Pi_{i \in I}(\mathbb{T})$.

We would like to compare $\bar{\mathbb{Q}}_{i \in I}^{\phi_1}$. $\ln \text{that } \Phi$

 $i\epsilon_I H_i$ with the infinite direct product as defined $i\epsilon_I H_i$ with the infinite direct product as defined in [\[20\]](#page-27-3), when $\{H_i\}_{i\in I}$ is a family of Hilbert spaces. Let us first recall from Defia Hilbert space isomorphism $\Psi : \bigotimes_{i \in I} \bigcup \rightarrow \ell^2(\Omega_{I; \mathbb{C}}^{\times})$ s
 $\varphi(\beta)\delta_{[\beta]_{\sim}} (\beta \in \Pi_{i \in I} \mathbb{T}).$

We would like to compare $\overline{\bigotimes}_{i \in I}^{\phi_1} H_i$ with the infinite d

in [\[20\]](#page-27-3), when $\{H_i\}_{i \in I}$ is a family of H $\lim_{i \in I} ||x_i|| - 1$ converges. We would like to compare $\bar{\bigotimes}_{i\in I}^{\phi_1} H_i$ with the infinite direct procin [\[20\]](#page-27-3), when $\{H_i\}_{i\in I}$ is a family of Hilbert spaces. Let us first rention 3.3.1 in [20] that $x \in \Pi_{i\in I} H_i$ is a C_0 -sequence if $\sum_{i\$ $\sum_{i\in I} |\langle x_i, y_i\rangle - 1|$

converges, then we write $x \approx y$. Denote by $[x]_{\approx}$ the equivalence class of x under \approx , and by $\Gamma_{I:H}$ the set of all such equivalence classes (see Definition 3.3.3 in [\[20\]](#page-27-3)).

Let $\prod \otimes_{i \in I} H_i$ be the infinite direct product Hilbert space as defined in [\[20\]](#page-27-3), and let $\prod \otimes_{i\in I} x_i$ be the element in $\prod \otimes_{i\in I} H_i$ corresponding to a C_0 -sequence x as in Theorem IV of [\[20\]](#page-27-3). Notice that if $x \in \Pi_{i \in I}^{\text{eu}} H_i$, then x is a C_0 -sequence, and we have a multilinear man we have a multilinear map e direct product Hilber
it in $\prod \otimes_{i \in I} H_i$ corresponding that if $x \in \prod_{i \in I}^{\text{eu}} H_i$, therefore $\prod_{i \in I} H_i \longrightarrow \prod \otimes_{i \in I} H_i$. as in Theorem IV of [20]. Notice that if $x \in \Pi_{i \in I}^{\text{eu}} H_i$, then
we have a multilinear map
 $\Upsilon : \Pi_{i \in I}^{\text{eu}} H_i \longrightarrow \prod \otimes_{i \in I} H_i$.
On the other hand, for any $\mathfrak{C} \in \Gamma_{I;H}$, we denote by $\prod \otimes_{i \in I} \otimes_{i \in I}^{\mathfrak{$

$$
\Upsilon: \Pi_{i\in I}^{\mathrm{eu}} H_i \longrightarrow \prod \otimes_{i\in I} H_i.
$$

 $\sum_{i\in I}^{\mathfrak{C}} H_i$ the closed subspace
nition 4.1.1 in [20]) we have a multilinear map

T

On the other hand, for any α

of $\prod \otimes_{i \in I} H_i$ generated by { of $\prod \otimes_{i \in I} H_i$ generated by $\{\prod \otimes_{i \in I} x_i : x \in \mathfrak{C}\}\$ (see Definition 4.1.1 in [\[20\]](#page-27-3)).

Proposition 4.6. *Let* $\{H_i\}_{i\in I}$ *be a family of Hilbert spaces.*

(a) $[x]_{\sim} \mapsto [x]_{\approx}$ ($x \in \Pi_{i \in I}$ $\mathfrak{S}_1(H_i)$) gives a well defined surjection $\kappa_H : \Omega_{I;H}^{\text{unit}} \rightarrow$
Manageum for any $x \in \Pi$ $\mathfrak{S}_n(H)$, there is a bijection letween $u^{-1}([\infty])$ $\Gamma_{I;H}$ *. Moreover, for any* $x, y \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$ *, there is a bijection between* $\kappa_H^{-1}([x]_{\approx})$ $and \kappa_H^{-1}([y]_{\approx}).$ **position 4.6.** Let $\{H_i\}_{i\in I}$ be a family of Hilbert spaces.

(a) $[x]_{\sim} \mapsto [x]_{\approx}$ $(x \in \Pi_{i\in I} \mathfrak{S}_1(H_i))$ gives a well defined surjection $\kappa_H : \Omega_{I;H}^{\text{unit}} \to$
 H . Moreover, for any $x, y \in \Pi_{i\in I} \mathfrak{S}_1(H_i)$,

 $\Theta_H|_{\Pi_{i\in I}^{\text{eu}}H_i}$ and $\tilde{\Upsilon}|_{\bigotimes_{i\in I}^{\mu}H_i}$ extends to a Hilbert space isomorphism $\tilde{\Upsilon}^{\mu}: \bar{\bigotimes}_{i\in I}^{\mu}H_i \to$ $\prod \otimes_{i \in I}^{\kappa_H(\mu)} H_i \ (\mu \in \Omega_{I;H}^{\text{unit}}).$

Proof. (a) Clearly, if $x \sim z$, then $x \approx z$ and κ_H is well defined. Lemma 3.3.7 in [\[20\]](#page-27-3) tells us that κ_H is surjective. Furthermore, there exists a unitary $u_i \in \mathcal{L}(H_i)$ such that $u \cdot x_i = u_i$ $(i \in I)$ and $[u_i]_{i \in I}$ induces the required bijective correspondence in that $u_i x_i = y_i$ ($i \in I$), and $[u_i]_{i \in I}$ induces the required bijective correspondence in the second statement the second statement. *T too]*. (a) Crearly, if $x \approx z$ and k_H is wen defined. Lemma 3.3.7 in [20]
tells us that κ_H is surjective. Furthermore, there exists a unitary $u_i \in \mathcal{L}(H_i)$ such
that $u_i x_i = y_i$ ($i \in I$), and $[u_i]_{i \in I}$ induc

(b) By the argument of Theorem 4.4 (b), one can construct a linear map Υ such $\prod_{i\in I} \otimes_{i\in I}^{[u]_{\approx}} H_i$ $(u \in \Pi_{i\in I} \mathfrak{S}_1(H_i))$. Furthermore, by Lemma [4.2](#page-13-0) (a), Proposition [4.1](#page-13-1) (b) and Theorem IV in [\[20\]](#page-27-3), we see that $\tilde{\Upsilon}|_{\bigotimes_{i \in I} [u]_{\sim}^{\sim} H_i}$ is an isometry. Finally, $\tilde{\Upsilon}|_{\bigotimes_{i \in I} [u]_{\sim}^{\sim} H_i}$ has dense range (by Lemma $4.1.2$ of $[20]$). $\omega_{i\in I}$ H_i (*u* ∈ H_{i∈I}O₁(H_i)). Furthermore, by Lemma 4.2 (a), I roposition 4.1 (b)
d Theorem IV in [20], we see that $\tilde{\Upsilon}|_{\bigotimes_{i\in I} [u]_{\Upsilon}^{\infty} H_i}$ is an isometry. Finally, $\tilde{\Upsilon}|_{\bigotimes_{i\in I} [u]_{\Upsilon}^{\in$

Notice that $\tilde{\Upsilon}$ is, in general, unbounded but Remark [4.7](#page-16-0) (b) below tells us that ¯ ^φ¹ says that it is possible to construct $\prod_{i\in I} \otimes_{i\in I} H_i$ in a way similar to $\overline{\mathbb{Q}}_{i\in I}^{\phi_1}$
says that it is possible to construct $\prod_{i\in I} \otimes_{i\in I} H_i$ in a way similar to $\overline{\mathbb{Q}}_{i\in I}^{\phi_1}$. says that it is possible to construct $\prod \otimes_{i\in I} H_i$ in a way similar to $\bar{\bigotimes}_{i\in I}^{\phi_1} H_i$. Note Notice that $\tilde{\Upsilon}$ is, in general, unbounded but Remark 4.7 (b) below tel
 $\bar{\mathcal{Q}}_{i\in I}^{\phi_1}H_i$ is a "natural dilation" of $\prod_{i\in I} \otimes_{i\in I} H_i$. On the other hand, Rema

says that it is possible to construct \prod_{i however, that the construction of $\bar{\otimes}_{i\in I}^{\phi_1}H_i$ is totally algebraic and $\bar{\otimes}_{i\in I}^{\phi_1}H_i$ itself seems to be more natural (see Theorem [4.8](#page-17-0) and Example [5.6](#page-23-0) below).

Remark 4.7. Suppose that $\{H_i\}_{i\in I}$ is a family of Hilbert spaces.

(a) ∼ and ≈ are different even in the case when $I = N$ and $H_i = \mathbb{C}$ ($i \in N$) because one can find $x, y \in \Pi_{i \in \mathbb{N}} \mathbb{T}$ with $x_i \neq y_i$ for all $i \in \mathbb{N}$ but for which $\sum_{i=1}^{\infty} |x_i, y_i - 1|$ converges. In fact $\kappa^{-1}([x])$ is an infinite set $\sum_{i=1}^{\infty} |\langle x_i, y_i \rangle - 1|$ converges. In fact, $\kappa_H^{-1}([x]_{\approx})$ is an infinite set. $y \in \Pi_{i \in \mathbb{N}} \mathbb{T}$ with
ges. In fact, κ_{H}^{-1}
 Γ [20], we have
 $\otimes_{i \in I} H_i = \overline{\bigoplus}$

(b) By Lemma $4.1.1$ in [\[20\]](#page-27-3), we have

$$
\prod \otimes_{i \in I} H_i \ = \ \bar{\bigoplus}_{\mathfrak{C} \in \Gamma_{I;H}}^{\ell^2} \prod \otimes_{i \in I}^{\mathfrak{C}} H_i.
$$

Therefore, Theorem [4.4](#page-15-0) (a) and Proposition [4.6](#page-16-1) tell us that for a fixed $\gamma_0 \in \Gamma_{I,H}$, one has a canonical Hilbert space isomorphism one has a canonical Hilbert space isomorphism

4.4 (a) and Proposition 4.6 tell us that f
Hilbert space isomorphism

$$
\overline{\bigotimes}_{i\in I}^{\phi_1} H_i \cong \ell^2(\kappa_H^{-1}(\gamma_0)) \overline{\otimes} (\prod \otimes_{i\in I} H_i).
$$

(c) For each $i \in I$, let K_i be an inner product space such that H_i is the $\overline{\bigotimes}_{i\in I}^{\phi_1} H_i \cong \ell^2(\kappa_H^{-1}(\gamma_0)) \overline{\otimes} (\prod \otimes_{i\in I} H_i).$

(c) For each $i \in I$, let K_i be an inner product space such that H_i is the completion of K_i . Then $\overline{\bigotimes}_{i\in I}^{\phi_1} K_i$ is, in general, not canonical $\bar{\mathcal{S}}_{i\in I}^{\phi_1}H_i$ because $\Omega_{I;K}^{\text{unit}}\subsetneq \Omega_{I;H}^{\text{unit}}$ if $K_i\subsetneq H_i$ for an infinite number of $i\in I$. On the other hand if I is countable for any $x\in \Pi_{i\in I}\mathfrak{S}_1(H_i)$ there exists $u\in \Pi_{i\in I}\mathfrak{S}_1(K_i)$ other hand, if I is countable, for any $x \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$, there exists $y \in \Pi_{i \in I} \mathfrak{S}_1(K_i)$ such that $x \approx y$. This shows that the restriction, $\kappa_{H,K}$, of κ_H to $\Omega_{IK}^{\text{unit}}$ is also a
guidation onto Γ . However, we do not know if the condinguity of κ^{-1} (*C*) are surjection onto $\Gamma_{I;H}$. However, we do not know if the cardinality of $\kappa_{H;K}^{-1}(\mathfrak{C})$ are the same for different $\mathfrak{O} \subseteq \Gamma_{I;H}$. the same for different $\mathfrak{C} \in \Gamma_{I:H}$.

(d) If ϕ_0 is as in Example [2.2](#page-3-2)(b), it is easy to see that $\overline{}$

$$
\langle \prod \otimes u_i, \prod \otimes v_i \rangle = \phi_0\big(\langle \otimes_{i \in I} u_i, \otimes_{i \in I} v_i \rangle_{\mathbb{C}^{\otimes I}}\big) \quad (u, v \in \Pi_{i \in I}^{\text{unit}} H_i).
$$

Thus, the sesquilinear form $\phi_0(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}})$
contained alternative construction for Π easy to see that
 $\mathbb{D}_{i \in I} v_i \rangle_{\mathbb{C}^{\otimes I}}$ $(u, v \in \Pi_{i \in I}^{\text{unit}} H_i)$.

produces $\prod_{i \in I} \otimes H_i$. If one wants a self-
 H_i one needs to establish the positivity (d) If ϕ_0 is as in Example 2.2(b), it is easy to see that
 $\langle \prod \otimes u_i, \prod \otimes v_i \rangle = \phi_0(\langle \otimes_{i \in I} u_i, \otimes_{i \in I} v_i \rangle_{\mathbb{C}^{\otimes I}})$ $(u, v \in \Pi_{i \in I}^{\text{unit}} H_i)$.

Thus, the sesquilinear form $\phi_0(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}})$ produ of $\phi_0(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}})$, which can be reduced to showing the positivity when all H_i are of the same finite dimension the same finite dimension. In the sesquilinear form $\phi_0(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}})$ produces $\prod \otimes H_i$. If one wants a self-
tained alternative construction for $\prod \otimes H_i$, one needs to establish the positivity
 $\phi_0(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}})$, which can ed alternative construction for $\prod_{i} \otimes H_i$, one needs to establish $\langle \cdot \rangle_{\mathbb{C}^{\otimes I}}$, which can be reduced to showing the positivity when
the finite dimension.
the remainder of this section, we show that $\bigotimes_{i \in I}$

a $C^*(\Omega_{I;\mathbb{C}}^{ut})$ -module, which gives many pre-inner products on $\bigotimes_{i\in I}^{unit}H_i$ including
 $\langle \ldots \rangle$ in the following we use the convention that the 4-valued inner product of $\langle \cdot, \cdot \rangle_{\phi_1}$. In the following, we use the convention that the A-valued inner product of an inner product A-module is A-linear in the first variable (where A is a pre- C^* algebra). On the other hand, we recall that if G is a group and λ_g is the canonical
image of g in $\mathbb{C}[G]$ the map $\sum_{\alpha} \alpha_{\alpha} \geq \alpha_{\alpha}$ ($\alpha \in \mathbb{C}$) where $e \in G$ is the im the remainder or this section
a $C^*(\Omega_{I;\mathbb{C}}^{\text{ut}})$ -module, which gives
 $\langle \cdot, \cdot \rangle_{\phi_1}$. In the following, we use the
an inner product A-module is A-
algebra). On the other hand, we i
image of g in $\mathbb{C}[G]$, the image of g in $\mathbb{C}[G]$, the map $\sum_{g\in G} \alpha_g \lambda_g \mapsto \alpha_e$ $(\alpha_g \in \mathbb{C})$, where $e \in G$ is the identity, extends to a faithful tracial state χ_G on $C^*(G)$.

Theorem 4.8. (a) *There exists an inner product* $\mathbb{C}[\Omega_{I;\mathbb{C}}^{ut}]$ *-module structure on* unit ightharpoontage of g in $\mathbb{C}[G]$, the map $\sum_{g \in G} \alpha_g \lambda_g \mapsto \alpha_e$ ($\alpha_g \in \mathbb{C}$), where $e \in G$ is the entity, extends to a faithful tracial state χ_G on $C^*(G)$.
 neorem 4.8. (a) There exists an inner product $\mathbb{C$ *this* $\mathbb{C}[\Omega_{I;\mathbb{C}}^{ut}]$ -module, we have a canonical Hilbert space isomorphism **Theorem 4.8.** (a) There exists an inner p
 $\bigotimes_{i \in I}^{\text{unit}} H_i$. If $\bigotimes_{i \in I}^{\text{mod}} H_i$ is the Hilbert $C^*(\Omega_{I;\mathbb{C}}^{\text{ut}})$

this $\mathbb{C}[\Omega_{I;\mathbb{C}}^{\text{ut}}]$ -module, we have a canonical Hill

(4.2) $\bigotimes_{i \in I}^{\phi_1} H_i \con$

(4.2)
$$
\overline{\bigotimes}_{i\in I}^{\phi_1} H_i \cong \left(\overline{\bigotimes}_{i\in I}^{\text{mod}} H_i \right) \overline{\otimes}_{\chi_{\Omega_{I;\mathbb{C}}^{\text{ut}}}} \mathbb{C}.
$$

(b) If $G \subseteq \Omega_{I;\mathbb{C}}^{ut}$ is a subgroup and $\mathcal{E}_G : C^*(\Omega_{I;\mathbb{C}}^{ut}) \to C^*(G)$ is the canonical con-(4.2) $\overline{\mathbb{Q}}_{i \in I}^{\phi_1} H_i \cong (\overline{\mathbb{Q}}_{i \in I}^{\text{mod}} H_i) \overline{\mathbb{Q}}_{\chi_{\Omega_{I\backslash i}^{\text{ut}}}} \mathbb{C}.$

(b) If $G \subseteq \Omega_{I; \mathbb{C}}^{\text{ut}}$ is a subgroup and $\mathcal{E}_G : C^*(\Omega_{I; \mathbb{C}}^{\text{ut}}) \to C^*(G)$ is the canonical conditional expectatio (4.2) $\bigotimes_{i \in I}^{\varphi^*} H_i \cong (\bigotimes_{i \in I}^{\text{mod}} H_i) \bar{\otimes}_{\chi_{\Omega_{I;\mathbb{C}}^{ut}}} \mathbb{C}.$
 (b) If $G \subseteq \Omega_{I;\mathbb{C}}^{ut}$ is a subgroup and $\mathcal{E}_G : C^*(\Omega_{I;\mathbb{C}}^{ut}) \to C^*(G)$ is the ditional expectation, there is an inner product $\mathbb{C}[$ $\sum_{i\in I}^{\text{mod}} H_i$) $\bar{\otimes}_{\mathcal{E}_G} C^*(G)$. (b) If $G \subseteq \Omega_{I;C}^{\text{ut}}$ is a su
ditional expectation, there
whose completion coincid
Proof. (a) Clearly, $\bigotimes_{i \in I}^{\text{unit}}$ a subgroup and $\mathcal{E}_G : C^*(\Omega_{I;\mathbb{C}}^{ut}) \to C^*(G)$ is the canonical con-
here is an inner product $\mathbb{C}[G]$ -module structure on $\bigotimes_{i \in I}^{\text{unit}} H_i$,
ncides with the Hilbert $C^*(G)$ -module $(\overline{\bigotimes}_{i \in I}^{\text{mod}} H_i) \overline{\otimes}_{\mathcal$ ditional expectation, the

Proposition [2.3](#page-4-0) (c)). Moreover, one has a linear "truncation" E from $\mathbb{C}^{\otimes I} =$
 $(\mathbb{A} \qquad \mathbb{R}^{\omega} \mathbb{C}) \oplus \mathbb{C}^{\otimes I}$ to $\mathbb{C}^{\otimes I}$ sending (α, β) to β . Define of. (a) Clearly, $\bigotimes_{i\in I}^{\text{unit}} H_i$ is a $\mathbb{C}_{\text{ut}}^{\otimes I}$ -submodule of the $\mathbb{C}^{\otimes I}$ -mod
position 2.3(c)). Moreover, one has a linear "truncation"
 $\omega \in \Omega_{I;\mathbb{C}} \setminus \Omega_{I;\mathbb{C}}^{\text{out}} \bigotimes_{i\in I}^{\omega} \mathbb{C}$ \oplus $\$

$$
\langle \xi, \eta \rangle_{\mathbb{C}_{\text{ut}}^{\otimes I}} := E\big(\langle \xi, \eta \rangle_{\mathbb{C}^{\otimes I}}\big) \quad \big(\xi, \eta \in \bigotimes_{i \in I}^{\text{unit}} H_i\big),
$$

which is a Hermitian $\mathbb{C}_{\rm ut}^{\otimes I}$ -form because by (3.1) , we have

$$
E(ab) = E(a)b
$$
 and $E(a^*) = E(a)^*$ $(a \in \mathbb{C}^{\otimes I}; b \in \mathbb{C}_{\text{ut}}^{\otimes I}).$

For any $u, v \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$, we write $u \sim_s v$ if there exists $\beta \in \Pi_{i \in I} \mathbb{T}$ such that $u_i = \beta_i v_i$ e.f. Then \sim_s is an equivalence relation on $\Pi_{i \in I} \mathfrak{S}_1(H_i)$ satisfying any $u, v \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$, we write $u \sim_s v$ if there exists $\beta \in \Pi_{i \in I} \mathbb{T}$ such that
 $\beta_i v_i$ e.f. Then \sim_s is an equivalence relation on $\Pi_{i \in I} \mathfrak{S}_1(H_i)$ satisfying

(a)
 $u \sim_s v$ if and only if $\langle \otimes_{i \$

(4.3)
$$
u \sim_s v
$$
 if and only if $\langle \otimes_{i \in I} u_i, \otimes_{i \in I} v_i \rangle_{\mathbb{C}^{\otimes I}} \in \mathbb{C}_{\text{ut}}^{\otimes I}$.

 $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $u^{(1)}, \ldots, u^{(n)} \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$. We first show that $\Phi(\langle \xi, \xi \rangle_{\mathbb{C} \otimes I} \in \mathbb{C}^* | G_{\mathbb{C} \otimes I})$ $C^*(\Omega_{I;\mathbb{C}}^{\text{ut}})_+$. As in the proof of Lemma [4.2](#page-13-0) (b), it suffices to consider the case when $u^{(k)} \sim_{s} u^{(1)}$ for any $k \in \{1,\ldots,n\}$ (because of relation (4.3)). Let $F \in \mathfrak{F}$ and $\beta^{(1)}, \ldots, \beta^{(n)} \in \Pi_{i \in I} \mathbb{T}$ be such that $u_i^{(k)} = \beta_i^{(k)} u_i^{(1)}$ $(i \in I \setminus F; k = 1, \ldots, n)$. For any $k, l \in \{1, \ldots, n\}$, we have $\begin{align} &\in \Pi_{i \in I} \mathbb{T} \ &\ldots, n \}, \ &\in F \langle u_i^{(k)} \rangle &:= \end{align}$

$$
\Phi\big((\Pi_{i\in F}\langle u_i^{(k)}, u_i^{(l)}\rangle_i)(\otimes_{i\in I\setminus F}\beta_i^{(k)}\overline{\beta_i^{(l)}})\big) = \langle \tilde{\varphi}_F(u^{(k)}), \tilde{\varphi}_F(u^{(l)})\rangle_F,
$$

where $\tilde{\varphi}_F(u^{(k)})$
canonical $\mathbb{C}[\Omega]$ $\varphi(\beta^{(k)})\Pi_{i\in F}\beta_i^{(k)}$
ued inner produc where $\tilde{\varphi}_F(u^{(k)}) := (\varphi(\beta^{(k)}) \Pi_{i \in F} \beta_i^{(k)})^{-1} (\otimes_{i \in F} u_i^{(k)}) \otimes \lambda_{[\beta^{(k)}]_{\sim}}$ and $\langle \cdot, \cdot \rangle_F$ is the canonical $\mathbb{C}[\Omega_{I;\mathbb{C}}^{\text{ut}}]$ -valued inner product on $(\bigotimes_{i \in F} H_i) \otimes \mathbb{C}[\Omega_{I;\mathbb{C}}^{\text{ut}}]$. Therefore, $(\varphi^{(k)}, u_i^{(l)})_i)(\otimes$
 $(\varphi(\beta^{(k)})\Pi_i)$

alued inner

$$
\Phi(\langle \xi, \xi \rangle_{\mathbb{C}_{\rm ut}^{\otimes I}}) = \left\langle \sum\nolimits_{k=1}^n \alpha_k \tilde{\varphi}_F(u^{(k)}), \sum\nolimits_{k=1}^n \alpha_k \tilde{\varphi}_F(u^{(k)}) \right\rangle_F \geq 0.
$$

Next, we show that $\chi_{\Omega_{\text{HC}}^{\text{ut}}} \circ \Phi \circ E = \phi_1$. Let $\alpha \in \Pi_{i \in I} \mathbb{C}^\times$. If $\alpha \approx 1$, then $\chi_{\Omega_{\text{HC}}^{\text{ut}}} \circ$ $\Phi \circ E(\otimes_{i \in I} \alpha_i) = 0$ (as $\Phi(E(\otimes_{i \in I} \alpha_i)) \notin \mathbb{C} \cdot \lambda_{[1]_{\sim}} \setminus \{0\}$, whether or not $[\alpha]_{\sim} \in \Omega_{\text{IC}}^{\text{ut}}$)
and we also have $\phi_1(\otimes_{i \in I} \alpha_i) = 0$ If $\alpha \sim 1$, then $\otimes_{i \in I} \alpha_i = (\Pi_{i \in I} \alpha_i)(\otimes_{i \in I} 1)$ and we also have $\phi_1(\otimes_{i\in I}\alpha_i)=0$. If $\alpha \sim 1$, then $\otimes_{i\in I}\alpha_i = (\prod_{i\in I}\alpha_i)(\otimes_{i\in I}1)$ $(\Pi_{i\in I}\alpha_i)\lambda_{[1]_{\sim}}$, which implies that $\chi_{\Omega_{I;\mathbb{C}}^{\text{ut}}}(\Phi(\otimes_{i\in I}\alpha_i)) = \Pi_{i\in I}\alpha_i = \phi_1(\otimes_{i\in I}\alpha_i)$.
Thus, we have

Thus, we have

(4.4)
$$
\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}}(\Phi(\langle \xi,\eta \rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}})) = \langle \xi,\eta \rangle_{\phi_1} (\xi,\eta \in \bigotimes_{i \in I}^{\mathrm{unit}} H_i).
$$

As a consequence, if $\Phi(\langle \xi, \xi \rangle_{\mathbb{C}_{\text{cut}}^{\otimes I}}) = 0$, we know from Lemma [4.2](#page-13-0) (b) that $\xi = 0$. Thus, we have
 $(4.4) \qquad \chi_{\Omega_{I;\mathbb{C}}^{\text{ut}}}(\Phi(\langle \xi, \eta \rangle_{\mathbb{C}_{\text{ut}}^{\otimes I}})) = \langle \xi, \eta \rangle_{\phi_1} \quad (\xi, \eta \in \bigotimes_{i \in I}^{\text{unit}} H_i).$

As a consequence, if $\Phi(\langle \xi, \xi \rangle_{\mathbb{C}_{\text{ut}}^{\otimes I}}) = 0$, we know from Lemma 4.2 (b) that $\xi = 0$.

T (4.4) $\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}}(\Phi(\langle \xi, \eta \rangle_{\mathbb{C}}))$
As a consequence, if $\Phi(\langle \xi, \xi \rangle_{\mathbb{C}_{\zeta}^{\zeta}})$
This gives an inner product \mathbb{C}
the required isomorphism $\bar{\mathbb{Q}}_{i\in I}^{\phi_{1}}$ $\phi_1^{\phi_1} H_i \cong (\bar{\bigotimes}^{\text{mod}}_{i \in I} H_i) \bar{\otimes}_{\chi_{\Omega_{I;\mathbb{C}}^{\text{ut}}}} \mathbb{C}$ also follows from [\(4.4\)](#page-18-1). a consequence, if $\Phi(\langle \xi, \xi \rangle_{\mathbb{C}_{\text{ut}}^{\otimes I}}) = 0$, we know from Lemma 4

s gives an inner product $\mathbb{C}[\Omega_{I;\mathbb{C}}^{\text{ut}}]$ -module structure on $\bigotimes_{i \in I}^{\text{unit}}$

required isomorphism $\overline{\bigotimes}_{i \in I}^{\phi_1} H_i \cong (\overline{\bigotimes$ \sim

 $\bigoplus_{\omega \in G} \bigotimes_{i \in I}^{\omega} \mathbb{C}$ under the ^{*}-isomorphism Φ of Corollary [3.4\)](#page-10-0), every element in $(\bigotimes_{i\in I}^{\text{unit}} H_i) \otimes_{\mathbb{C}[G]} \mathbb{C}[G]$ is the required isomorphism $\overline{\mathbb{Q}}_{i\in I}^{\phi_1} H_i \cong (\overline{\mathbb{Q}}_{i\in I} H_i) \overline{\mathbb{Q}}_{\chi_{\Omega_{I;\mathbb{C}}^{\text{ut}}}} \mathbb{C}$ also follows f

(b) Since $\mathbb{Q}_{i\in I}^{\text{unit}} H_i$ is a $\mathbb{C}[G]$ -module (we identify $\mathbb{C}[G]$ with $\bigoplus_{\omega \in G} \$ $\prod_{i\in I}^{\text{unit}} H$, then the *-isomorphism Φ of Corollary 3.4), every element in ((
of the form $\xi \otimes_{\mathbb{C}[G]} 1$ for some $\xi \in \bigotimes_{i \in I}^{\text{unit}} H_i$. Moreover, if ξ
(4.5) $\langle \xi \otimes_{\mathbb{C}[G]} 1, \eta \otimes_{\mathbb{C}[G]} 1 \rangle_{(\overline{\bigotimes}_{i \in I}^{\text{mod}} \mathbb{C}) \overline{\ot$

$$
(4.5) \langle \xi \otimes_{\mathbb{C}[G]} 1, \eta \otimes_{\mathbb{C}[G]} 1 \rangle_{(\bar{\mathcal{O}}_{i \in I}^{\text{mod}} \mathbb{C}) \bar{\otimes}_{\mathcal{E}_G} C^*(G)} = \mathcal{E}_G(\Phi(\langle \xi, \eta \rangle_{\mathbb{C}^{\otimes I}_{\text{ut}}})) = \Phi(E_G(\langle \xi, \eta \rangle_{\mathbb{C}^{\otimes I}})),
$$

 $\bigoplus_{\omega \in G} \bigotimes_{i \in I}^{\omega} \mathbb{C}$ defined as where E_G is the linear truncation map from \mathcal{C} to $\bigoplus_{\omega \in G} \bigotimes_{i \in I} \mathcal{C}$ defined as
in part (a). Therefore, $\Phi(E_G(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}}))$ is a positive Hermitian $\mathbb{C}[G]$ -form on $\lim_{i \in I} H_i$. Obviously, $\chi_{\Omega_{I;\mathbb{C}}^{\text{ut}}} = \chi_G \circ \mathcal{E}_G$, and by [\(4.4\)](#page-18-1), fine
corn
H)

$$
\chi_G(\Phi(E_G(\langle \xi, \eta \rangle_{\mathbb{C}^{\otimes I}}))) = \chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}}(\Phi(\langle \xi, \eta \rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}})) = \langle \xi, \eta \rangle_{\phi_1} \quad (\xi, \eta \in \bigotimes_{i \in I}^{\mathrm{unit}} H).
$$

This implies that $\Phi(E_G(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}}))$ is non-degenerate (since $\langle \cdot, \cdot \rangle_{\phi_1}$ is non-degenerate by Lemma [4.2](#page-13-0)(b)). Now, equation [\(4.5\)](#page-18-2) tells us that the Hilbert $C^*(G)$ -module $\bar{\mathbf{\infty}}^{\operatorname{mod}}$ H . C⋅K⋅Nα

s that $\Phi(E_G(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}}))$ is non-degenerate (since $\langle \cdot, \cdot \rangle_{\phi_1}$ is non-degenerate

4.2 (b)). Now, equation (4.5) tells us that the Hilbert $C^*(G)$ -module
 $\bar{\otimes}_{\mathcal{E}_G} C^*(G)$ is the completion o

ich *C*[*G*]-valued inner product $\Phi(E_G(\langle \cdot, \cdot \rangle_{\mathbb{C}^{\otimes I}}))$. △

Let {*e*} be the trivial subgroup of $\Omega_{I; \mathbb{C}}^{ut}$. Since one can identify $E_{\{e\}}$ with ϕ_1

(through the argument of Theorem 4.8 (b)), one has Let $\{e\}$ be the trivial subgroup of $\Omega_{I,C}^{\text{ut}}$. Since one can identify $E_{\{e\}}$ with ϕ_1 cough the argument of Theorem 4.8(b)) one has (through the argument of Theorem 4.8 (b)), one has

$$
\bar{\bigotimes}_{i\in I}^{\phi_1} H_i \cong \ (\bar{\bigotimes}_{i\in I}^{\text{mod}} H_i) \bar{\otimes}_{\mathcal{E}_{\{e\}}} \mathbb{C}.
$$

Remark 4.9. (a) For any subgroup $G \subseteq \Omega_{I;\mathbb{C}}^{\text{ut}}$ and any faithful state φ on $C^*(G)$, the Hilbert space the Hilbert space **Remark 4.9.** (a) For any subgroup C
the Hilbert space
 $\left((\bigotimes_{i \in I}^{mod} H_i \right)$
induces an inner product on $\bigotimes_{i \in I}^{ind} H_i$.

$$
\Big(\big(\bar{\bigotimes}_{i\in I}^{\rm mod} H_i\big)\bar{\otimes}_{\mathcal{E}_G} C^*(G)\Big)\bar{\otimes}_{\varphi}\mathbb{C}
$$

(b) If $x \in \Pi_{i\in I}^0\mathbb{C}$ (see Example [2.2](#page-3-2)(b)), then $\sup_{i\in I}|x_i| < \infty$. This, together with the surjectivity of $\kappa_{\mathbb{C}}$ (see Proposition [4.6](#page-16-1) (a)), tells us that $\Gamma_{I;\mathbb{C}}$ is a group
under the multiplication: $[x]_{\mathbb{C}}$, $[y]_{\mathbb{C}} := [xy]_{\mathbb{C}}$ (where $(xy) := x, y$ for any $i \in I$) under the multiplication: $[x]_{\approx} \cdot [y]_{\approx} := [xy]_{\approx}$ (where $(xy)_i := x_i y_i$ for any $i \in I$). Moreover, $\kappa_{\mathbb{C}} : \Omega_{I;\mathbb{C}}^{\text{ut}} = \Omega_{I;\mathbb{C}}^{\text{unit}} \to \Gamma_{I;\mathbb{C}}$ is a group homomorphism, which induces a group of $\kappa_{\mathbb{C}}$ is a group homomorphism, which induces a surjective *-homomorphism $\bar{\kappa}_{\mathbb{C}} : C^*(\Omega_{I;\mathbb{C}}^{\text{ut}}) \to C^*(\Gamma_{I;\mathbb{C}}).$ (c) It is natural to ask whether $((\overline{\mathbb{Q}}_{\text{ref}}^{i})_{\text{ref}} + \mathbb{Z}_{\text{ref}}^{i})_{\text{ref}}$ is a group ler the multiplication: $[x]_{\approx} \cdot [y]_{\approx} := [xy]_{\approx}$ (where $(xy)_{i} := x_{i}y_{i}$ for any $i \in I$).

reover, $\kappa_{\mathbb{C}} : \Omega_{I; \mathbb{C}}^{u} = \$ under the multiplication: $[x]_{\approx} : [y]_{\approx} := [xy]_{\approx}$ (where $(xy)_i := x_i y_i$ for any $i \in I$).

Moreover, $\kappa_{\mathbb{C}} : \Omega_{I;\mathbb{C}}^{\text{ut}} \to \Gamma_{I;\mathbb{C}}$ is a group homomorphism, which induces a

surjective *-homomorphism $\bar{\kappa}_{\mathbb{C}} :$

 $x, y \in \Pi_{i \in I}^{\text{unit}} H_i$, we write $x \approx_{\mathbb{T}} y$ if there exists $\alpha \in \Pi_{i \in I} \mathbb{T}$ with $\alpha \approx 1$ such that $x = \alpha \cdot y$ of It is easy to check that $\approx_{\mathbb{T}}$ is an equivalence relation in general $x_i = \alpha_i y_i$ e.f. It is easy to check that $\approx_{\mathbb{T}}$ is an equivalence relation in general standing strictly between \sim and \approx Moreover one has standing strictly between ∼ and ≈. Moreover, one has $\prod \otimes_{i \in I} H_i$ canonically. Unfortunately, this is not t $x_i = \alpha_i y_i$ e.f. It is easy to check that $\approx_{\mathbb{T}}$ is an equivalence relation in general
standing strictly between \sim and \approx . Moreover, one has
 $\langle ((\otimes_{i \in I} x_i) \otimes_{\bar{\kappa}_{\mathbb{C}}} 1) \otimes_{\chi_{\Gamma_{I; \mathbb{C}}}} 1, ((\otimes_{i \in I} y_i) \ot$

$$
\big\langle ((\otimes_{i\in I} x_i) \otimes_{\bar{\kappa}_\mathbb{C}} 1) \otimes_{\chi_{\Gamma_{I;\mathbb{C}}}} 1, ((\otimes_{i\in I} y_i) \otimes_{\bar{\kappa}_\mathbb{C}} 1) \otimes_{\chi_{\Gamma_{I;\mathbb{C}}}} 1 \big\rangle = 0 \text{ whenever } x \not\approx_{\mathbb{T}} y,
$$

 $H_i = \mathbb{C}$, then $\approx_{\mathbb{T}}$ and \approx coincide, and one can show that the two Hilbert spaces $((\bar{\bigotimes}^{\rm mod}_{i\in I} \mathbb{C}) \bar{\otimes}_{\kappa_{\mathbb{C}}} C^{*}(\Gamma_{I;\mathbb{C}}))$ $I_{I,C}$ 1, $((\otimes_{i\in I} y_i) \otimes_{\bar{\kappa}_C} 1) \otimes_{\chi_{\Gamma_{I,C}}} 1$ = 0 whenever $x \not\approx y$. Note howev
 $\in I$ y_i = 0 whenever $x \not\approx y$. Note howev
 $\in \infty$ coincide, and one can show that the two l
 $\bar{\otimes}_{\chi_{\Gamma_{I,C}}} \mathbb{C}$ and $\prod \$ while $\langle \prod_{i \in I} \otimes_{i \in I} x_i, \prod_{i \in I} \otimes_{i \in I} y_i \rangle = 0$ wheneve
 $H_i = \mathbb{C}$, then $\approx_{\mathbb{T}}$ and \approx coincide, and one $((\bar{\otimes}_{i \in I}^{\text{mod}} \mathbb{C}) \bar{\otimes}_{\kappa_{\mathbb{C}}} C^*(\Gamma_{I; \mathbb{C}})) \bar{\otimes}_{\chi_{\Gamma_{I; \mathbb{C}}}} \mathbb{C}$ and $\prod_{i \in I} \otimes$

(a) It is clear that $\overline{\hat{\otimes}}_{i\in I}^{\text{mod}}\mathbb{C} = C^*(\Omega_{I;\mathbb{C}}^{ut})$. For any state φ on $C^*(\Omega_{I;\mathbb{C}}^{\text{ut}})$, the Hilbert space $(\bar{\otimes}_{i\in I}^{\text{mod}}\mathbb{C})\bar{\otimes}_{\varphi}\mathbb{C}$ is the GNS construction of φ .

(b) If G is a subgroup of $\Omega_{I;\mathbb{C}}^{\text{ut}}$, we have

$$
\big(\bar{\bigotimes}_{i\in I}^{\mathrm{mod}}\mathbb{C}\big)\bar{\otimes}_{\mathcal{E}_G}C^*(G)\;\cong\; \ell^2(\Omega_{I;\mathbb{C}}^{\mathrm{ut}}/G)\bar{\otimes}C^*(G).
$$

In fact, let $q : \Omega_{I;C}^{\text{ut}} \to \Omega_{I;C}^{\text{ut}}/G$ be the quotient map and $\sigma : \Omega_{I;C}^{\text{ut}}/G \to \Omega_{I;C}^{\text{ut}}$ be a group section. One has a hijection from $\Omega_{I;C}^{\text{ut}} \to (\Omega_{I;C}^{\text{ut}}/G) \times G$ conding ω to a cross section. One has a bijection from $\Omega_{I;\mathbb{C}}^{\text{ut}}$ to $(\Omega_{I;\mathbb{C}}^{\text{ut}}/G) \times G$ sending ω to $(a(\cdot), \pi(a(\cdot), -1), \cdot)$ $(q(\omega), \sigma(q(\omega)^{-1})\omega).$

GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS
This gives a bijective linear map $\Delta : \mathbb{C}[\Omega_{I;\mathbb{C}}^{ut}] \to \bigoplus$ $\bigoplus_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}/G}\mathbb{C}[G]$ such that for any $\omega \in \Omega_{I;\mathbb{C}}^{\mathrm{ut}}$ and $\varepsilon \in \Omega_{I;\mathbb{C}}^{\mathrm{ut}}/G$, MEGEBRAIC TEN

Ve linear map \angle
 $/G,$
 $\Delta(\lambda_\omega)_\varepsilon$:= $\Bigg\{$

$$
\Delta(\lambda_{\omega})_{\varepsilon} := \begin{cases} \lambda_{\sigma(\varepsilon^{-1})\omega} & \text{if } q(\omega) = \varepsilon \\ 0 & \text{otherwise.} \end{cases}
$$

 $\Delta(\lambda_{\omega})_{\varepsilon} := \begin{cases} \lambda_{\sigma(\varepsilon^{-1})\omega} & \text{if } q(\omega) = \varepsilon \\ 0 & \text{otherwise.} \end{cases}$

Let $\Phi : \bigotimes_{i \in I}^{\text{unit}} \mathbb{C} = \mathbb{C}_{ut}^{\otimes I} \to \mathbb{C}[\Omega_{I; \mathbb{C}}^{ut}]$ and $\varphi : \Pi_{i \in I} \mathbb{T} \to \mathbb{T}$ be as in Corollary [3.4.](#page-10-0)

Suppose that $\alpha, \beta \in \Pi_{t \in$ Suppose that $\alpha, \beta \in \Pi_{i \in I} \mathbb{C}^{\times}$. If $[\alpha \beta^{-1}]_{\sim}$ does not belong to G, then we have $E_G(\langle \otimes_{i \in I} \alpha_i, \otimes_{i \in I} \beta_i \rangle_{\mathbb{C} \otimes I}) = 0$ and

$$
\langle \Delta \circ \Phi(\otimes_{i \in I} \alpha_i), \Delta \circ \Phi(\otimes_{i \in I} \beta_i) \rangle_{\bigoplus_{\Omega_{I;C}^{\mathrm{out}}/G}^{\ell^2} \mathbb{C}[G]} = 0.
$$

On the other hand, if $[\alpha\beta^{-1}]_{\sim} \in G$, then

$$
\langle \Delta \circ \Phi(\otimes_{i \in I} \alpha_i), \Delta \circ \Phi(\otimes_{i \in I} \beta_i) \rangle_{\bigoplus_{\Omega_{1;C/G}^{\mathrm{pt}} \cap G}^{\ell^2} \mathbb{C}[G]}
$$
\n
$$
= \varphi(\alpha \beta^{-1}) \lambda_{[\alpha \beta^{-1}]_{\sim}} = \Phi(\otimes_{i \in I} \alpha_i \beta_i^{-1}) = \Phi(E_G(\langle \otimes_{i \in I} \alpha_i, \otimes_{i \in I} \beta_i \rangle_{\mathbb{C}^{\otimes I}})).
$$
\nThis shows that $\Delta \circ \Phi$ is an inner product $\mathbb{C}[G]$ -module isomorphism from $\bigotimes_{i \in I}^{\mathrm{unit}} \mathbb{C}[G]$ (equipped with the inner product $\mathbb{C}[G]$ -module structure as in Theorem 4.8(b))

(equipped with the inner product $\mathbb{C}[G]$ -module structure as in Theorem [4.8](#page-17-0)(b)) = $\varphi(\alpha \beta^{-1})\lambda$ [
This shows that Δ c
(equipped with the
onto $\bigoplus_{\Omega_{I;C}^{\text{out}}/G}^{\ell^2}$

5. Tensor products of *[∗]*-representations of *[∗]*-algebras

In this section, $\{(A_i, H_i, \Psi_i)\}_{i \in I}$ *is a family of unital* ^{*}-representations, in the *sense that* A_i *is a unital* ^{*}*-algebra,* H_i *is a Hilbert space and* $\Psi_i : A_i \to \mathcal{L}(H_i)$ *is a unital* **-homomorphism* $(i \in I)$ this section, $\{(A_i, H_i, \Psi_i)\}_{i \in I}$ is a family of unital *-representations, in the
se that A_i is a unital *-algebra, H_i is a Hilbert space and $\Psi_i : A_i \to \mathcal{L}(H_i)$ is a
tal *-homomorphism $(i \in I)$.
Suppose that $\Psi_0 := \til$

unital *-*homomorphism* ($i \in I$).

Suppose that $\Psi_0 := \tilde{\otimes}_{i \in I} \Psi_i$

sition 2.3 (c). It is easy to check

(5.1) $\langle \Psi_0(a)\xi, \eta \rangle_{\mathbb{C} \otimes I} = \langle \xi, \Psi_i \rangle$ sition [2.3](#page-4-0) (c). It is easy to check that v_{11} is a different component component $(i \in \mathfrak{S}_i)$

c). It is easy to check that $\Psi_0 := \tilde{\bigotimes}_{i \in \mathfrak{S}_i}$
 $\Psi_0(a)\xi, \eta \rangle_{\mathbb{C}^{\otimes I}} = \langle \eta \rangle$

(5.1)
$$
\langle \Psi_0(a)\xi, \eta \rangle_{\mathbb{C}^{\otimes I}} = \langle \xi, \Psi_0(a^*)\eta \rangle_{\mathbb{C}^{\otimes I}} \quad (a \in \bigotimes_{i \in I} A_i; \xi, \eta \in \bigotimes_{i \in I} H_i).
$$

Furthermore, one has the following result (which is more or less well known).

(5.1) $\langle \Psi_0(a)\xi, \eta \rangle_{\mathbb{C}^{\otimes I}} = \langle \xi, \Psi_0(a^*)\eta \rangle_{\mathbb{C}^{\otimes I}} \quad (a \in \bigotimes_{i \in I} A_i; \xi, \eta \in \bigotimes_{i \in I} H_i).$
Furthermore, one has the following result (which is more or less well known).
Proposition 5.1. *For any* $\mu \in \Omega_{I$ sentation $\bigotimes_{i\in I}^{\mu} \Psi_i : \bigotimes_{i\in I}^e A_i \to \mathcal{L}(\overline{\bigotimes}_{i\in I}^{\mu} H_i)$. If all the Ψ_i are injective, then so is $\bigotimes_{i\in I}^{\mu}\Psi_i$. **oposition 5.1.** For any $\mu \in \Omega_{I;H}^{\text{unit}}$, the map $\tilde{\bigotimes}_{i \in I} \Psi_i$
tation $\bigotimes_{i \in I}^{\mu} \Psi_i : \bigotimes_{i \in I}^e A_i \to \mathcal{L}(\bigotimes_{i \in I}^{\mu} H_i)$. If all the Ψ
 $\in I^{\Psi_i}$.
Consequently, one has a unital *-representation of \mathbf{r}

Consequently, one has a unital *-representation of $\bigotimes_{i\in I}^{e} A_i$ on the Hilbert space $\bar{\mathbf{\mathcal{S}}}^{\phi_1}_{i\in I}H_i$. However, it seems impossible to extend it to a unital *-representation of if $\bigotimes_{i\in I}^{\mu} \Psi_i$.

Consequently, one has a unital *-representation of $\bigotimes_{i\in I}^{\ell} A_i$ on the Hilbert space $\overline{\bigotimes}_{i\in I}^{\phi_1} H_i$. However, it seems impossible to extend it to a unital *-representation of $\bigotimes_{$ which such an extension is possible, is the subalgebra $\bigotimes_{i\in I}^{ut}A_i$. Example [5.6](#page-23-0) (a) also tells us that it is probably the right subalgebra to consider.

C.K. No

Let us digress a little bit and give another [∗]-representation of $\bigotimes_{i \in I}^{\text{ut}} A_i$, which

direct consequence of Proposition 5.1 Theorem 3.2(a) and Theorem 4.1 in [5] is a direct consequence of Proposition [5.1,](#page-20-0) Theorem [3.2](#page-9-0) (a) and Theorem 4.1 in [\[5\]](#page-26-10) (it is not hard to verify that the representation as given in Theorem 4.1 of [\[5\]](#page-26-10) is ³⁵⁰ C. K. Note that the sum of $\mathbb{Q}_{i\in I}^{\text{ut}}A_i$, which is a direct consequence of Proposition 5.1, Theorem 3.2 (a) and Theorem 4.1 in [5] (it is not hard to verify that the representation as given in Theorem 4.1 of not canonical since it depends on the choice of a cross section $c : \Omega_{I;A}^{\text{ut}} \to \Pi_{i \in I} U_{A_i}$
(see Bemark 3.3.(a)) (see Remark $3.3(a)$ $3.3(a)$).

Corollary 5.2. *Suppose that the* Ψ_i *are injective. For any* $\mu \in \Omega_{I;H}^{\text{unit}}$, the injection Ω_{μ}^{μ} is the interesting mitring in $\Omega_{\mu}^{\text{unit}}$, $\Lambda_{I;H}$ ($\overline{\Omega}_{I}^{\mu}$, H) Ω $\frac{1}{26}$ induces an injective unital *-representation of $\mathbb{Q}_{i\in I}^{\text{ut}}\Psi_i$ is impective). Note however, that such a *-representation is
t canonical since it depends on the choice of a cross section $c : \Omega_{I;A}^{\text{ut}} \to \Pi_{i\in I}U$ $\ell^2(\widetilde{\Omega}_{I;A}^{\mathrm{ut}})$.

Let us now return to the discussion of the tensor product type representation Corollary 5.2. Suppose that the Ψ_i are injective. For any $\mu \in \Omega_{I;H}^{\text{unif}}$, the injection $\bigotimes_{i \in I}^{\mu} I_i$ induces an injective unital *-representation of $\bigotimes_{i \in I}^{\text{ut}} A_i$ on $(\overline{\bigotimes}_{i \in I}^{\mu} H_i) \otimes$ $\ell^2(\Omega_{$ Let us now return to the discussion of the tensor product type representation
of $\bigotimes_{i\in I}^{\text{ut}} A_i$. Observe that $\{\Psi_i\}_{i\in I}$ induces a canonical action $\alpha^{\Psi} : \Omega_{I;A}^{\text{unt}} \times \Omega_{I;H}^{\text{unit}} \rightarrow$
 $\Omega_{I;H}^{\text{unit}}$. For sim ty, we will denote $\alpha^{\Psi}_{\omega}(\mu)$ by $\omega \cdot \mu$ ($\omega \in \Omega^{\text{ut}}_{I;A}; \mu \in \Omega$

 $\prod_{i\in I}^{\varphi_1} H_i$. $\sum_{i=1}^{n} E_i$. For simplicity, we will denote $\alpha_{\omega}^{\Psi}(\mu)$ by $\omega \cdot \mu$ (*i*
 eorem 5.3. (a) *The map* $\tilde{\otimes}_{i \in I} \Psi_i$ *induces a unitional i* $\sum_{i \in I} A_i \to \mathcal{L}(\bar{\otimes}_{i \in I}^{\phi_1} H_i)$.

(b) $(\bar{\otimes}_{i \in I}^{\phi_1} H_i, (\bar{\$ **Theorem 5.3.** (a) The map $\tilde{\otimes}_{i \in I} \Psi_i$ induces a unital *-representation $\otimes_{i \in I}^{\phi_1} \Psi_i$: $_{i\in I}^{\mu}H_i, \bigotimes_{i\in I}^{\mu}\Psi_i$. (e) **IF** $\sum_{i \in I} A_i \rightarrow \mathcal{L}(\bar{\otimes}_{i \in I}^{\phi_1} H_i)$ **.**

(b) $(\bar{\otimes}_{i \in I}^{\phi_1} H_i, (\bar{\otimes}_{i \in I}^{\phi_1} \Psi_i)|_{\bar{\otimes}_{i \in I}^e A_i}) = \bigoplus_{\mu \in I}$

(c) If all Ψ_i are injective, then so is $\bar{\otimes}_{i \in I}^{\phi_1}$ $\bigotimes_{i \in I} A_i \to \bigcup_{i \in I} B_i$.

(b) $\big(\bar{\bigotimes}_{i \in I}^{\phi_1} H_i, \big(\bar{\bigotimes}_{i \in I}^{\phi_1} H_i\big)$.

(c) If all Ψ_i are injection.
 Proof. (a) Set $\Psi_0 := \tilde{\bigotimes}$

so is $\bigotimes_{i\in I}^{\phi_1}\Psi_i$.

 $i\in I$ **Ψ**_{*i*}. For any $\mu \in \Omega_{I;H}^{\text{unit}}$, $\omega \in \Omega_{I;A}^{\text{ut}}$ and $a \in \Pi_{i\in I}^{\omega}A_i$, it is clear that Ψ_i .

{\vdots} \vdots \vdots $\bigotimes{i=1}^{\omega \cdot \mu}$

(5.2)
$$
\Psi_0(\otimes_{i\in I} a_i) \big(\bigotimes_{i\in I}^{\mu} H_i\big) \subseteq \bigotimes_{i\in I}^{\omega \cdot \mu} H_i.
$$

(5.2) $\Psi_0(\otimes_{i \in I} a_i) (\bigotimes_{i \in I}^{\mu} H_i) \subseteq \bigotimes_{i \in I}^{\omega \cdot \mu} H_i$

Suppose that $u \in \omega$ and $F \in \mathfrak{F}$ are such that $a_i = u_i$ for i where $x \in \mu$, $F' \in \mathfrak{F}$ with $F \subseteq F'$ and $\xi_0 \in \bigotimes_{i \in F'} H_i$, then

Suppose that
$$
u \in \omega
$$
 and $F \in \mathfrak{F}$ are such that $a_i = u_i$ for $i \in I \setminus F$. If $\xi = J_{F'}^x(\xi_0)$
where $x \in \mu$, $F' \in \mathfrak{F}$ with $F \subseteq F'$ and $\xi_0 \in \bigotimes_{i \in F'} H_i$, then

$$
\langle \Psi_0(\otimes_{i \in I} a_i)\xi, \Psi_0(\otimes_{i \in I} a_i)\xi \rangle_{\mathbb{C}^{\otimes I}} = \langle (\bigotimes_{i \in F} \Psi_i(a_i) \otimes \text{id})\xi_0, (\bigotimes_{i \in F} \Psi_i(a_i) \otimes \text{id})\xi_0 \rangle(\otimes_{i \in I} 1).
$$
This implies that $\Psi_0(\otimes_{i \in I} a_i)$ is bounded on $(\bigotimes_{i \in I}^{\text{unit}} H_i, \langle \cdot, \cdot \rangle_{\phi_1})$ (see Theorem 4.4 (a) and Proposition 4.1 (b)) and produces a unital homomorphism $\bigotimes_{i \in I}^{\phi_1} \Psi_i : \bigotimes_{i \in I}^{\text{unit}} A_i$

 $\psi_{i\in I}^{\phi_1}\Psi_i : \bigotimes_{i\in I}^{\mathrm{ut}}$ $\prod_{i=1}^{i}$ $\langle (\bigotimes_{i \in F} \Psi_i(a_i) \otimes id)\xi_0, (\bigotimes_{i \in F} \Psi_i(a_i) \otimes id)\xi_0 \rangle (\otimes_{i \in I} 1).$
This implies that $\Psi_0(\otimes_{i \in I} a_i)$ is bounded on $(\bigotimes_{i \in I}^{\text{unit}} H_i, \langle \cdot, \cdot \rangle_{\phi_1})$ (see Theorem 4.4 (a) and Proposition 4.1 (b)) and produces a unit $\langle (\bigotimes_{i \in F} \Psi_i(a_i) \otimes id)\xi_0, (\bigotimes_{i \in I} A_i)$
at $\Psi_0(\otimes_{i \in I} a_i)$ is bounded on $(\bigotimes_{i \in I}^{\text{unit}} H_i)$
n 4.1(b)) and produces a unital homo
. Now, relation [\(5.1\)](#page-20-2) tells us that $\bigotimes_{i \in I}^{\phi_1}$ (b) This part follows directly from the argument of part (a). s implies that $\Psi_0(\otimes)$

[Proposition 4.1 (b)
 $\mathcal{L}(\bar{\otimes}_{i\in I}^{\phi_1} H_i)$. Now,

(b) This part follow

(c) Set $\Psi := \bigotimes_{i\in I}^{\phi_1}$

 $\psi_{i\in I}\Psi_i$. Suppose that $v^{(1)},\ldots,v^{(n)}\in \Pi_{i\in I}U_{A_i}$ are mutually and Proposition 4.1(b)) and produces a unital nor
 $\rightarrow \mathcal{L}(\bar{\otimes}_{i\in I}^{\phi_1}H_i)$. Now, relation (5.1) tells us that \otimes

(b) This part follows directly from the argument

(c) Set $\Psi := \bigotimes_{i\in I}^{\phi_1} \Psi_i$. Suppose that $i \in F A_i$ and $a^{(k)} := J_F^{v^{(k)}}(b^{(k)})$ $(k = 1, \ldots, n)$ are such that

$$
\Psi\bigl(\sum\nolimits_{k=1}^n a^{(k)}\bigr) = 0.
$$

By induction, it suffices to show that $a^{(1)} = 0$.

By replacing $a^{(k)}$ with $(v^{(1)})^{-1}a^{(k)}$ if necessary, we may assume that $v_i^{(1)} = e_i$
i *I*) If $n = 1$ we take an arbitrary $\xi \in \prod_{i \in I} G_i(H_i)$ If $n > 1$ we claim that $(i \in I)$. If $n = 1$, we take an arbitrary $\xi \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$. If $n > 1$, we claim that there exists $\xi \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$ such that

(5.3)
$$
\xi \sim [V_i^{(k)} \xi_i]_{i \in I} \qquad (k = 2, \ldots, n),
$$

where $V_i^{(k)} := \Psi_i(v_i^{(k)})$. In fact, if $k \in \{2, ..., n\}$ and $i \in I^k := \{i \in I : v_i^{(k)} \neq e_i\}$ (which is an infinite set), the subset $\mathfrak{S}_1(H_i) \cap \ker(V_i^{(k)} - id_{H_i})$ is nowhere dense in $\mathfrak{S}_1(H_i)$ as ker $(V_i^{(k)} - id_{H_i})$ is a proper closed subspace of H_i (note that Ψ_i is injective). For any $i \in I$, we consider $N_i := \{k \in \{2, ..., n\} : i \in I^k\}$. By the Baire category theorem, for every $i \in I$, one can c injective). For any $i \in I$, we consider $N_i := \{k \in \{2,\ldots,n\} : i \in I^k\}$. By the Baire category theorem, for every $i \in I$, one can choose $\xi_i \in \mathfrak{S}_1(H_i) \setminus \bigcup_{k \in N_i} \ker(V_i^{(k)} - \mathrm{id}_L)$ Now $\xi := [\xi_i]_{i \in I}$ will satisfy relation (5.3) id_{H_i}). Now, $\xi := [\xi_i]_{i \in I}$ will satisfy relation [\(5.3\)](#page-22-1). very $i \in I$, c
will satisfy
 $H_i) \subseteq \bigotimes_{i \in I}^{\xi}$
 $H_i \cap \sum_{k}^{n}$

Since $\Psi(a^{(1)})\big(\bigotimes_{i\in I}^{\xi}H_i\big) \subseteq \bigotimes_{i\in I}^{\xi}H_i$ and

$$
\bigotimes_{i \in I}^{\xi} H_i \cap \sum_{k=2}^n \Psi(a^{(k)}) \big(\bigotimes_{i \in I}^{\xi} H_i \big) = \{0\}
$$

(because of Theorem [2.5](#page-5-1) as well as [\(5.2\)](#page-21-1) and [\(5.3\)](#page-22-1)), we have $\Psi(a^{(1)})|_{\bigotimes_{i\in I}^{s}H_i}=0$. Therefore, part (b) and Proposition [5.1](#page-20-0) tells us that $a^{(1)} = 0$.

Remark 5.4. (a) By the argument proving Theorem [5.3](#page-21-0) (c), if all the Ψ_i are injective, then α^{Ψ} is *strongly faithful* in the sense that for any finite subset $F \subseteq$ $\Omega_{I;H}^{\text{ut}} \setminus \{e\}$, there exists $\mu \in \Omega_{I;H}^{\text{unit}}$ with $\omega \cdot \mu \neq \mu \ (\omega \in F)$.

(b) If $y, z \in \Pi_{i \in I} H_i$ are C_0 -sequences and $u, v \in \Pi_{i \in I} U_{A_i}$, then

(5.4)
$$
y \approx z
$$
 if and only if $[\Psi_i(u_i)y_i]_{i \in I} \approx [\Psi_i(u_i)z_i]_{i \in I}$

and $[\Psi_i(u_i)y_i]_{i\in I} \approx [\Psi_i(v_i)y_i]_{i\in I}$ whenever $u \sim v$. Thus, $\{\Psi_i\}_{i\in I}$ induces an action $\tilde{\alpha}^{\Psi} : \Omega_{I;A}^{\text{uf}} \times \Gamma_{I;H} \to \Gamma_{I;H}$. Again, we write $\omega \cdot \gamma$ for $\tilde{\alpha}_{\omega}^{\Psi}(\gamma)$ $(\omega \in \Omega_{I;A}^{\text{uf}}; \gamma \in \Gamma_{I;A})$. The map κ_H in Proposition [4.6](#page-16-1) (a) is *equivariant* in the sense that $\kappa_H \circ \alpha_\omega^{\Psi} = \tilde{\alpha}_\omega^{\Psi} \circ \kappa_H$
 $(\omega \in \Omega^{\mathfrak{U}})$ $(\omega \in \Omega_{I;A}^{\text{ut}}).$

(c) If all the A_i are C^{*}-algebras and all the Ψ_i are irreducible, then α^{Ψ} is transitive. $C(\omega \in \Omega_{I,A}^{ut})$.

Corollary 5.5. *There exists a unital* ^{*}*-representation* $\prod_{i\in I} \otimes_{i\in I} \Psi_i$: $\otimes_{i\in I}^{ut} A_i \rightarrow$
 $C(\prod_{i\in I} \otimes_{i\in I} H_i)$ each that for any $u \in \Omega^{unit}$ are Ω^{ut} and $h \in \Omega^{ut}$ at Ω^{ut} .

(c) If all the A_i are C^* -algebras and all the Ψ_i are irreducible,
transitive.
Corollary 5.5. There exists a unital *-representation $\prod_{i\in I} \otimes_{i\in I} \Psi_i$:
 $\mathcal{L}(\prod_{i\in I} A_i)$ such that for any $\mu \in \Omega_{I;H}^{\text{unit$ transitive.

Corollary 5.5. There e $\mathcal{L}(\prod \otimes_{i \in I} H_i) \text{ such that } j$ (5.5) $(\prod \otimes_{i \in I} \Psi_i))$

(5.5)
$$
\left(\prod \otimes_{i \in I} \Psi_i\right)(b) \circ \tilde{\Upsilon}^{\mu} = \tilde{\Upsilon}^{\omega \cdot \mu} \circ \left(\bigotimes_{i \in I}^{\phi_1} \Psi_i\right)(b)\big|_{\tilde{\bigotimes}_{i \in I}^{\mu} H_i},
$$

where $\tilde{\Upsilon}^{\mu}$ *is as in Proposition* [4.6](#page-16-1) (b).

Proof. By Proposition [4.6](#page-16-1)(b), there is a bounded linear map

in Proposition 4.6 (b).
\nsition 4.6 (b), there is a bounded linear map
\n
$$
\left(\prod \otimes_{i\in I} \Psi_i\right)(b) : \prod \otimes_{i\in I}^{\kappa_H(\mu)} H_i \to \prod \otimes_{i\in I}^{\omega \cdot \kappa_H(\mu)} H_i
$$

such that equality (5.5) holds (see also Remark 5.4 (b)). Since we have Z.

$$
\sup_{\mu \in \Omega_{I;H}^{\text{unit}}} ||(\bigotimes_{i \in I}^{\phi_1} \Psi_i)(b)|_{\bar{\bigotimes}_{i \in I}^{\mu} H_i} || < \infty
$$

(because of Theorem [5.3](#page-21-0) (a)), we know from Proposition [4.6](#page-16-1) (a) and Lemma 4.1.1 in [\[20\]](#page-27-3) that $(\prod \otimes_{i \in I} \Psi_i)(b)$ induces an element in $\mathcal{L}(\prod \otimes_{i \in I} H_i)$, which clearly gives a $*$ -representation. \Box cause of Theorem 5.3(a)), we know from Proposition 4.6(a) and Lemma 4.1.1

20] that $(\prod \otimes_{i\in I} \Psi_i)(b)$ induces an element in $\mathcal{L}(\prod \otimes_{i\in I} H_i)$, which clearly gives

representation.

It is natural to ask if $\prod \otimes_{i\$

is never injective as can be seen from Example 5.6 (b) and the discussion following it.

Example 5.6. For any $i \in I$, let $A_i = \mathbb{C} = H_i$ and let $\iota_i : A_i \to \mathcal{L}(H_i)$ be the canonical map. Suppose that Φ φ and $\hat{\Phi}$ are as in Example 4.5 canonical map. Suppose that Φ , φ and $\tilde{\Phi}$ are as in Example [4.5.](#page-15-1)

(a) Let $\Lambda : \mathbb{C}[\Omega_{I;\mathbb{C}}^{\text{ut}}] \to \mathcal{L}(\ell^2(\Omega_{I;\mathbb{C}}^{\text{ut}}))$ be the left regular representation. For every $\lambda \in \Pi$ or \mathbb{R}^n one has $\alpha, \beta \in \Pi_{i \in I} \mathbb{T}$, one has **9.0.** For any $i \in$
map. Suppose tha
 $i \Lambda : \mathbb{C}[\Omega_{I;\mathbb{C}}^{ut}] \to \mathcal{L}(\Omega_{I}^{T}, \text{one has})$
 $\hat{\Phi}^* \circ \Lambda(\lambda_{[\alpha]_{\sim}}) \circ \hat{\Phi}$

(a) Let
$$
\Lambda : \mathbb{C}[\Omega_{I;\mathbb{C}}^{ut}] \to \mathcal{L}(\ell^2(\Omega_{I;\mathbb{C}}^{ut}))
$$
 be the left regular representation.
\n $\alpha, \beta \in \Pi_{i \in I} \mathbb{T}$, one has
\n
$$
(\hat{\Phi}^* \circ \Lambda(\lambda_{[\alpha]_{\sim}}) \circ \hat{\Phi})(\otimes_{i \in I} \beta_i) = \varphi(\alpha^{-1}) \otimes_{i \in I} \alpha_i \beta_i
$$
\n
$$
= (\bigotimes_{i \in I}^{\phi_1} \iota_i) (\Phi^{-1}(\lambda_{[\alpha]_{\sim}}))(\otimes_{i \in I} \beta_i).
$$
\nConsequently, $\bigotimes_{i \in I}^{\phi_1} \iota_i$ can be identified with Λ (under Φ and $\hat{\Phi})$).

(b) Let $\alpha \in \Pi_{i \in I} \mathbb{T}$ be such that $\alpha \approx 1$ but $\alpha \approx 1$ with $\Pi_{i \in I} \alpha_i = 1$. If $\beta \in \Pi_{i \in I} \mathbb{C}$
Co-sequence with $\|\Pi \otimes_{i \in I} \beta_i\| = 1$ one has $\|\Pi \otimes_{i \in I} \alpha_i \beta_i\| = 1$ and is a C_0 -sequence with $\|\prod \otimes_{i\in I} \beta_i\| = 1$, one has $\|\prod \otimes_{i\in I} \alpha_i \beta_i\| = 1$ and (b) Let $\alpha \in \Pi_{i\in I}$ is besteen that $\alpha \approx 1$ but $\alpha \approx 1$ which $\Pi_{i\in I}$ is a C_0 -sequence with $\|\prod_{i\in I} \beta_i\| = 1$, one has $\|\prod_{i\in I} \alpha_i \alpha_i \beta_i\|$
 $\langle \prod_{i\in I} \alpha_i \beta_i, \prod_{i\in I} \beta_i \rangle = 1$, which imply that $\prod_{i\in I} \alpha_i$

$$
\langle \prod \otimes_{i \in I} \alpha_i \beta_i, \prod \otimes_{i \in I} \beta_i \rangle = 1,
$$

= $\prod \otimes_{i \in I} \beta_i$. Therefore, $(\prod \otimes_{i \in I} \iota_i)(\otimes_{i \in I} \alpha_i)$ = id but $\otimes_{i\in I} \alpha_i \neq \otimes_{i\in I} 1$. Consequently, $\prod \otimes_{i\in I} \iota_i$ is not injective (actually, $(\prod \otimes_{i \in I} \iota_i) \circ \Phi^{-1}$ is not injective as a group representation of $\Omega_{I;\mathbb{C}}^{\mathrm{ut}}$. ch imply that $\prod_{i \in I} \alpha_i \beta_i$
but $\otimes_{i \in I} \alpha_i \neq \otimes_{i \in I} 1$. Contains $\otimes_{i \in I} \iota_i$ o Φ^{-1} is not injective
In general, even $(\prod_{i \in I} \otimes_{i \in I} \Psi_i)$ $\overline{\cdot}$

 $\int_{\mathbb{R}^d \in \mathbb{R}^d} \int_{\mathbb{R}^d} e_i e^{-\frac{1}{2} \pi i}$ is not injective. In fact, suppose that α is as above. For any C_0 -sequence $\xi \in \Pi_{i \in I} H_i$, with $\|\prod_{i \in I} \otimes_{i \in I} \xi_i\| = 1$, the same
prounded as Example 5.6(b) tells us that $\Pi \otimes_{i \in I} \alpha_i \xi_i = \Pi \otimes_{i \in I} \xi_i$. Thus argument as Example [5.6](#page-23-0) (b) tells us that $\prod \otimes_{i\in I} \alpha_i \xi_i = \prod \otimes_{i\in I} \xi_i$.

argument as Example 5.6 (b) tells us that $\prod \otimes_{i\in I} \alpha_i \xi_i = \prod \otimes_{i\in I} \xi_i$. Thus, \overline{H} $\overline{$

$$
\left(\prod \otimes_{i\in I} \Psi_i\right)(\otimes_{i\in I} e_i - \otimes_{i\in I} \alpha_i e_i) = 0.
$$

On the other hand, by Theorem [5.3](#page-21-0) and Corollary [5.5,](#page-22-3) there exist canonical [∗]-homomorphisms

On the other hand, by Theorem 5.3 and Corollary 5.5, there exist canonic
homomorphisms

$$
J^{\phi_1}: \bigotimes_{i \in I}^{\mathrm{ut}} \mathcal{L}(H_i) \to \mathcal{L}(\overline{\bigotimes}_{i \in I}^{\phi_1} H_i) \text{ and } J^{\Pi}: \bigotimes_{i \in I}^{\mathrm{ut}} \mathcal{L}(H_i) \to \mathcal{L}(\prod \otimes_{i \in I} H_i).
$$

Notice that J^{ϕ_1} is injective but J^{Π} is never injective.

Corollary 5.7. Let $\pi_i : G_i \to U_{\mathcal{L}(H_i)}$ be a unitary representation of a group G_i , *for each* $i \in I$. GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS
 Tollary 5.7. Let π_i : $G_i \rightarrow U_{\mathcal{L}(H_i)}$ be a unitary representation of a group G_i ,

each $i \in I$.

(a) There exist canonical unitary representations $\bigotimes_{i \in I}^{\phi_1} \pi_i$ a

Π**i orollary** 5.7.
 i e *ach* $i \in I$.

(a) *There* e
 $i \in I$ *G*_{*i*} on $\overline{\mathbb{Q}}_{i \in I}^{0}$ 7. Let $\pi_i: G_i \to U_{\mathcal{L}(H_i)}$ be a unit
 exist canonical unitary represent
 $\phi_1^{\phi_1}$
 ϕ_i^{ϵ} and $\prod \otimes_{i \in I} H_i$ respectively.

(b) *If the induced* *-representation $\hat{\pi}_i : \mathbb{C}[G_i] \to \mathcal{L}(H_i)$ *is injective for all* $i \in I$ *, the induced* **-exist canonical unitary representations* $\bigotimes_{i \in I}^{\phi_1} \pi_i$ and $\Pi_{i \in I}G_i$ on $\bar{\bigotimes}_{i \in I}^{\phi_1}H_i$ and $\Pi \otimes_{i \in I}H_i$ respectively.

(b) If the induced *-representation $\hat{\pi}_i : \mathbb{C}[G_i] \to \mathcal{L}(H_i$ *Proof.* (a) Let $\overline{\otimes}_{i\in I}^{\phi_1} H_i$ and $\prod \otimes_{i\in I} H_i$ respectively.

(b) *If the induced* *-*representation* $\hat{\pi}_i : \mathbb{C}[G_i] \to \mathcal{L}(H_i)$
 Proof. (a) Let $\overline{\otimes}_{i\in I}^{\text{ut}} \pi_i := \Theta_{\mathcal{L}(H)} \circ \Pi_{i\in I} \pi_i : \Pi_{i\in I} G_i$

Proof. (a) Let $\bigotimes_{i\in I}^{\text{ut}}\pi_i:=\Theta_{\mathcal{L}(H)}\circ \Pi_{i\in I}\pi_i:\Pi_{i\in I}G_i\to \bigotimes_{i\in I}^{\text{ut}}\mathcal{L}(H_i)$. Then

\n
$$
\text{used *-representation } \widehat{\otimes_{i \in I}^{\mathfrak{q}_i}} \text{ of } \widehat{\mathbb{C}}[\Pi_{i \in I}G_i] \text{ is also injective.}
$$
\n

\n\n (a) Let $\bigotimes_{i \in I}^{\mathfrak{u}_t} \pi_i := \Theta_{\mathcal{L}(H)} \circ \Pi_{i \in I} \pi_i : \Pi_{i \in I}G_i \to \bigotimes_{i \in I}^{\mathfrak{u}_t} \mathcal{L}(H_i).$ Then\n

\n\n $\bigotimes_{i \in I}^{\phi_1} \pi_i := J^{\phi_1} \circ \bigotimes_{i \in I}^{\mathfrak{u}_t} \pi_i \text{ and } \prod \otimes_{i \in I} \pi_i := J^{\Pi} \circ \bigotimes_{i \in I}^{\mathfrak{u}_t} \pi_i$ \n

are the required representations.

 $\bigotimes_{i\in I}^{\phi_1} \pi_i := J^{\phi_1} \circ \bigotimes_{i\in I}^{\text{ut}} \pi_i$ and $\prod \otimes_{i\in I} \pi_i := J^{\Pi} \circ \bigotimes_{i\in I}^{\text{ut}} \pi_i$
the required representations.
(b) By Theorem [5.3](#page-21-0) (c), $\bigotimes_{i\in I}^{\phi_1} \hat{\pi}_i$ is injective. As $\bigotimes_{i\in I}^{\phi_1} \pi_i$ is t $\phi_1 \phi_1 \hat{\pi}_i$ on $\mathbb{C}[\Pi_{i \in I} G_i]$ (see Example [3.1](#page-8-0)(a)), it is also injective. \Box are the required representations.

(b) By Theorem 5.3 (c), $\bigotimes_{i \in I}^{\phi_1} \hat{\pi}_i$ is injective. As $\widehat{\bigotimes}_{i \in I}^{\phi_1} \pi_i$ is the rest
 $\bigotimes_{i \in I}^{\phi_1} \hat{\pi}_i$ on $\mathbb{C}[\Pi_{i \in I}G_i]$ (see Example 3.1(a)), it is also i

Proof. Let $\tau_i : \mathbb{C} \to A_i$ be the canonical unital map and set $\check{\Psi}_i := \Psi_i \circ \tau_i$ $(i \in I)$.
Suppose that $\alpha, \beta \in \Pi$ is I^{w} with $\alpha \not\approx \beta$ and $\xi \in \Pi^{\text{unit}} H$. Then $[\alpha, \xi]_{i \in I} \not\approx [\beta, \xi]_{i \in I}$ Suppose that $\alpha, \beta \in \prod_{i \in I} \mathbb{T}$ with $\alpha \not\approx \beta$ and $\xi \in \prod_{i \in I}^{\text{unit}} H_i$. Then $[\alpha_i \xi_i]_{i \in I} \not\approx [\beta_i \xi_i]_{i \in I}$
and the two unit vectors and the two unit vectors and $\xi \in \Pi_{i \in I}^{\text{unit}} H_i$.

and $\xi \in \Pi_{i \in I}^{\text{unit}} H_i$.

Suppose that
$$
\alpha, \beta \in \Pi_{i \in I} \mathbb{T}
$$
 with $\alpha \not\approx \beta$ and $\xi \in \Pi_{i \in I}^{\text{unif}} H_i$. Then $[\alpha_i \xi_i]_{i \in I} \not\approx [\beta_i \xi_i]_{i \in I}$
and the two unit vectors

$$
(\prod \otimes_{i \in I} \check{\Psi}_i)(\otimes_{i \in I} \alpha_i)(\prod \otimes_{i \in I} \xi_i) \text{ and } (\prod \otimes_{i \in I} \check{\Psi}_i)(\otimes_{i \in I} \beta_i)(\prod \otimes_{i \in I} \xi_i)
$$

are orthogonal. Consequently, $\dim (\prod \otimes_{i \in I} \check{\Psi}_i)(\mathbb{C}_{\text{out}}^{\otimes I}) > 1$. As $(\prod \otimes_{i \in I} \Psi_i) \circ$
 $(\bigotimes_{i \in I} \tau_i) = \prod \otimes_{i \in I} \check{\Psi}_i$, we have $(\prod \otimes_{i \in I} \check{\Psi}_i)(\mathbb{C}_{\text{out}}^{\otimes I}) \subset Z((\prod \otimes_{i \in I} \Psi_i)(\bigotimes_{i \in I} \mathcal{H}_i))$

if the two unit vectors
 $\prod \otimes_{i \in I} \check{\Psi}_i$ $(\otimes_{i \in I} \alpha_i)$ $(\prod \otimes_{i \in I} \xi_i)$ and $(\prod \otimes_{i \in I} \check{\Psi}_i)$ $(\otimes_{i \in I} \check{\Psi}_i)$

orthogonal. Consequently, dim $(\prod \otimes_{i \in I} \check{\Psi}_i)$ $(\mathbb{C}_{\text{ut}}^{\otimes I}) >$
 $\iota \in I^{\tau_i}$ $=\prod \otimes_{i \in$ $\otimes_{i\in I} \beta_i$) ($\prod \otimes_{i\in I} \xi_i$)

1. As ($\prod \otimes_{i\in I} \Psi_i$) ($\bigcap_{i\in I} \otimes_{i\in I} \xi_i$)

uppert also shows that $(\prod \otimes_{i\in I} \check{\Psi}_i)(\otimes_{i\in I} \alpha_i)(\prod \otimes_{i\in I} \xi_i)$ and $(\prod \otimes_{i\in I} \check{\Psi}_i)(\otimes_{i\in I} \beta_i)(\prod \otimes_{i\in I} \xi_i)$
are orthogonal. Consequently, dim $(\prod \otimes_{i\in I} \check{\Psi}_i)(\mathbb{C}_{ut}^{\otimes I}) > 1$. As $(\prod \otimes_{i\in I} \Psi_i) \circ$
 $(\bigotimes_{i\in I} \tau_i) = \prod \otimes_{i$ $\psi_{i\in I}\Psi_i$ is not irreducible. \square

For any C^* -algebra A, we denote by $S(A)$ and $(H_\rho, \pi_\omega, \xi_\omega)$ the state space of A the GNS construction of $\omega \in S(A)$ respectively. We would like to consider a and the GNS construction of $\omega \in S(A)$, respectively. We would like to consider a
natural injective *-representation of \mathbb{R}^{nt} 4. defined in terms of $(H - \pi)$ and $\prod_{i\in I} \otimes_{i\in I} \Psi_i$ is not irreducible. A similar but easier argument also show $\bigotimes_{i\in I}^{\phi_1} \Psi_i$ is not irreducible.
For any C^* -algebra A, we denote by $S(A)$ and $(H_\rho, \pi_\omega, \xi_\omega)$ the state space and the GNS by C -algebra A, we denote by $D(A)$ and (H_{ρ}, n_{ω}) ;
SNS construction of $\omega \in S(A)$, respectively. We igective *-representation of $\bigotimes_{i \in I}^{\text{ut}} A_i$ defined in te
 $\Pi_{i \in I} S(A_i)$ and $\check{\rho}$ is defined as
 $\check{\rho}(a) := \langle (\bigot$

If $\rho \in \Pi_{i \in I} S(A_i)$ and $\check{\rho}$ is defined as

If
$$
\rho \in \Pi_{i \in I}S(A_i)
$$
 and $\check{\rho}$ is defined as
\n
$$
\check{\rho}(a) := \langle (\bigotimes_{i \in I}^{\phi_0} \pi_{\rho_i})(a)(\otimes_{i \in I} \xi_{\rho_i}), (\otimes_{i \in I} \xi_{\rho_i}) \rangle \quad (a \in \bigotimes_{i \in I}^{\text{ut}} A_i),
$$
\nthen the closure of $(\bigotimes_{i \in I}^{\phi_1} \pi_{\rho_i})(\bigotimes_{i \in I}^{\text{ut}} A_i)(\otimes_{i \in I} \xi_{\rho_i})$ will coincide with

 r_{ℓ}

$$
\langle (\bigotimes_{i \in I}^{\phi_0} \pi_{\rho_i}) (a) (\otimes_{i \in I} \xi_{\rho_i}), (\otimes_{i \in I} \xi_{\rho_i}) \rangle \quad (a \in \bigotimes
$$

of $(\bigotimes_{i \in I}^{\phi_1} \pi_{\rho_i}) (\bigotimes_{i \in I}^{\text{ut}} A_i) (\otimes_{i \in I} \xi_{\rho_i})$ will coincide v

$$
H_{\tilde{\rho}} := \tilde{\bigoplus}_{\omega \in \Omega_{I;A}^{\text{ut}}} \overline{\bigotimes}_{i \in I}^{\omega \cdot [\xi_{\rho}]_{\sim}} H_{\rho_i} \subseteq \overline{\bigotimes}_{i \in I}^{\phi_1} H_{\rho_i}.
$$

then the closure of $(\otimes$
 $H_{\check{\rho}}$:
We set $\pi_{\check{\rho}}(a) := (\otimes_{i \infty}^{\phi_1}$ $i \in I^{\mu} \rho_i$ $(a)|_{H_{\rho}}$. Notice that if all the ρ_i are pure states, then $H_{\check{\rho}} := \overline{\bigoplus}_{\omega \in \Omega^{\text{ut}}_{I;A}} \mathcal{C}$
We set $\pi_{\check{\rho}}(a) := (\bigotimes_{i \in I}^{\phi_1} \pi_{\rho_i})(a)|_{H_{\check{\rho}}}$. I
 $H_{\check{\rho}} = \overline{\bigotimes}_{i \in I}^{\phi_1} H_{\rho_i}$ (see Remark [5.4](#page-22-0) (c)).

Corollary 5.9. *Let* A_i *be a* C^* *-algebra* $(i \in I)$ *. The* **-representation* $\Psi_A := \bigoplus_{i=1}^{\infty} (H_i \pi_i)$ *is injective. Consequently the* **-representation* $\bigoplus_{\rho \in \Pi_i \in IS(A_i)} (H_{\tilde{\rho}}, \pi_{\tilde{\rho}})$ *is injective. Consequently, the* *-representation et A_i be a
 ϕ) is inject
 $A := \bigoplus$

$$
\Phi_A \; := \; \bigoplus\nolimits_{\rho \in \Pi_{i \in I} S(A_i)} (\bar{\bigotimes}_{i \in I}^{\phi_1} H_{\rho_i}, \bigotimes\nolimits_{i \in I}^{\phi_1} \pi_{\rho_i})
$$

is also injective.

Proof. Suppose that (H_i, Ψ_i) is a universal ^{*}-representation of A_i ($i \in I$). Let F, $u^{(1)} = u^{(n)} b^{(1)}$ and $a^{(1)} = a^{(n)}$ be as in the proof of Theorem 5.3(c) $u^{(1)}, \ldots, u^{(n)}, b^{(1)}, \ldots, b^{(n)},$ and $a^{(1)}, \ldots, a^{(n)}$ be as in the proof of Theorem [5.3](#page-21-0) (c) with $\Psi_A(\sum_{i=1}^n a^{(k)}) = 0$. Again, it suffices to show that $a^{(1)} = 0$ and we may with $\Psi_A(\sum_{k=1}^n a^{(k)}) = 0$. Again, it suffices to show that $a^{(1)} = 0$, and we may
assume that $u_i^{(1)} = e_i$ $(i \in I)$. If $n = 1$, we take any $x \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$. If $n > 1$, we
take an element $x \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$ assume that $u_i^{(1)} = e_i$ $(i \in I)$. If $n = 1$, we take any $x \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$. If $n > 1$, we take an element $x \in \Pi_{i \in I} \mathfrak{S}_1(H_i)$ satisfying take an element $x \in \prod_{i \in I} \mathfrak{S}_1(H_i)$ satisfying

$$
x \sim \left[\Psi_i\big(u_i^{(k)}\big)x_i\right]_{i \in I} \quad (k = 2, \dots, n)
$$

(the argument of Theorem [5.3](#page-21-0) (c) ensures its existence). Let us set $\rho_i(a) := \langle \Psi_i(a)x, x \rangle$ when $i \in I \setminus F$ and pick any $a_i \in S(A_i)$ when $i \in F$. For every $\langle \Psi_i(a)x_i, x_i \rangle$ when $i \in I \setminus F$, and pick any $\rho_i \in S(A_i)$ when $i \in F$. For every $x \approx$
(the argument of Theorem $\langle \Psi_i(a)x_i, x_i \rangle$ when $i \in I \setminus I$, one may regard ($\xi \in H$ is identified with (the argument of Theorem 5.3 (c) ensures its existence). Let us set $\rho_i(a) := \langle \Psi_i(a)x_i, x_i \rangle$ when $i \in I \setminus F$, and pick any $\rho_i \in S(A_i)$ when $i \in F$. For every $i \in I \setminus F$, one may regard $(H_{\rho_i}, \pi_{\rho_i})$ as a subrepresentation $i \in I \setminus I$, and pick any $p_i \in D(I_i)$ when $i \in I$. For every
regard $(H_{\rho_i}, \pi_{\rho_i})$ as a subrepresentation of (H_i, Ψ_i) such that
ied with $r_i \in H$. Then r can be considered as an element in $H_{\check{\rho}}$. Since $x \nsim \left[\pi_{\rho_i} \left(u_i^{(k)} \right) \right]$ $\ddot{}$ $\big]_{i\in I}$ for all $2 \leq k \leq n$, the argument of Theorem [5.3](#page-21-0) (c) tells us that (π, π_{ρ_i}) as a subre
 $\in H_i$. Then x of or all $2 \le k \le 3$
 $(a^{(1)})\eta = 0$ ($H_{\check{\rho}}$. Since $x \sim [\pi_{\rho_i}(u_i^{(k)})x_i]_{i \in I}$
tells us that
 $(\bigotimes_{i \in I}^{[x]_{\sim}} \pi_{\rho_i})$
Consequently, $(\bigotimes_{i \in F}^{[x]_{\sim}} \pi_{\rho_i})$ (b⁽¹⁾)
The second statement follows re

$$
\big(\bigotimes\nolimits_{i \in I}^{[x]_\sim} \pi_{\rho_i}\big)(a^{(1)})\eta = 0 \quad \big(\eta \in \bigotimes\nolimits_{i \in I}^x H_{\rho_i}\big).
$$

The second statement follows readily from the first one. \Box Consequently, $(\bigotimes_{i \in F} \pi_{\rho_i}) (b^{(1)}) = 0$ and $b^{(1)} = 0$ (as ρ_i is arbitrary when $i \in F$).
The second statement follows readily from the first one Solution $(\bigotimes_{i \in I} \pi_{\rho_i})(a^{\vee})\eta = 0$ ($\eta \in \bigotimes_{i \in I} \pi_{\rho_i}$).

Sequently, $(\bigotimes_{i \in F} \pi_{\rho_i})(b^{(1)}) = 0$ and $b^{(1)} = 0$ (as ρ_i is arbitrary when $i \in F$).

Second statement follows readily from the first one.
 \Box

N

include also that $(\bigvee_{i\in I}h_{\rho_i}, \bigvee_{i\in I}h_{\rho_i})$ is in general in and $(H_{\rho_i}, \pi_{\rho_i})$ can be regarded as a cyclic analogue of it.

We end this paper with the following result concerning tensor products of Hilbert algebras.

Corollary 5.10. *Let* $\{A_i\}_{i\in I}$ *be a family of unital Hilbert algebras (see, e.g., Definition* VI.1.1 *in* [\[18\]](#page-27-7)) *such that* $||e_i|| = 1$ ($i \in I$)*. Then* $A := \bigotimes_{i \in I}^{ut} A_i$ *is also a pointal Hilbert algebras* (*see, e.g., Definition* VI.1.1 *in* [18]) *such that* $||e_i|| = 1$ ($i \in I$)*. Then* $A := \bigotimes_{i \$ *unital Hilbert algebra with* $\|\otimes_{i\in I} e_i\| = 1$ *.* **Corollary 5.10.** Let
Definition VI.1.1 in [18]
unital Hilbert algebra w
Proof. Note that since
have $\bigotimes_{i \in I}^{\text{ut}} A_i \subseteq \bigotimes_{i \in I}^{\text{unit}}$

Proof. Note that since $||e_i|| = 1$, one has $||u_i|| = 1$ for any $u_i \in U_{A_i}$. Thus, we *i*
inital Hilbert algebra with $\|\hat{\omega}_{i\in I}e_i\| = 1$.
Proof. Note that since $||e_i|| = 1$, one has $||u_i|| = 1$ for any $u_i \in U_{A_i}$. Thus, we
have $\bigotimes_{i\in I}^{\text{ut}} A_i \subseteq \bigotimes_{i\in I}^{\text{unit}} A_i$, which gives an inner product $\langle \cdot$ Filtert algebra with $\|\otimes_{i\in I}e_i\|=1$.

Note that since $||e_i||=1$, one has $||u_i||=1$ for any $u_i \in U_{A_i}$. Thus, we
 $\sum_{i\in I}^{\text{out}} A_i \subseteq \bigotimes_{i\in I}^{\text{unit}} A_i$, which gives an inner product $\langle \cdot, \cdot \rangle_A$ on A. Observe
 $\bigotimes_{i\in$ distinct elements in $\Omega_{I,A}^{\text{ut}}$. Thus, in order to show that the involution of A is an isometry it suffices to shock that $||x^*|| = ||x||$ whenever $x \in \mathbb{R}^{\omega}$. A and $\omega \in \Omega^{\text{ut}}$ *Proof.* Note that since $||e_i|| = 1$, one has $||u_i|| = 1$ for any u_i
have $\bigotimes_{i \in I}^{\text{ut}} A_i \subseteq \bigotimes_{i \in I}^{\text{unit}} A_i$, which gives an inner product $\langle \cdot, \cdot \rangle$
that $\bigotimes_{i \in I}^{\omega} A_i$ is orthogonal to $\bigotimes_{i \in I}^{\omega'} A_i$ (in te $\sum_{i\in I}^{\omega} A_i$ and $\omega \in \Omega_{I;A}^{\text{ut}}$. Here $\bigotimes_{i\in I}^{\text{int}} A_i \subseteq \bigotimes_{i\in I}^{\text{unit}} A_i$, which gives an inner
that $\bigotimes_{i\in I}^{\omega} A_i$ is orthogonal to $\bigotimes_{i\in I}^{\omega'} A_i$ (in terms
distinct elements in $\Omega_{I;A}^{\text{ut}}$. Thus, in order to sh
isometry, it suffices to In fact, for any $u \in \Pi_{i \in I} U_{A_i}$, $F \in \mathfrak{F}$ and $a \in \bigotimes_{i \in F} A_i$, we have

$$
||J_F^u(a)^*|| = ||J_F^{u^*}(a^*)|| = ||a^*|| = ||a|| = ||J_F^u(a)||,
$$

ON GENUINE INFINITE ALGEBRAIC TENSOR PRODUCTS 355
because the involution of $\bigotimes_{i \in F} A_i$ is an isometry. Let H_i be the completion of A_i
(with respect to the inner product) and let $\Psi_i : A_i \to \mathcal{L}(H_i)$ be the canonica (with respect to the inner product) and let $\Psi_i : A_i \to \mathcal{L}(H_i)$ be the canonical unital ^{*}-representation ($i \in I$). Since

$$
\bigotimes_{i\in I}^{\phi_1} \Psi_i(a)b = ab \quad (a, b \in A),
$$

Theorem [5.3](#page-21-0) (a) tells us that for each $x \in A$, one has $\langle xy, z \rangle_A = \langle y, x^*z \rangle_A$ $(y, z \in A)$ and $\sup_{\|u\| \leq 1} \|xy\| < \infty$. Finally, as A is unital, we see that A is a Hilbert algebra (with $\|\hat{\otimes}_{i\in I}e_i\|=1$). Theorem 5.3 (a) tells us that for each x and sup $||xy|| \leq \infty$. Finally, as A (with $||\otimes_{i \in I} e_i|| = 1$).
Consequently, if all the A_i are weal Neumann algebras, then so is $\bigotimes_{i \in I}^{ut} A_i$.

Consequently, if all the A_i are weakly dense unital ^{*}-subalgebras of finite von

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Received May 27, 2011; revised November 2, 2011.

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This work is supported by the National Natural Science Foundation of China (11071126).