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On genuine infinite algebraic tensor products

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Abstract. In this paper, we study genuine infinite tensor products of some algebraic structures. By a genuine infinite tensor product of vector spaces, we mean a vector space $\bigotimes_{i\in I} X_i$ whose linear maps coincide with multilinear maps on an infinite family $\{X_i\}_{i\in I}$ of vector spaces. After establishing its existence, we give a direct sum decomposition of $\bigotimes_{i\in I} X_i$ over a set $\Omega_{I;X}$, through which we obtain a more concrete description and some properties of $\bigotimes_{i\in I} X_i$. If $\{A_i\}_{i\in I}$ is a family of unital *-algebras, we define, through a subgroup $\Omega^{\rm ut}_{I;A} \subseteq \Omega_{I;A}$, an interesting subalgebra $\bigotimes^{\rm ut}_{i\in I} A_i$. When all A_i are C^*-algebras or group algebras, it is the linear span of the tensor products of unitary elements of A_i. Moreover, it is shown that $\bigotimes^{\rm ut}_{i\in I}\mathbb{C}$ is the group algebra of $\Omega^{\rm ut}_{I;\mathbb{C}}$. In general, $\bigotimes^{\rm ut}_{i\in I} A_i$ can be identified with the algebraic crossed product of a cocycle twisted action of $\Omega^{\rm ut}_{I;A}$. On the other hand, if $\{H_i\}_{i\in I}$ is a family of inner product spaces, we define a Hilbert $C^*(\Omega^{\rm ut}_{I;\mathbb{C}})$-module $\bar{\bigotimes}^{\rm mod}_{i\in I} H_i$, which is the completion of a subspace $\bigotimes^{\rm unit}_{i\in I} H_i$ of $\bigotimes_{i\in I} H_i$. If $\chi_{\Omega^{\rm ut}_{I;\mathbb{C}}}$ is the canonical tracial state on $C^*(\Omega^{\rm ut}_{I;\mathbb{C}})$, then $\bar{\bigotimes}^{\rm mod}_{i\in I} H_i \otimes_{\chi_{\Omega^{\rm ut}_{I;\mathbb{C}}}} \mathbb{C}$ coincides with the Hilbert space $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ given by a very elementary algebraic construction and is a natural dilation of the infinite direct product $\prod\otimes_{i\in I} H_i$ as defined by J. von Neumann. We will show that the canonical representation of $\bigotimes^{\rm ut}_{i\in I}\mathcal{L}(H_i)$ on $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ is injective (note that the canonical representation of $\bigotimes^{\rm ut}_{i\in I}\mathcal{L}(H_i)$ on $\prod\otimes_{i\in I} H_i$ is not injective). We will also show that if $\{A_i\}_{i\in I}$ is a family of unital Hilbert algebras, then so is $\bigotimes^{\rm ut}_{i\in I} A_i$.

1. Introduction

In this paper, we study infinite tensor products of some algebraic structures. In the literature, infinite tensor products are often defined as inductive limit of finite tensor products (see, e.g., [4], [21] [9], [14], and [15]). As far as we know, the only

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alternative approach so far is the one by J. von Neumann, concerning *infinite direct products of Hilbert spaces* (see [20]). Some authors used this approach to define infinite tensor products of other functional analytic structures (see, e.g., [3], [11] and [13]). The work of von Neumann attracted the attention of many physicists who are interested in "quantum mechanics with infinite degrees of freedom", as well as mathematicians whose interest is in the field of operator algebras (see, e.g., [1], [2], [3], [8], [12], [17], and [19]).

However, von Neumann's approach is not appropriate for purely algebraic objects. The aim of this article is to study "genuine infinite algebraic tensor products" (i.e. ones that are defined in terms of multilinear maps instead of through inductive limits) of some algebraic structures. There are several motivations behind this study.

1. Conceptually speaking, it is natural to define "infinite tensor products" as the object that produces a unique linear map from a multilinear map on a given infinite family of objects (see Definition 2.1). As infinite direct products of Hilbert spaces are important in both physics and mathematics, it is believed that such infinite tensor products of algebraic structures are also important.

2. We want to construct an infinite tensor product of Hilbert spaces that is easier for non-analyst to grasp (compare with the infinite direct product as defined by J. von Neumann; see Lemma 4.2 and Remark 4.7 (d)) and is more natural (see Theorem 4.8, Example 4.10 and Example 5.6).

3. Given a family of groups $\{G_i\}_{i\in I}$, it is well known that the group algebra of the group

$$\bigoplus\nolimits_{i\in I} G_i := \big\{[g_i]_{i\in I} \in \Pi_{i\in I} G_i : g_i = e \text{ except for finite number of } i \in I\big\}$$

is an inductive limit of finite tensor products. However, if one wants to consider the group algebra $\mathbb{C}[\Pi_{i\in I} G_i]$, one is forced to consider a "bigger version of tensor products" (see Example 3.1).

In this article, the algebraic structures that we consider are vector spaces, unital *-algebras, inner product spaces, and *-representations of unital *-algebras on Hilbert spaces. In our study, we discovered some interesting phenomena about infinite tensor products that do not have counterparts in the case of finite tensor products. Most of these phenomena relate to a certain object, $\Omega_{I;X}$, defined as in Remark 2.4 (d), which "encodes the asymptotic information" of a given family $\{X_i\}_{i\in I}$.

In Section 2, we will begin our study by defining the infinite tensor product $(\bigotimes_{i\in I} X_i, \Theta_X)$ of a family $\{X_i\}_{i\in I}$ of vector spaces. Two particular concerns are bases of $\bigotimes_{i\in I} X_i$ and the relationship between $\bigotimes_{i\in I} X_i$ and inductive limits of finite tensor products of $\{X_i\}_{i\in I}$ (which depend on choices of fixed elements in $\Pi_{i\in I} X_i$). In order to do this, we obtain a direct sum decomposition of $\bigotimes_{i\in I} X_i$ indexed by a set $\Omega_{I;X}$ (see Theorem 2.5) with all the direct summands being inductive limits of finite tensor products (see Proposition 2.6 (b)). From this, we also obtain that the canonical map

$$\Psi : \bigotimes\nolimits_{i\in I} L(X_i; Y_i) \to L\big(\bigotimes\nolimits_{i\in I} X_i; \bigotimes\nolimits_{i\in I} Y_i\big)$$

is injective (but not surjective). As a consequence, $\bigotimes_{i\in I} X_i$ is automatically a faithful module over the big unital commutative algebra $\bigotimes_{i\in I}\mathbb{C}$ (see Corollary 2.9 and Example 2.10). Moreover, one may regard the canonical map

$$\Theta_{\mathbb{C}} : \Pi_{i\in I}\mathbb{C} \to \bigotimes_{i\in I}\mathbb{C}$$

as a generalized multiplication (see Example 2.10 (a)). In this sense, one can make sense of infinite products like $(-1)^I$.

Clearly, $\bigotimes_{i\in I} A_i$ is a unital *-algebra if all A_i are unital *-algebras. We will study in Section 3, a natural *-subalgebra $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$ of $\bigotimes_{i\in I} A_i$ which is a direct sum over a subgroup $\Omega^{\mathrm{ut}}_{I;A}$ of the semigroup $\Omega_{I;A}$. The reasons for considering this subalgebra are that it has good representations (see the discussion after Proposition 5.1), and it is big enough to contain $\mathbb{C}[\Pi_{i\in I} G_i]$ when $A_i = \mathbb{C}[G_i]$ for all $i \in I$ (see Example 3.1 (a)). Moreover, if all A_i are generated by their unitary elements (in particular, if A_i are group algebras or C^*-algebras), then $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$ is the linear span of the tensor products of unitary elements in A_i. We will show that $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$ can be identified with the crossed products of some twisted actions in the sense of Busby and Smith (i.e., a cocycle action with a 2-cocycle) of $\Omega^{\mathrm{ut}}_{I;A}$ on $\bigotimes^{e}_{i\in I} A_i$ (the unital *-algebra inductive limit of finite tensor products of A_i). Moreover, it is shown that $\bigotimes^{\mathrm{ut}}_{i\in I}\mathbb{C}$ can be identified with the group algebra of $\Omega^{\mathrm{ut}}_{I;\mathbb{C}}$ (Corollary 3.4). We will also study the center of $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$ in the case when A_i is generated by its unitary elements (for all $i \in I$).

In Section 4, we will consider tensor products of inner product spaces. If $\{H_i\}_{i\in I}$ is a family of inner product spaces, we define a natural inner product on a subspace $\bigotimes^{\mathrm{unit}}_{i\in I} H_i$ of $\bigotimes_{i\in I} H_i$ (see Lemma 4.2 (b)). In the case of Hilbert spaces, the completion $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ of $\bigotimes^{\mathrm{unit}}_{i\in I} H_i$ is a "natural dilation" of the infinite direct product $\prod \otimes_{i\in I} H_i$ as defined by J. von Neumann in [20] (see Remark 4.7 (b)). Note that the construction for $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ is totally algebraical and is more natural (see Example 4.10 and Example 5.6). Note also that one can construct $\prod \otimes_{i\in I} H_i$ in a similar way as $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ (see Remark 4.7 (d)). On the other hand, there is an inner product $\mathbb{C}[\Omega^{\mathrm{ut}}_{I;\mathbb{C}}]$-module structure on $\bigotimes^{\mathrm{unit}}_{i\in I} H_i$ which produces $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ (see Theorem 4.8), as well as many other pre-inner products on $\bigotimes^{\mathrm{unit}}_{i\in I} H_i$ (see Remark 4.9 (a)).

Section 5 will be devoted to the study of *-representations of unital *-algebras. More precisely, if $\Psi_i : A_i \to \mathcal{L}(H_i)$ is a unital *-representation ($i \in I$), we define a canonical *-representation

$$\bigotimes^{\phi_1}_{i\in I}\Psi_i \ : \ \bigotimes^{\mathrm{ut}}_{i\in I} A_i \ \to \ \mathcal{L}(\bar{\bigotimes}^{\phi_1}_{i\in I} H_i).$$

We will show in Theorem 5.3 (c) that if all the Ψ_i are injective, then $\bigotimes^{\phi_1}_{i\in I}\Psi_i$ is also injective. This is equivalent to the canonical *-representations of $\bigotimes^{\mathrm{ut}}_{i\in I}\mathcal{L}(H_i)$ on $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ being injective, and is related to the "strong faithfulness" of the canonical action of $\Omega^{\mathrm{ut}}_{I;\mathcal{L}(H)}$ on $\Omega^{\mathrm{unit}}_{I;H}$ (see Remark 5.4 (b)). Note however, that the corresponding tensor type representation of $\bigotimes^{\mathrm{ut}}_{i\in I}\mathcal{L}(H_i)$ on $\prod \otimes_{i\in I} H_i$ is not injective.

Consequently, if (H_i, π_i) is a unitary representation of a group G_i that induces an injective *-representation of $\mathbb{C}[G_i]$ on H_i $(i \in I)$, then we obtain an injective "tensor type" *-representation of $\mathbb{C}[\Pi_{i\in I} G_i]$ on $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ (see Corollary 5.7). On the other hand, we will show that $\bigoplus_{\rho\in\Pi_{i\in I} S(A_i)} \big(\bar{\bigotimes}^{\phi_1}_{i\in I} H_{\rho_i}, \bigotimes^{\phi_1}_{i\in I} \pi_{\rho_i}\big)$ is an injective *-representation of $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$ when all the A_i are C^*-algebras (Corollary 5.9). Finally, we show that if all the A_i are unital Hilbert algebras, then so is $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$.

Notation 1.1. i) In this article, all the vector spaces, algebras as well as inner product spaces are over the complex field $\mathbb{C}$, although some results remain valid if one considers the real field instead.

ii) Throughout this article, I is an infinite set, and $\mathfrak{F}$ is the set of all non-empty finite subsets of I.

iii) For any vector space X, we write $X^\times := X \setminus \{0\}$ and define X^* to be the set of linear functionals on X. If Y is another vector space, we denote by $X \otimes Y$ and $L(X;Y)$ respectively, the algebraic tensor product of X and Y, and the set of linear maps from X to Y. We also write $L(X) := L(X;X)$.

iv) If $\{X_i\}_{i\in I}$ is a family of vector spaces and $x \in \Pi_{i\in I} X_i$, we denote by x_i the "i^{th}-coordinate" of x (i.e. $x = [x_i]_{i\in I}$). If $x, y \in \Pi_{i\in I} X_i$ are such that $x_i = y_i$ except for a finite number of $i \in I$, we write

$$x_i = y_i \quad \text{e.f.}$$

v) If V is a normed space, we denote by $\mathcal{L}(V)$ and V' the set of bounded linear operators and the set of bounded linear functionals, respectively, on V. Moreover, we set $\mathfrak{S}_1(V) := \{x \in V : \|x\| = 1\}$ and $B_1(V) := \{x \in V : \|x\| \le 1\}$.

vi) If A is a unital *-algebra, we denote by e_A the identity of A and write $U_A := \{a \in A : a^*a = e_A = aa^*\}$.

2. Tensor products of vector spaces

In this section, $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ are families of non-zero vector spaces.

Definition 2.1. Let Y be a vector space. A map $\Phi : \Pi_{i\in I} X_i \to Y$ is said to be *multilinear* if Φ is linear on each variable. Suppose that $\bigotimes_{i\in I} X_i$ is a vector space and $\Theta_X : \Pi_{i\in I} X_i \to \bigotimes_{i\in I} X_i$ is a multilinear map such that for any vector space Y and any multilinear map $\Phi : \Pi_{i\in I} X_i \to Y$, there exists a unique linear map $\tilde{\Phi} : \bigotimes_{i\in I} X_i \to Y$ with $\Phi = \tilde{\Phi} \circ \Theta_X$. Then $\big(\bigotimes_{i\in I} X_i, \Theta_X\big)$ is called the *tensor product* of $\{X_i\}_{i\in I}$. We will denote $\otimes_{i\in I} x_i := \Theta_X(x)$ $(x \in \Pi_{i\in I} X_i)$ and set $X^{\otimes I} := \bigotimes_{i\in I} X_i$ if all X_i are equal to the same vector space X.

Let us first give the following simple example showing that non trivial multilinear maps with an infinite number of variables do exist. They are also crucial for some constructions later on.

Example 2.2. (a) Let $\Pi^1_{i\in I}\mathbb{C} := \{\beta \in \Pi_{i\in I}\mathbb{C} : \beta_i = 1 \text{ e.f.}\}$ and set

$$\varphi_1(\beta) \ := \ \begin{cases} \Pi_{i\in I}\beta_i & \text{if } \beta \in \Pi^1_{i\in I}\mathbb{C}, \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to check that φ_1 is a non-zero multilinear map from $\Pi_{i\in I}\mathbb{C}$ to $\mathbb{C}$. If $\phi_1 : \bigotimes_{i\in I}\mathbb{C} \to \mathbb{C}$ is the linear functional induced by φ_1 (the existence of $\bigotimes_{i\in I}\mathbb{C}$ will be established in Proposition 2.3 (a)), then ϕ_1 is an involutive unital map.

(b) Let $\Pi^0_{i\in I}\mathbb{C} := \{\beta \in \Pi_{i\in I}\mathbb{C} : \sum_{i\in I} |\beta_i - 1| < \infty\}$. For each $\beta \in \Pi^0_{i\in I}\mathbb{C}$, the net $\{\Pi_{i\in F}\beta_i\}_{F\in\mathfrak{F}}$ converges to a complex number, denoted by $\Pi_{i\in I}\beta_i$ (see, e.g., 2.4.1 in [20]). We define $\varphi_0(\beta) := \Pi_{i\in I}\beta_i$ whenever $\beta \in \Pi^0_{i\in I}\mathbb{C}$ and set $\varphi_0|_{\Pi_{i\in I}\mathbb{C}\setminus\Pi^0_{i\in I}\mathbb{C}} \equiv 0$. As in part (a), φ_0 induces an involutive unital linear functional ϕ_0 on $\bigotimes_{i\in I}\mathbb{C}$.

Clearly, infinite tensor products are unique (up to linear bijections) if they exist. The existence of infinite tensor products follows from a similar argument as that for finite tensor products, but we give an outline here for future reference.

Proposition 2.3. (a) *The tensor product* $\left(\bigotimes_{i\in I} X_i, \Theta_X\right)$ *exists.*

(b) *If* $\{A_i\}_{i\in I}$ *is a family of algebras (respectively,* **-algebras), then* $\bigotimes_{i\in I} A_i$ *is an algebra (respectively, a* **-algebra) with* $(\otimes_{i\in I} a_i)(\otimes_{i\in I} b_i) := \otimes_{i\in I} a_i b_i$ *(and* $(\otimes_{i\in I} a_i)^* := (\otimes_{i\in I} a_i^*)$*) for* $a, b \in \Pi_{i\in I}A_i$.

(c) *If* $\Psi_i : A_i \to L(X_i)$ *is a homomorphism for each* $i \in I$*, there is a canonical homomorphism* $\tilde{\bigotimes}_{i\in I}\Psi_i : \bigotimes_{i\in I} A_i \to L\left(\bigotimes_{i\in I} X_i\right)$ *such that*

$$(\tilde{\bigotimes}_{i\in I}\Psi_i)(\otimes_{i\in I} a_i)\otimes_{i\in I} x_i \ = \ \otimes_{i\in I} \Psi_i(a_i)x_i \quad (a \in \Pi_{i\in I}A_i; x \in \Pi_{i\in I}X_i).$$

(d) *If* $A = \bigoplus_{n=0}^\infty A_n$ *is a graded algebra and* $\bigoplus_{n=0}^\infty M_n$ *is a graded left* A*-module, then* $\bigoplus_{n=0}^\infty \bigotimes_{k\geq n} M_k$ *is a graded* A*-module with*

$$a_m(\otimes_{k\geq n}x_k) = \otimes_{k\geq n}a_m x_k \in \bigotimes_{k\geq m+n} M_k(a_m \in A_m; x \in \Pi_{k\geq n}M_k).$$

Proof. Parts (b), (c) and (d) follow from the universal property of tensor products, and we will only give a brief account for part (a). Let V be the free vector space generated by elements in $\Pi_{i\in I}X_i$ and let $\Theta_0 : \Pi_{i\in I}X_i \to V$ be the canonical map. Suppose that $W := \operatorname{span} W_0$, where

$$\begin{aligned} W_0 \ := \ \{&\lambda\Theta_0(u) + \Theta_0(v) - \Theta_0(w) : \lambda \in \mathbb{C}; u, v, w \in \Pi_{i\in I}X_i; \exists i_0 \in I \text{ with} \\ &\lambda u_{i_0} + v_{i_0} = w_{i_0} \text{ and } u_j = v_j = w_j, \forall j \in I \setminus \{i_0\}\}. \end{aligned} \tag{2.1}$$

If we put $\bigotimes_{i\in I} X_i := V/W$, and set Θ_X to be the composition of Θ_0 with the quotient map from V to V/W, then they will satisfy the requirement in Definition 2.1. □

In the following remark, we list some observations that will be used implicitly throughout this article.

Remark 2.4. (a) As Θ_X is multilinear, $\bigotimes_{i\in I} X_i = \operatorname{span}\Theta_X\big(\Pi_{i\in I} X_i^\times\big)$.

(b) If I_1 and I_2 are non-empty disjoint subsets of I with $I = I_1 \cup I_2$, it follows, from the universal property, that $\bigotimes_{i\in I} X_i \cong \big(\bigotimes_{i\in I_1} X_i\big) \otimes \big(\bigotimes_{j\in I_2} X_j\big)$ canonically.

(c) $\bigotimes_{i\in I}(X_i \otimes Y_i) \cong (\bigotimes_{i\in I} X_i) \otimes (\bigotimes_{i\in I} Y_i)$ canonically.

(d) For any $x, y \in \Pi_{i\in I} X_i^\times$, we write

$$x \sim y \quad \text{if} \quad x_i = y_i \ \ e.f.$$

Obviously, $\sim$ is an equivalence relation on $\Pi_{i\in I} X_i^\times$, and we set $[x]_\sim$ to be the equivalence class of $x \in \Pi_{i\in I} X_i^\times$. Let $\Omega_{I;X}$ be the collection of such equivalence classes. It is not hard to see that $\Omega_{I;\mathbb{C}}$ is a quotient group of $\Pi_{i\in I}\mathbb{C}^\times$, and that it acts freely on $\Omega_{I;X}$.

(e) The element $\otimes_{i\in I} 1 \in \mathbb{C}^{\otimes I}$ is non-zero. In fact, if $\otimes_{i\in I} 1 = 0$, then $\mathbb{C}^{\otimes I} = (0)$ (by Proposition 2.3 (b)), and this implies the only multilinear map from $\Pi_{i\in I}\mathbb{C}$ to $\mathbb{C}$ is zero, which contradicts Example 2.2.

The "asymptotic object" $\Omega_{I;X}$ defined in (d) above is crucial in the study of genuine infinite tensor products, as can be seen from our next result. Let us first give some more notation here. For every $u \in \Pi_{i\in I} X_i^\times$, we set

$$\Pi_{i\in I}^u X_i := \{x \in \Pi_{i\in I} X_i : x \sim u\} \quad \text{and} \quad \bigotimes_{i\in I}^u X_i := \operatorname{span}\Theta_X(\Pi_{i\in I}^u X_i).$$

If $u \sim v$, then $\Pi_{i\in I}^u X_i = \Pi_{i\in I}^v X_i$, and we will also write $\Pi_{i\in I}^{[u]_\sim} X_i := \Pi_{i\in I}^u X_i$ and $\bigotimes_{i\in I}^{[u]_\sim} X_i := \bigotimes_{i\in I}^u X_i$.

Theorem 2.5. $\bigotimes_{i\in I} X_i = \bigoplus_{\omega\in\Omega_{I;X}} \bigotimes_{i\in I}^\omega X_i$.

Proof. Suppose that $x^{(1)}, \dots, x^{(n)} \in \Pi_{i\in I} X_i^\times$ and that $0 = n_0 < \cdots < n_N = n$ is a sequence satisfying $x^{(n_k+1)} \sim \cdots \sim x^{(n_{k+1})}$ for $k \in \{0, \dots, N-1\}$, but $x^{(n_k)} \nsim x^{(n_l)}$ whenever $1 \le k \ne l \le N$. We first show that if $\nu_1, \dots, \nu_n \in \mathbb{C}$ with $\sum_{l=1}^n \nu_l \Theta_X(x^{(l)}) = 0$, then

$$\sum\nolimits_{l=n_k+1}^{n_{k+1}} \nu_l \Theta_X(x^{(l)}) = 0 \quad (k = 0, \dots, N-1).$$

In fact, by the proof of Proposition 2.3 (a), there exist $m \in \mathbb{N}$, $\mu_1, \dots, \mu_m \in \mathbb{C}$ and $\lambda_k\Theta_0(u^{(k)}) + \Theta_0(v^{(k)}) - \Theta_0(w^{(k)}) \in W_0$ $(k = 1, \dots, m)$ such that

$$\sum\nolimits_{l=1}^n \nu_l \Theta_0(x^{(l)}) = \sum\nolimits_{k=1}^m \mu_k\big(\lambda_k\Theta_0(u^{(k)}) + \Theta_0(v^{(k)}) - \Theta_0(w^{(k)})\big).$$

Observe that if one of the elements in $\{u^{(k)}, v^{(k)}, w^{(k)}\}$ is equivalent to $x^{(1)}$ (under $\sim$), then so are the other two (see (2.1)). After renaming, one may assume that $u^{(k)} \sim v^{(k)} \sim w^{(k)} \sim x^{(1)}$ for $k = 1, \dots, m_1$, but none of $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ is equivalent to $x^{(1)}$ when $k \in \{m_1 + 1, \dots, m\}$.

Since the two sets

$$\{x^{(n_1+1)},\ldots,x^{(n)}\}\cup\{u^{(m_1+1)},\ldots,u^{(m)}\}\cup\{v^{(m_1+1)},\ldots,v^{(m)}\}\cup\{w^{(m_1+1)},\ldots,w^{(m)}\}$$

and

$$\{x^{(1)},\ldots,x^{(n_1)}\}\cup\{u^{(1)},\ldots,u^{(m_1)}\}\cup\{v^{(1)},\ldots,v^{(m_1)}\}\cup\{w^{(1)},\ldots,w^{(m_1)}\}$$

are disjoint and elements in $\Theta_0\left(\Pi_{i\in I}X_i\right)$ are linearly independent in V, we have

$$\sum\nolimits_{l=1}^{n_1}\nu_l\Theta_0(x^{(l)})\ -\ \sum\nolimits_{k=1}^{m_1}\mu_k\big(\lambda_k\Theta_0(u^{(k)})+\Theta_0(v^{(k)})-\Theta_0(w^{(k)})\big)\ =\ 0.$$

This implies that $\sum_{l=1}^{n_1}\nu_l\Theta_X(x^{(l)}) = 0$. Similarly, $\sum_{l=n_k+1}^{n_{k+1}}\nu_l\Theta_X(x^{(l)}) = 0$ for $k=1,\ldots,N-1$.

The above shows that

$$\Big(\bigotimes_{i\in I}^{\omega_M}X_i\Big)\cap\Big(\sum_{k=1}^{M-1}\bigotimes_{i\in I}^{\omega_k}X_i\Big)=\{0\}$$

when $\omega_1,\ldots,\omega_M$ are distinct elements in $\Omega_{I;X}$. On the other hand, for every $x\in\Pi_{i\in I}X_i^\times$, one has $\Theta_X(x)\in\bigotimes_{i\in I}^{[x]_\sim}X_i$. These give the required equality. □

For any $F\in\mathfrak{F}$ and $u\in\Pi_{i\in I}X_i^\times$, one has a linear map

$$J_F^u\ :\ \bigotimes\nolimits_{i\in F}X_i\ \longrightarrow\ \bigotimes\nolimits_{i\in I}^{u}X_i$$

given by $J_F^u(\otimes_{i\in F}x_i) := \otimes_{j\in I}\tilde{x}_j$ $(x_i\in X_i)$, where $\tilde{x}_j := x_j$ when $j\in F$, and $\tilde{x}_j := u_j$ when $j\in I\setminus F$.

For any $F,G\in\mathfrak{F}$ with $F\subseteq G$, a similar construction gives a linear map $J_{G;F}^u : \bigotimes_{i\in F}X_i\to\bigotimes_{i\in G}X_i$. It is clear that $\big(\bigotimes_{i\in F}X_i, J_{G;F}^u\big)_{F\subseteq G\in\mathfrak{F}}$ is an inductive system in the category of vector spaces with linear maps as morphisms.

Proposition 2.6. (a) *J_F^u is injective for any $u\in\Pi_{i\in I}X_i^\times$ and $F\in\mathfrak{F}$. Consequently, $\Theta_X(u)\neq 0$.*

(b) *The inductive limit of $\big(\bigotimes_{i\in F}X_i, J_{G;F}^u\big)_{F\subseteq G\in\mathfrak{F}}$ is $\big(\bigotimes_{i\in I}^u X_i, \{J_F^u\}_{F\in\mathfrak{F}}\big)$.*

Proof. (a) Suppose that $a\in\ker J_F^u$ and $\psi\in(\bigotimes_{i\in F}X_i)^*$. For each $j\in I\setminus F$, choose $f_j\in X_j^*$ with $f_j(u_j)=1$. Remark 2.4 (b) and the universal property give a linear map $\check{\psi}:\bigotimes_{i\in I}X_i\to\mathbb{C}^{\otimes I}$ satisfying

$$\check{\psi}(\otimes_{i\in I}x_i)=\psi(\otimes_{i\in F}x_i)\left(\otimes_{j\in I\setminus F}f_j(x_j)\right)\qquad(x\in\Pi_{i\in I}X_i).$$

Thus, $\psi(a)(\otimes_{i\in I}1)=\check{\psi}(J_F^u(a))=0$, which implies that $a=0$ (as ψ is arbitrary) as required. On the other hand, if $i_0\in I$, then $\Theta_X(u)=J_{\{i_0\}}^u(u_{i_0})\neq 0$.

(b) This follows directly from part (a). □

Part (b) of the above implies that $\Theta_X(C^\omega)$ is a basis for $\bigotimes_{i\in I}^\omega X_i$, where C^ω is as defined in the following result.

Corollary 2.7. (a) *Let $c:\Omega_{I;X}\to\Pi_{i\in I}X_i^\times$ be a cross section. For each $\omega\in\Omega_{I;X}$ and $i\in I$, we pick a basis B_i^ω of X_i that contains $c(\omega)_i$ and set*

$$C^\omega := \{x\in\Pi_{i\in I}^\omega X_i : x_i\in B_i^\omega, \forall i\in I\}.$$

If $C:=\bigcup_{\omega\in\Omega_{I;X}} C^\omega$, then $\Theta_X(C)$ is a basis for $\bigotimes_{i\in I} X_i$.

(b) *If $\Phi_i : X_i\to Y_i$ is an injective linear map ($i\in I$), the induced linear map $\bigotimes_{i\in I}\Phi_i:\bigotimes_{i\in I}X_i\to\bigotimes_{i\in I}Y_i$ is injective.*

Proposition 2.8. *The map $\Psi:\bigotimes_{i\in I}L(X_i;Y_i)\to L(\bigotimes_{i\in I}X_i;\bigotimes_{i\in I}Y_i)$ (given by the universal property) is injective.*

Proof. Suppose that $T^{(1)},\dots,T^{(n)}\in\Pi_{i\in I}L(X_i;Y_i)^\times$ are mutually inequivalent elements (under $\sim$), $F\in\mathfrak{F}$, $R^{(1)},\dots,R^{(n)}\in\bigotimes_{i\in F}L(X_i;Y_i)$ with $S^{(k)}:=J_F^{T^{(k)}}(R^{(k)})$ ($k=1,\dots,n$) satisfying

$$\Psi\Big(\sum\nolimits_{k=1}^n S^{(k)}\Big) \;=\; 0.$$

Using an induction argument, it suffices to show that $S^{(1)}=0$.

If $n=1$, we take any $x\in\Pi_{i\in I}X_i^\times$ with $T_i^{(1)}x_i\neq 0$ ($i\in I$). If $n>1$, we claim that there is $x\in\Pi_{i\in I}X_i^\times$ such that

$$[T_i^{(1)}x_i]_{i\in I}\in\Pi_{i\in I}Y_i^\times \quad\text{and}\quad [T_i^{(k)}x_i]_{i\in I}\nsim[T_i^{(1)}x_i]_{i\in I}\ \ (k=2,\dots,n).$$

In fact, let $I^k:=\{i\in I:T_i^{(k)}\neq T_i^{(1)}\}$, which is an infinite set for any $k=2,\dots,n$. For any $i\in I$, we put $N_i:=\{k\in\{2,\dots,n\}:i\in I^k\}$ and take any $x_i\in X_i\setminus\big(\bigcup_{k\in N_i}\ker(T_i^{(k)}-T_i^{(1)})\cup\ker T_i^{(1)}\big)$ (note that X_i cannot be a finite union of proper subspaces). Thus, $T_i^{(1)}x_i\neq 0$ (for each $i\in I$) and $T_i^{(k)}x_i\neq T_i^{(1)}x_i$ (for $k\in\{2,\dots,n\}$ and $i\in I^k$).

Now, we have

$$\Psi(S^{(1)})\big(\bigotimes\nolimits_{i\in I}^x X_i\big)\cap\Big(\sum\nolimits_{k=2}^n\Psi(S^{(k)})\big(\bigotimes\nolimits_{i\in I}^x X_i\big)\Big)=(0)$$

by Theorem 2.5 and the fact that $\Psi(S^{(l)})\big(\bigotimes_{i\in I}^x X_i\big)\in\bigotimes_{i\in I}^{y^{(l)}}Y_i$, where $y_i^{(l)}=T_i^{(l)}x_i$ ($i\in I;l=1,\dots,n$). Consequently, $\Psi(S^{(1)})|_{\bigotimes_{i\in I}^x X_i}=0$. As $T_i^{(1)}x_i\neq 0$ ($i\in I$), it is easy to see that $R^{(1)}=0$ as required. □

Note that Ψ is not surjective even if $X_i=Y_i=\mathbb{C}$ ($i\in I$) since in this case, Ψ is a homomorphism and $\bigotimes_{i\in I}\mathbb{C}$ is commutative while $L(\bigotimes_{i\in I}\mathbb{C})$ is not.

The following result follows from Propositions 2.3 (c) and 2.8 as well as Corollary 2.7 (b). It says that an infinite tensor product of vector spaces is automatically a faithful module over a big commutative algebra.

Corollary 2.9. *If X_i is a faithful A_i-module ($i \in I$), then $\bigotimes_{i\in I} X_i$ is a faithful $\bigotimes_{i\in I} A_i$-module. In particular, $\bigotimes_{i\in I} Y_i$ is a faithful unital $\mathbb{C}^{\otimes I}$-module.*

Example 2.10. (a) If $\beta \in \Pi_{i\in I}\mathbb{C}^\times$, then $\bigotimes_{i\in I}^{\beta}\mathbb{C} = \mathbb{C}\cdot \otimes_{i\in I}\beta_i$. In fact, for any $F \in \mathfrak{F}$ and $\mu_i \in \mathbb{C}$ ($i \in F$), we have $J_F^\beta(\otimes_{i\in F}\mu_i) = (\Pi_{i\in F}\mu_i/\beta_i)(\otimes_{i\in I}\beta_i)$.

(b) For $n \in \mathbb{N}$, let $I_1, \dots, I_n$ be infinite disjoint subsets of I with $I = \bigcup_{k=1}^n I_k$ and $\overline{\beta} = (\beta_1, \dots, \beta_n) \in (\mathbb{C}^\times)^n$. Define $\widetilde{\beta} \in \Pi_{i\in I}\mathbb{C}^\times$ by $\widetilde{\beta}_i = \beta_k$ whenever $i \in I_k$. Then $\overline{\beta} \mapsto [\widetilde{\beta}]_\sim$ is an injective group homomorphism from $(\mathbb{C}^\times)^n$ to $\Omega_{I;\mathbb{C}}$.

(c) Let G be a subgroup of $\mathbb{T}^n \subseteq (\mathbb{C}^\times)^n$ (where $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$). If $\overline{\beta^{(1)}}, \dots, \overline{\beta^{(m)}}$ are distinct elements in G and $\widetilde{\beta^{(1)}}, \dots, \widetilde{\beta^{(m)}} \in \Pi_{i\in I}\mathbb{C}^\times$ are as in part (b), then $\otimes_{i\in I}\widetilde{\beta_i^{(1)}}, \dots, \otimes_{i\in I}\widetilde{\beta_i^{(m)}}$ are linearly independent in $\mathbb{C}^{\otimes I}$. Therefore, the *-subalgebra of $\mathbb{C}^{\otimes I}$ generated by $\{\otimes_{i\in I}\widetilde{\beta}_i : \overline{\beta} \in G\}$ is *-isomorphic to the group algebra $\mathbb{C}[G]$.

As $\otimes_{i\in I}\alpha_i = (\Pi_{i\in I}\alpha_i)(\otimes_{i\in I} 1)$ if $\alpha_i = 1$ e.f., one may regard $\otimes_{i\in I}\alpha_i$ as a generalization of the product. In this case, one can consider infinite products like $(-1)^I$.

3. Tensor products of unital *-algebras

Throughout this section, A_i is a unital *-algebra with identity e_i ($i \in I$), and we set $\Omega_{I;A}^{\mathrm{ut}} := \Pi_{i\in I} U_{A_i}/\sim$.

Notice that in this case, $\Omega_{I;A}$ is a *-semigroup with identity and $\Omega_{I;A}^{\mathrm{ut}}$ can be regarded as a subgroup of $\Omega_{I;A}$ with the inverse being the involution on $\Omega_{I;A}$. Moreover, $\bigotimes_{i\in I} A_i$ is a $\Omega_{I;A}$-graded *-algebra in the sense that for any $\omega, \omega' \in \Omega_{I;A}$,

$$\left(\bigotimes_{i\in I}^{\omega} A_i\right)\cdot\left(\bigotimes_{i\in I}^{\omega'} A_i\right) \subseteq \bigotimes_{i\in I}^{\omega\omega'} A_i \quad\text{and}\quad \left(\bigotimes_{i\in I}^{\omega} A_i\right)^* \subseteq \bigotimes_{i\in I}^{\omega^*} A_i. \tag{3.1}$$

By Proposition 2.6 (b), $\bigotimes_{i\in I}^{e} A_i$ can be identified with the unital *-algebra inductive limit of finite tensor products of A_i. We will study the following *-subalgebra that contains $\bigotimes_{i\in I}^{e} A_i$:

$$\bigotimes_{i\in I}^{\mathrm{ut}} A_i \;:=\; \bigoplus_{\omega\in\Omega_{I;A}^{\mathrm{ut}}} \bigotimes_{i\in I}^{\omega} A_i.$$

The motivation for considering this subalgebra is partially Example 3.1 (a) below, and partially because it has good representations (see the discussion after Proposition 5.1 below). Moreover, if all the A_i are linear spans of U_{A_i} (in particular, if they are C^*-algebras or group algebras), then $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ is the linear span of $\Theta_A(\Pi_{i\in I} U_{A_i})$. If $A_i = A$ for all $i \in I$, we write $A_{\mathrm{ut}}^{\otimes I} := \bigotimes_{i\in I}^{\mathrm{ut}} A_i$.

Example 3.1. (a) Let G_i be a group and $\mathbb{C}[G_i]$ be its group algebra ($i \in I$). If $\Lambda : \Pi_{i\in I} G_i \to \Pi_{i\in I} U_{\mathbb{C}[G_i]}$ is the canonical map, then $\lambda := \Theta_{\mathbb{C}[G]} \circ \Lambda$ gives a *-isomorphism from $\mathbb{C}[\Pi_{i\in I} G_i]$ to the *-subalgebra

$$\bigotimes\nolimits_{i\in I}^{\Lambda(\Pi_{i\in I} G_i)} \mathbb{C}[G_i] \ := \ \sum\nolimits_{t\in\Pi_{i\in I} G_i} \bigotimes\nolimits_{i\in I}^{\Lambda(t)} \mathbb{C}[G_i] \ \subseteq \ \bigotimes\nolimits_{i\in I}^{\mathrm{ut}} \mathbb{C}[G_i].$$

In fact, λ induces a *-homomorphism from $\mathbb{C}[\Pi_{i\in I} G_i]$ to $\bigotimes_{i\in I}^{\mathrm{ut}} \mathbb{C}[G_i]$. Let $q : \Pi_{i\in I} G_i \to \Pi_{i\in I} G_i / \oplus_{i\in I} G_i$ be the quotient map. For a fixed $s \in \Pi_{i\in I} G_i$, if we set

$$\bigoplus\nolimits_{i\in I}^{s} G_i \ := \ \big\{ t \in \Pi_{i\in I} G_i : q(t) = q(s) \big\},$$

then $s^{-1}\big(\bigoplus_{i\in I}^{s} G_i\big) = \bigoplus_{i\in I} G_i$. Thus, $\{\lambda(t) : t \in \bigoplus_{i\in I}^{s} G_i\}$ is a set of linearly independent elements in $\bigotimes_{i\in I} \mathbb{C}[G_i]$ (as $\lambda|_{\mathbb{C}[\bigoplus_{i\in I} G_i]}$ is a bijection onto $\bigotimes_{i\in I}^{e} \mathbb{C}[G_i]$). On the other hand, if $s^{(1)}, \dots, s^{(n)} \in \Pi_{i\in I} G_i$ are such that $q(s^{(k)}) \neq q(s^{(l)})$ whenever $k \neq l$, then $\lambda(s^{(1)}), \dots, \lambda(s^{(n)})$ are linearly independent in $\bigotimes_{i\in I} \mathbb{C}[G_i]$ (see Theorem 2.5). Consequently, $\{\lambda(t) : t \in \Pi_{i\in I} G_i\}$ form a basis for $\bigotimes_{i\in I}^{\Lambda(\Pi_{i\in I} G_i)} \mathbb{C}[G_i]$.

(b) It is well known that there is a twisted action (α, u), in the sense of Busby and Smith, of $\Omega_{I;G} := \Pi_{i\in I} G_i / \oplus_{i\in I} G_i$ on $\mathbb{C}[\bigoplus_{i\in I} G_i] \cong \bigotimes_{i\in I}^{e} \mathbb{C}[G_i]$ (see 2.1 in [5]) such that $\mathbb{C}[\Pi_{i\in I} G_i]$ is *-isomorphic to the algebraic crossed product $\bigotimes_{i\in I}^{e} \mathbb{C}[G_i] \rtimes_{\alpha,u} \Omega_{I;G}$.

There exists a canonical action Ξ of $\Pi_{i\in I} U_{A_i}$ on $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ given by inner automorphisms, i.e.

$$\Xi_u(a) := (\otimes_{i\in I} u_i) \cdot a \cdot (\otimes_{i\in I} u_i^*) \qquad \big(u \in \Pi_{i\in I} U_{A_i}; a \in \bigotimes\nolimits_{i\in I}^{\mathrm{ut}} A_i\big).$$

This induces an action Ξ^e of $\Pi_{i\in I} U_{A_i}$ on the subalgebra $\bigotimes_{i\in I}^{e} A_i$. The following result gives an identification of $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ as the algebraic crossed product (see, e.g., page 166 of [16]) of a cocycle twisted action (i.e., a twisted action in the sense of Busby and Smith) of $\Omega_{I;A}^{\mathrm{ut}}$ on $\bigotimes_{i\in I}^{e} A_i$ induced by Ξ^e.

Before we give this result, let us recall that an abelian group G is *divisible* if for any $g \in G$ and $n \in \mathbb{N}$, there is $h \in G$ with $g = h^n$.

Theorem 3.2. (a) *There is a cocycle twisted action $(\check{\Xi}, m)$ of $\Omega_{I;A}^{\mathrm{ut}}$ on $\bigotimes_{i\in I}^{e} A_i$ such that $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ is $\Omega_{I;A}^{\mathrm{ut}}$-graded *-isomorphic to $(\bigotimes_{i\in I}^{e} A_i) \rtimes_{\check{\Xi},m} \Omega_{I;A}^{\mathrm{ut}}$.*

(b) *Suppose that all the A_i are commutative. If $\bigotimes_{i\in I}^{e} A_i$ is a unital *-subalgebra of a commutative *-algebra B with U_B being divisible, $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ is $\Omega_{I;A}^{\mathrm{ut}}$-graded *-isomorphic to a unital *-subalgebra of $B \otimes \mathbb{C}[\Omega_{I;A}^{\mathrm{ut}}]$. If $U_{\bigotimes_{i\in I}^{e} A_i}$ is itself divisible, $\bigotimes_{i\in I}^{\mathrm{ut}} A_i \cong (\bigotimes_{i\in I}^{e} A_i) \otimes \mathbb{C}[\Omega_{I;A}^{\mathrm{ut}}]$ as $\Omega_{I;A}^{\mathrm{ut}}$-graded *-algebras.*

Proof. Let $c : \Omega_{I;A}^{\mathrm{ut}} \to \Pi_{i\in I} U_{A_i}$ be a cross section with $c([e]_\sim) = e$.

(a) For any $\mu, \nu \in \Omega_{I;A}^{\mathrm{ut}}$, we set

$$\check{\Xi}_\mu \ := \ \Xi_{c(\mu)}^{e} \quad \text{and} \quad m(\mu, \nu) \ := \ \otimes_{i\in I}\, c(\mu)_i c(\nu)_i c(\mu\nu)_i^{-1}.$$

As $c(\mu)c(\nu) \sim c(\mu\nu)$, we have $m(\mu,\nu) \in \bigotimes^e_{i\in I} A_i$. It is easy to check that $(\check{\Xi}, m)$ is a twisted action in the sense of Busby and Smith. Furthermore, we define $\Psi : (\bigotimes^e_{i\in I} A_i) \rtimes_{\check{\Xi},m} \Omega^{\mathrm{ut}}_{I;A} \to \bigotimes^{\mathrm{ut}}_{i\in I} A_i$ by

$$\Psi(f) := \sum\nolimits_{\omega\in\Omega^{\mathrm{ut}}_{I;A}} f(\omega)(\otimes_{i\in I} c(\omega)_i) \quad \big(f \in (\bigotimes\nolimits^e_{i\in I} A_i) \rtimes_{\check{\Xi},m} \Omega^{\mathrm{ut}}_{I;A}\big).$$

It is not hard to verify that Ψ is a bijective $\Omega^{\mathrm{ut}}_{I;A}$-graded *-homomorphism.

(b) Let $\Pi^e_{i\in I} U_{A_i} := \Pi^e_{i\in I} A_i \cap \Pi_{i\in I} U_{A_i}$. By Baer's theorem, $\Theta_A|_{\Pi^e_{i\in I} U_{A_i}}$ can be extended to a group homomorphism $\varphi : \Pi_{i\in I} U_{A_i} \to U_B$. Since

$$\varphi(c(\mu))\varphi(c(\nu))\varphi(c(\mu\nu))^{-1} = \otimes_{i\in I}\, c(\mu)_i c(\nu)_i c(\mu\nu)_i^{-1} \quad (\mu,\nu \in \Omega^{\mathrm{ut}}_{I;A}),$$

the map $\Phi : \bigotimes^{\mathrm{ut}}_{i\in I} A_i \to B \otimes \mathbb{C}[\Omega^{\mathrm{ut}}_{I;A}]$ given by

$$\Phi(a) := (a \cdot \otimes_{i\in I}\, c(\omega)_i^{-1})\varphi(c(\omega)) \otimes \lambda(\omega) \quad \big(a \in \bigotimes\nolimits^\omega_{i\in I} A_i; \omega \in \Omega^{\mathrm{ut}}_{I;A}\big) \tag{3.2}$$

is a $\Omega^{\mathrm{ut}}_{I;A}$-graded *-homomorphism. If $\sum_{\omega\in\Omega^{\mathrm{ut}}_{I;A}} a^\omega \in \ker\Phi$ (with $a^\omega \in \bigotimes^\omega_{i\in I} A_i$), then for every $\omega \in \Omega^{\mathrm{ut}}_{I;A}$, one has $(a^\omega \cdot \otimes_{i\in I}\, c(\omega)_i^{-1})\varphi(c(\omega)) = 0$, which implies $a^\omega = 0$, and hence Φ is injective. The image of Φ is the linear span of

$$\big\{b\varphi(c(\omega)) \otimes \lambda(\omega) : b \in \bigotimes\nolimits^e_{i\in I} A_i; \omega \in \Omega^{\mathrm{ut}}_{I;A}\big\},$$

and it is clear that Φ is surjective if $B = \bigotimes^e_{i\in I} A_i$. □

Remark 3.3. (a) The cocycle twisted action $(\check{\Xi}, m)$ depends on the choice of a cross section, and different cross sections may give different twisted actions (although their crossed products are all isomorphic). On the other hand, the map Φ in part (b) also depends on the choice of a cross section as well as the choice of an extension of $\Theta_A|_{\Pi^e_{i\in I} U_{A_i}}$.

(b) If S_i is a set and A_i is a *-subalgebra of $\ell^\infty(S_i)$ $(i \in I)$, then by Theorem 3.2 (b), $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$ is a *-subalgebra of $\ell^\infty(\Pi_{i\in I} S_i) \otimes \mathbb{C}[\Omega^{\mathrm{ut}}_{I;A}]$. Our first proof for this fact use 18.4 in [6] and 7.1 in [7].

(c) If all the A_i are commutative, then $\bigotimes^{\mathrm{ut}}_{i\in I} A_i \cong (\bigotimes^e_{i\in I} A_i) \otimes \mathbb{C}[\Omega^{\mathrm{ut}}_{I;A}]$ as $\Omega^{\mathrm{ut}}_{I;A}$-graded *-algebras if and only if there is a group homomorphism $\pi : \Omega^{\mathrm{ut}}_{I;A} \to U_{\bigotimes^{\mathrm{ut}}_{i\in I} A_i}$ such that $\pi(\omega) \in \bigotimes^\omega_{i\in I} A_i$ $(\omega \in \Omega^{\mathrm{ut}}_{I;A})$. In fact, if such a π exists, one may replace $(a \cdot \otimes_{i\in I}\, c(\omega)_i^{-1})\varphi(c(\omega))$ in (3.2) with $a\pi(\omega^{-1})$ and show that the corresponding Φ is a *-isomorphism.

Clearly, the second statement of Theorem 3.2 (b) applies to the case when $A_i = \mathbb{C}^{n_i}$ for some $n_i \in \mathbb{N}$ $(i \in I)$. In particular, Theorem 3.2 (b) and its argument give the following corollary.

Corollary 3.4. *If φ_1 is as in Example 2.2 (a) and $\varphi : \Pi_{i\in I}\mathbb{T} \to \mathbb{T}$ is a group homomorphism that extends $\varphi_1|_{\Pi^1_{i\in I}\mathbb{T}}$ (its existence is guaranteed by Baer's theorem), then $\Phi(\otimes_{i\in I}\, \alpha_i) := \varphi(\alpha)\lambda([\alpha]_\sim)$ $(\alpha \in \Pi_{i\in I}\mathbb{T})$ is a well-defined *-isomorphism from $\mathbb{C}^{\otimes I}_{\mathrm{ut}}$ onto $\mathbb{C}[\Omega^{\mathrm{ut}}_{I;\mathbb{C}}]$.*

Conversely, it is clear that if $\varphi : \Pi_{i\in I}\mathbb{T} \to \mathbb{T}$ is any map such that Φ as defined in the above is a well-defined *-isomorphism, then φ is a group homomorphism extending $\varphi_1|_{\Pi^1_{i\in I}\mathbb{T}}$. On the other hand, there is a simpler proof for Corollary 3.4. In fact, for $\alpha, \beta \in \Pi_{i\in I}\mathbb{T}$ with $\alpha \sim \beta$, one has $\varphi(\alpha)^{-1} \cdot \otimes_{i\in I} \alpha_i = \varphi(\beta)^{-1} \cdot \otimes_{i\in I} \beta_i$. Thus, $[\alpha]_\sim \mapsto \varphi(\alpha)^{-1} \cdot \otimes_{i\in I} \alpha_i$ is a well-defined group homomorphism from $\Omega^{\mathrm{ut}}_{I;\mathbb{C}}$ to $U_{\mathbb{C}^{\otimes I}_{\mathrm{ut}}}$ such that $\{\varphi(\alpha)^{-1} \cdot \otimes_{i\in I} \alpha_i : [\alpha]_\sim \in \Omega^{\mathrm{ut}}_{I;\mathbb{C}}\}$ is a basis for $\mathbb{C}^{\otimes I}_{\mathrm{ut}}$.

Example 3.5. For any subgroup $G \subseteq \mathbb{T}^n$, the algebra defined as in Example 2.10 (c) is a *-subalgebras of $\mathbb{C}^{\otimes I}_{\mathrm{ut}}$.

In the remainder of this section, we will show that the center of $\bigotimes^{\mathrm{ut}}_{i\in I} A_i$ is the tensor product of centers of the A_i when $A_i = \operatorname{span} U_{A_i}$ for all $i \in I$.

If A is an algebra and G is a group, we denote by $Z(A)$ and $Z(G)$ the center of A and the center of G respectively. Clearly, the inclusion $\Pi_{i\in I} U_{Z(A_i)} \subseteq \Pi_{i\in I} U_{A_i}$ induces an injective group homomorphism from $\Omega^{\mathrm{ut}}_{I;Z(A)}$ to $\Omega^{\mathrm{ut}}_{I;A}$ and we regard the former as a subgroup of the latter.

Theorem 3.6. *Suppose that there is $F_0 \in \mathfrak{F}$ with $A_i = \operatorname{span} U_{A_i}$ for all $i \in I_0 := I \setminus F_0$.*

(a) $Z(\Omega^{\mathrm{ut}}_{I;A}) = \Omega^{\mathrm{ut}}_{I;Z(A)}$. *Moreover, $Z(\Omega^{\mathrm{ut}}_{I;A}) = \Omega^{\mathrm{ut}}_{I;A}$ if and only if all but a finite number of the A_i are commutative.*

(b) *Every element in $\Omega^{\mathrm{ut}}_{I;A} \setminus Z(\Omega^{\mathrm{ut}}_{I;A})$ has an infinite conjugacy class.*

(c) $Z\big(\bigotimes^{\mathrm{ut}}_{i\in I} A_i\big) = \bigotimes^{\mathrm{ut}}_{i\in I} Z(A_i)$.

Proof. (a) It is obvious that $\Omega^{\mathrm{ut}}_{I;Z(A)} \subseteq Z(\Omega^{\mathrm{ut}}_{I;A})$. Suppose $u \in \Pi_{i\in I} U_{A_i}$ with $[u]_\sim \notin \Omega^{\mathrm{ut}}_{I;Z(A)}$. There is an infinite subset $J \subseteq I_0$ such that $u_i \notin Z(A_i)$ $(i \in J)$. For each $i \in J$, one can find $v_i \in U_{A_i}$ such that $u_i v_i \neq v_i u_i$. For any $i \in I \setminus J$, we put $v_i = e_i$. Then $[v]_\sim \in \Omega^{\mathrm{ut}}_{I;A}$ and $[u]_\sim [v]_\sim \neq [v]_\sim [u]_\sim$. Consequently, $[u]_\sim \notin Z(\Omega^{\mathrm{ut}}_{I;A})$. This argument also shows that if the set $\{i \in I : Z(A_i) \neq A_i\}$ is infinite, then $Z(\Omega^{\mathrm{ut}}_{I;A}) \neq \Omega^{\mathrm{ut}}_{I;A}$. Conversely, it is clear that $\Omega^{\mathrm{ut}}_{I;Z(A)} = \Omega^{\mathrm{ut}}_{I;A}$ if all but a finite numbers of the A_i are commutative.

(b) Suppose that $[u]_\sim \in \Omega^{\mathrm{ut}}_{I;A} \setminus Z(\Omega^{\mathrm{ut}}_{I;A})$ and $\{i_n\}_{n\in\mathbb{N}}$ is a sequence of distinct elements in I_0 such that $u_{i_n} \notin Z(A_{i_n})$ $(n \in \mathbb{N})$. For each $n \in \mathbb{N}$, choose $v_{i_n} \in U_{A_{i_n}}$ with $v_{i_n} u_{i_n} v^*_{i_n} \neq u_{i_n}$. For any prime number p, we set $w^{(p)}_{i_n} := v_{i_n}$ $(n \in \mathbb{N}p)$, and $w^{(p)}_i := e_i$ if $i \in I \setminus \{i_n : n \in \mathbb{N}p\}$. If p and q are distinct prime numbers, then

$$w^{(q)}_{i_n} u_{i_n} (w^{(q)}_{i_n})^* = u_{i_n} \neq w^{(p)}_{i_n} u_{i_n} (w^{(p)}_{i_n})^* \quad (n \in \mathbb{N}p \setminus \mathbb{N}q).$$

Consequently, $w^{(q)} u (w^{(q)})^* \nsim w^{(p)} u (w^{(p)})^*$, and the conjugacy class of $[u]_\sim$ is infinite.

(c) Since $Z(\bigotimes^{\mathrm{ut}}_{i\in I} A_i) = \bigotimes_{i\in F_0} Z(A_i) \otimes Z(\bigotimes^{\mathrm{ut}}_{i\in I_0} A_i)$, we may assume that $A_i = \operatorname{span} U_{A_i}$ for all $i \in I$. In this case, $Z(\bigotimes^{\mathrm{ut}}_{i\in I} A_i) = \big(\bigotimes^{\mathrm{ut}}_{i\in I} A_i\big)^\Xi$, where $\big(\bigotimes^{\mathrm{ut}}_{i\in I} A_i\big)^\Xi$ is the fixed point algebra of the action Ξ as defined above. Moreover, one has $\bigotimes^{\mathrm{ut}}_{i\in I} Z(A_i) \subseteq Z(\bigotimes^{\mathrm{ut}}_{i\in I} A_i)$ and it remains to show that $\big(\bigotimes^{\mathrm{ut}}_{i\in I} A_i\big)^\Xi \subseteq \bigotimes^{\mathrm{ut}}_{i\in I} Z(A_i)$.

Let $v^{(1)}, \dots, v^{(n)} \in \Pi_{i\in I} U_{A_i}$ be mutually inequivalent elements, let $F \in \mathfrak{F}$, and let $b_1, \dots, b_n \in \bigotimes_{i\in F} A_i \setminus \{0\}$ be such that $a := \sum_{k=1}^n J_F^{v^{(k)}}(b_k) \in \big(\bigotimes_{i\in I}^{\mathrm{ut}} A_i\big)^{\Xi}$. We first claim that $[v^{(k)}]_\sim \in \Omega^{\mathrm{ut}}_{I;Z(A)}$ $(k = 1, \dots, n)$. Suppose, to the contrary, that $[v^{(1)}]_\sim \notin \Omega^{\mathrm{ut}}_{I;Z(A)} = Z(\Omega^{\mathrm{ut}}_{I;A})$. For every $u \in \Pi_{i\in I} U_{A_i}$, one has

$$\Xi_u\big(J_F^{v^{(1)}}(b_k)\big) \in \big(\bigotimes_{i\in I}^{[uv^{(1)}u^*]_\sim} A_i\big) \setminus \{0\}.$$

As $\Xi_u(a) = a$, we see that $[uv^{(1)}u^*]_\sim \in \{[v^{(1)}]_\sim, \dots, [v^{(n)}]_\sim\}$, which contradicts the fact that $\{[uv^{(1)}u^*]_\sim : [u]_\sim \in \Omega^{\mathrm{ut}}_{I;A}\}$ is an infinite set (by part (b)).

By enlarging F, we may assume that $v^{(k)} \in \Pi_{i\in I} U_{Z(A_i)}$ $(k = 1, \dots, n)$. For each $u \in \Pi_{i\in I} U_{A_i}$ and $k \in \{1, \dots, n\}$, one has $\Xi_u(J_F^{v^{(k)}}(b_k)) = J_F^{v^{(k)}}(b_k)$ and so, $b_k \in Z(\bigotimes_{i\in F} A_i)$. Therefore, $a \in \bigotimes_{i\in I}^{\mathrm{ut}} Z(A_i)$, as expected. □

The reader should notice that $\bigotimes_{i\in I}^{\mathrm{ut}} Z(A_i)$ equals $\bigoplus_{\omega\in Z(\Omega^{\mathrm{ut}}_{I;A})} \bigotimes_{i\in I}^{\omega} Z(A_i)$ instead of $\bigoplus_{\omega\in\Omega^{\mathrm{ut}}_{I;A}} \bigotimes_{i\in I}^{\omega} Z(A_i)$ (strictly speaking, the latter object does not make sense).

Example 3.7. (a) If $n_i \in \mathbb{N}$ $(i \in I)$, then $Z\big(\bigotimes_{i\in I}^{\mathrm{ut}} M_{n_i}(\mathbb{C})\big) \cong \mathbb{C}_{\mathrm{ut}}^{\otimes I}$.

(b) If G_i are icc groups, then $Z(\bigotimes_{i\in I}^{\mathrm{ut}} \mathbb{C}[G_i]) \cong \mathbb{C}_{\mathrm{ut}}^{\otimes I}$ canonically.

We end this section with the following brief discussion on the non-unital case. Suppose that $\{A_i\}_{i\in I}$ is a family of *-algebras, not necessarily unital. If $M(A_i)$ is the double centraliser algebra of A_i $(i \in I)$, we define an ideal, $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$, of $\bigotimes_{i\in I}^{\mathrm{ut}} M(A_i)$ as follows:

$$\bigotimes_{i\in I}^{\mathrm{ut}} A_i \ := \ \mathrm{span}\,\big\{J_F^u(a) : F \in \mathfrak{F}; a \in \bigotimes\nolimits_{i\in F} A_i; u \in \Pi_{i\in I} U_{M(A_i)}\big\}.$$

In general, $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ is not a subset of $\bigotimes_{i\in I} A_i$. In a similar fashion, we define

$$\bigotimes_{i\in I}^{e} A_i \ := \ \mathrm{span}\,\big\{J_F^u(a) : F \in \mathfrak{F}; a \in \bigotimes\nolimits_{i\in F} A_i; u \in \Pi_{i\in I} U_{M(A_i)}; u \sim e\big\},$$

which is an ideal of $\bigotimes_{i\in I}^{e} M(A_i)$. By the proof of Theorem 3.2 (a), one may identify $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ as the ideal of $(\bigotimes_{i\in I}^{e} M(A_i)) \rtimes_{\check{\Xi},m} \Omega^{\mathrm{ut}}_{I;M(A)}$ consisting of functions from $\Omega^{\mathrm{ut}}_{I;M(A)}$ to $\bigotimes_{i\in I}^{e} A_i$ having finite supports.

4. Tensor products of inner product spaces

Throughout this section, $(H_i, \langle\cdot,\cdot\rangle)$ is a non-zero inner product space $(i \in I)$. Moreover, we denote $\Omega^{\mathrm{unit}}_{I;H} := \Pi_{i\in I}\mathfrak{S}_1(H_i)/\sim$.

If B is a unital *-algebra and X is a unital left B-module, a map $\langle\cdot,\cdot\rangle_B : X \times X \to B$ is called a (*left*) *Hermitian B-form on X* if $\langle ax+y, z\rangle_B = a\langle x, z\rangle_B + \langle y, z\rangle_B$ and $\langle x, y\rangle_B^* = \langle y, x\rangle_B$ $(x, y, z \in X; a \in B)$. It is easy to see that a Hermitian B-form on X can be regarded as a B-bimodule map $\theta : X \otimes \tilde{X} \to B$ satisfying $\theta(x \otimes \tilde{y})^* = \theta(y \otimes \tilde{x})$ (where $\tilde{X}$ is the conjugate vector space of X regarded as a unital right B-module in the canonical way). Consequently, part (a) of the following result follows readily from the universal property of tensor products, while part (b) is easily verified.

Proposition 4.1. (a) *There is a Hermitian $\mathbb{C}^{\otimes I}$-form on $\bigotimes_{i\in I} H_i$ such that $\langle \otimes_{i\in I} x_i, \otimes_{i\in I} y_i\rangle_{\mathbb{C}^{\otimes I}} := \otimes_{i\in I} \langle x_i, y_i\rangle$ $(x, y \in \Pi_{i\in I} H_i)$.*

(b) *For a fixed $\mu \in \Omega^{\rm unit}_{I;H}$, one has $\langle \Theta_H(x), \Theta_H(y)\rangle_{\mathbb{C}^{\otimes I}} = \Pi_{i\in I}\langle x_i, y_i\rangle(\otimes_{i\in I} 1)$ $(x, y \in \Pi^{\mu}_{i\in I} H_i)$. This induces an inner product on $\bigotimes^{\mu}_{i\in I} H_i$ that coincides with the one given by the inductive limit of $\big(\bigotimes_{i\in F} H_i, J^{\mu}_{G;F}\big)_{F\subseteq G\in\mathfrak{F}}$, in the category of inner product spaces with isometries as morphisms.*

We want to construct a nice inner product space from the above Hermitian $\mathbb{C}^{\otimes I}$-form. A naive idea is to appeal to a construction for Hilbert C^*-modules that produces a Hilbert space from a positive linear functional on $\mathbb{C}^{\otimes I}$. However, the difficulty is that there is no canonical order structure on $\mathbb{C}^{\otimes I}$. Nevertheless, we will make a similar construction using the functional ϕ_1 in Example 2.2 (a). In this case, one can only consider a subspace of $\bigotimes_{i\in I} H_i$ (see Example 4.3 below).

Lemma 4.2. *Define $\langle \xi, \eta\rangle_{\phi_1} := \phi_1(\langle \xi, \eta\rangle_{\mathbb{C}^{\otimes I}})$ $(\xi, \eta \in \bigotimes_{i\in I} H_i)$ and set*

$$\bigotimes^{\rm ct}_{i\in I} H_i := {\rm span}\, \Theta_H(\Pi_{i\in I} B_1(H_i))$$

as well as $\bigotimes^{\rm unit}_{i\in I} H_i := {\rm span}\, \Theta_H(\Pi_{i\in I} \mathfrak{S}_1(H_i))$.

(a) *For any $\mu \in \Omega^{\rm unit}_{I;H}$, the restriction of $\langle\cdot,\cdot\rangle_{\phi_1}$ to $\bigotimes^{\mu}_{i\in I} H_i \times \bigotimes^{\mu}_{i\in I} H_i$ coincides with the inner product in Proposition 4.1 (b).*

(b) *$\langle\cdot,\cdot\rangle_{\phi_1}$ is a positive sesquilinear form on $\bigotimes^{\rm ct}_{i\in I} H_i$ and is an inner product on $\bigotimes^{\rm unit}_{i\in I} H_i$. Moreover, if*

$$K := \Big\{ y \in \bigotimes^{\rm ct}_{i\in I} H_i : \langle x, y\rangle_{\phi_1} = 0, \forall x \in \bigotimes^{\rm ct}_{i\in I} H_i \Big\},$$

then $\bigotimes^{\rm ct}_{i\in I} H_i = K \oplus \bigotimes^{\rm unit}_{i\in I} H_i$ (as vector spaces).

(c) *If $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$, then $\bigotimes^{\rm unit}_{i\in I} H_i = (\bigotimes^{\rm unit}_{i\in I_1} H_i) \otimes (\bigotimes^{\rm unit}_{j\in I_2} H_j)$ as inner product spaces.*

Proof. (a) This part is clear.

(b) It is obvious that $\langle\cdot,\cdot\rangle_{\phi_1}$ is a sesquilinear form on $\bigotimes^{\rm ct}_{i\in I} H_i$. Let

$$E := \big\{ x \in \Pi_{i\in I} B_1(H_i) : \|x_i\| < 1 \text{ for an infinite number of } i \in I \big\}$$

and $\tilde{K} := {\rm span}\, \Theta_H(E)$. Clearly, $\bigotimes^{\rm ct}_{i\in I} H_i = \tilde{K} \oplus \bigotimes^{\rm unit}_{i\in I} H_i$. Moreover, if $u \in \Pi_{i\in I} B_1(H_i)$ and $v \in E$, then $\langle u_i, v_i\rangle \neq 1$ for an infinite number of $i \in I$, which implies that $\langle \otimes_{i\in I} u_i, \otimes_{i\in I} v_i\rangle_{\phi_1} = 0$. Consequently, $\tilde{K} \subseteq K$.

We claim that $\langle\xi,\xi\rangle_{\phi_1} \geq 0$ ($\xi \in \bigotimes^{\rm ct}_{i\in I} H_i$). Suppose that $\xi = \sum_{k=1}^n \lambda_k \otimes_{i\in I} u_i^{(k)}$ with $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and $u^{(1)},\dots,u^{(n)} \in \Pi_{i\in I} B_1(H_i)$. Then

$$\langle\xi,\xi\rangle_{\phi_1} \;=\; \sum\nolimits_{k,l=1}^n \lambda_k\bar\lambda_l\phi_1\big(\otimes_{i\in I}\langle u_i^{(k)},u_i^{(l)}\rangle\big).$$

As in the above, $\phi_1\big(\otimes_{i\in I}\langle u_i^{(k)},u_i^{(l)}\rangle\big) = 0$ if either $u^{(k)}$ or $u^{(l)}$ is in E. Thus, by rescaling, we may assume that

$$u^{(1)},\dots,u^{(n)} \in \Pi_{i\in I}\mathfrak{S}_1(H_i).$$

Furthermore, we assume that there exist $0 = n_0 < \cdots < n_m = n$ such that $u^{(n_p+1)} \sim \cdots \sim u^{(n_{p+1})}$ for all $p \in \{0,\dots,m-1\}$, but $u^{(n_p)} \nsim u^{(n_q)}$ whenever $1 \leq p \neq q \leq m$. It is not hard to check that $u^{(k)} \sim u^{(l)}$ if and only if $\langle u_i^{(k)},u_i^{(l)}\rangle = 1$ e.f. (as $\|u_i^{(k)}\|,\|u_i^{(l)}\| \leq 1$). Consequently, if $1 \leq p \neq q \leq m$,

(4.1) $$\phi_1\big(\otimes_{i\in I}\langle u_i^{(k)},u_i^{(l)}\rangle\big) = 0 \quad \text{when } n_p < k \leq n_{p+1} \text{ and } n_q < l \leq n_{q+1}.$$

Therefore, in order to show $\langle\xi,\xi\rangle_{\phi_1} \geq 0$, it suffices to consider the case when $u^{(k)} \sim u^{(l)}$ for all $k,l \in \{1,\dots,n\}$, which is the same as $\xi \in \bigotimes^{u^{(1)}}_{i\in I} H_i$. Thus, $\langle\xi,\xi\rangle_{\phi_1} \geq 0$ by part (a).

Next, we show that $\langle\cdot,\cdot\rangle_{\phi_1}$ is an inner product on $\bigotimes^{\rm unit}_{i\in I} H_i$. Suppose that $\xi = \sum_{k=1}^n \lambda_k\otimes_{i\in I} u_i^{(k)}$ with $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ and $u^{(1)},\dots,u^{(n)} \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$ such that $\langle\xi,\xi\rangle_{\phi_1} = 0$. If $n_0,\dots,n_m$ are as above, then

$$\phi_1\Big(\big\langle\sum\nolimits_{k=n_p+1}^{n_{p+1}}\lambda_k\otimes_{i\in I} u_i^{(k)}, \sum\nolimits_{l=n_q+1}^{n_{q+1}}\lambda_l\otimes_{i\in I} u_i^{(l)}\big\rangle_{\mathbb{C}^{\otimes I}}\Big) \;=\; 0,$$

because of (4.1) and the positivity of $\langle\cdot,\cdot\rangle_{\phi_1}$. Hence, we may assume $u^{(k)} \sim u^{(l)}$ for all $k,l \in \{1,\dots,n\}$, and apply part (a) to conclude that $\xi = 0$.

Finally, as $\langle\cdot,\cdot\rangle_{\phi_1}$ is an inner product on $\bigotimes^{\rm unit}_{i\in I} H_i$ and we have both $\bigotimes^{\rm ct}_{i\in I} H_i = \tilde K \oplus \bigotimes^{\rm unit}_{i\in I} H_i$ and $\tilde K \subseteq K$, we obtain $K \subseteq \tilde K$ as well.

(c) Observe that the linear bijection $\Psi : (\bigotimes_{i\in I_1} H_i) \otimes (\bigotimes_{j\in I_2} H_j) \to \bigotimes_{i\in I} H_i$ as in Remark 2.4 (b) restricts to a surjection from $(\bigotimes^{\rm unit}_{i\in I_1} H_i) \otimes (\bigotimes^{\rm unit}_{j\in I_2} H_j)$ to $\bigotimes^{\rm unit}_{i\in I} H_i$. Moreover, for any $u,u' \in \Pi_{i\in I_1}\mathfrak{S}_1(H_i)$ and $v,v' \in \Pi_{j\in I_2}\mathfrak{S}_1(H_j)$, we have $(u,u') \sim (v,v')$ as elements in $\Pi_{i\in I}\mathfrak{S}_1(H_i)$ if and only if $u \sim u'$ and $v \sim v'$. Thus, the argument in part (b) tells us that

$$\begin{aligned}&\big\langle(\otimes_{i\in I_1}u_i)\otimes(\otimes_{j\in I_2}v_j),(\otimes_{i\in I_1}u_i')\otimes(\otimes_{j\in I_2}v_j')\big\rangle_{\phi_1}\\ &\qquad\qquad =\; \langle\otimes_{i\in I_1}u_i,\otimes_{i\in I_1}u_i'\rangle_{\phi_1}\langle\otimes_{j\in I_2}v_j,\otimes_{j\in I_2}v_j'\rangle_{\phi_1}.\end{aligned}$$

This shows that $\Psi|_{(\bigotimes^{\rm unit}_{i\in I_1} H_i)\otimes(\bigotimes^{\rm unit}_{j\in I_2} H_j)}$ is inner product preserving. □

We denote by $\bar\bigotimes^{\mu}_{i\in I} H_i$ *and* $\bar\bigotimes^{\phi_1}_{i\in I} H_i$ *the completions of* $\bigotimes^{\mu}_{i\in I} H_i$ *and* $\bigotimes^{\rm unit}_{i\in I} H_i$*, respectively, under the norms induced by* $\langle\cdot,\cdot\rangle_{\phi_1}$.

Example 4.3. If $H_i = \mathbb{C}$ ($i \in I$), then the sesquilinear form $\langle \cdot, \cdot \rangle_{\phi_1}$ is not positive on the whole space $\bigotimes_{i\in I} H_i$ since $\big\langle (\otimes_{i\in I} 1/2 - \otimes_{i\in I} 2), (\otimes_{i\in I} 1/2 - \otimes_{i\in I} 2)\big\rangle_{\phi_1} = -2$.

Set $\Pi^{\mathrm{eu}}_{i\in I} H_i := \{x \in \Pi_{i\in I} H_i : x_i \in \mathfrak{S}_1(H_i)$ except for a finite number of $i\}$ and let K be an inner product space. A multilinear map $\Phi : \Pi^{\mathrm{eu}}_{i\in I} H_i \to K$ (i.e. Φ is coordinatewise linear) is said to be *componentwise inner product preserving* if for any $\mu, \nu \in \Omega^{\mathrm{unit}}_{I;H}$,

$$\langle \Phi(x), \Phi(y)\rangle = \delta_{\mu,\nu} \Pi_{i\in I} \langle x_i, y_i\rangle \quad (x \in \Pi^{\mu}_{i\in I} H_i; y \in \Pi^{\nu}_{i\in I} H_i),$$

where $\delta_{\mu,\nu}$ is the Kronecker delta.

Theorem 4.4. (a) $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i \cong \bar{\bigoplus}^{\ell^2}_{\mu\in\Omega^{\mathrm{unit}}_{I;H}} \bar{\bigotimes}^{\mu}_{i\in I} H_i$ *canonically as Hilbert spaces.*

(b) $\Theta_H|_{\Pi^{\mathrm{eu}}_{i\in I} H_i} : \Pi^{\mathrm{eu}}_{i\in I} H_i \to \bigotimes^{\mathrm{unit}}_{i\in I} H_i$ *is a componentwise inner product preserving multilinear map. For any inner product space K and any componentwise inner product preserving multilinear map $\Phi : \Pi^{\mathrm{eu}}_{i\in I} H_i \to K$, there is a unique isometry $\tilde{\Phi} : \bigotimes^{\mathrm{unit}}_{i\in I} H_i \to K$ such that $\Phi = \tilde{\Phi} \circ \Theta_H|_{\Pi^{\mathrm{eu}}_{i\in I} H_i}$.*

Proof. (a) Clearly, $\bigotimes^{\mathrm{unit}}_{i\in I} H_i = \sum_{\mu\in\Omega^{\mathrm{unit}}_{I;H}} \bigotimes^{\mu}_{i\in I} H_i$. Moreover, as in the proof of Lemma 4.2 (b), the two subspaces $\bigotimes^{\mu}_{i\in I} H_i$ and $\bigotimes^{\nu}_{i\in I} H_i$ are orthogonal if μ and ν are distinct elements in $\Omega^{\mathrm{unit}}_{I;H}$. The rest of the argument is standard.

(b) It is easy to see that $\Theta_H|_{\Pi^{\mathrm{eu}}_{i\in I} H_i}$ is componentwise inner product preserving. The uniqueness of $\tilde{\Phi}$ follows from the fact that $\Theta_H(\Pi^{\mathrm{eu}}_{i\in I} H_i)$ generates $\bigotimes^{\mathrm{unit}}_{i\in I} H_i$. To show the existence of $\tilde{\Phi}$, we first define a multilinear map $\Phi_0 : \Pi_{i\in I} H_i \to K$ by setting $\Phi_0 = \Phi$ on $\Pi^{\mathrm{eu}}_{i\in I} H_i$ and $\Phi_0 = 0$ on $\Pi_{i\in I} H_i \setminus \Pi^{\mathrm{eu}}_{i\in I} H_i$. Let $\tilde{\Phi}_0 : \bigotimes_{i\in I} H_i \to K$ be the induced linear map and set $\tilde{\Phi} := \tilde{\Phi}_0|_{\bigotimes^{\mathrm{unit}}_{i\in I} H_i}$. Suppose that $u, v \in \Pi_{i\in I} \mathfrak{S}_1(H_i)$, $\xi \in \bigotimes^{u}_{i\in I} H_i$ and $\eta \in \bigotimes^{v}_{i\in I} H_i$. If $u \nsim v$, then $\langle \xi, \eta\rangle_{\phi_1} = 0 = \langle \tilde{\Phi}(\xi), \tilde{\Phi}(\eta)\rangle$. Otherwise, there exist $F \in \mathfrak{F}$ and $\xi_0, \eta_0 \in \bigotimes_{i\in F} H_i$ such that $\xi = J^u_F(\xi_0)$, $\eta = J^v_F(\eta_0)$ and $u_i = v_i$ if $i \in I \setminus F$. In this case, $\langle \tilde{\Phi}(\xi), \tilde{\Phi}(\eta)\rangle = \langle \xi_0, \eta_0\rangle = \langle \xi, \eta\rangle_{\phi_1}$. □

Example 4.5. Suppose that Φ and φ are as in Corollary 3.4, and $\{\delta_\mu\}_{\mu\in\Omega^{\mathrm{unit}}_{I;\mathbb{C}}}$ is the canonical orthonormal basis for $\ell^2\big(\Omega^{\mathrm{unit}}_{I;\mathbb{C}}\big)$. Note that $\Omega^{\mathrm{ut}}_{I;\mathbb{C}} = \Omega^{\mathrm{unit}}_{I;\mathbb{C}}$ and consider the linear bijection $J : \mathbb{C}[\Omega^{\mathrm{ut}}_{I;\mathbb{C}}] \to \mathbb{C}[\Omega^{\mathrm{unit}}_{I;\mathbb{C}}]$ given by $J(\lambda([\alpha]_\sim)) := \delta_{[\alpha]_\sim}$ ($\alpha \in \Pi_{i\in I}\mathbb{T}$). By Example 2.10 (a) and Theorem 4.4 (a), the map $J \circ \Phi$ induces a Hilbert space isomorphism $\hat{\Phi} : \bar{\bigotimes}^{\phi_1}_{i\in I} \mathbb{C} \to \ell^2\big(\Omega^{\mathrm{unit}}_{I;\mathbb{C}}\big)$ such that $\hat{\Phi}(\otimes_{i\in I} \beta_i) = \varphi(\beta)\delta_{[\beta]_\sim}$ ($\beta \in \Pi_{i\in I}\mathbb{T}$).

We would like to compare $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ with the infinite direct product as defined in [20], when $\{H_i\}_{i\in I}$ is a family of Hilbert spaces. Let us first recall from Definition 3.3.1 in [20] that $x \in \Pi_{i\in I} H_i$ is a C_0*-sequence* if $\sum_{i\in I} \big|\|x_i\| - 1\big|$ converges. As in Definition 3.3.2 in [20], if x and y are C_0-sequences such that $\sum_{i\in I} \big|\langle x_i, y_i\rangle - 1\big|$

converges, then we write $x \approx y$. Denote by $[x]_\approx$ the equivalence class of x under $\approx$, and by $\Gamma_{I;H}$ the set of all such equivalence classes (see Definition 3.3.3 in [20]).

Let $\prod \otimes_{i\in I} H_i$ be the infinite direct product Hilbert space as defined in [20], and let $\prod \otimes_{i\in I} x_i$ be the element in $\prod \otimes_{i\in I} H_i$ corresponding to a C_0-sequence x as in Theorem IV of [20]. Notice that if $x \in \Pi^{\mathrm{eu}}_{i\in I} H_i$, then x is a C_0-sequence, and we have a multilinear map

$$\Upsilon : \Pi^{\mathrm{eu}}_{i\in I} H_i \longrightarrow \prod \otimes_{i\in I} H_i.$$

On the other hand, for any $\mathfrak{C} \in \Gamma_{I;H}$, we denote by $\prod \otimes^{\mathfrak{C}}_{i\in I} H_i$ the closed subspace of $\prod \otimes_{i\in I} H_i$ generated by $\{\prod \otimes_{i\in I} x_i : x \in \mathfrak{C}\}$ (see Definition 4.1.1 in [20]).

Proposition 4.6. *Let $\{H_i\}_{i\in I}$ be a family of Hilbert spaces.*

(a) $[x]_\sim \mapsto [x]_\approx$ ($x \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$) *gives a well defined surjection $\kappa_H : \Omega^{\mathrm{unit}}_{I;H} \to \Gamma_{I;H}$. Moreover, for any $x, y \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$, there is a bijection between $\kappa_H^{-1}([x]_\approx)$ and $\kappa_H^{-1}([y]_\approx)$.*

(b) *There exists a linear map $\tilde{\Upsilon} : \bigotimes^{\mathrm{unit}}_{i\in I} H_i \to \prod \otimes_{i\in I} H_i$ such that $\Upsilon = \tilde{\Upsilon} \circ \Theta_H|_{\Pi^{\mathrm{eu}}_{i\in I} H_i}$ and $\tilde{\Upsilon}|_{\bigotimes^{\mu}_{i\in I} H_i}$ extends to a Hilbert space isomorphism $\tilde{\Upsilon}^\mu : \bar{\bigotimes}^{\mu}_{i\in I} H_i \to \prod \otimes^{\kappa_H(\mu)}_{i\in I} H_i$ ($\mu \in \Omega^{\mathrm{unit}}_{I;H}$).*

Proof. (a) Clearly, if $x \sim z$, then $x \approx z$ and κ_H is well defined. Lemma 3.3.7 in [20] tells us that κ_H is surjective. Furthermore, there exists a unitary $u_i \in \mathcal{L}(H_i)$ such that $u_i x_i = y_i$ ($i \in I$), and $[u_i]_{i\in I}$ induces the required bijective correspondence in the second statement.

(b) By the argument of Theorem 4.4 (b), one can construct a linear map $\tilde{\Upsilon}$ such that $\Upsilon = \tilde{\Upsilon} \circ \Theta_H|_{\Pi^{\mathrm{eu}}_{i\in I} H_i}$. By the argument of part (a), we see that $\tilde{\Upsilon}(\bigotimes^{[u]_\sim}_{i\in I} H_i) \subseteq \prod \otimes^{[u]_\approx}_{i\in I} H_i$ ($u \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$). Furthermore, by Lemma 4.2 (a), Proposition 4.1 (b) and Theorem IV in [20], we see that $\tilde{\Upsilon}|_{\bigotimes^{[u]_\sim}_{i\in I} H_i}$ is an isometry. Finally, $\tilde{\Upsilon}|_{\bigotimes^{[u]_\sim}_{i\in I} H_i}$ has dense range (by Lemma 4.1.2 of [20]). □

Notice that $\tilde{\Upsilon}$ is, in general, unbounded but Remark 4.7 (b) below tells us that $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ is a "natural dilation" of $\prod \otimes_{i\in I} H_i$. On the other hand, Remark 4.7 (d) says that it is possible to construct $\prod \otimes_{i\in I} H_i$ in a way similar to $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$. Note however, that the construction of $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ is totally algebraic and $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ itself seems to be more natural (see Theorem 4.8 and Example 5.6 below).

Remark 4.7. Suppose that $\{H_i\}_{i\in I}$ is a family of Hilbert spaces.

(a) $\sim$ and $\approx$ are different even in the case when $I = \mathbb{N}$ and $H_i = \mathbb{C}$ ($i \in \mathbb{N}$) because one can find $x, y \in \Pi_{i\in\mathbb{N}}\mathbb{T}$ with $x_i \neq y_i$ for all $i \in \mathbb{N}$ but for which $\sum_{i=1}^\infty \big|\langle x_i, y_i\rangle - 1\big|$ converges. In fact, $\kappa_H^{-1}([x]_\approx)$ is an infinite set.

(b) By Lemma 4.1.1 in [20], we have

$$\prod \otimes_{i\in I} H_i \;=\; \bar{\bigoplus}^{\ell^2}_{\mathfrak{C}\in\Gamma_{I;H}} \prod \otimes^{\mathfrak{C}}_{i\in I} H_i.$$

Therefore, Theorem 4.4 (a) and Proposition 4.6 tell us that for a fixed $\gamma_0 \in \Gamma_{I;H}$, one has a canonical Hilbert space isomorphism

$$\bar{\bigotimes}^{\phi_1}_{i\in I} H_i \cong \ell^2\big(\kappa_H^{-1}(\gamma_0)\big)\bar{\otimes}\big(\prod \otimes_{i\in I} H_i\big).$$

(c) For each $i \in I$, let K_i be an inner product space such that H_i is the completion of K_i. Then $\bar{\bigotimes}^{\phi_1}_{i\in I} K_i$ is, in general, not canonically isomorphic to $\bar{\bigotimes}^{\phi_1}_{i\in I} H_i$ because $\Omega^{\rm unit}_{I;K} \subsetneq \Omega^{\rm unit}_{I;H}$ if $K_i \subsetneq H_i$ for an infinite number of $i \in I$. On the other hand, if I is countable, for any $x \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$, there exists $y \in \Pi_{i\in I}\mathfrak{S}_1(K_i)$ such that $x \approx y$. This shows that the restriction, $\kappa_{H;K}$, of κ_H to $\Omega^{\rm unit}_{I;K}$ is also a surjection onto $\Gamma_{I;H}$. However, we do not know if the cardinality of $\kappa_{H;K}^{-1}(\mathfrak{C})$ are the same for different $\mathfrak{C} \in \Gamma_{I;H}$.

(d) If ϕ_0 is as in Example 2.2 (b), it is easy to see that

$$\langle \prod \otimes u_i, \prod \otimes v_i\rangle = \phi_0\big(\langle \otimes_{i\in I} u_i, \otimes_{i\in I} v_i\rangle_{\mathbb{C}^{\otimes I}}\big) \quad (u, v \in \Pi^{\rm unit}_{i\in I} H_i).$$

Thus, the sesquilinear form $\phi_0\big(\langle\cdot,\cdot\rangle_{\mathbb{C}^{\otimes I}}\big)$ produces $\prod \otimes H_i$. If one wants a self-contained alternative construction for $\prod \otimes H_i$, one needs to establish the positivity of $\phi_0\big(\langle\cdot,\cdot\rangle_{\mathbb{C}^{\otimes I}}\big)$, which can be reduced to showing the positivity when all H_i are of the same finite dimension.

In the remainder of this section, we show that $\bigotimes^{\rm unit}_{i\in I} H_i$ can be completed into a $C^*(\Omega^{\rm ut}_{I;\mathbb{C}})$-module, which gives many pre-inner products on $\bigotimes^{\rm unit}_{i\in I} H_i$ including $\langle\cdot,\cdot\rangle_{\phi_1}$. In the following, we use the convention that the A-valued inner product of an inner product A-module is A-linear in the first variable (where A is a pre-C^*-algebra). On the other hand, we recall that if G is a group and λ_g is the canonical image of g in $\mathbb{C}[G]$, the map $\sum_{g\in G}\alpha_g\lambda_g \mapsto \alpha_e$ ($\alpha_g \in \mathbb{C}$), where $e \in G$ is the identity, extends to a faithful tracial state χ_G on $C^*(G)$.

Theorem 4.8. (a) *There exists an inner product $\mathbb{C}[\Omega^{\rm ut}_{I;\mathbb{C}}]$-module structure on $\bigotimes^{\rm unit}_{i\in I} H_i$. If $\bar{\bigotimes}^{\rm mod}_{i\in I} H_i$ is the Hilbert $C^*(\Omega^{\rm ut}_{I;\mathbb{C}})$-module given by the completion of this $\mathbb{C}[\Omega^{\rm ut}_{I;\mathbb{C}}]$-module, we have a canonical Hilbert space isomorphism*

$$\bar{\bigotimes}^{\phi_1}_{i\in I} H_i \cong \Big(\bar{\bigotimes}^{\rm mod}_{i\in I} H_i\Big)\bar{\otimes}_{\chi_{\Omega^{\rm ut}_{I;\mathbb{C}}}}\mathbb{C}. \tag{4.2}$$

(b) *If $G \subseteq \Omega^{\rm ut}_{I;\mathbb{C}}$ is a subgroup and $\mathcal{E}_G : C^*(\Omega^{\rm ut}_{I;\mathbb{C}}) \to C^*(G)$ is the canonical conditional expectation, there is an inner product $\mathbb{C}[G]$-module structure on $\bigotimes^{\rm unit}_{i\in I} H_i$, whose completion coincides with the Hilbert $C^*(G)$-module $\big(\bar{\bigotimes}^{\rm mod}_{i\in I} H_i\big)\bar{\otimes}_{\mathcal{E}_G} C^*(G)$.*

Proof. (a) Clearly, $\bigotimes^{\rm unit}_{i\in I} H_i$ is a $\mathbb{C}^{\otimes I}_{\rm ut}$-submodule of the $\mathbb{C}^{\otimes I}$-module $\bigotimes_{i\in I} H_i$ (see Proposition 2.3 (c)). Moreover, one has a linear "truncation" E from $\mathbb{C}^{\otimes I} = \big(\bigoplus_{\omega\in\Omega_{I;\mathbb{C}}\setminus\Omega^{\rm ut}_{I;\mathbb{C}}}\bigotimes^{\omega}_{i\in I}\mathbb{C}\big) \oplus \mathbb{C}^{\otimes I}_{\rm ut}$ to $\mathbb{C}^{\otimes I}_{\rm ut}$ sending (α, β) to β. Define

$$\langle \xi, \eta\rangle_{\mathbb{C}^{\otimes I}_{\rm ut}} := E\big(\langle \xi, \eta\rangle_{\mathbb{C}^{\otimes I}}\big) \quad \Big(\xi, \eta \in \bigotimes^{\rm unit}_{i\in I} H_i\Big),$$

which is a Hermitian $\mathbb{C}_{\mathrm{ut}}^{\otimes I}$-form because by (3.1), we have

$$E(ab) = E(a)b \quad \text{and} \quad E(a^*) = E(a)^* \quad (a \in \mathbb{C}^{\otimes I}; b \in \mathbb{C}_{\mathrm{ut}}^{\otimes I}).$$

For any $u, v \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$, we write $u \sim_{\mathrm{s}} v$ if there exists $\beta \in \Pi_{i\in I}\mathbb{T}$ such that $u_i = \beta_i v_i$ e.f. Then $\sim_{\mathrm{s}}$ is an equivalence relation on $\Pi_{i\in I}\mathfrak{S}_1(H_i)$ satisfying

$$u \sim_{\mathrm{s}} v \quad \text{if and only if} \quad \langle \otimes_{i\in I} u_i, \otimes_{i\in I} v_i\rangle_{\mathbb{C}^{\otimes I}} \in \mathbb{C}_{\mathrm{ut}}^{\otimes I}. \tag{4.3}$$

Let Φ and φ be as in Corollary 3.4. Suppose that $\xi = \sum_{k=1}^n \alpha_k \otimes_{i\in I} u_i^{(k)}$ with $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $u^{(1)}, \dots, u^{(n)} \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$. We first show that $\Phi(\langle\xi,\xi\rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}}) \in C^*(\Omega_{I;\mathbb{C}}^{\mathrm{ut}})_+$. As in the proof of Lemma 4.2 (b), it suffices to consider the case when $u^{(k)} \sim_{\mathrm{s}} u^{(1)}$ for any $k \in \{1, \dots, n\}$ (because of relation (4.3)). Let $F \in \mathfrak{F}$ and $\beta^{(1)}, \dots, \beta^{(n)} \in \Pi_{i\in I}\mathbb{T}$ be such that $u_i^{(k)} = \beta_i^{(k)} u_i^{(1)}$ ($i \in I \setminus F$; $k = 1, \dots, n$). For any $k, l \in \{1, \dots, n\}$, we have

$$\Phi\big((\Pi_{i\in F}\langle u_i^{(k)}, u_i^{(l)}\rangle_i)(\otimes_{i\in I\setminus F}\beta_i^{(k)}\overline{\beta_i^{(l)}})\big) \;=\; \langle \tilde\varphi_F(u^{(k)}), \tilde\varphi_F(u^{(l)})\rangle_F,$$

where $\tilde\varphi_F(u^{(k)}) := \big(\varphi(\beta^{(k)})\Pi_{i\in F}\beta_i^{(k)}\big)^{-1}(\otimes_{i\in F} u_i^{(k)}) \otimes \lambda_{[\beta^{(k)}]_\sim}$ and $\langle\cdot,\cdot\rangle_F$ is the canonical $\mathbb{C}[\Omega_{I;\mathbb{C}}^{\mathrm{ut}}]$-valued inner product on $(\bigotimes_{i\in F} H_i) \otimes \mathbb{C}[\Omega_{I;\mathbb{C}}^{\mathrm{ut}}]$. Therefore,

$$\Phi(\langle\xi,\xi\rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}}) \;=\; \Big\langle \sum_{k=1}^n \alpha_k\tilde\varphi_F(u^{(k)}), \sum_{k=1}^n \alpha_k\tilde\varphi_F(u^{(k)})\Big\rangle_F \;\geq\; 0.$$

Next, we show that $\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}} \circ \Phi \circ E = \phi_1$. Let $\alpha \in \Pi_{i\in I}\mathbb{C}^\times$. If $\alpha \nsim 1$, then $\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}} \circ \Phi \circ E(\otimes_{i\in I}\alpha_i) = 0$ (as $\Phi(E(\otimes_{i\in I}\alpha_i)) \notin \mathbb{C}\cdot\lambda_{[1]_\sim} \setminus \{0\}$, whether or not $[\alpha]_\sim \in \Omega_{I;\mathbb{C}}^{\mathrm{ut}}$) and we also have $\phi_1(\otimes_{i\in I}\alpha_i) = 0$. If $\alpha \sim 1$, then $\otimes_{i\in I}\alpha_i = (\Pi_{i\in I}\alpha_i)(\otimes_{i\in I} 1) = (\Pi_{i\in I}\alpha_i)\lambda_{[1]_\sim}$, which implies that $\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}}(\Phi(\otimes_{i\in I}\alpha_i)) = \Pi_{i\in I}\alpha_i = \phi_1(\otimes_{i\in I}\alpha_i)$.

Thus, we have

$$\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}}\big(\Phi(\langle\xi,\eta\rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}})\big) = \langle\xi,\eta\rangle_{\phi_1} \quad \big(\xi,\eta \in \bigotimes_{i\in I}^{\mathrm{unit}} H_i\big). \tag{4.4}$$

As a consequence, if $\Phi(\langle\xi,\xi\rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}}) = 0$, we know from Lemma 4.2 (b) that $\xi = 0$. This gives an inner product $\mathbb{C}[\Omega_{I;\mathbb{C}}^{\mathrm{ut}}]$-module structure on $\bigotimes_{i\in I}^{\mathrm{unit}} H_i$. Furthermore, the required isomorphism $\bar\bigotimes_{i\in I}^{\phi_1} H_i \cong (\bar\bigotimes_{i\in I}^{\mathrm{mod}} H_i)\bar\otimes_{\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}}}\mathbb{C}$ also follows from (4.4).

(b) Since $\bigotimes_{i\in I}^{\mathrm{unit}} H_i$ is a $\mathbb{C}[G]$-module (we identify $\mathbb{C}[G]$ with $\bigoplus_{\omega\in G}\bigotimes_{i\in I}^{\omega}\mathbb{C}$ under the *-isomorphism Φ of Corollary 3.4), every element in $(\bigotimes_{i\in I}^{\mathrm{unit}} H_i) \otimes_{\mathbb{C}[G]} \mathbb{C}[G]$ is of the form $\xi \otimes_{\mathbb{C}[G]} 1$ for some $\xi \in \bigotimes_{i\in I}^{\mathrm{unit}} H_i$. Moreover, if $\xi, \eta \in \bigotimes_{i\in I}^{\mathrm{unit}} H$, then

$$\langle \xi\otimes_{\mathbb{C}[G]} 1, \eta\otimes_{\mathbb{C}[G]} 1\rangle_{(\bar\bigotimes_{i\in I}^{\mathrm{mod}}\mathbb{C})\bar\otimes_{\mathcal{E}_G} C^*(G)} = \mathcal{E}_G(\Phi(\langle\xi,\eta\rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}})) = \Phi(E_G(\langle\xi,\eta\rangle_{\mathbb{C}^{\otimes I}})), \tag{4.5}$$

where E_G is the linear "truncation" map from $\mathbb{C}^{\otimes I}$ to $\bigoplus_{\omega\in G}\bigotimes_{i\in I}^{\omega}\mathbb{C}$ defined as in part (a). Therefore, $\Phi(E_G(\langle\cdot,\cdot\rangle_{\mathbb{C}^{\otimes I}}))$ is a positive Hermitian $\mathbb{C}[G]$-form on $\bigotimes_{i\in I}^{\mathrm{unit}} H_i$. Obviously, $\chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}} = \chi_G \circ \mathcal{E}_G$, and by (4.4),

$$\chi_G(\Phi(E_G(\langle\xi,\eta\rangle_{\mathbb{C}^{\otimes I}}))) = \chi_{\Omega_{I;\mathbb{C}}^{\mathrm{ut}}}(\Phi(\langle\xi,\eta\rangle_{\mathbb{C}_{\mathrm{ut}}^{\otimes I}})) = \langle\xi,\eta\rangle_{\phi_1} \quad \big(\xi,\eta \in \bigotimes_{i\in I}^{\mathrm{unit}} H\big).$$

This implies that $\Phi(E_G(\langle\cdot,\cdot\rangle_{\mathbb{C}^{\otimes I}}))$ is non-degenerate (since $\langle\cdot,\cdot\rangle_{\phi_1}$ is non-degenerate by Lemma 4.2 (b)). Now, equation (4.5) tells us that the Hilbert $C^*(G)$-module $(\bar{\bigotimes}^{\mathrm{mod}}_{i\in I} H_i)\bar{\otimes}_{\mathcal{E}_G} C^*(G)$ is the completion of $\bigotimes^{\mathrm{unit}}_{i\in I} H_i$ under the norm induced by the $\mathbb{C}[G]$-valued inner product $\Phi(E_G(\langle\cdot,\cdot\rangle_{\mathbb{C}^{\otimes I}}))$. □

Let $\{e\}$ be the trivial subgroup of $\Omega^{\mathrm{ut}}_{I;\mathbb{C}}$. Since one can identify $E_{\{e\}}$ with ϕ_1 (through the argument of Theorem 4.8 (b)), one has

$$\bar{\bigotimes}^{\phi_1}_{i\in I} H_i \;\cong\; (\bar{\bigotimes}^{\mathrm{mod}}_{i\in I} H_i)\bar{\otimes}_{\mathcal{E}_{\{e\}}}\mathbb{C}.$$

Remark 4.9. (a) For any subgroup $G\subseteq\Omega^{\mathrm{ut}}_{I;\mathbb{C}}$ and any faithful state φ on $C^*(G)$, the Hilbert space

$$\Big((\bar{\bigotimes}^{\mathrm{mod}}_{i\in I} H_i)\bar{\otimes}_{\mathcal{E}_G} C^*(G)\Big)\bar{\otimes}_{\varphi}\mathbb{C}$$

induces an inner product on $\bigotimes^{\mathrm{unit}}_{i\in I} H_i$.

(b) If $x\in\Pi^0_{i\in I}\mathbb{C}$ (see Example 2.2 (b)), then $\sup_{i\in I}|x_i|<\infty$. This, together with the surjectivity of $\kappa_{\mathbb{C}}$ (see Proposition 4.6 (a)), tells us that $\Gamma_{I;\mathbb{C}}$ is a group under the multiplication: $[x]_\approx\cdot[y]_\approx := [xy]_\approx$ (where $(xy)_i := x_iy_i$ for any $i\in I$). Moreover, $\kappa_{\mathbb{C}}:\Omega^{\mathrm{ut}}_{I;\mathbb{C}}=\Omega^{\mathrm{unit}}_{I;\mathbb{C}}\to\Gamma_{I;\mathbb{C}}$ is a group homomorphism, which induces a surjective *-homomorphism $\bar{\kappa}_{\mathbb{C}}: C^*(\Omega^{\mathrm{ut}}_{I;\mathbb{C}})\to C^*(\Gamma_{I;\mathbb{C}})$.

(c) It is natural to ask whether $\big((\bar{\bigotimes}^{\mathrm{mod}}_{i\in I} H_i)\bar{\otimes}_{\bar{\kappa}_{\mathbb{C}}} C^*(\Gamma_{I;\mathbb{C}})\big)\bar{\otimes}_{\chi_{\Gamma_{I;\mathbb{C}}}}\mathbb{C}$ is isomorphic to $\prod\otimes_{i\in I} H_i$ canonically. Unfortunately, this is not the case. In fact, for any $x,y\in\Pi^{\mathrm{unit}}_{i\in I}H_i$, we write $x\approx_{\mathbb{T}} y$ if there exists $\alpha\in\Pi_{i\in I}\mathbb{T}$ with $\alpha\approx 1$ such that $x_i=\alpha_i y_i$ e.f. It is easy to check that $\approx_{\mathbb{T}}$ is an equivalence relation in general standing strictly between $\sim$ and $\approx$. Moreover, one has

$$\big\langle((\otimes_{i\in I}x_i)\otimes_{\bar{\kappa}_{\mathbb{C}}}1)\otimes_{\chi_{\Gamma_{I;\mathbb{C}}}}1,((\otimes_{i\in I}y_i)\otimes_{\bar{\kappa}_{\mathbb{C}}}1)\otimes_{\chi_{\Gamma_{I;\mathbb{C}}}}1\big\rangle = 0 \quad \text{whenever } x\not\approx_{\mathbb{T}} y,$$

while $\big\langle\prod\otimes_{i\in I}x_i,\prod\otimes_{i\in I}y_i\big\rangle=0$ whenever $x\not\approx y$. Note however, that if all $H_i=\mathbb{C}$, then $\approx_{\mathbb{T}}$ and $\approx$ coincide, and one can show that the two Hilbert spaces $\big((\bar{\bigotimes}^{\mathrm{mod}}_{i\in I}\mathbb{C})\bar{\otimes}_{\kappa_{\mathbb{C}}}C^*(\Gamma_{I;\mathbb{C}})\big)\bar{\otimes}_{\chi_{\Gamma_{I;\mathbb{C}}}}\mathbb{C}$ and $\prod\otimes_{i\in I}\mathbb{C}$ coincide canonically.

Example 4.10. (a) It is clear that $\bar{\bigotimes}^{\mathrm{mod}}_{i\in I}\mathbb{C}=C^*(\Omega^{\mathrm{ut}}_{I;\mathbb{C}})$. For any state φ on $C^*(\Omega^{\mathrm{ut}}_{I;\mathbb{C}})$, the Hilbert space $(\bar{\bigotimes}^{\mathrm{mod}}_{i\in I}\mathbb{C})\bar{\otimes}_{\varphi}\mathbb{C}$ is the GNS construction of φ.

(b) If G is a subgroup of $\Omega^{\mathrm{ut}}_{I;\mathbb{C}}$, we have

$$(\bar{\bigotimes}^{\mathrm{mod}}_{i\in I}\mathbb{C})\bar{\otimes}_{\mathcal{E}_G}C^*(G)\;\cong\;\ell^2(\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G)\bar{\otimes}C^*(G).$$

In fact, let $q:\Omega^{\mathrm{ut}}_{I;\mathbb{C}}\to\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G$ be the quotient map and $\sigma:\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G\to\Omega^{\mathrm{ut}}_{I;\mathbb{C}}$ be a cross section. One has a bijection from $\Omega^{\mathrm{ut}}_{I;\mathbb{C}}$ to $(\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G)\times G$ sending ω to $(q(\omega),\sigma(q(\omega)^{-1})\omega)$.

This gives a bijective linear map $\Delta : \mathbb{C}[\Omega^{\mathrm{ut}}_{I;\mathbb{C}}] \to \bigoplus_{\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G} \mathbb{C}[G]$ such that for any $\omega \in \Omega^{\mathrm{ut}}_{I;\mathbb{C}}$ and $\varepsilon \in \Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G$,

$$\Delta(\lambda_\omega)_\varepsilon \ := \ \begin{cases} \lambda_{\sigma(\varepsilon^{-1})\omega} & \text{if } q(\omega) = \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Phi : \bigotimes^{\mathrm{unit}}_{i\in I}\mathbb{C} = \mathbb{C}^{\otimes I}_{\mathrm{ut}} \to \mathbb{C}[\Omega^{\mathrm{ut}}_{I;\mathbb{C}}]$ and $\varphi : \Pi_{i\in I}\mathbb{T} \to \mathbb{T}$ be as in Corollary 3.4. Suppose that $\alpha, \beta \in \Pi_{i\in I}\mathbb{C}^\times$. If $[\alpha\beta^{-1}]_\sim$ does not belong to G, then we have $E_G(\langle \otimes_{i\in I}\alpha_i, \otimes_{i\in I}\beta_i\rangle_{\mathbb{C}^{\otimes I}}) = 0$ and

$$\big\langle \Delta\circ\Phi\big(\otimes_{i\in I}\alpha_i\big), \Delta\circ\Phi(\otimes_{i\in I}\beta_i)\big\rangle_{\bigoplus^{\ell^2}_{\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G}\mathbb{C}[G]} \ = \ 0.$$

On the other hand, if $[\alpha\beta^{-1}]_\sim \in G$, then

$$\begin{aligned} &\big\langle \Delta\circ\Phi\big(\otimes_{i\in I}\alpha_i\big), \Delta\circ\Phi(\otimes_{i\in I}\beta_i)\big\rangle_{\bigoplus^{\ell^2}_{\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G}\mathbb{C}[G]} \\ &= \ \varphi(\alpha\beta^{-1})\lambda_{[\alpha\beta^{-1}]_\sim} \ = \ \Phi(\otimes_{i\in I}\alpha_i\beta_i^{-1}) \ = \ \Phi(E_G(\langle\otimes_{i\in I}\alpha_i, \otimes_{i\in I}\beta_i\rangle_{\mathbb{C}^{\otimes I}})). \end{aligned}$$

This shows that $\Delta\circ\Phi$ is an inner product $\mathbb{C}[G]$-module isomorphism from $\bigotimes^{\mathrm{unit}}_{i\in I}\mathbb{C}$ (equipped with the inner product $\mathbb{C}[G]$-module structure as in Theorem 4.8 (b)) onto $\bigoplus^{\ell^2}_{\Omega^{\mathrm{ut}}_{I;\mathbb{C}}/G}\mathbb{C}[G]$.

5. Tensor products of *-representations of *-algebras

In this section, $\{(A_i, H_i, \Psi_i)\}_{i\in I}$ is a family of unital *-representations, in the sense that A_i is a unital *-algebra, H_i is a Hilbert space and $\Psi_i : A_i \to \mathcal{L}(H_i)$ is a unital *-homomorphism $(i \in I)$.

Suppose that $\Psi_0 := \tilde{\bigotimes}_{i\in I}\Psi_i : \bigotimes_{i\in I}A_i \to L(\bigotimes_{i\in I}H_i)$ is the map as in Proposition 2.3 (c). It is easy to check that

$$\text{(5.1)} \qquad \big\langle \Psi_0(a)\xi, \eta\big\rangle_{\mathbb{C}^{\otimes I}} = \big\langle \xi, \Psi_0(a^*)\eta\big\rangle_{\mathbb{C}^{\otimes I}} \quad \big(a \in \bigotimes\nolimits_{i\in I}A_i; \xi, \eta \in \bigotimes\nolimits_{i\in I}H_i\big).$$

Furthermore, one has the following result (which is more or less well known).

Proposition 5.1. *For any $\mu \in \Omega^{\mathrm{unit}}_{I;H}$, the map $\tilde{\bigotimes}_{i\in I}\Psi_i$ induces a unital *-representation $\bigotimes^\mu_{i\in I}\Psi_i : \bigotimes^e_{i\in I}A_i \to \mathcal{L}(\bar{\bigotimes}^\mu_{i\in I}H_i)$. If all the Ψ_i are injective, then so is $\bigotimes^\mu_{i\in I}\Psi_i$.*

Consequently, one has a unital *-representation of $\bigotimes^e_{i\in I}A_i$ on the Hilbert space $\bar{\bigotimes}^{\phi_1}_{i\in I}H_i$. However, it seems impossible to extend it to a unital *-representation of $\bigotimes_{i\in I}A_i$ on $\bar{\bigotimes}^{\phi_1}_{i\in I}H_i$. The biggest *-subalgebra $\bigotimes_{i\in I}A_i$ that we can think of, for which such an extension is possible, is the subalgebra $\bigotimes^{\mathrm{ut}}_{i\in I}A_i$. Example 5.6 (a) also tells us that it is probably the right subalgebra to consider.

Let us digress a little bit and give another *-representation of $\bigotimes^{\rm ut}_{i\in I} A_i$, which is a direct consequence of Proposition 5.1, Theorem 3.2 (a) and Theorem 4.1 in [5] (it is not hard to verify that the representation as given in Theorem 4.1 of [5] is injective when $\bigotimes^{\mu}_{i\in I}\Psi_i$ is injective). Note however, that such a *-representation is not canonical since it depends on the choice of a cross section $c: \Omega^{\rm ut}_{I;A} \to \Pi_{i\in I} U_{A_i}$ (see Remark 3.3 (a)).

Corollary 5.2. *Suppose that the Ψ_i are injective. For any $\mu \in \Omega^{\rm unit}_{I;H}$, the injection $\bigotimes^{\mu}_{i\in I}\Psi_i$ induces an injective unital *-representation of $\bigotimes^{\rm ut}_{i\in I} A_i$ on $(\bar{\bigotimes}^{\mu}_{i\in I} H_i)\otimes \ell^2(\Omega^{\rm ut}_{I;A})$.*

Let us now return to the discussion of the tensor product type representation of $\bigotimes^{\rm ut}_{i\in I} A_i$. Observe that $\{\Psi_i\}_{i\in I}$ induces a canonical action $\alpha^\Psi : \Omega^{\rm ut}_{I;A}\times \Omega^{\rm unit}_{I;H} \to \Omega^{\rm unit}_{I;H}$. For simplicity, we will denote $\alpha^\Psi_\omega(\mu)$ by $\omega\cdot\mu$ ($\omega\in\Omega^{\rm ut}_{I;A}; \mu\in\Omega^{\rm unit}_{I;H}$).

Theorem 5.3. (a) *The map $\tilde{\bigotimes}_{i\in I}\Psi_i$ induces a unital *-representation $\bigotimes^{\phi_1}_{i\in I}\Psi_i : \bigotimes^{\rm ut}_{i\in I} A_i \to \mathcal{L}(\bar{\bigotimes}^{\phi_1}_{i\in I} H_i)$.*

(b) $\big(\bar{\bigotimes}^{\phi_1}_{i\in I} H_i, (\bigotimes^{\phi_1}_{i\in I}\Psi_i)|_{\otimes^e_{i\in I} A_i}\big) = \bigoplus_{\mu\in\Omega^{\rm unit}_{I;H}} \big(\bar{\bigotimes}^{\mu}_{i\in I} H_i, \bigotimes^{\mu}_{i\in I}\Psi_i\big)$.

(c) *If all Ψ_i are injective, then so is $\bigotimes^{\phi_1}_{i\in I}\Psi_i$.*

Proof. (a) Set $\Psi_0 := \tilde{\bigotimes}_{i\in I}\Psi_i$. For any $\mu\in\Omega^{\rm unit}_{I;H}$, $\omega\in\Omega^{\rm ut}_{I;A}$ and $a\in\Pi^{\omega}_{i\in I} A_i$, it is clear that

$$\Psi_0(\otimes_{i\in I} a_i)\Big(\bigotimes^{\mu}_{i\in I} H_i\Big) \ \subseteq\ \bigotimes^{\omega\cdot\mu}_{i\in I} H_i. \tag{5.2}$$

Suppose that $u\in\omega$ and $F\in\mathfrak{F}$ are such that $a_i = u_i$ for $i\in I\setminus F$. If $\xi = J^x_{F'}(\xi_0)$ where $x\in\mu$, $F'\in\mathfrak{F}$ with $F\subseteq F'$ and $\xi_0\in\bigotimes_{i\in F'} H_i$, then

$$\langle\Psi_0(\otimes_{i\in I} a_i)\xi, \Psi_0(\otimes_{i\in I} a_i)\xi\rangle_{\mathbb{C}^{\otimes I}} = \Big\langle\Big(\bigotimes_{i\in F}\Psi_i(a_i)\otimes{\rm id}\Big)\xi_0, \Big(\bigotimes_{i\in F}\Psi_i(a_i)\otimes{\rm id}\Big)\xi_0\Big\rangle(\otimes_{i\in I} 1).$$

This implies that $\Psi_0(\otimes_{i\in I} a_i)$ is bounded on $\big(\bigotimes^{\rm unit}_{i\in I} H_i, \langle\cdot,\cdot\rangle_{\phi_1}\big)$ (see Theorem 4.4 (a) and Proposition 4.1 (b)) and produces a unital homomorphism $\bigotimes^{\phi_1}_{i\in I}\Psi_i : \bigotimes^{\rm ut}_{i\in I} A_i \to \mathcal{L}(\bar{\bigotimes}^{\phi_1}_{i\in I} H_i)$. Now, relation (5.1) tells us that $\bigotimes^{\phi_1}_{i\in I}\Psi_i$ preserves the involution.

(b) This part follows directly from the argument of part (a).

(c) Set $\Psi := \bigotimes^{\phi_1}_{i\in I}\Psi_i$. Suppose that $v^{(1)},\dots,v^{(n)}\in\Pi_{i\in I} U_{A_i}$ are mutually inequivalent elements, $F\in\mathfrak{F}$, $b^{(1)},\dots,b^{(n)}\in\bigotimes_{i\in F} A_i$ and $a^{(k)} := J^{v^{(k)}}_F(b^{(k)})$ ($k=1,\dots,n$) are such that

$$\Psi\Big(\sum\nolimits_{k=1}^n a^{(k)}\Big) \ =\ 0.$$

By induction, it suffices to show that $a^{(1)} = 0$.

By replacing $a^{(k)}$ with $(v^{(1)})^{-1}a^{(k)}$ if necessary, we may assume that $v_i^{(1)} = e_i$ $(i \in I)$. If $n = 1$, we take an arbitrary $\xi \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$. If $n > 1$, we claim that there exists $\xi \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$ such that

$$\xi \not\sim [V_i^{(k)}\xi_i]_{i\in I} \qquad (k = 2,\dots,n), \tag{5.3}$$

where $V_i^{(k)} := \Psi_i(v_i^{(k)})$. In fact, if $k \in \{2,\dots,n\}$ and $i \in I^k := \{i \in I : v_i^{(k)} \neq e_i\}$ (which is an infinite set), the subset $\mathfrak{S}_1(H_i) \cap \ker(V_i^{(k)} - \mathrm{id}_{H_i})$ is nowhere dense in $\mathfrak{S}_1(H_i)$ as $\ker(V_i^{(k)} - \mathrm{id}_{H_i})$ is a proper closed subspace of H_i (note that Ψ_i is injective). For any $i \in I$, we consider $N_i := \{k \in \{2,\dots,n\} : i \in I^k\}$. By the Baire category theorem, for every $i \in I$, one can choose $\xi_i \in \mathfrak{S}_1(H_i) \setminus \bigcup_{k\in N_i}\ker(V_i^{(k)} - \mathrm{id}_{H_i})$. Now, $\xi := [\xi_i]_{i\in I}$ will satisfy relation (5.3).

Since $\Psi(a^{(1)})\big(\bigotimes_{i\in I}^{\xi} H_i\big) \subseteq \bigotimes_{i\in I}^{\xi} H_i$ and

$$\bigotimes_{i\in I}^{\xi} H_i \ \cap\ \sum\nolimits_{k=2}^{n}\Psi(a^{(k)})\big(\bigotimes_{i\in I}^{\xi} H_i\big) \ = \ \{0\}$$

(because of Theorem 2.5 as well as (5.2) and (5.3)), we have $\Psi(a^{(1)})|_{\bigotimes_{i\in I}^{\xi} H_i} = 0$. Therefore, part (b) and Proposition 5.1 tells us that $a^{(1)} = 0$. □

Remark 5.4. (a) By the argument proving Theorem 5.3 (c), if all the Ψ_i are injective, then α^Ψ is *strongly faithful* in the sense that for any finite subset $F \subseteq \Omega_{I;A}^{\mathrm{ut}} \setminus \{e\}$, there exists $\mu \in \Omega_{I;H}^{\mathrm{unit}}$ with $\omega\cdot\mu \neq \mu$ $(\omega \in F)$.

(b) If $y, z \in \Pi_{i\in I}H_i$ are C_0-sequences and $u, v \in \Pi_{i\in I}U_{A_i}$, then

$$y \approx z \quad \text{if and only if} \quad [\Psi_i(u_i)y_i]_{i\in I} \approx [\Psi_i(u_i)z_i]_{i\in I} \tag{5.4}$$

and $[\Psi_i(u_i)y_i]_{i\in I} \approx [\Psi_i(v_i)y_i]_{i\in I}$ whenever $u \sim v$. Thus, $\{\Psi_i\}_{i\in I}$ induces an action $\tilde\alpha^\Psi : \Omega_{I;A}^{\mathrm{ut}} \times \Gamma_{I;H} \to \Gamma_{I;H}$. Again, we write $\omega\cdot\gamma$ for $\tilde\alpha_\omega^\Psi(\gamma)$ $(\omega \in \Omega_{I;A}^{\mathrm{ut}}; \gamma \in \Gamma_{I;A})$. The map κ_H in Proposition 4.6 (a) is *equivariant* in the sense that $\kappa_H \circ \alpha_\omega^\Psi = \tilde\alpha_\omega^\Psi \circ \kappa_H$ $(\omega \in \Omega_{I;A}^{\mathrm{ut}})$.

(c) If all the A_i are C^*-algebras and all the Ψ_i are irreducible, then α^Ψ is transitive.

Corollary 5.5. *There exists a unital *-representation* $\prod\otimes_{i\in I}\Psi_i : \bigotimes_{i\in I}^{\mathrm{ut}} A_i \to \mathcal{L}\big(\prod\otimes_{i\in I} H_i\big)$ *such that for any* $\mu \in \Omega_{I;H}^{\mathrm{unit}}$, $\omega \in \Omega_{I;A}^{\mathrm{ut}}$ *and* $b \in \bigotimes_{i\in I}^{\omega} A_i$,

$$\big(\prod\otimes_{i\in I}\Psi_i\big)(b) \circ \tilde\Upsilon^\mu \ = \ \tilde\Upsilon^{\omega\cdot\mu} \circ \big(\bigotimes_{i\in I}^{\phi_1}\Psi_i\big)(b)\big|_{\bar\otimes_{i\in I}^{\mu} H_i}, \tag{5.5}$$

where $\tilde\Upsilon^\mu$ *is as in Proposition* 4.6 (b).

Proof. By Proposition 4.6 (b), there is a bounded linear map

$$\big(\prod\otimes_{i\in I}\Psi_i\big)(b) : \prod\otimes_{i\in I}^{\kappa_H(\mu)} H_i \to \prod\otimes_{i\in I}^{\omega\cdot\kappa_H(\mu)} H_i$$

such that equality (5.5) holds (see also Remark 5.4 (b)). Since we have

$$\sup_{\mu\in\Omega_{I;H}^{\mathrm{unit}}}\big\|\big(\bigotimes_{i\in I}^{\phi_1}\Psi_i\big)(b)|_{\bar{\otimes}_{i\in I}^{\mu}H_i}\big\| < \infty$$

(because of Theorem 5.3 (a)), we know from Proposition 4.6 (a) and Lemma 4.1.1 in [20] that $(\prod\otimes_{i\in I}\Psi_i)(b)$ induces an element in $\mathcal{L}(\prod\otimes_{i\in I}H_i)$, which clearly gives a *-representation. □

It is natural to ask if $\prod\otimes_{i\in I}\Psi_i$ is injective if all the Ψ_i are. However, $\prod\otimes_{i\in I}\Psi_i$ is never injective as can be seen from Example 5.6 (b) and the discussion following it.

Example 5.6. For any $i\in I$, let $A_i = \mathbb{C} = H_i$ and let $\iota_i : A_i \to \mathcal{L}(H_i)$ be the canonical map. Suppose that Φ, φ and $\hat{\Phi}$ are as in Example 4.5.

(a) Let $\Lambda : \mathbb{C}[\Omega_{I;\mathbb{C}}^{\mathrm{ut}}] \to \mathcal{L}(\ell^2(\Omega_{I;\mathbb{C}}^{\mathrm{ut}}))$ be the left regular representation. For every $\alpha,\beta\in\Pi_{i\in I}\mathbb{T}$, one has

$$\begin{aligned}\big(\hat{\Phi}^*\circ\Lambda(\lambda_{[\alpha]_\sim})\circ\hat{\Phi}\big)(\otimes_{i\in I}\beta_i) &= \varphi(\alpha^{-1})\otimes_{i\in I}\alpha_i\beta_i\\ &= \big(\bigotimes_{i\in I}^{\phi_1}\iota_i\big)(\Phi^{-1}(\lambda_{[\alpha]_\sim}))(\otimes_{i\in I}\beta_i).\end{aligned}$$

Consequently, $\bigotimes_{i\in I}^{\phi_1}\iota_i$ can be identified with Λ (under Φ and $\hat{\Phi}$).

(b) Let $\alpha\in\Pi_{i\in I}\mathbb{T}$ be such that $\alpha\nsim 1$ but $\alpha\approx 1$ with $\Pi_{i\in I}\alpha_i = 1$. If $\beta\in\Pi_{i\in I}\mathbb{C}$ is a C_0-sequence with $\|\prod\otimes_{i\in I}\beta_i\| = 1$, one has $\|\prod\otimes_{i\in I}\alpha_i\beta_i\| = 1$ and

$$\big\langle\prod\otimes_{i\in I}\alpha_i\beta_i, \prod\otimes_{i\in I}\beta_i\big\rangle \ = \ 1,$$

which imply that $\prod\otimes_{i\in I}\alpha_i\beta_i = \prod\otimes_{i\in I}\beta_i$. Therefore, $(\prod\otimes_{i\in I}\iota_i)(\otimes_{i\in I}\alpha_i) = \mathrm{id}$ but $\otimes_{i\in I}\alpha_i \neq \otimes_{i\in I}1$. Consequently, $\prod\otimes_{i\in I}\iota_i$ is not injective (actually, $(\prod\otimes_{i\in I}\iota_i)\circ\Phi^{-1}$ is not injective as a group representation of $\Omega_{I;\mathbb{C}}^{\mathrm{ut}}$).

In general, even $\big(\prod\otimes_{i\in I}\Psi_i\big)|_{\otimes_{i\in I}^{\mathrm{ut}}\mathbb{C}e_i}$ is not injective. In fact, suppose that α is as above. For any C_0-sequence $\xi\in\Pi_{i\in I}H_i$, with $\|\prod\otimes_{i\in I}\xi_i\| = 1$, the same argument as Example 5.6 (b) tells us that $\prod\otimes_{i\in I}\alpha_i\xi_i = \prod\otimes_{i\in I}\xi_i$. Thus,

$$\big(\prod\otimes_{i\in I}\Psi_i\big)(\otimes_{i\in I}e_i - \otimes_{i\in I}\alpha_i e_i) = 0.$$

On the other hand, by Theorem 5.3 and Corollary 5.5, there exist canonical *-homomorphisms

$$J^{\phi_1} : \bigotimes_{i\in I}^{\mathrm{ut}}\mathcal{L}(H_i) \to \mathcal{L}\big(\bar{\bigotimes}_{i\in I}^{\phi_1}H_i\big) \quad\text{and}\quad J^{\Pi} : \bigotimes_{i\in I}^{\mathrm{ut}}\mathcal{L}(H_i) \to \mathcal{L}\big(\prod\otimes_{i\in I}H_i\big).$$

Notice that J^{ϕ_1} is injective but J^{Π} is never injective.

Corollary 5.7. *Let $\pi_i : G_i \to U_{\mathcal{L}(H_i)}$ be a unitary representation of a group G_i, for each $i \in I$.*

(a) *There exist canonical unitary representations $\bigotimes_{i\in I}^{\phi_1}\pi_i$ and $\prod \otimes_{i\in I}\, \pi_i$ of $\Pi_{i\in I} G_i$ on $\bar{\bigotimes}_{i\in I}^{\phi_1} H_i$ and $\prod \otimes_{i\in I}\, H_i$ respectively.*

(b) *If the induced *-representation $\hat{\pi}_i : \mathbb{C}[G_i] \to \mathcal{L}(H_i)$ is injective for all $i \in I$, the induced *-representation $\widehat{\bigotimes_{i\in I}^{\phi_1}\pi_i}$ of $\mathbb{C}[\Pi_{i\in I} G_i]$ is also injective.*

Proof. (a) Let $\bigotimes_{i\in I}^{\mathrm{ut}}\pi_i := \Theta_{\mathcal{L}(H)} \circ \Pi_{i\in I}\pi_i : \Pi_{i\in I} G_i \to \bigotimes_{i\in I}^{\mathrm{ut}}\mathcal{L}(H_i)$. Then

$$\bigotimes_{i\in I}^{\phi_1}\pi_i \ := \ J^{\phi_1} \circ \bigotimes_{i\in I}^{\mathrm{ut}}\pi_i \quad \text{and} \quad \prod \otimes_{i\in I}\, \pi_i \ := \ J^{\Pi} \circ \bigotimes_{i\in I}^{\mathrm{ut}}\pi_i$$

are the required representations.

(b) By Theorem 5.3 (c), $\bigotimes_{i\in I}^{\phi_1}\hat{\pi}_i$ is injective. As $\widehat{\bigotimes_{i\in I}^{\phi_1}\pi_i}$ is the restriction of $\bigotimes_{i\in I}^{\phi_1}\hat{\pi}_i$ on $\mathbb{C}[\Pi_{i\in I} G_i]$ (see Example 3.1 (a)), it is also injective. □

Corollary 5.8. *$\prod \otimes_{i\in I}\, \Psi_i$ is never irreducible, and neither is $\bigotimes_{i\in I}^{\phi_1}\Psi_i$.*

Proof. Let $\tau_i : \mathbb{C} \to A_i$ be the canonical unital map and set $\check{\Psi}_i := \Psi_i \circ \tau_i$ $(i \in I)$. Suppose that $\alpha, \beta \in \Pi_{i\in I}\mathbb{T}$ with $\alpha \not\approx \beta$ and $\xi \in \Pi_{i\in I}^{\mathrm{unit}} H_i$. Then $[\alpha_i\xi_i]_{i\in I} \not\approx [\beta_i\xi_i]_{i\in I}$ and the two unit vectors

$$\big(\prod \otimes_{i\in I}\, \check{\Psi}_i\big)(\otimes_{i\in I}\, \alpha_i)\big(\prod \otimes_{i\in I}\, \xi_i\big) \quad \text{and} \quad \big(\prod \otimes_{i\in I}\, \check{\Psi}_i\big)(\otimes_{i\in I}\, \beta_i)\big(\prod \otimes_{i\in I}\, \xi_i\big)$$

are orthogonal. Consequently, $\dim\, (\prod \otimes_{i\in I}\, \check{\Psi}_i)(\mathbb{C}_{\mathrm{ut}}^{\otimes I}) > 1$. As $(\prod \otimes_{i\in I}\, \Psi_i) \circ (\bigotimes_{i\in I}\tau_i) = \prod \otimes_{i\in I}\, \check{\Psi}_i$, we have $(\prod \otimes_{i\in I}\, \check{\Psi}_i)(\mathbb{C}_{\mathrm{ut}}^{\otimes I}) \subseteq Z\big((\prod \otimes_{i\in I}\, \Psi_i)(\bigotimes_{i\in I}^{\mathrm{ut}} A_i)\big)$ and $\prod \otimes_{i\in I}\, \Psi_i$ is not irreducible. A similar but easier argument also shows that $\bigotimes_{i\in I}^{\phi_1}\Psi_i$ is not irreducible. □

For any C^*-algebra A, we denote by $S(A)$ and $(H_\rho, \pi_\omega, \xi_\omega)$ the state space of A and the GNS construction of $\omega \in S(A)$, respectively. We would like to consider a natural injective *-representation of $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$ defined in terms of $(H_{\omega_i}, \pi_{\omega_i})$.

If $\rho \in \Pi_{i\in I} S(A_i)$ and $\check{\rho}$ is defined as

$$\check{\rho}(a) := \big\langle \big(\bigotimes_{i\in I}^{\phi_0}\pi_{\rho_i}\big)(a)(\otimes_{i\in I}\, \xi_{\rho_i}), (\otimes_{i\in I}\, \xi_{\rho_i})\big\rangle \quad \big(a \in \bigotimes_{i\in I}^{\mathrm{ut}} A_i\big),$$

then the closure of $\big(\bigotimes_{i\in I}^{\phi_1}\pi_{\rho_i}\big)\big(\bigotimes_{i\in I}^{\mathrm{ut}} A_i\big)(\otimes_{i\in I}\, \xi_{\rho_i})$ will coincide with

$$H_{\check{\rho}} \ := \ \bar{\bigoplus}_{\omega\in\Omega_{I;A}^{\mathrm{ut}}} \bar{\bigotimes}_{i\in I}^{\omega\cdot[\xi_\rho]_\sim} H_{\rho_i} \ \subseteq \ \bar{\bigotimes}_{i\in I}^{\phi_1} H_{\rho_i}.$$

We set $\pi_{\check{\rho}}(a) := \big(\bigotimes_{i\in I}^{\phi_1}\pi_{\rho_i}\big)(a)|_{H_{\check{\rho}}}$. Notice that if all the ρ_i are pure states, then $H_{\check{\rho}} = \bar{\bigotimes}_{i\in I}^{\phi_1} H_{\rho_i}$ (see Remark 5.4 (c)).

Corollary 5.9. *Let A_i be a C^*-algebra ($i \in I$). The *-representation $\Psi_A := \bigoplus_{\rho\in\Pi_{i\in I}S(A_i)}(H_{\check\rho}, \pi_{\check\rho})$ is injective. Consequently, the *-representation*

$$\Phi_A := \bigoplus_{\rho\in\Pi_{i\in I}S(A_i)}\Big(\bar{\bigotimes}^{\phi_1}_{i\in I} H_{\rho_i}, \bigotimes^{\phi_1}_{i\in I}\pi_{\rho_i}\Big)$$

is also injective.

Proof. Suppose that (H_i, Ψ_i) is a universal *-representation of A_i ($i \in I$). Let F, $u^{(1)}, \dots, u^{(n)}$, $b^{(1)}, \dots, b^{(n)}$, and $a^{(1)}, \dots, a^{(n)}$ be as in the proof of Theorem 5.3 (c) with $\Psi_A\big(\sum_{k=1}^n a^{(k)}\big) = 0$. Again, it suffices to show that $a^{(1)} = 0$, and we may assume that $u_i^{(1)} = e_i$ ($i \in I$). If $n = 1$, we take any $x \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$. If $n > 1$, we take an element $x \in \Pi_{i\in I}\mathfrak{S}_1(H_i)$ satisfying

$$x \not\sim \big[\Psi_i\big(u_i^{(k)}\big)x_i\big]_{i\in I} \quad (k = 2, \dots, n)$$

(the argument of Theorem 5.3 (c) ensures its existence). Let us set $\rho_i(a) := \langle\Psi_i(a)x_i, x_i\rangle$ when $i \in I \setminus F$, and pick any $\rho_i \in S(A_i)$ when $i \in F$. For every $i \in I \setminus F$, one may regard $\big(H_{\rho_i}, \pi_{\rho_i}\big)$ as a subrepresentation of (H_i, Ψ_i) such that $\xi_{\rho_i} \in H_{\rho_i}$ is identified with $x_i \in H_i$. Then x can be considered as an element in $H_{\check\rho}$. Since $x \not\sim \big[\pi_{\rho_i}\big(u_i^{(k)}\big)x_i\big]_{i\in I}$ for all $2 \le k \le n$, the argument of Theorem 5.3 (c) tells us that

$$\Big(\bigotimes^{[x]_\sim}_{i\in I}\pi_{\rho_i}\Big)(a^{(1)})\eta = 0 \quad \Big(\eta \in \bigotimes^{x}_{i\in I} H_{\rho_i}\Big).$$

Consequently, $\big(\bigotimes_{i\in F}\pi_{\rho_i}\big)\big(b^{(1)}\big) = 0$ and $b^{(1)} = 0$ (as ρ_i is arbitrary when $i \in F$). The second statement follows readily from the first one. □

Notice also that $\big(\bar{\bigotimes}^{\phi_1}_{i\in I} H_{\rho_i}, \bigotimes^{\phi_1}_{i\in I}\pi_{\rho_i}\big)$ is in general not a cyclic representation, and $(H_{\check\rho}, \pi_{\check\rho})$ can be regarded as a cyclic analogue of it.

We end this paper with the following result concerning tensor products of Hilbert algebras.

Corollary 5.10. *Let $\{A_i\}_{i\in I}$ be a family of unital Hilbert algebras (see, e.g., Definition* VI.1.1 *in* [18]*) such that $\|e_i\| = 1$ ($i \in I$). Then $A := \bigotimes^{\mathrm{ut}}_{i\in I} A_i$ is also a unital Hilbert algebra with $\|\otimes_{i\in I} e_i\| = 1$.*

Proof. Note that since $\|e_i\| = 1$, one has $\|u_i\| = 1$ for any $u_i \in U_{A_i}$. Thus, we have $\bigotimes^{\mathrm{ut}}_{i\in I} A_i \subseteq \bigotimes^{\mathrm{unit}}_{i\in I} A_i$, which gives an inner product $\langle\cdot,\cdot\rangle_A$ on A. Observe that $\bigotimes^{\omega}_{i\in I} A_i$ is orthogonal to $\bigotimes^{\omega'}_{i\in I} A_i$ (in terms of $\langle\cdot,\cdot\rangle_A$) whenever ω and ω' are distinct elements in $\Omega^{\mathrm{ut}}_{I;A}$. Thus, in order to show that the involution of A is an isometry, it suffices to check that $\|x^*\| = \|x\|$ whenever $x \in \bigotimes^{\omega}_{i\in I} A_i$ and $\omega \in \Omega^{\mathrm{ut}}_{I;A}$. In fact, for any $u \in \Pi_{i\in I} U_{A_i}$, $F \in \mathfrak{F}$ and $a \in \bigotimes_{i\in F} A_i$, we have

$$\|J^u_F(a)^*\| = \|J^{u^*}_F(a^*)\| = \|a^*\| = \|a\| = \|J^u_F(a)\|,$$

because the involution of $\bigotimes_{i\in F} A_i$ is an isometry. Let H_i be the completion of A_i (with respect to the inner product) and let $\Psi_i : A_i \to \mathcal{L}(H_i)$ be the canonical unital *-representation ($i \in I$). Since

$$\bigotimes_{i\in I}^{\phi_1} \Psi_i(a)b \;=\; ab \quad (a, b \in A),$$

Theorem 5.3 (a) tells us that for each $x \in A$, one has $\langle xy, z\rangle_A = \langle y, x^*z\rangle_A$ ($y, z \in A$) and $\sup_{\|y\|\leq 1} \|xy\| < \infty$. Finally, as A is unital, we see that A is a Hilbert algebra (with $\|\otimes_{i\in I} e_i\| = 1$). □

Consequently, if all the A_i are weakly dense unital *-subalgebras of finite von Neumann algebras, then so is $\bigotimes_{i\in I}^{\mathrm{ut}} A_i$.

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