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Minimal smoothness conditions for bilinear Fourier multipliers

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Abstract. The problem of finding the differentiability conditions for bilinear Fourier multipliers that are as small as possible to ensure the boundedness of the corresponding operators from products of Hardy spaces $H^{p_1} \times H^{p_2}$ to L^p, $1/p_1 + 1/p_2 = 1/p$, is considered. The minimal conditions in terms of the product type Sobolev norms are given for the whole range $0 < p_1, p_2 \leq \infty$.

1. Introduction

For $m \in L^\infty(\mathbb{R}^{2n})$, the bilinear Fourier multiplier operator T_m is defined by

$$T_m(f_1, f_2)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)}\, m(\xi)\, \widehat{f_1}(\xi_1)\, \widehat{f_2}(\xi_2)\, d\xi_1\, d\xi_2$$

for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, where $x \in \mathbb{R}^n$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n$.

Coifman and Meyer (see [3], [4] and [15]) proved that if the multiplier $m(\xi)$ satisfies the condition

$$\big|\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)\big| \leq C_{\alpha_1,\alpha_2} \big(|\xi_1| + |\xi_2|\big)^{-(|\alpha_1|+|\alpha_2|)}, \tag{1.1}$$

then T_m extends to a bounded operator $L^{p_1} \times L^{p_2} \to L^p$ for p_1, p_2 and p satisfying $1 < p_1, p_2, p < \infty$ and $1/p_1 + 1/p_2 = 1/p$. They also proved the boundedness $L^p \times L^\infty \to L^p$ for $1 < p < \infty$. The boundedness of $T_m : L^\infty \times L^\infty \to \mathrm{BMO}$ is also implicitly given in [4], [15]. Kenig–Stein [14] proved weak type estimate for the case $p_1 = p_2 = 2p = 1$ and extended the results of Coifman–Meyer to the range $p \leq 1$. Grafakos–Torres [10] gave a general theory for multilinear Calderón–Zygmund operators and generalized the results of [3], [4], [15], and [14]. Grafakos–Kalton [7] proved that the boundedness of $T_m : L^{p_1} \times L^{p_2} \to L^p$ can be extended to $p_1 \leq 1$ or $p_2 \leq 1$ if we replace L^{p_1} and L^{p_2} by the Hardy spaces H^{p_1} and H^{p_2}

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respectively. In fact, the above papers include several general results, not all of which can be mentioned here.

To ensure the above mentioned boundedness of T_m, it is not necessary to assume the condition (1.1) for all derivatives, but it is sufficient to assume it for derivatives up to certain order. In this paper we shall consider the problem of finding the differentiability conditions of the type (1.1) that are "as small as possible" to ensure the boundedness of $T_m : H^{p_1} \times H^{p_2} \to L^p$.

Before we state our result in detail, we shall recall some previously known results. Coifman–Meyer [4], [15] proved the boundedness of T_m by reducing it to linear Calderón–Zygmund operators. They considered the linear operator T_{f_2} defined by

$$T_{f_2}(f_1)(x) = T_m(f_1, f_2)(x) = \int_{\mathbb{R}^n} K_{f_2}(x, y_1)\, f_1(y_1)\, dy_1.$$

They showed that the kernel $K_{f_2}(x, y_1)$ of this operator is a Calderón–Zygmund kernel and then used the $T1$-theorem to deduce the boundedness of T_m. In their proof, to ensure the kernel $K_{f_2}(x, y_1)$ be a Calderón–Zygmund kernel, they had to assume the condition (1.1) up to order $2n + 1$. (The number of derivatives assumed on m in the statement of p. 22 in [4] seems to be an error. At least, the proof given in pp. 22–23 of [4] requires (1.1) up to order $2n + 1$.) Grafakos–Torres [10] gave a different proof by using the bilinear $T1$-theorem. In this case, to ensure that the kernel of T_m be a Calderón–Zygmund kernel in the bilinear sense, they had to assume (1.1) up to the same order $2n + 1$. Coifman–Meyer [3] used the paraproduct operator to deduce the boundedness of T_m. In this method, they had to assume (1.1) up to an order much higher than $2n+1$. The differentiability conditions for m assumed in these papers seem to be too strong if we compare them with the conditions occurring in the case of linear Fourier multiplier operators. In more recent papers [20], [9], and [8], results under much weaker assumptions are given, which we shall mention later.

Recall the case of linear Fourier multiplier operators. To distinguish it from the bilinear operator T_m, we denote the linear operator by $m(D)$: for $m \in L^\infty(\mathbb{R}^n)$,

$$m(D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, m(\xi)\, \widehat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

It is well known that $m(D)$ can be extended to a bounded operator in H^p if $m(\xi)$ satisfies

$$|\partial_\xi^\alpha m(\xi)| \le C_\alpha\, |\xi|^{-|\alpha|}.$$

Hörmander (Theorem 2.5 in [12]) essentially proved the following: $m(D)$ can be extended to a bounded operator in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if the multiplier $m(\xi)$ satisfies

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\Psi\|_{W^s(\mathbb{R}^n)} < \infty \tag{1.2}$$

with an $s > n/2$, where Ψ is a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}\Psi \subset \{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\}, \quad \sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}, \tag{1.3}$$

with $d = n$ and where $\|\cdot\|_{W^s(\mathbb{R}^n)}$ denotes the usual Sobolev norm,

$$\|f\|_{W^s(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 \, d\xi \Big)^{1/2}, \tag{1.4}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. Calderón–Torchinsky (Theorem 4.6 of [2]) proved the following: if $0 < p \le 1$ and $s > n/p - n/2$, and if the multiplier $m(\xi)$ satisfies (1.2), then $m(D)$ can be extended to a bounded operator in the Hardy space $H^p(\mathbb{R}^n)$. It is known that the numbers $n/2$ and $n/p - n/2$ in these results are minimal, that is, they cannot be replaced by smaller numbers (see Remark 1.3 below). The purpose of the present paper is to find such minimal conditions for the case of bilinear Fourier multipliers.

To explain our main results in detail, we introduce some notation. We shall write

$$\|T_m\|_{H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}$$

to denote the smallest constant C that satisfies

$$\|T_m(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \le C, \|f_1\|_{H^{p_1}(\mathbb{R}^n)} \, \|f_2\|_{H^{p_2}(\mathbb{R}^n)}$$

for all $f_1 \in \mathcal{S}(\mathbb{R}^n) \cap H^{p_1}(\mathbb{R}^n)$ and $f_2 \in \mathcal{S}(\mathbb{R}^n) \cap H^{p_2}(\mathbb{R}^n)$. We define

$$\|T_m\|_{L^\infty(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)}$$

in the same way by replacing the norms $\|\cdot\|_{H^{p_1}}$, $\|\cdot\|_{H^{p_2}}$ and $\|\cdot\|_{L^p}$ by $\|\cdot\|_{L^\infty}$, $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{\mathrm{BMO}}$, respectively. We use the convention that $H^{p_i} = L^{p_i}$ for $1 < p_i \le \infty$. For $s_1, s_2 \in \mathbb{R}$ and for $F \in \mathcal{S}'(\mathbb{R}^{2n})$, the product type Sobolev norm $\|F\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}$ is defined by

$$\|F\|_{W^{(s_1,s_2)}} = \Big(\int_{\mathbb{R}^{2n}} \langle \xi_1 \rangle^{2s_1} \langle \xi_2 \rangle^{2s_2} |\widehat{F}(\xi_1, \xi_2)|^2 \, d\xi_1 d\xi_2 \Big)^{1/2},$$

where $\xi_i \in \mathbb{R}^n$. We take a function $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ that satisfies (1.3) with $d = 2n$ and, for $m \in L^\infty(\mathbb{R}^{2n})$ and $j \in \mathbb{Z}$, define

$$m_j(\xi) = m(2^j \xi_1, 2^j \xi_2) \, \Psi(\xi_1, \xi_2), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.5}$$

Now, for bilinear Fourier multiplier operators, Grafakos–Miyachi–Tomita [8] have obtained some results with minimal conditions by using the product type Sobolev norms. The results of [8] are as follows. First,

$$s_1 > n/2, \ s_2 > n/2 \implies \|T_m\|_{L^2(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}. \tag{1.6}$$

Second, for $0 < p \le 1$,

$$s_1 > n/2, \ s_2 > n/p - n/2 \implies \|T_m\|_{L^\infty(\mathbb{R}^n) \times H^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}. \tag{1.7}$$

In addition, the numbers $n/2$ and $n/p - n/2$ in (1.6) and (1.7) are minimal. (See Theorems 1.1 and 1.2, and Propositions 7.1 and 7.2 in [8].)

The purpose of the present paper is to extend these results of [8]. We use the product type Sobolev norm for the multipliers and we shall find minimal conditions, for the whole range $0 < p_1, p_2 \le \infty$, for the boundedness of T_m from $H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. The fact (1.7) is one of the keys in the proofs of the results of this paper. The fact (1.6) will also be a key tool in our arguments.

The main results of this paper are given in the following two theorems:

Theorem 1.1. *Let $0 < p_1, p_2, p \le \infty$ and $1/p_1 + 1/p_2 = 1/p$. If*

$$s_1 > \max\left\{\frac{n}{2}, \frac{n}{p_1} - \frac{n}{2}\right\}, \quad s_2 > \max\left\{\frac{n}{2}, \frac{n}{p_2} - \frac{n}{2}\right\}, \quad \text{and} \quad s_1 + s_2 > \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{2},$$

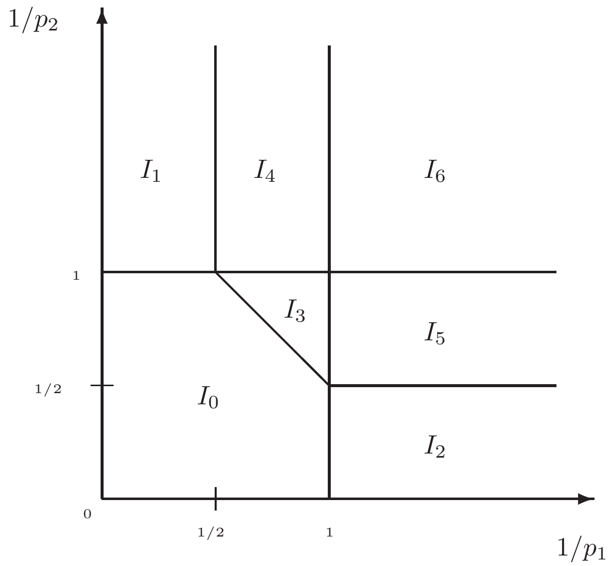
then

$$\|T_m\|_{H^{p_1}(\mathbb{R}^n)\times H^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}, \tag{1.8}$$

where $H^{p_1} \times H^{p_2} \to L^p$ is replaced by $L^\infty \times L^\infty \to \mathrm{BMO}$ if $p_1 = p_2 = p = \infty$.

Theorem 1.2. *Let $0 < p_1, p_2, p \le \infty$ and $1/p_1 + 1/p_2 = 1/p$. Then the estimate (1.8), where $H^{p_1} \times H^{p_2} \to L^p$ is replaced by $L^\infty \times L^\infty \to \mathrm{BMO}$ if $p_1 = p_2 = p = \infty$, holds only if*

$$s_1 \ge \max\left\{\frac{n}{2}, \frac{n}{p_1} - \frac{n}{2}\right\}, \quad s_2 \ge \max\left\{\frac{n}{2}, \frac{n}{p_2} - \frac{n}{2}\right\}, \quad \text{and} \quad s_1 + s_2 \ge \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{2}.$$



To visualize easily the various conditions of Theorem 1.1, we divide the region of $(1/p_1, 1/p_2)$ into seven regions $I_0, \ldots, I_6$ as in the figure. The assumptions on s_1

and s_2 of Theorem 1.1 are written as follows:

$$
\begin{array}{ll}
s_1 > n/2, \quad s_2 > n/2 & \text{if} \quad (1/p_1, 1/p_2) \in I_0; \\
s_1 > n/2, \quad s_2 > n/p_2 - n/2 & \text{if} \quad (1/p_1, 1/p_2) \in I_1; \\
s_1 > n/p_1 - n/2, \quad s_2 > n/2 & \text{if} \quad (1/p_1, 1/p_2) \in I_2; \\
\begin{cases} s_1 > n/2, \quad s_2 > n/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} & \text{if} \quad (1/p_1, 1/p_2) \in I_3; \\
\begin{cases} s_1 > n/2, \quad s_2 > n/p_2 - n/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} & \text{if} \quad (1/p_1, 1/p_2) \in I_4; \\
\begin{cases} s_1 > n/p_1 - n/2, \quad s_2 > n/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} & \text{if} \quad (1/p_1, 1/p_2) \in I_5; \\
\begin{cases} s_1 > n/p_1 - n/2, \quad s_2 > n/p_2 - n/2, \\ s_1 + s_2 > n/p_1 + n/p_2 - n/2 \end{cases} & \text{if} \quad (1/p_1, 1/p_2) \in I_6.
\end{array}
$$

Notice that the condition $s_1 + s_2 > n/p_1 + n/p_2 - n/2$ is necessary only in the regions I_3, I_4, I_5, and I_6.

Next, we observe some interesting features of the results of Theorems 1.1 and 1.2.

First, we see that simple interpolation of minimal conditions does not necessarily give a minimal condition. Consider for example the bound for $H^p(\mathbb{R}^n) \times H^p(\mathbb{R}^n) \to L^{p/2}(\mathbb{R}^n)$ in the range $p \le 1$. By interpolating (1.7) and its variant with f_1 and f_2 interchanged, we obtain

$$
\begin{aligned}
(1.9) \quad & s_1 > n/p, \ s_2 > n/p \\
& \qquad \Longrightarrow \quad \|T_m\|_{H^p(\mathbb{R}^n) \times H^p(\mathbb{R}^n) \to L^{p/2}(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})}
\end{aligned}
$$

(cf. Theorem 6.1 of [8]). Although the assertion (1.7) gives a minimal condition, the condition s_1, $s_2 > n/p$ in (1.9) is not minimal. As given in Theorems 1.1 and 1.2, we can obtain the conclusion of (1.9) under the assumptions s_1, $s_2 > n/p - n/2$, $s_1 + s_2 > 2n/p - n/2$, and these are the minimal conditions.

Second, we observe that the situation is not so simple even in the range $1 < p_i < \infty$. Consider for simplicity the estimate

$$
\|T_m\|_{L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n) \to L^{p/2}(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1, s_2)}(\mathbb{R}^{2n})},
$$

in the range $1 < p < \infty$. As Theorems 1.1 and 1.2 assert, if $p \ge 4/3$ then this estimate holds for s_1, $s_2 > n/2$, but if $p < 4/3$ then we have to assume the additional condition $s_1 + s_2 > 2n/p - n/2$ or, to be precise, at least $s_1 + s_2 \ge 2n/p - n/2$.

The problem of the minimal condition for bilinear Fourier multipliers can also be formulated in terms of the usual Sobolev norm, (1.2), with n replaced by $2n$.

In this direction, Tomita [20] proved that if $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p$ then

$$s > n,\ p > 1 \implies \|T_m\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^s(\mathbb{R}^{2n})}. \tag{1.10}$$

Grafakos–Si [9] generalized this result to the range $p \le 1$ by using the L^r-type Sobolev norm, $1 < r \le 2$. In the present paper, we shall not consider the problem with the usual Sobolev norm. Here, however, we only mention that we can relax the restriction $p > 1$ of (1.10) to $p > 2/3$ by virtue of Theorem 1.1.

Bilinear and multilinear Fourier multiplier operators are widely investigated and have many applications. For other results on these operators and related topics, see Muscalu–Pipher–Tao–Thiele [17], Bernicot–Germain [1], and the references therein.

The contents of this paper are as follows. In Section 2, we recall some preliminary facts. We prove Theorem 1.1 in Sections 3–6. In Section 3, we treat the case $0 < p_1, p_2 \le 1$. In Section 4, we treat the case $0 < p_1 \le 1$, $p_2 = 2$. In Section 5, we treat the case $p_1 = p_2 = p = \infty$. In Section 6, we complete the proof of Theorem 1.1 combining the results of Sections 3, 4 and 5, and the result (1.7) by interpolation. Finally in Section 7, we prove Theorem 1.2.

We make a remark concerning the arguments of this paper. Since we are interested in the estimate for operator norms, we give the proofs by assuming that all the functions, including the multipliers, that appear in our argument are of the Schwartz class and we omit the limiting arguments that are necessary for rigorous proof. For example, in our argument we repeatedly write f_1 as a series of H^{p_1}-atoms $a_{1,k}$,

$$f_1 = \sum_k \lambda_{1,k}\, a_{1,k}, \quad \sum_k |\lambda_{1,k}|^{p_1} \lesssim \|f_1\|_{H^{p_1}}^{p_1}\,, \tag{1.11}$$

and we write

$$T_m(f_1, f_2) = \sum_k \lambda_{1,k}\, T_m(a_{1,k}, f_2). \tag{1.12}$$

Some limiting argument is necessary to ensure the convergence of the series (1.12). One way to make the argument precise is to use the fact that the first series of (1.11) can be taken so that it converges in L^2 if $f_1 \in L^2 \cap H^{p_1}$ and to use the L^2 estimate of T_m given in (1.6) to deduce the convergence of the series of (1.12). Another way is to consider at first only those f_1 that can be written as (1.11) with a finite sum and then use some limiting argument to treat general f_1. We leave such detailed arguments to the reader.

For two nonnegative quantities A and B, the notation $A \lesssim B$ means that $A \le CB$ for some unspecified constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

Remark 1.3. One way to see the minimality of the numbers $n/2$ and $n/p - n/2$ of the theorems of Hörmander (Theorem 2.5 of [12]) and Calderón–Torchinsky (Theorem 4.6 of [2]) mentioned above is to use the multiplier

$$m_{a,b}(\xi) = \psi(\xi)\,|\xi|^{-b} \exp(i|\xi|^a),$$

where $a > 0$, $a \neq 1$, $b > 0$, and $\psi(\xi)$ is a smooth function which vanishes in a neighborhood of $\xi = 0$ and is equal to 1 for $|\xi|$ large. It is easy to see that $m_{a,b}$ satisfies (1.2) for $s = b/a$. On the other hand, it is known that $m(D)$ is bounded in $H^p(\mathbb{R}^n)$, $0 < p < \infty$, only if $b/a \geq |n/p - n/2|$ (see comments after Theorem 3c in [11], Part II of [22], or Theorem 3 in [16]). Another way to see the minimality will be given in Section 7 of the present paper.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

The Hardy–Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |f(y)|\,dy,$$

where f is a locally integrable function on $\mathbb{R}^n$. We also use the notation $M_q f(x) = M(|f|^q)(x)^{1/q}$.

We recall the definition and some properties of Hardy spaces on $\mathbb{R}^n$ (see Chapter 3 of [18]). Let $0 < p \leq \infty$, and let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$. Then the Hardy space $H^p(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H^p} = \Big\| \sup_{0<t<\infty} |\phi_t * f| \Big\|_{L^p} < \infty,$$

where $\phi_t(x) = t^{-n}\phi(x/t)$. It is known that $H^p(\mathbb{R}^n)$ does not depend on the choice of the function ϕ (see Chapter 3, Theorem 1, in [18]). If $1 < p \leq \infty$, then $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ (see Chapter 3, Section 1.2, in [18]). For $0 < p \leq 1$, a function a on $\mathbb{R}^n$ is called an H^p-atom if there exists a cube $Q = Q_a$ such that

$$\operatorname{supp} a \subset Q, \quad \|a\|_{L^\infty} \leq |Q|^{-1/p}, \quad \int_{\mathbb{R}^n} x^\alpha\, a(x)\,dx = 0, \quad |\alpha| \leq N,$$

where $|Q|$ is the Lebesgue measure of Q and N is any fixed integer satisfying $N \geq [n(1/p - 1)]$ (see p. 112 of [18]). It is known that every $f \in H^p(\mathbb{R}^n)$ can be written as

$$f = \sum_{i=1}^{\infty} \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where $\{a_i\}$ is a collection of H^p-atoms and $\{\lambda_i\}$ is a sequence of complex numbers with $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$. Moreover,

$$\|f\|_{H^p} \approx \inf \Big(\sum_{i=1}^{\infty} |\lambda_i|^p \Big)^{1/p},$$

where the infimum is taken over all representations of f (see Theorem 2 in Chapter 3 of [18]).

Let ϕ_0 be a C^∞-function on $[0,\infty)$ satisfying

$$\phi_0(t) = 1 \quad \text{on } [0, 1/8], \quad \operatorname{supp} \phi_0 \subset [0, 1/4].$$

We set $\phi_1(t) = 1 - \phi_0(t)$, and define the functions $\Phi_{(i_1,i_2)}$ on $\mathbb{R}^{2n} \setminus \{0\}$, $(i_1, i_2) \in \{0,1\}^2$, by

$$\Phi_{(i_1,i_2)}(\xi_1, \xi_2) = \phi_{i_1}(|\xi_1|/|\xi|)\phi_{i_2}(|\xi_2|/|\xi|), \tag{2.1}$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n$ and $|\xi| = \sqrt{|\xi_1|^2 + |\xi_2|^2}$. We note that $\Phi_{(0,0)} = 0$.

Lemma 2.1 ([6], Lemma 3.1; [20], Section 5). 1) *For* $(\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$,

$$\Phi_{(1,1)}(\xi_1, \xi_2) + \Phi_{(0,1)}(\xi_1, \xi_2) + \Phi_{(1,0)}(\xi_1, \xi_2) = 1.$$

2) *Each* $\Phi_{(i_1,i_2)}$ *satisfies*

$$\left|\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \Phi_{(i_1,i_2)}(\xi_1, \xi_2)\right| \le C_{(i_1,i_2)}^{\alpha_1,\alpha_2} \big(|\xi_1| + |\xi_2|\big)^{-(|\alpha_1|+|\alpha_2|)}$$

for all multi-indices α_1, α_2.

3) $\operatorname{supp} \Phi_{(1,1)} \subset \big\{|\xi_1|/8 \le |\xi_2| \le 8|\xi_1|\big\}$, $\operatorname{supp} \Phi_{(0,1)} \subset \big\{|\xi_1| \le |\xi_2|/2\big\}$ *and* $\operatorname{supp} \Phi_{(1,0)} \subset \big\{|\xi_2| \le |\xi_1|/2\big\}$.

Lemma 2.2 (Lemma 3.2 in [6], Lemma 3.3 in [8]). *Let* $s > n/2$, $\max\{1, n/s\} < q < 2$ *and* $r > 0$. *Then there exists a constant* $C > 0$ *such that*

$$\left|T_{m(\cdot/2^j)}(f_1, f_2)(x)\right| \le C \|m\|_{W^{(s,s)}} M_q f_1(x) M_q f_2(x)$$

for all $j \in \mathbb{Z}$, *all* $m \in W^{(s,s)}(\mathbb{R}^{2n})$ *with* $\operatorname{supp} m \subset \{(|\xi_1|^2 + |\xi_2|^2)^{1/2} \le r\}$ *and all* $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$.

For a function $F(x_1, x_2)$ on $\mathbb{R}^n \times \mathbb{R}^n$, we denote by $\|F(x_1, x_2)\|_{L^p_{x_i}}$ the L^p-norm of $F(x_1, x_2)$ with respect to the variable x_i, $i = 1, 2$. The proof of the following lemma can be reduced to Theorem 1.4.1 in [21], but we shall give a proof for the reader's convenience.

Lemma 2.3. *Let* $2 \le q \le \infty$, $r > 0$ *and* $s_1, s_2 \in \mathbb{R}$. *Assume that* $\operatorname{supp} m \subset \{(|\xi_1|^2 + |\xi_2|^2)^{1/2} \le r\}$, *and set* $K = \mathcal{F}^{-1} m$. *Then there exists a constant* $C > 0$ *such that*

$$\|\langle x_1\rangle^{s_1} \langle x_2\rangle^{s_2} K(x_1, x_2)\|_{L^q_{x_2}} \le C \,\|\langle x_1\rangle^{s_1} \langle x_2\rangle^{s_2} K(x_1, x_2)\|_{L^2_{x_2}} \quad \textit{for all } x_1 \in \mathbb{R}^n,$$

$$\|\langle x_1\rangle^{s_1} \langle x_2\rangle^{s_2} K(x_1, x_2)\|_{L^q_{x_1}} \le C \,\|\langle x_1\rangle^{s_1} \langle x_2\rangle^{s_2} K(x_1, x_2)\|_{L^2_{x_1}} \quad \textit{for all } x_2 \in \mathbb{R}^n,$$

where C *depends only on* q, r, s_1 *and* s_2.

Proof. We only consider the first estimate, since our argument works also for the second one.

First, let us prove the case $q = \infty$. Using $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\widehat{\varphi} = 1$ on $\{|\xi_2| \le r\}$, we can write $m(\xi_1,\xi_2) = m(\xi_1,\xi_2)\widehat{\varphi}(\xi_2)$. Then, by Schwarz's inequality,

$$\begin{aligned}
\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}|K(x_1,x_2)| &= \langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}\Big|\int_{\mathbb{R}^n} K(x_1,x_2-y_2)\varphi(y_2)\,dy_2\Big| \\
&\lesssim \int_{\mathbb{R}^n} \langle x_1\rangle^{s_1}\langle x_2-y_2\rangle^{s_2}\big|K(x_1,x_2-y_2)\big|\langle y_2\rangle^{|s_2|}|\varphi(y_2)|\,dy_2 \\
&\le \Big(\int_{\mathbb{R}^n}\big|\langle x_1\rangle^{s_1}\langle x_2-y_2\rangle^{s_2}K(x_1,x_2-y_2)\big|^2\,dy_2\Big)^{1/2} \\
&\qquad \times \Big(\int_{\mathbb{R}^n}\big|\langle y_2\rangle^{|s_2|}\varphi(y_2)\big|^2\,dy_2\Big)^{1/2} \\
&\approx \|\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}K(x_1,x_2)\|_{L^2_{x_2}}.
\end{aligned}$$

Hence,

$$\|\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}K(x_1,x_2)\|_{L^\infty_{x_2}} \lesssim \|\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}K(x_1,x_2)\|_{L^2_{x_2}}.$$

The case $q = 2$ is obvious, and the case $2 < q < \infty$ follows from interpolation. □

Lemma 2.4 (Lemma 3.4 in [8]). *Let $s_1, s_2 \in \mathbb{R}$, and let $\Psi' \in \mathcal{S}(\mathbb{R}^{2n})$ be such that* $\operatorname{supp}\Psi'$ *is a compact subset of $\mathbb{R}^{2n}\setminus\{0\}$. Assume that $\Phi \in C^\infty(\mathbb{R}^{2n}\setminus\{0\})$ satisfies*

$$\big|\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2}\Phi(\xi_1,\xi_2)\big| \le C_{\alpha_1,\alpha_2}\big(|\xi_1|+|\xi_2|\big)^{-(|\alpha_1|+|\alpha_2|)}$$

for all multi-indices α_1, α_2. Then there exists a constant $C > 0$ such that

$$\sup_{j\in\mathbb{Z}}\big\|\,m(2^j\cdot)\,\Phi(2^j\cdot)\,\Psi'\big\|_{W^{(s_1,s_2)}} \le C\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,s_2)}}$$

for all $m \in L^\infty(\mathbb{R}^{2n})$ satisfying $\sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,s_2)}} < \infty$, where m_j is defined by (1.5).

The condition $s_1, s_2 > n/2$ was assumed in Lemma 3.4 of [8], but it is easy to modify the argument there to cover all $s_1, s_2 \in \mathbb{R}$.

We end this section with the following remark which will be used in the sequel.

Remark 2.5. By Lemma 2.4, we have

$$\big\|\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}\,\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}K_j(x_1,x_2)\big\|_{L^2_{x_1,x_2}} \lesssim \sup_{j\in\mathbb{Z}}\|m_j\|_{W^{(s_1,s_2)}},$$

where $s_1, s_2 \in \mathbb{R}$, $K_j = \mathcal{F}^{-1}m_j$ and m_j is defined by (1.5). In fact, since

$$\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}K_j(x_1,x_2) = i^{|\alpha_1|+|\alpha_2|}\,\mathcal{F}^{-1}\big[\,m(2^j\cdot)\,\xi_1^{\alpha_1}\,\xi_2^{\alpha_2}\,\Psi\big](x_1,x_2),$$

the estimate follows from Lemma 2.4 with $\Phi \equiv 1$ and $\Psi' = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\Psi$.

3. The boundedness from $H^{p_1} \times H^{p_2}$ to L^p for $0 < p_1, p_2 \le 1$

In this section, we shall prove Theorem 1.1 with $0 < p_1, p_2 \le 1$. That is, in the case $0 < p_1, p_2 \le 1$ and $1/p_1 + 1/p_2 = 1/p$, under the assumptions

$$s_1 > \frac{n}{p_1} - \frac{n}{2}, \quad s_2 > \frac{n}{p_2} - \frac{n}{2}, \quad s_1 + s_2 > \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{2}, \tag{3.1}$$

we show that

$$\|T_m\|_{H^{p_1} \times H^{p_2} \to L^p} \lesssim \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}. \tag{3.2}$$

Let a_i, $i = 1, 2$, be H^{p_i}-atoms with vanishing moments up to order $N_i - 1$ and $\operatorname{supp} a_i \subset Q_i$, where the N_i are large enough. We denote by c_i the center of Q_i, by $\ell(Q_i)$ the side length of Q_i, and by Q_i^* the cube with the same center as Q_i but expanded by a factor of $2\sqrt{n}$. In order to obtain (3.2), we shall prove that there exist a function b_1 depending only on a_1 and a function b_2 depending only on a_2 such that

$$\begin{aligned} &|T_m(a_1, a_2)(x)| \chi_{(Q_1^* \cap Q_2^*)^c}(x) \lesssim A b_1(x) b_2(x), \\ &\|b_1\|_{L^{p_1}} \lesssim 1, \quad \|b_2\|_{L^{p_2}} \lesssim 1, \end{aligned} \tag{3.3}$$

where $A = \sup_{j \in \mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}$.

Before proving (3.3), let us observe that this implies (3.2). To do this, we write f_i as a sum of H^{p_i}-atoms as $f_i = \sum_{k_i} \lambda_{i,k_i} a_{i,k_i}$ with $\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \lesssim \|f_i\|_{H^{p_i}}^{p_i}$ for $i = 1, 2$, and divide $T_m(f_1, f_2)$ as follows:

$$\begin{aligned} T_m(f_1, f_2) &= \sum_{k_1,k_2} \lambda_{1,k_1} \lambda_{2,k_2} T_m(a_{1,k_1}, a_{2,k_2}) \\ &= \sum_{k_1,k_2} \lambda_{1,k_1} \lambda_{2,k_2} T_m(a_{1,k_1}, a_{2,k_2}) \chi_{Q_{1,k_1}^* \cap Q_{2,k_2}^*} \\ &\quad + \sum_{k_1,k_2} \lambda_{1,k_1} \lambda_{2,k_2} T_m(a_{1,k_1}, a_{2,k_2}) \chi_{(Q_{1,k_1}^* \cap Q_{2,k_2}^*)^c}. \end{aligned}$$

The first term can be handled by the method of Grafakos–Kalton [7]. In fact, since $s_1, s_2 > n/2$, (1.6) gives

$$\|T_m\|_{L^2 \times L^\infty \to L^2} + \|T_m\|_{L^\infty \times L^2 \to L^2} \lesssim A. \tag{3.4}$$

Then, by using the inequality

$$\Big\| \sum_\nu |f_\nu| \chi_{Q_\nu} \Big\|_{L^p} \lesssim \Big\| \sum_\nu \frac{1}{|Q_\nu|} \Big(\int_{Q_\nu} |f_\nu(y)|\, dy \Big) \chi_{Q_\nu} \Big\|_{L^p}$$

(which holds for all $0 < p \le 1$) and the L^2-estimate (3.4), we can prove

$$\Big\| \sum_{k_1,k_2} \lambda_{1,k_1} \lambda_{2,k_2} T_m(a_{1,k_1}, a_{2,k_2}) \chi_{Q_{1,k_1}^* \cap Q_{2,k_2}^*} \Big\|_{L^p} \lesssim A \, \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}$$

in the same way as in pp. 173–174 of [7] (here we do not need (3.3)). On the other hand, for each a_{1,k_1} and a_{2,k_2}, let us take b_{1,k_1} and b_{2,k_2} satisfying (3.3). Then, since

$$\Big| \sum_{k_1,k_2} \lambda_{1,k_1}\lambda_{2,k_2}\, T_m(a_{1,k_1}, a_{2,k_2})\, \chi_{(Q^*_{1,k_1}\cap Q^*_{2,k_2})^c} \Big| \lesssim A \prod_{i=1}^{2} \Big(\sum_{k_i} |\lambda_{i,k_i}|\, b_{i,k_i} \Big),$$

we have, by Hölder's inequality,

$$\begin{aligned} \Big\| \sum_{k_1,k_2} \lambda_{1,k_1}\, \lambda_{2,k_2}\, T_m(a_{1,k_1}, a_{2,k_2})\, \chi_{(Q^*_{1,k_1}\cap Q^*_{2,k_2})^c} \Big\|_{L^p} &\lesssim A \prod_{i=1}^{2} \Big\| \sum_{k_i} |\lambda_{i,k_i}|\, b_{i,k_i} \Big\|_{L^{p_i}} \\ &\le A \prod_{i=1}^{2} \Big(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \|b_{i,k_i}\|_{L^{p_i}}^{p_i} \Big)^{1/p_i} \lesssim A\, \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}}. \end{aligned}$$

Hence, we obtain (3.2).

In order to obtain (3.3), we shall prove the following:

$$|T_m(a_1,a_2)(x)|\, \chi_{(Q_1^*)^c\cap(Q_2^*)^c}(x) \lesssim A u(x) v(x), \quad \|u\|_{L^{p_1}} \lesssim 1, \quad \|v\|_{L^{p_2}} \lesssim 1, \tag{3.5}$$

$$|T_m(a_1,a_2)(x)|\, \chi_{(Q_1^*)^c\cap Q_2^*}(x) \lesssim A u'(x) v'(x), \quad \|u'\|_{L^{p_1}} \lesssim 1, \quad \|v'\|_{L^{p_2}} \lesssim 1, \tag{3.6}$$

$$|T_m(a_1,a_2)(x)|\, \chi_{Q_1^*\cap(Q_2^*)^c}(x) \lesssim A u''(x) v''(x), \quad \|u''\|_{L^{p_1}} \lesssim 1, \quad \|v''\|_{L^{p_2}} \lesssim 1, \tag{3.7}$$

where u, u' and u'' depend only on a_1, and v, v' and v'' depend on only a_2. Once (3.5)–(3.7) are proved, we can take $u + u' + u''$ and $v + v' + v''$ as b_1 and b_2 in (3.3).

Let $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ be as in (1.3) with $d = 2n$, and write $m_j(\xi) = m(2^j\xi)\Psi(\xi)$ and $K_j = \mathcal{F}^{-1} m_j$. Then $T_m(a_1,a_2)(x) = \sum_{j\in\mathbb{Z}} g_j(x)$ with

$$\begin{aligned} g_j(x) &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)}\, m(\xi)\, \Psi(\xi/2^j)\, \widehat{a_1}(\xi_1)\, \widehat{a_2}(\xi_2)\, d\xi_1 d\xi_2 \\ &= \int_{\mathbb{R}^{2n}} 2^{2jn}\, K_j\big(2^j(x-y_1), 2^j(x-y_2)\big)\, a_1(y_1)\, a_2(y_2)\, dy_1 dy_2. \end{aligned} \tag{3.8}$$

Using the moment condition for a_1 and Taylor's formula, we can write

$$\begin{aligned} g_j(x) = 2^{2jn} \sum_{|\alpha_1|=N_1} C_{\alpha_1} \int_{\substack{0<\theta_1<1 \\ y_1\in Q_1, y_2\in Q_2}} &(1-\theta_1)^{N_1-1} K_j^{(\alpha_1,0)}\big(2^j x^{\theta_1}_{c_1,y_1}, 2^j(x-y_2)\big) \\ &\times \big(2^j(y_1-c_1)\big)^{\alpha_1}\, a_1(y_1)\, a_2(y_2)\, d\theta_1 dy_1 dy_2, \end{aligned} \tag{3.9}$$

where

$$x^{\theta_1}_{c_1,y_1} = x - c_1 - \theta_1(y_1 - c_1) \quad \text{and} \quad K_j^{(\alpha_1,\alpha_2)}(x_1,x_2) = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} K_j(x_1,x_2).$$

We note that the moment condition of a_2 gives the similar representation of g_j with the variables y_1 and y_2 interchanged.

Proof of (3.5). Under the assumption (3.1), we can take α_1 and α_2 such that

$$\begin{cases} s_1 > \alpha_1 n \\ \alpha_1 > 1/p_1 - 1/2, \end{cases} \qquad \begin{cases} s_2 > \alpha_2 n \\ \alpha_2 > 1/p_2 - 1/2, \end{cases} \qquad \alpha_1 + \alpha_2 = 1/p_1 + 1/p_2 - 1/2.$$

We define β_1 and β_2 by $\beta_1/2 = 1/p_1 - \alpha_1$ and $\beta_2/2 = 1/p_2 - \alpha_2$. Notice that $\beta_1/2 = \alpha_2 - 1/p_2 + 1/2 > 0$, and similarly, $\beta_2/2 > 0$ and $\beta_1 + \beta_2 = 1$.

In order to obtain u and v satisfying (3.5), we shall prove that for each $j \in \mathbb{Z}$ there exist a function u_j depending only on a_1 and a function v_j depending only on a_2 such that

$$|g_j(x)|\chi_{(Q_1^*)^c \cap (Q_2^*)^c}(x) \lesssim A u_j(x) v_j(x), \tag{3.10}$$

$$\|u_j\|_{L^{p_1}} \lesssim \begin{cases} (2^j \ell(Q_1))^{-n/p_1+n+N_1\beta_1} & \text{if } \ 2^j\ell(Q_1) \le 1, \\ (2^j \ell(Q_1))^{-n/p_1+n-(s_1-\alpha_1 n)} & \text{if } \ 2^j\ell(Q_1) > 1, \end{cases} \tag{3.11}$$

$$\|v_j\|_{L^{p_2}} \lesssim \begin{cases} (2^j \ell(Q_2))^{-n/p_2+n+N_2\beta_2} & \text{if } \ 2^j\ell(Q_2) \le 1, \\ (2^j \ell(Q_2))^{-n/p_2+n-(s_2-\alpha_2 n)} & \text{if } \ 2^j\ell(Q_2) > 1. \end{cases} \tag{3.12}$$

Before proving (3.10)–(3.12), let us observe that these imply (3.5). First, (3.10) gives

$$\begin{aligned} |T_m(a_1,a_2)|\chi_{(Q_1^*)^c \cap (Q_2^*)^c} &\le \sum_{j\in\mathbb{Z}} |g_j|\chi_{(Q_1^*)^c \cap (Q_2^*)^c} \\ &\lesssim A \sum_{j\in\mathbb{Z}} u_j v_j \le A\Big(\sum_{j\in\mathbb{Z}} u_j\Big)\Big(\sum_{j\in\mathbb{Z}} v_j\Big). \end{aligned}$$

Second, if we set $u = \sum_{j\in\mathbb{Z}} u_j$, then $\|u\|_{L^{p_1}} \lesssim 1$. In fact, since $-n/p_1+n+N_1\beta_1 > 0$ and $-n/p_1 + n - (s_1 - \alpha_1 n) < 0$, where we have used that N_1 is large enough and $p_1 \le 1$, we have, by (3.11),

$$\begin{aligned} \|u\|_{L^{p_1}}^{p_1} \le \sum_{j\in\mathbb{Z}} \|u_j\|_{L^{p_1}}^{p_1} &= \Bigg(\sum_{2^j\ell(Q_1)\le 1} + \sum_{2^j\ell(Q_1)>1} \Bigg) \|u_j\|_{L^{p_1}}^{p_1} \\ &\lesssim \sum_{2^j\ell(Q_1)\le 1} (2^j\ell(Q_1))^{(-n/p_1+n+N_1\beta_1)p_1} \\ &\quad + \sum_{2^j\ell(Q_1)>1} (2^j\ell(Q_1))^{(-n/p_1+n-(s_1-\alpha_1 n))p_1} \lesssim 1. \end{aligned}$$

Similarly, if we set $v = \sum_{j\in\mathbb{Z}} v_j$, then $\|v\|_{L^{p_2}} \lesssim 1$. Hence, we obtain u and v satisfying (3.5).

Let us prove (3.10)–(3.12). We assume $x \in (Q_1^*)^c \cap (Q_2^*)^c$. Note that

$$|x - c_1| \approx |x - y_1| \quad \text{and} \quad |x - c_2| \approx |x - y_2| \quad \text{for } y_1 \in Q_1 \text{ and } y_2 \in Q_2.$$

Then, it follows from (3.8) and Lemma 2.3 that

$$\begin{aligned}
&\langle 2^j(x-c_1)\rangle^{s_1}\langle 2^j(x-c_2)\rangle^{s_2}|g_j(x)| \\
&\quad\lesssim 2^{2jn}\int_{\substack{y_1\in Q_1\\ y_2\in Q_2}}\langle 2^j(x-y_1)\rangle^{s_1}\langle 2^j(x-y_2)\rangle^{s_2}|K_j(2^j(x-y_1),2^j(x-y_2))| \\
&\qquad\qquad\times \ell(Q_1)^{-n/p_1}\ell(Q_2)^{-n/p_2}\,dy_1dy_2 \\
&\quad\le 2^{2jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2+n} \\
&\qquad\times\Big(\int_{y_1\in Q_1}\sup_{z_2\in\mathbb{R}^n}\big(\langle 2^j(x-y_1)\rangle^{s_1}\langle z_2\rangle^{s_2}|K_j(2^j(x-y_1),z_2)|\big)\,\ell(Q_1)^{-n}\,dy_1\Big) \\
&\quad\lesssim 2^{2jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2+n} \\
&\qquad\times\Big(\int_{y_1\in Q_1}\big\|\langle 2^j(x-y_1)\rangle^{s_1}\langle z_2\rangle^{s_2}K_j(2^j(x-y_1),z_2)\big\|_{L^2_{z_2}}\,\ell(Q_1)^{-n}\,dy_1\Big) \\
&\quad= 2^{2jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2+n}h_j^{(Q_1,0,0)}(x),
\end{aligned}$$

where

$$h_j^{(Q_1,0,0)}(x)=\int_{y_1\in Q_1}\big\|\langle 2^j(x-y_1)\rangle^{s_1}\langle z_2\rangle^{s_2}K_j(2^j(x-y_1),z_2)\big\|_{L^2_{z_2}}\ell(Q_1)^{-n}\,dy_1. \tag{3.13}$$

Thus,

$$\begin{aligned}
|g_j(x)|\lesssim{}& 2^{2jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2+n} \\
&\times\langle 2^j(x-c_1)\rangle^{-s_1}\langle 2^j(x-c_2)\rangle^{-s_2}h_j^{(Q_1,0,0)}(x).
\end{aligned}\tag{3.14}$$

By Minkowski's inequality for integrals,

$$\begin{aligned}
\|h_j^{(Q_1,0,0)}\|_{L^2}&\le\int_{y_1\in Q_1}\Big\|\,\big\|\langle 2^j(x-y_1)\rangle^{s_1}\langle z_2\rangle^{s_2}K_j(2^j(x-y_1),z_2)\big\|_{L^2_{z_2}}\Big\|_{L^2_x}\ell(Q_1)^{-n}dy_1 \\
&=2^{-jn/2}\,\|\langle z_1\rangle^{s_1}\langle z_2\rangle^{s_2}K_j(z_1,z_2)\|_{L^2_{z_1,z_2}} \\
&=2^{-jn/2}\,\|m_j\|_{W^{(s_1,s_2)}}\le A2^{-jn/2}. \qquad (3.15)
\end{aligned}$$

On the other hand, since

$$|x-c_1|\approx|x-c_1-\theta_1(y_1-c_1)|=|x_{c_1,y_1}^{\theta_1}|\quad\text{for } 0<\theta_1<1 \text{ and } y_1\in Q_1,$$

replacing (3.8) by (3.9) in the argument above, we obtain

$$\begin{aligned}
|g_j(x)|\lesssim{}& 2^{2jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2+n} \\
&\times\langle 2^j(x-c_1)\rangle^{-s_1}\langle 2^j(x-c_2)\rangle^{-s_2}h_j^{(Q_1,N_1,0)}(x),
\end{aligned}\tag{3.16}$$

where

$$\begin{aligned}
h_j^{(Q_1,N_1,0)}(x)={}&\sum_{|\alpha_1|=N_1}\int_{\substack{0<\theta_1<1\\ y_1\in Q_1}}\big\|\langle 2^jx_{c_1,y_1}^{\theta_1}\rangle^{s_1}\langle z_2\rangle^{s_2}K_j^{(\alpha_1,0)}(2^jx_{c_1,y_1}^{\theta_1},z_2)\big\|_{L^2_{z_2}} \\
&\times(2^j\ell(Q_1))^{N_1}\ell(Q_1)^{-n}\,d\theta_1dy_1, \qquad (3.17)
\end{aligned}$$

and we also have, by Remark 2.5,

$$\|h_j^{(Q_1,N_1,0)}\|_{L^2} \lesssim A2^{-jn/2}(2^j\ell(Q_1))^{N_1}. \tag{3.18}$$

It follows from (3.14) and (3.16) that

$$\begin{aligned} |g_j(x)| &\lesssim 2^{2jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2+n}\langle 2^j(x-c_1)\rangle^{-s_1}\langle 2^j(x-c_2)\rangle^{-s_2} \\ &\quad\times \min\big\{h_j^{(Q_1,0,0)}(x), h_j^{(Q_1,N_1,0)}(x)\big\}. \end{aligned} \tag{3.19}$$

By interchanging the roles of y_1 and y_2 in the argument above, we can also prove, for $x \in (Q_1^*)^c \cap (Q_2^*)^c$,

$$\begin{aligned} |g_j(x)| &\lesssim 2^{2jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2+n}\langle 2^j(x-c_1)\rangle^{-s_1}\langle 2^j(x-c_2)\rangle^{-s_2} \\ &\quad\times \min\big\{h_j^{(Q_2,0,0)}(x), h_j^{(Q_2,0,N_2)}(x)\big\}, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} h_j^{(Q_2,0,0)}(x) &= \int_{y_2\in Q_2} \big\|\langle z_1\rangle^{s_1}\langle 2^j(x-y_2)\rangle^{s_2} K_j(z_1, 2^j(x-y_2))\big\|_{L^2_{z_1}}\, \ell(Q_2)^{-n}\, dy_2, \\ h_j^{(Q_2,0,N_2)}(x) &= \sum_{|\alpha_2|=N_2} \int_{\substack{0<\theta_2<1 \\ y_2\in Q_2}} \big\|\langle z_1\rangle^{s_1}\langle 2^j x_{c_2,y_2}^{\theta_2}\rangle^{s_2} K_j^{(0,\alpha_2)}(z_1, 2^j x_{c_2,y_2}^{\theta_2})\big\|_{L^2_{z_1}} \\ &\qquad\qquad \times (2^j\ell(Q_2))^{N_2}\ell(Q_2)^{-n}\, d\theta_2 dy_2 \end{aligned}$$

and $x_{c_2,y_2}^{\theta_2} = x - c_2 - \theta_2(y_2-c_2)$.

By (3.19) and (3.20), we see that

$$\begin{aligned} |g_j(x)|\chi_{(Q_1^*)^c\cap(Q_2^*)^c}(x) &= A\times A^{-\beta_1}|g_j(x)|^{\beta_1}\chi_{(Q_1^*)^c}(x)\times A^{-\beta_2}|g_j(x)|^{\beta_2}\chi_{(Q_2^*)^c}(x) \\ &\lesssim A\times A^{-\beta_1}\, 2^{jn}\,\ell(Q_1)^{-n/p_1+n}\chi_{(Q_1^*)^c}(x)\langle 2^j(x-c_1)\rangle^{-s_1} \\ &\quad\times \big(\min\big\{h_j^{(Q_1,0,0)}(x), h_j^{(Q_1,N_1,0)}(x)\big\}\big)^{\beta_1} \\ &\quad\times A^{-\beta_2}\, 2^{jn}\,\ell(Q_2)^{-n/p_2+n}\chi_{(Q_2^*)^c}(x)\langle 2^j(x-c_2)\rangle^{-s_2} \\ &\quad\times \big(\min\big\{h_j^{(Q_2,0,0)}(x), h_j^{(Q_2,0,N_2)}(x)\big\}\big)^{\beta_2} \\ &= A\times u_j(x)\times v_j(x). \end{aligned}$$

It should be emphasized that u_j depends only on Q_1 (namely, a_1) and v_j depends only on Q_2 (namely, a_2), and we obtain (3.10). Let us check that u_j satisfies (3.11). By Hölder's inequality with $1/p_1 = \alpha_1 + \beta_1/2$,

$$\begin{aligned} \|u_j\|_{L^{p_1}} &\le A^{-\beta_1}\, 2^{jn}\,\ell(Q_1)^{-n/p_1+n}\,\big\|\langle 2^j(\cdot - c_1)\rangle^{-s_1}\big\|_{L^{1/\alpha_1}((Q_1^*)^c)} \\ &\quad\times \big\|\big(\min\big\{h_j^{(Q_1,0,0)}, h_j^{(Q_1,N_1,0)}\big\}\big)^{\beta_1}\big\|_{L^{2/\beta_1}}. \end{aligned}$$

Since $s_1/\alpha_1 > n$,

$$\big\|\langle 2^j(\cdot - c_1)\rangle^{-s_1}\big\|_{L^{1/\alpha_1}((Q_1^*)^c)} \approx \begin{cases} 2^{-jn\alpha_1} & \text{if } \ 2^j\ell(Q_1) \le 1, \\ 2^{-jn\alpha_1}(2^j\ell(Q_1))^{-s_1+\alpha_1 n} & \text{if } \ 2^j\ell(Q_1) > 1, \end{cases}$$

$$= \begin{cases} 2^{-jn(1/p_1-\beta_1/2)} & \text{if } \ 2^j\ell(Q_1) \le 1, \\ 2^{-jn(1/p_1-\beta_1/2)}(2^j\ell(Q_1))^{-s_1+\alpha_1 n} & \text{if } \ 2^j\ell(Q_1) > 1. \end{cases}$$

By (3.15) and (3.18), we also have

$$\begin{aligned} \big\|\big(\min\{h_j^{(Q_1,0,0)}, h_j^{(Q_1,N_1,0)}\}\big)^{\beta_1}\big\|_{L^{2/\beta_1}} &\le \min\big\{\big\|h_j^{(Q_1,0,0)}\big\|_{L^2}^{\beta_1}, \big\|h_j^{(Q_1,N_1,0)}\big\|_{L^2}^{\beta_1}\big\} \\ &\lesssim \begin{cases} \big(A\,2^{-jn/2}(2^j\ell(Q_1))^{N_1}\big)^{\beta_1} & \text{if } 2^j\ell(Q_1) \le 1, \\ \big(A\,2^{-jn/2}\big)^{\beta_1} & \text{if } 2^j\ell(Q_1) > 1.. \end{cases} \end{aligned}$$

Therefore, u_j satisfies (3.11). In the same way, we can check that v_j satisfies (3.12). □

Proof of (3.6). In order to obtain u' and v' satisfying (3.6), we shall prove that for each $j \in \mathbb{Z}$ there exist a function u'_j depending only on a_1 and a function v' depending only on a_2 such that

$$|g_j(x)|\chi_{(Q_1^*)^c\cap Q_2^*}(x) \lesssim Au'_j(x)v'(x), \tag{3.21}$$

$$\|u'_j\|_{L^{p_1}} \lesssim \begin{cases} (2^j\ell(Q_1))^{-n/p_1+n+N_1} & \text{if } \ 2^j\ell(Q_1) \le 1 \\ (2^j\ell(Q_1))^{-s_1+n/2} & \text{if } \ 2^j\ell(Q_1) > 1, \end{cases} \tag{3.22}$$

$$\|v'\|_{L^{p_2}} \lesssim 1. \tag{3.23}$$

Once these are proved, we can take $\sum_{j\in\mathbb{Z}} u'_j$ and v' as u' and v' in (3.6).

Let us prove (3.21)–(3.23). We assume $x \in (Q_1^*)^c \cap Q_2^*$. Since $|x-c_1| \approx |x-y_1|$ for $y_1 \in Q_1$ and $s_2 > n/2$, we use (3.8) and Schwarz's inequality to obtain

$$\begin{aligned} &\langle 2^j(x - c_1)\rangle^{s_1}|g_j(x)| \\ &\lesssim 2^{2jn}\int_{\substack{y_1\in Q_1\\ y_2\in\mathbb{R}^n}} \langle 2^j(x-y_1)\rangle^{s_1}|K_j(2^j(x-y_1), 2^j(x-y_2))| \\ &\qquad\qquad \times \ell(Q_1)^{-n/p_1}\ell(Q_2)^{-n/p_2}\,dy_1dy_2 \\ &= 2^{jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2} \\ &\qquad \times \int_{\substack{y_1\in Q_1\\ z_2\in\mathbb{R}^n}} \langle 2^j(x-y_1)\rangle^{s_1}|K_j(2^j(x-y_1), z_2)|\ell(Q_1)^{-n}\,dy_1dz_2 \\ &\lesssim 2^{jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2} \\ &\qquad \times \int_{y_1\in Q_1} \big\|\langle 2^j(x-y_1)\rangle^{s_1}\langle z_2\rangle^{s_2}K_j(2^j(x-y_1), z_2)\big\|_{L^2_{z_2}}\,\ell(Q_1)^{-n}\,dy_1 \\ &= 2^{jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2}{h_j}^{(Q_1,0,0)}(x), \end{aligned}$$

where ${h_j}^{(Q_1,0,0)}$ is defined by (3.13).

Thus,

$$|g_j(x)| \lesssim 2^{jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2}\langle 2^j(x-c_1)\rangle^{-s_1} h_j{}^{(Q_1,0,0)}(x). \tag{3.24}$$

On the other hand, since $|x-c_1| \approx |x-c_1-\theta_1(y_1-c_1)| = |x^{\theta_1}_{c_1,y_1}|$ for $0<\theta_1<1$ and $y_1 \in Q_1$, replacing (3.8) by (3.9) in the argument above, we obtain

$$|g_j(x)| \lesssim 2^{jn}\ell(Q_1)^{-n/p_1+n}\ell(Q_2)^{-n/p_2}\langle 2^j(x-c_1)\rangle^{-s_1} h_j^{(Q_1,N_1,0)}(x), \tag{3.25}$$

where $h_j^{(Q_1,N_1,0)}$ is defined by (3.17).

Now, (3.24) and (3.25) imply (3.21) with

$$\begin{aligned} u_j'(x) &= A^{-1}2^{jn}\ell(Q_1)^{-n/p_1+n}\chi_{(Q_1^*)^c}(x) \\ &\qquad \times \langle 2^j(x-c_1)\rangle^{-s_1} \min\left\{h_j^{(Q_1,0,0)}(x), h_j^{(Q_1,N_1,0)}(x)\right\}, \\ v'(x) &= \ell(Q_2)^{-n/p_2}\chi_{Q_2^*}(x). \end{aligned}$$

It is clear that v' satisfies (3.23). Let us check that u_j' satisfies (3.22). By Hölder's inequality with $1/p_1 = 1/q_1 + 1/2$,

$$\begin{aligned} \|u_j'\|_{L^{p_1}} &\lesssim A^{-1}2^{jn}\ell(Q_1)^{-n/p_1+n} \\ &\qquad \times \left\|\langle 2^j(\cdot - c_1)\rangle^{-s_1}\right\|_{L^{q_1}((Q_1^*)^c)} \left\|\min\left\{h_j^{(Q_1,0,0)}, h_j^{(Q_1,N_1,0)}\right\}\right\|_{L^2}. \end{aligned}$$

Since $s_1q_1 > n$,

$$\begin{aligned} &\left\|\langle 2^j(\cdot - c_1)\rangle^{-s_1}\right\|_{L^{q_1}((Q_1^*)^c)} \\ &\qquad \approx \begin{cases} 2^{-jn(1/p_1-1/2)} & \text{if } \ 2^j\ell(Q_1) \le 1 \\ 2^{-jn(1/p_1-1/2)}(2^j\ell(Q_1))^{-s_1+n(1/p_1-1/2)} & \text{if } \ 2^j\ell(Q_1) > 1. \end{cases} \end{aligned}$$

By (3.15) and (3.18),

$$\left\|\min\left\{h_j^{(Q_1,0,0)}, h_j^{(Q_1,N_1,0)}\right\}\right\|_{L^2} \lesssim \begin{cases} A2^{-jn/2}(2^j\ell(Q_1))^{N_1} & \text{if } \ 2^j\ell(Q_1) \le 1 \\ A2^{-jn/2} & \text{if } \ 2^j\ell(Q_1) > 1. \end{cases}$$

Therefore, u_j' satisfies (3.22). □

Proof of (3.7). This can be proved in the same way as in the proof of (3.6) only by interchanging the roles of y_1 and y_2. This completes the proof of (3.3) and thus (3.1)–(3.2) is proved. □

Remark 3.1. Notice that the proof of (3.6) works under the weaker assumption that $s_1 > n/p_1 - n/2$ and $s_2 > n/2$. Similarly we can prove (3.7) under the assumption that $s_1 > n/2$ and $s_2 > n/p_2 - n/2$.

4. The boundedness from $H^{p_1} \times L^2$ to L^p for $0 < p_1 \le 1$

In this section, we shall prove Theorem 1.1 with $0 < p_1 \le 1$ and $p_2 = 2$. That is, in the case $0 < p_1 \le 1$ and $1/p_1 + 1/2 = 1/p$, we show that

$$s_1 > n/p_1 - n/2,\ s_2 > n/2 \implies \|T_m\|_{H^{p_1}\times L^2 \to L^p} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}. \tag{4.1}$$

It should be pointed out that by interchanging the roles of p_1 and p_2 in the proof of (4.1) we can also prove, for $0 < p_2 \le 1$, $1/2 + 1/p_2 = 1/p$,

$$s_1 > n/2,\ s_2 > n/p_2 - n/2 \implies \|T_m\|_{L^2\times H^{p_2} \to L^p} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}. \tag{4.2}$$

By Lemma 2.1, we can decompose m as follows:

$$m = m\,\Phi_{(1,1)} + m\,\Phi_{(0,1)} + m\,\Phi_{(1,0)} = m^{(1)} + m^{(2)} + m^{(3)}.$$

Then

$$\begin{aligned}
&\operatorname{supp} m^{(1)} \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; |\xi_1|/8 \le |\xi_2| \le 8|\xi_1|\},\\
&\operatorname{supp} m^{(2)} \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; |\xi_1| \le |\xi_2|/2\},\\
&\operatorname{supp} m^{(3)} \subset \{(\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^n \;:\; |\xi_2| \le |\xi_1|/2\}.
\end{aligned}$$

We use the following notation: $\mathcal{A}_0$ denotes the set of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\varphi$ is compact and $\varphi = 1$ on some neighborhood of the origin; $\mathcal{A}_1$ denotes the set of $\psi' \in \mathcal{S}(\mathbb{R}^n)$ for which $\operatorname{supp}\psi'$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$.

In the rest of this section, we assume $0 < p_1 \le 1$, $1/p_1 + 1/2 = 1/p$, $s_1 > n/p_1 - n/2$, and $s_2 > n/2$. We shall prove

$$\|T_{m^{(i)}}\|_{H^{p_1}\times L^2 \to L^p} \lesssim \sup_{j\in\mathbb{Z}} \|m_j^{(i)}\|_{W^{(s_1,s_2)}}$$

for $i = 1,2,3$, where the $m_j^{(i)}$ are defined by (1.5) with m replaced by $m^{(i)}$. Once these are proved, (4.1) follows from 2) of Lemma 2.1 and Lemma 2.4. Let $s = \min\{s_1,s_2\}$. Then, since $n/s < 2$, we can take q satisfying $\max\{1, n/s\} < q < 2$. We consider first $m^{(1)}$.

Estimate for $m^{(1)}$. We write simply m instead of $m^{(1)}$. In order to obtain the boundedness of T_m, we shall prove that for an H^{p_1}-atom a_1 and an L^2-function f_2 there exist a function b_1 depending only on a_1 and a function b_2 depending only on f_2 such that

$$|T_m(a_1,f_2)(x)| \lesssim A b_1(x) b_2(x), \quad \|b_1\|_{L^{p_1}} \lesssim 1, \quad \|b_2\|_{L^2} \lesssim \|f_2\|_{L^2}, \tag{4.3}$$

where

$$A = \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}.$$

Let us observe that (4.3) implies the boundedness of T_m. To see this, we decompose $f_1 \in H^{p_1}(\mathbb{R}^n)$ as

$$f_1 = \sum_k \lambda_{1,k} a_{1,k},$$

with H^{p_1}-atoms $a_{1,k}$ and with

$$\sum_k |\lambda_{1,k}|^{p_1} \lesssim \|f_1\|_{H^{p_1}}^{p_1}.$$

Then by taking the functions $b_{1,k}$ and b_2 satisfying (4.3) for $a_1 = a_{1,k}$, we have

$$\begin{aligned}\|T_m(f_1,f_2)\|_{L^p} &= \Big\| \sum_k \lambda_{1,k} T_m(a_{1,k}, f_2) \Big\|_{L^p} \lesssim A \Big\| \Big(\sum_k |\lambda_{1,k}| b_{1,k} \Big) b_2 \Big\|_{L^p} \\ &\le A \Big\| \sum_k |\lambda_{1,k}| b_{1,k} \Big\|_{L^{p_1}} \|b_2\|_{L^2} \lesssim A \Big(\sum_k |\lambda_{1,k}|^{p_1} \Big)^{1/p_1} \|f_2\|_{L^2} \\ &\lesssim A \|f_1\|_{H^{p_1}} \|f_2\|_{L^2}.\end{aligned}$$

To obtain (4.3), we shall prove

$$|T_m(a_1,f_2)(x)| \chi_{(Q_1^*)^c}(x) \lesssim A u(x) v(x), \quad \|u\|_{L^{p_1}} \lesssim 1, \quad \|v\|_{L^2} \lesssim \|f_2\|_{L^2}, \tag{4.4}$$

$$|T_m(a_1,f_2)(x)| \chi_{Q_1^*}(x) \lesssim A u'(x) v'(x), \quad \|u'\|_{L^{p_1}} \lesssim 1, \quad \|v'\|_{L^2} \lesssim \|f_2\|_{L^2}, \tag{4.5}$$

where u and u' depend only on a_1, and v and v' depend only on f_2. Once (4.4) and (4.5) are proved, we can take $u + u'$ and $v + v'$ as b_1 and b_2 in (4.3). In order to prove (4.4) and (4.5), we decompose $T_m(a_1, f_2)(x)$ as

$$T_m(a_1,f_2)(x) = \sum_{j \in \mathbb{Z}} g_j(x),$$

where $g_j(x)$ is defined by (3.8) with a_2 replaced by f_2.

Proof of (4.4). We shall prove that for each $j \in \mathbb{Z}$ there exists a function u_j depending only on a_1 such that

$$|g_j(x)| \chi_{(Q_1^*)^c}(x) \lesssim A u_j(x) M_q f_2(x), \tag{4.6}$$

$$\|u_j\|_{L^{p_1}} \lesssim \begin{cases} (2^j \ell(Q_1))^{-n/p_1 + n + N_1} & \text{if} \quad 2^j \ell(Q_1) \le 1 \\ (2^j \ell(Q_1))^{-s_1 + n/2} & \text{if} \quad 2^j \ell(Q_1) > 1. \end{cases} \tag{4.7}$$

Once these are proved, we can take $\sum_{j \in \mathbb{Z}} u_j$ and $M_q f_2$ as u and v in (4.4). Here, notice that M_q is bounded on $L^2(\mathbb{R}^n)$ since $q < 2$.

We assume that $x \in (Q_1^*)^c$. Since $|x - c_1| \approx |x - y_1|$ for $y_1 \in Q_1$, $s_2 q > n$ and $q' > 2$, we have by (3.8), Hölder's inequality and Lemma 2.3,

$$
\begin{aligned}
&\langle 2^j(x-c_1)\rangle^{s_1} |g_j(x)| \\
&\lesssim 2^{2jn} \int_{\substack{y_1 \in Q_1 \\ y_2 \in \mathbb{R}^n}} \langle 2^j(x-y_1)\rangle^{s_1} \, |K_j(2^j(x-y_1), 2^j(x-y_2))| \ell(Q_1)^{-n/p_1} |f_2(y_2)| \, dy_1 dy_2 \\
&= 2^{2jn} \ell(Q_1)^{-n/p_1+n} \int_{\substack{y_1 \in Q_1 \\ y_2 \in \mathbb{R}^n}} \langle 2^j(x-y_1)\rangle^{s_1} \langle 2^j(x-y_2)\rangle^{s_2} \\
&\qquad \times |K_j(2^j(x-y_1), 2^j(x-y_2))| \ell(Q_1)^{-n} \frac{|f_2(y_2)|}{\langle 2^j(x-y_2)\rangle^{s_2}} \, dy_1 dy_2 \\
&\lesssim 2^{jn} \ell(Q_1)^{-n/p_1+n} \Big(2^{jn} \int_{\mathbb{R}^n} \frac{|f_2(y_2)|^q}{\langle 2^j(x-y_2)\rangle^{s_2 q}} \, dy_2 \Big)^{1/q} \\
&\qquad \times \int_{y_1 \in Q_1} \big\| \langle 2^j(x-y_1)\rangle^{s_1} \langle z_2 \rangle^{s_2} K_j(2^j(x-y_1), z_2) \big\|_{L^{q'}_{z_2}} \ell(Q_1)^{-n} \, dy_1 \\
&\lesssim 2^{jn} \ell(Q_1)^{-n/p_1+n} M_q f_2(x) \\
&\qquad \times \int_{y_1 \in Q_1} \big\| \langle 2^j(x-y_1)\rangle^{s_1} \langle z_2 \rangle^{s_2} K_j(2^j(x-y_1), z_2) \big\|_{L^{2}_{z_2}} \ell(Q_1)^{-n} \, dy_1 \\
&= 2^{jn} \ell(Q_1)^{-n/p_1+n} h_j^{(Q_1,0,0)}(x) M_q f_2(x),
\end{aligned}
$$

where $h_j^{(Q_1,0,0)}$ is defined by (3.13). Thus

$$
|g_j(x)| \lesssim 2^{jn} \, \ell(Q_1)^{-n/p_1+n} \, \langle 2^j(x-c_1)\rangle^{-s_1} \, h_j^{(Q_1,0,0)}(x) \, M_q f_2(x). \tag{4.8}
$$

On the other hand, since $|x - c_1| \approx |x - c_1 - \theta_1(y_1 - c_1)| = |x^{\theta_1}_{c_1,y_1}|$ for $0 < \theta_1 < 1$ and $y_1 \in Q_1$, replacing (3.8) by (3.9) in the argument above, we obtain

$$
|g_j(x)| \lesssim 2^{jn} \, \ell(Q_1)^{-n/p_1+n} \, \langle 2^j(x-c_1)\rangle^{-s_1} \, h_j^{(Q_1,N_1,0)}(x) \, M_q f_2(x), \tag{4.9}
$$

where $h_j^{(Q_1,N_1,0)}$ is defined by (3.17).

Now, (4.8) and (4.9) imply (4.6) with

$$
\begin{aligned}
u_j(x) = A^{-1} \, 2^{jn} \, \ell(Q_1)^{-n/p_1+n} \, &\chi_{(Q_1^*)^c}(x) \\
&\times \langle 2^j(x-c_1)\rangle^{-s_1} \, \min\big\{ h_j^{(Q_1,0,0)}(x), h_j^{(Q_1,N_1,0)}(x) \big\}.
\end{aligned}
$$

This u_j is the same as the u_j' in the proof of (3.6). Thus we have already checked that u_j satisfies (4.7) in the proof of (3.6) (cf. also Remark 3.1). □

Proof of (4.5). We shall prove that

$$
|g_j(x)| \chi_{Q_1^*}(x) \lesssim A \, M_q(\psi(D/2^j)a_1)(x) \, \chi_{Q_1^*}(x) \, M_q(\psi'(D/2^j)f_2)(x), \tag{4.10}
$$

where $\psi, \psi' \in \mathcal{A}_1$. Once this is proved, we obtain (4.5). In fact, (4.10) implies the first inequality of (4.5) with

$$u'(x) = \Big(\sum_{j\in\mathbb{Z}} M_q(\psi(D/2^j)a_1)(x)^2 \Big)^{1/2} \chi_{Q_1^*}(x),$$
$$v'(x) = \Big(\sum_{j\in\mathbb{Z}} M_q(\psi'(D/2^j)f_2)(x)^2 \Big)^{1/2}.$$

Since $q < 2$, we have, by the vector-valued maximal inequality of Fefferman–Stein and the Littlewood–Paley inequality,

$$\begin{aligned} \|u'\|_{L^{p_1}} &= \Big\| \Big(\sum_{j\in\mathbb{Z}} M_q(\psi(D/2^j)a_1)^2 \Big)^{1/2} \chi_{Q_1^*} \Big\|_{L^{p_1}} \\ &\le \Big\| \Big(\sum_{j\in\mathbb{Z}} M_q(\psi(D/2^j)a_1)^2 \Big)^{1/2} \Big\|_{L^2} |Q_1^*|^{1/p_1-1/2} \\ &= \Big\| \Big(\sum_{j\in\mathbb{Z}} M(|\psi(D/2^j)a_1|^q)^{2/q} \Big)^{q/2} \Big\|_{L^{2/q}}^{1/q} |Q_1^*|^{1/p_1-1/2} \\ &\lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)a_1|^2 \Big)^{1/2} \Big\|_{L^2} |Q_1^*|^{1/p_1-1/2} \lesssim \|a_1\|_{L^2} |Q_1^*|^{1/p_1-1/2} \lesssim 1, \end{aligned}$$

and similarly $\|v'\|_{L^2} \lesssim \|f_2\|_{L^2}$.

Let us prove (4.10). Since $\operatorname{supp} \Psi(\cdot/2^j) \subset \{2^{j-1} \le (|\xi_1|^2 + |\xi_2|^2)^{1/2} \le 2^{j+1}\}$ and $\operatorname{supp} m \subset \{|\xi_2|/8 \le |\xi_1| \le 8|\xi_2|\}$, where Ψ is as in (1.3) with $d = 2n$, if $(\xi_1, \xi_2) \in \operatorname{supp} m(\cdot)\Psi(\cdot/2^j)$, then $|\xi_1| \approx |\xi_2| \approx 2^j$. Hence, we can find $\psi, \psi' \in \mathcal{A}_1$ independent of j such that

$$\begin{aligned} g_j(x) &= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)} m_j(\xi_1/2^j, \xi_2/2^j) \\ &\qquad\qquad \times \psi(\xi_1/2^j)\, \widehat{a_1}(\xi_1)\, \psi'(\xi_2/2^j)\, \widehat{f_2}(\xi_2)\, d\xi_1 d\xi_2 \\ &= T_{m_j(\cdot/2^j)}\big(\psi(D/2^j)a_1, \psi'(D/2^j)f_2\big)(x), \end{aligned}$$

where

$$m_j(\xi_1, \xi_2) = m(2^j\xi_1, 2^j\xi_2)\, \Psi(\xi_1, \xi_2).$$

Since $\operatorname{supp} m_j$ is included in a compact subset independent of j, (4.10) follows from Lemma 2.2. This completes the proof of (4.5). □

We next consider $m^{(2)}$.

Estimate for $m^{(2)}$. We write simply m instead of $m^{(2)}$. In order to obtain the boundedness of T_m, we shall use the Littlewood–Paley function

$$G(F)(x) = \Big(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)F(x)|^2 \Big)^{1/2},$$

where ψ is as in (1.3) with $d = n$. Since $\|F\|_{L^p} \lesssim \|F\|_{H^p} \approx \|G(F)\|_{L^p}$, the boundedness of T_m will follow if we prove the estimate

$$\|G(T_m(f_1, f_2))\|_{L^p} \lesssim A\,\|f_1\|_{H^{p_1}}\,\|f_2\|_{L^2}, \tag{4.11}$$

where $A = \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}$.

To prove (4.11), we shall prove that for an H^{p_1}-atom a_1 and for an L^2-function f_2 there exist a function b_1 depending only on a_1 and a function b_2 depending only on f_2 such that

$$G(T_m(a_1, f_2))(x) \lesssim Ab_1(x)b_2(x), \quad \|b_1\|_{L^{p_1}} \lesssim 1, \quad \|b_2\|_{L^2} \lesssim \|f_2\|_{L^2}. \tag{4.12}$$

Let us observe that (4.12) implies (4.11). To see this, we decompose f_1 as

$$f_1 = \sum_k \lambda_{1,k}\, a_{1,k},$$

with H^{p_1}-atoms $a_{1,k}$ and with $\sum_k |\lambda_{1,k}|^{p_1} \lesssim \|f_1\|_{H^{p_1}}^{p_1}$. Then by taking the functions $b_{1,k}$ and b_2 satisfying (4.12) for $a_1 = a_{1,k}$, we have

$$\begin{aligned} G(T_m(f_1, f_2))(x) &= G\Big(\sum_k \lambda_{1,k}\, T_m(a_{1,k}, f_2)\Big)(x) \\ &\le \sum_k |\lambda_{1,k}|\, G(T_m(a_{1,k}, f_2))(x) \lesssim A \sum_k |\lambda_{1,k}|\, b_{1,k}(x)\, b_2(x). \end{aligned}$$

Hence, by Hölder's inequality,

$$\begin{aligned} \|G(T_m(f_1, f_2))\|_{L^p} &\lesssim A \Big\| \sum_k |\lambda_{1,k}|b_{1,k} \Big\|_{L^{p_1}} \|b_2\|_{L^2} \\ &\le A\, \Big(\sum_k |\lambda_{1,k}|^{p_1}\, \|b_{1,k}\|_{L^{p_1}}^{p_1} \Big)^{1/p_1} \|b_2\|_{L^2} \\ &\lesssim A \Big(\sum_k |\lambda_{1,k}|^{p_1} \Big)^{1/p_1} \|b_2\|_{L^2} \lesssim A\,\|f_1\|_{H^{p_1}}\,\|f_2\|_{L^2}, \end{aligned}$$

which is the estimate (4.11).

To prove (4.12), we prove that for each $j \in \mathbb{Z}$ there exists a function u_j depending only on a_1 such that

$$|\psi(D/2^j)\, T_m(a_1, f_2)(x)|\, \chi_{(Q_1^*)^c}(x) \lesssim A\, u_j(x)\, M_q f_2(x), \tag{4.13}$$

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} u_j^2 \Big)^{1/2} \Big\|_{L^{p_1}} \lesssim 1 \tag{4.14}$$

and also prove that there exists a $\psi' \in \mathcal{A}_1$ such that

$$|\psi(D/2^j)\, T_m(a_1, f_2)(x)|\, \chi_{Q_1^*}(x) \lesssim A\, M_q a_1(x)\, M_q(\psi'(D/2^j) f_2)(x). \tag{4.15}$$

We shall see that these estimates imply (4.12). In fact, (4.13) and (4.15) imply

$$\begin{aligned}G(T_m(a_1,f_2))(x) &= \Big(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)T_m(a_1,f_2)(x)|^2 \chi_{(Q_1^*)^c}(x)\Big)^{1/2} \\ &\quad + \Big(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)T_m(a_1,f_2)(x)|^2 \chi_{Q_1^*}(x)\Big)^{1/2} \\ &\lesssim A\Big(\sum_{j\in\mathbb{Z}} u_j(x)^2\Big)^{1/2} M_q f_2(x) + AM_q a_1(x)\chi_{Q_1^*}(x)\Big(\sum_{j\in\mathbb{Z}} M_q(\psi'(D/2^j)f_2)(x)^2\Big)^{1/2} \\ &= A(u(x)v(x)+u'(x)v'(x)),\end{aligned}$$

where

$$u(x) = \Big(\sum_{j\in\mathbb{Z}} u_j(x)^2\Big)^{1/2}, \quad v(x) = M_q f_2(x),$$

$$u'(x) = M_q a_1(x)\chi_{Q_1^*}(x), \quad v'(x) = \Big(\sum_{j\in\mathbb{Z}} M_q(\psi'(D/2^j)f_2)(x)^2\Big)^{1/2}.$$

We have $\|u\|_{L^{p_1}} \lesssim 1$ as in (4.14) and, since $M_q(a_1)(x) \le \|a_1\|_{L^\infty} \le |Q_1|^{-1/p_1}$,

$$\|u'\|_{L^{p_1}} \le |Q_1|^{-1/p_1}\|\chi_{Q_1^*}\|_{L^{p_1}} \lesssim 1.$$

Since $q<2$, we have $\|v\|_{L^2} \lesssim \|f_2\|_{L^2}$ and, by the vector-valued maximal inequality of Fefferman–Stein, we also have

$$\|v'\|_{L^2} \lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}} |\psi'(D/2^j)f_2|^2\Big)^{1/2}\Big\|_{L^2} \lesssim \|f_2\|_{L^2}.$$

Thus we obtain (4.12) with $b_1 = u+u'$ and $b_2 = v+v'$. We shall now prove (4.13)–(4.14) and (4.15).

Proof of (4.13)–(4.14). Since $\operatorname{supp} m \subset \{|\xi_1| \le |\xi_2|/2\}$, if $(\xi_1,\xi_2) \in \operatorname{supp} m$, then $|\xi_1+\xi_2| \approx |\xi_2|$. Hence, we can find $\varphi \in \mathcal{A}_0$ and $\psi' \in \mathcal{A}_1$ independent of j such that

$$m(\xi_1,\xi_2)\,\psi((\xi_1+\xi_2)/2^j) = m(\xi_1,\xi_2)\,\psi((\xi_1+\xi_2)/2^j)\,\varphi(\xi_1/2^j)\,\psi'(\xi_2/2^j).$$

Then, we can write

$$\begin{aligned}&\psi(D/2^j)\,T_m(a_1,f_2)(x) \\ &\qquad = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)}\, m(\xi_1,\xi_2)\,\psi((\xi_1+\xi_2)/2^j)\,\widehat{a_1}(\xi_1)\,\widehat{f_2}(\xi_2)\,d\xi_1 d\xi_2 \\ &\qquad = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)}\, m_{(j)}(\xi_1/2^j,\xi_2/2^j)\,\widehat{a_1}(\xi_1)\,\widehat{f_2}(\xi_2)\,d\xi_1 d\xi_2 \\ &\qquad = T_{m_{(j)}(\cdot/2^j)}(a_1,f_2)(x),\end{aligned}$$

where

$$m_{(j)}(\xi_1,\xi_2) = m(2^j\xi_1,2^j\xi_2)\,\psi(\xi_1+\xi_2)\,\varphi(\xi_1)\,\psi'(\xi_2). \tag{4.16}$$

This representation says that $\psi(D/2^j)T_m(a_1, f_2)$ is essentially the same as the g_j appearing in the proof of (4.4). Therefore, we can prove (4.13) and (4.14) in the same way as we proved (4.6) and (4.7). Notice that the inequality

$$\sup_{j\in\mathbb{Z}} \|m_{(j)}\|_{W^{(s_1,s_2)}} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}} = A$$

follows from Lemma 2.4, where the m_j are defined by (1.5), and (4.14) follows from (4.7) since

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} u_j^2\Big)^{1/2}\Big\|_{L^{p_1}} \le \Big\|\Big(\sum_{j\in\mathbb{Z}} u_j^{p_1}\Big)^{1/p_1}\Big\|_{L^{p_1}} = \Big(\sum_{j\in\mathbb{Z}} \|u_j\|_{L^{p_1}}^{p_1}\Big)^{1/p_1}. \qquad \square$$

Proof of (4.15). It follows from the argument in the proof of (4.13)–(4.14) that there exists a $\psi' \in \mathcal{A}_1$ such that

$$m(\xi_1, \xi_2)\, \psi((\xi_1 + \xi_2)/2^j) = m_{(j)}(\xi_1/2^j, \xi_2/2^j)\, \psi'(\xi_2/2^j),$$

where $m_{(j)}$ is defined by (4.16). Hence,

$$\begin{aligned}
&\psi(D/2^j)T_m(a_1, f_2)(x)\\
&\quad= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} e^{ix\cdot(\xi_1+\xi_2)}\, m_{(j)}(\xi_1/2^j, \xi_2/2^j)\, \widehat{a_1}(\xi_1)\, \psi'(\xi_2/2^j)\, \widehat{f_2}(\xi_2)\, d\xi_1 d\xi_2\\
&\quad= T_{m_{(j)}(\cdot/2^j)}(a_1, \psi'(D/2^j) f_2)(x),
\end{aligned}$$

Since $\operatorname{supp} m_{(j)}$ is included in a compact subset independent of j, (4.15) follows from Lemma 2.2. $\square$

We finally consider $m^{(3)}$.

Estimate for $m^{(3)}$. By the same argument as in the case of $m^{(2)}$, it is sufficient to prove that for an H^{p_1}-atom a_1 and an L^2-function f_2 there exist a function b_1 depending only on a_1 and a function b_2 depending only on f_2 satisfying (4.12). To prove this, we consider $\psi(D/2^j)(T_m(a_1, f_2))$. By interchanging the roles of ξ_1 and ξ_2 in the argument for $m^{(2)}$, we obtain the same estimates (4.13)–(4.14) for the part on $(Q_1^*)^c$ and, for the part on Q_1^*, we obtain

$$|\psi(D/2^j)T_m(a_1, f_2)(x)|\, \chi_{Q_1^*}(x) \lesssim A M_q(\psi'(D/2^j)a_1)(x)\, M_q(f_2)(x). \tag{4.17}$$

As in the case of $m^{(2)}$, these estimates imply

$$G(T_m(a_1, f_2))(x) \lesssim A\big(u(x)\, v(x) + u'(x)\, v(x)\big),$$

with

$$u(x) = \Big(\sum_{j\in\mathbb{Z}} u_j(x)^2\Big)^{1/2}, \quad v(x) = M_q f_2(x),$$

$$u'(x) = \Big(\sum_{j\in\mathbb{Z}} M_q(\psi'(D/2^j)a_1)(x)^2\Big)^{1/2} \chi_{Q_1^*}(x).$$

We have $\|u\|_{L^{p_1}} \lesssim 1$ and $\|v\|_{L^2} \lesssim \|f_2\|_{L^2}$ for the same reason as in the case of $m^{(2)}$. As for u', we use Hölder's inequality and the vector-valued maximal inequality of Fefferman–Stein to obtain

$$\begin{aligned}\|u'\|_{L^{p_1}} &\le \Big\| \Big(\sum_{j\in\mathbb{Z}} M_q(\psi'(D/2^j)a_1)^2 \Big)^{1/2} \Big\|_{L^2} |\, Q_1^*|^{1/p_1-1/2} \\ &\lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} |\psi'(D/2^j)a_1(x)|^2 \Big)^{1/2} \Big\|_{L^2} |\, Q_1^*|^{1/p_1-1/2} \\ &\lesssim \|a_1\|_{L^2}\ |\, Q_1^*|^{1/p_1-1/2} \lesssim 1.\end{aligned}$$

Thus we obtain (4.12) with $b_1 = u + u'$ and $b_2 = v$. The proof of (4.1) is complete.

5. The boundedness from $L^\infty \times L^\infty$ to BMO

In this section, we shall prove Theorem 1.1 with $p_1 = p_2 = \infty$. That is, we show that

$$s_1 > n/2,\ s_2 > n/2 \implies \|T_m\|_{L^\infty\times L^\infty\to \mathrm{BMO}} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}. \tag{5.1}$$

To do this, we need the following lemma:

Lemma 5.1. *Let $s_1, s_2 > n/2$. Then*

$$\int_{\substack{|y_1|>2|x| \\ |y_2|>2|x|}} |K(x+y_1, x+y_2) - K(y_1,y_2)|\, dy_1dy_2 \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}$$

for all $x \in \mathbb{R}^n$, where $K = \mathcal{F}^{-1}m$ and m_j is defined by (1.5).

Proof. We have

$$\begin{aligned}&\int_{\substack{|y_1|>2|x| \\ |y_2|>2|x|}} \big|K(x+y_1,x+y_2) - K(y_1,y_2)\big| dy_1dy_2 \\ &\qquad\le \int_{\substack{|y_1|>2|x| \\ |y_2|>2|x|}} \big|K(x+y_1,x+y_2) - K(y_1,x+y_2)\big| dy_1dy_2 \\ &\qquad\quad + \int_{\substack{|y_1|>2|x| \\ |y_2|>2|x|}} \big|K(y_1,x+y_2) - K(y_1,y_2)\big| dy_1dy_2 \\ &\qquad\le \int_{\substack{|y_1|>2|x| \\ y_2\in\mathbb{R}^n}} \big|K(x+y_1,y_2) - K(y_1,y_2)\big| dy_1dy_2 \\ &\qquad\quad + \int_{\substack{y_1\in\mathbb{R}^n \\ |y_2|>2|x|}} \big|K(y_1,x+y_2) - K(y_1,y_2)\big| dy_1dy_2.\end{aligned}$$

We only consider the first term; the argument works for the second term as well.

Since

$$K(x_1,x_2)=\sum_{j\in\mathbb{Z}}2^{2jn}K_j(2^jx_1,2^jx_2),$$

where $K_j=\mathcal{F}^{-1}m_j$, we have

$$\begin{aligned}
&\int_{\substack{|y_1|>2|x|\\ y_2\in\mathbb{R}^n}}\big|K(x+y_1,y_2)-K(y_1,y_2)\big|dy_1dy_2\\
&\qquad\le\sum_{j\in\mathbb{Z}}2^{2jn}\int_{\substack{|y_1|>2|x|\\ y_2\in\mathbb{R}^n}}\big|K_j(2^j(x+y_1),2^jy_2)-K_j(2^jy_1,2^jy_2)\big|dy_1dy_2\\
&\qquad=\sum_{j\in\mathbb{Z}}2^{jn}\int_{\substack{|y_1|>2|x|\\ y_2\in\mathbb{R}^n}}\big|K_j(2^j(x+y_1),y_2)-K_j(2^jy_1,y_2)\big|dy_1dy_2.
\end{aligned}$$

Using $s_1,s_2>n/2$, we see that

$$\begin{aligned}
&2^{jn}\int_{\substack{|y_1|>2|x|\\ y_2\in\mathbb{R}^n}}\big|K_j(2^j(x+y_1),y_2)-K_j(2^jy_1,y_2)\big|\,dy_1dy_2\\
&\quad\le 2\cdot 2^{jn}\int_{\substack{|y_1|>|x|\\ y_2\in\mathbb{R}^n}}\big|K_j(2^jy_1,y_2)\big|\,dy_1dy_2=2\int_{\substack{|y_1|>2^j|x|\\ y_2\in\mathbb{R}^n}}|K_j(y_1,y_2)|\,dy_1dy_2\\
&\quad\le 2\Big(\int_{\substack{|y_1|>2^j|x|\\ y_2\in\mathbb{R}^n}}\langle y_1\rangle^{-2s_1}\langle y_2\rangle^{-2s_2}\,dy_1dy_2\Big)^{1/2}\|\langle y_1\rangle^{s_1}\langle y_2\rangle^{s_2}K_j(y_1,y_2)\|_{L^2_{y_1,y_2}}\\
&\quad\lesssim\Big(\sup_{k\in\mathbb{Z}}\|m_k\|_{W^{(s_1,s_2)}}\Big)(2^j|x|)^{-s_1+n/2}.
\end{aligned}$$

On the other hand, it follows from Taylor's formula and Remark 2.5 that

$$\begin{aligned}
&2^{jn}\int_{\substack{|y_1|>2|x|\\ y_2\in\mathbb{R}^n}}\big|K_j(2^j(x+y_1),y_2)-K_j(2^jy_1,y_2)\big|dy_1dy_2\\
&\quad=2^{jn}\int_{\substack{|y_1|>2|x|\\ y_2\in\mathbb{R}^n}}\Big|\sum_{|\alpha_1|=1}(2^jx)^{\alpha_1}\int_0^1K_j^{(\alpha_1,0)}(2^j(\theta_1x+y_1),y_2)d\theta_1\Big|dy_1dy_2\\
&\quad\le 2^j|x|\sum_{|\alpha_1|=1}\int_{\mathbb{R}^{2n}}\big|K_j^{(\alpha_1,0)}(y_1,y_2)\big|dy_1dy_2\\
&\quad\lesssim 2^j|x|\Big\|\langle y_1\rangle^{s_1}\langle y_2\rangle^{s_2}K_j^{(\alpha_1,0)}(y_1,y_2)\Big\|_{L^2_{y_1,y_2}}\lesssim\Big(\sup_{k\in\mathbb{Z}}\|m_k\|_{W^{(s_1,s_2)}}\Big)2^j|x|.
\end{aligned}$$

Combining these estimates, we have

$$\sum_{j\in\mathbb{Z}}2^{jn}\int_{\substack{|y_1|>2|x|\\ y_2\in\mathbb{R}^n}}\big|K_j(2^j(x+y_1),y_2)-K_j(2^jy_1,y_2)\big|\,dy_1dy_2\lesssim\sup_{k\in\mathbb{Z}}\|m_k\|_{W^{(s_1,s_2)}}.$$

This completes the proof. □

We are now ready to prove (5.1).

Proof of (5.1). We assume $s_1 > n/2$ and $s_2 > n/2$. Since

$$\|T_m(f_1, f_2)\|_{\mathrm{BMO}} \approx \sup_{Q} \inf_{a\in\mathbb{C}} \frac{1}{|Q|} \int_Q |T_m(f_1, f_2)(x) - a|\, dx,$$

it is sufficient to prove that for each cube Q there exists a constant $a_Q \in \mathbb{C}$ such that

$$\frac{1}{|Q|} \int_Q |T_m(f_1, f_2)(x) - a_Q|\, dx \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}} \|f_1\|_{L^\infty} \|f_2\|_{L^\infty}.$$

Given a cube Q, we denote by c its center, and set

$$a_Q = \int_{\substack{y_1\in(Q^*)^c \\ y_2\in(Q^*)^c}} K(c-y_1, c-y_2)\, f_1(y_1)\, f_2(y_2)\, dy_1 dy_2,$$
$$f_i^{(0)} = f_i\, \chi_{Q^*} \quad \text{and} \quad f_i^{(1)} = f_i\, \chi_{(Q^*)^c}, \quad i = 1, 2.$$

Then

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T_m(f_1, f_2)(x) - a_Q|\, dx \\ &\le \frac{1}{|Q|} \int_Q |T_m(f_1^{(0)}, f_2^{(0)})(x)|\, dx + \frac{1}{|Q|} \int_Q |T_m(f_1^{(1)}, f_2^{(0)})(x)|\, dx \\ &\quad + \frac{1}{|Q|} \int_Q |T_m(f_1^{(0)}, f_2^{(1)})(x)|\, dx + \frac{1}{|Q|} \int_Q |T_m(f_1^{(1)}, f_2^{(1)})(x) - a_Q|\, dx. \end{aligned} \tag{5.2}$$

Since $s_1, s_2 > n/2$, we have by (1.6)

$$\|T_m\|_{L^2\times L^\infty\to L^2} + \|T_m\|_{L^\infty\times L^2\to L^2} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}.$$

Using this L^2-estimate of T_m, we can estimate the first three terms in (5.2). In fact, the third term can be estimated as

$$\begin{aligned} \frac{1}{|Q|} \int_Q |T_m(f_1^{(0)}, f_2^{(1)})(x)|\, dx &\le |Q|^{-1/2} \|T_m(f_1^{(0)}, f_2^{(1)})\|_{L^2} \\ &\le |Q|^{-1/2} \|T_m\|_{L^2\times L^\infty\to L^2} \|f_1^{(0)}\|_{L^2} \|f_2^{(1)}\|_{L^\infty} \\ &\lesssim \big(\sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}\big) \|f_1\|_{L^\infty} \|f_2\|_{L^\infty}, \end{aligned}$$

and the first and the second terms can be estimated in the same way.

Let us consider the last term in (5.2). Since $|y_i - c| > 2|x - c|$ if $x \in Q$

and $y_i \in (Q^*)^c$, it follows from Lemma 5.1 that

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q |T_m(f_1^{(1)}, f_2^{(1)})(x) - a_Q|\,dx \\
&\quad = \frac{1}{|Q|}\int_Q \Big| \int_{\substack{y_1\in(Q^*)^c\\ y_2\in(Q^*)^c}} \big(K(x-y_1, x-y_2) - K(c-y_1, c-y_2)\big) f_1(y_1) f_2(y_2)\,dy_1 dy_2\Big|\,dx \\
&\quad \le \frac{\|f_1\|_{L^\infty}\|f_2\|_{L^\infty}}{|Q|}\int_Q \Big(\int_{\substack{|y_1-c|>2|x-c|\\ |y_2-c|>2|x-c|}} \big|K(x-y_1, x-y_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - K(c-y_1, c-y_2)\big|\,dy_1 dy_2\Big) dx \\
&\quad = \frac{\|f_1\|_{L^\infty}\|f_2\|_{L^\infty}}{|Q|}\int_Q \Big(\int_{\substack{|y_1|>2|x-c|\\ |y_2|>2|x-c|}} \big|K(x-c+y_1, x-c+y_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - K(y_1, y_2)\big|\,dy_1 dy_2\Big) dx \\
&\quad \lesssim \big(\sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}\big)\|f_1\|_{L^\infty}\|f_2\|_{L^\infty}.
\end{aligned}
$$

The proof of (5.1) is complete. □

6. Completion of the proof of Theorem 1.1

In Sections 3–5, we have proved the following:

(3.1)–(3.2) for $1/p_1 \ge 1$, $1/p_2 \ge 1$;
(4.1) for $1/p_1 \ge 1$, $1/p_2 = 1/2$;
(4.2) for $1/p_1 = 1/2$, $1/p_2 \ge 1$;
(5.1) for $1/p_1 = 1/p_2 = 0$.

Recall that Theorem 1.2 of [8] gives the following: for $0 < p \le 1$,

$$s_1 > n/p - n/2,\ s_2 > n/2 \implies \|T_m\|_{H^p\times L^\infty\to L^p} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}, \tag{6.1}$$

$$s_1 > n/2,\ s_2 > n/p - n/2 \implies \|T_m\|_{L^\infty\times H^p\to L^p} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}}. \tag{6.2}$$

Notice that these are exactly the assertions of Theorem 1.1 for $(1/p_1, 1/p_2)$ in the respective ranges.

The assertions of Theorem 1.1 for I_0, I_1, and I_2 are derived from (4.1), (4.2), (5.1), (6.1), and (6.2) by means of interpolation. For this, it is sufficient to use the usual real or complex interpolation for bilinear operators in H^p and L^p spaces. In fact, the interpolation theorem for bilinear operator is necessary only to obtain the results for $(1/p_1, 1/p_2)$ on the line segment joining $(1/2, 1)$ and $(1, 1/2)$. In other parts of I_0, I_1, and I_2, it is sufficient to apply interpolation for linear operators to the linear operators obtained from $T_m(f_1, f_2)$ by freezing f_1 or f_2.

The assertion for I_6 is nothing but (3.1)–(3.2).

There remain the assertions for I_3, I_4, and I_5. To prove these assertions, we use the following lemma:

Lemma 6.1. *The set of points* $(1/p_1, 1/p_2, s_1, s_2) \in (0,\infty)^4$ *for which the estimate* (1.8) *holds is convex.*

This lemma can be proved by the use of the interpolation theorem for analytic families of operators (Stein–Weiss [19]) and the results for complex interpolation spaces between H^p and L^p spaces (see Janson–Jones [13]). For details, see Section 6 of [8].

By using Lemma 6.1, we can deduce the assertions of Theorem 1.1 for I_3, I_4, and I_5 from (3.1)–(3.2), (4.1), and (4.2). To prove the assertion for I_3, for example, consider the sets:

$$\begin{aligned}\mathcal{E} &= \big\{(1/p_1, 1/p_2, s_1, s_2) \in (0,\infty)^4 \mid (1/p_1, 1/p_2) \in I_3, \\ &\qquad\qquad s_1 > n/2,\ s_2 > n/2,\ s_1 + s_2 > n/p_1 + n/p_2 - n/2\big\}, \\ \mathcal{E}_0 &= \big\{(1/p_1, 1/p_2, s_1, s_2) \in \mathcal{E} \mid (1/p_1, 1/p_2) = (1,1) \text{ or } (1,1/2) \text{ or } (1/2,1)\big\}.\end{aligned}$$

The assertions (3.1)–(3.2), (4.1), and (4.2) imply that the estimate (1.8) holds for $(1/p_1, 1/p_2, s_1, s_2) \in \mathcal{E}_0$. It is easy to check that $\mathcal{E}$ is the convex hull of $\mathcal{E}_0$. Hence by Lemma 6.1, (1.8) holds for all $(1/p_1, 1/p_2, s_1, s_2) \in \mathcal{E}$, which is the assertion of Theorem 1.1 for $(1/p_1, 1/p_2) \in I_3$. The proofs for I_4 and I_5 are similar. This completes the proof of Theorem 1.1.

7. Sharpness of the conditions of Theorem 1.1

In this section, we shall prove Theorem 1.2. We assume that $0 < p_1, p_2, p \le \infty$, $1/p_1 + 1/p_2 = 1/p$, $s_1, s_2 > 0$, and the estimate

$$\|T_m(f_1, f_2)\|_{L^p} \lesssim \sup_{j\in\mathbb{Z}} \|m_j\|_{W^{(s_1,s_2)}} \|f_1\|_{H^{p_1}} \|f_2\|_{H^{p_2}} \tag{7.1}$$

holds, where L^p should be replaced by BMO in the case $p = \infty$, and we shall prove

$$s_1 \ge \max\Big\{\frac{n}{2}, \frac{n}{p_1} - \frac{n}{2}\Big\}, \qquad s_2 \ge \max\Big\{\frac{n}{2}, \frac{n}{p_2} - \frac{n}{2}\Big\} \tag{7.2}$$

and

$$s_1 + s_2 \ge \frac{n}{p_1} + \frac{n}{p_2} - \frac{n}{2}. \tag{7.3}$$

Before proving (7.2), we make the following remark:

Remark 7.1. If $f \in \mathcal{S}(\mathbb{R}^n)$ is a function with $\operatorname{supp} \widehat{f} \subset \{2^{-j_0} \le |\xi| \le 2^{j_0}\}$, then $C^{-1}\|f\|_{L^p} \le \|f\|_{H^p} \le C\|f\|_{L^p}$, where $C > 0$ depends only on j_0 and p. A proof goes as follows. In the case $p > 1$, this equivalence is obvious since $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

Suppose $p \le 1$. It is sufficient to prove $\|G(f)\|_{L^p} \le C\|f\|_{L^p}$ (see Section 4 for the definition of $G(f)$). By the condition on the support of $\widehat{f}$,

$$G(f)(x) = \Big(\sum_{j=-j_0}^{j_0} |\psi(D/2^j)\, f(x)|^2 \Big)^{1/2}.$$

On the other hand, it is known that there exists a constant $C = C_{j_0,p} > 0$ such that

$$\|g * h\|_{L^p} \le C\, \|g\|_{L^p}\, \|h\|_{L^p}$$

for all $g, h \in L^p(\mathbb{R}^n)$ with $\operatorname{supp} \widehat{g}$, $\operatorname{supp} \widehat{h} \subset \{|\xi| \le 2^{j_0+1}\}$ (Proposition 1.5.3 of [21]). These imply that $\|G(f)\|_{L^p} \le C\|f\|_{L^p}$.

We first prove the necessity of the condition (7.2).

Proof of (7.2). Our proof is based on the idea given in Section 7 of [8]. From the inequality (7.1), we shall deduce $s_1 \ge \max\{n/2, n/p_1 - n/2\}$. Interchanging the roles of ξ_1 and ξ_2 in our argument below, we can also prove $s_2 \ge \max\{n/2, n/p_2 - n/2\}$. First, we additionally assume that $p < \infty$.

Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ be such that

$$\begin{aligned}
&\operatorname{supp} \widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 1\},\\
&\operatorname{supp} \widehat{\psi} \subset \{\xi \in \mathbb{R}^n : 9/10 \le |\xi| \le 11/10\},\\
&\widehat{\psi}(\xi) = 1 \quad \text{if } 19/20 \le |\xi| \le 21/20.
\end{aligned}$$

Take a point ζ° in $\mathbb{R}^n$ satisfying $|\zeta^\circ| = 1/10$, and set, for sufficiently small $\epsilon > 0$,

$$m^{(\epsilon)}(\xi_1, \xi_2) = \widehat{\varphi}((\xi_1 - \zeta^\circ)/\epsilon)\, \widehat{\psi}(\xi_2).$$

For this $m^{(\epsilon)}$, we have

$$T_{m^{(\epsilon)}}(f_1, f_2)(x) = \mathcal{F}^{-1}\big[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\widehat{f_1}\big](x)\, \mathcal{F}^{-1}\big[\widehat{\psi}\, \widehat{f_2}\big](x),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on $\mathbb{R}^n$. Thus the inequality (7.1) implies

$$\big\|\mathcal{F}^{-1}\big[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\, \widehat{f_1}\big]\, \mathcal{F}^{-1}\big[\widehat{\psi}\widehat{f_2}\big]\, \big\|_{L^p} \lesssim \sup_{j \in \mathbb{Z}} \|m_j^{(\epsilon)}\|_{W^{(s_1,s_2)}}\, \|f_1\|_{H^{p_1}}\, \|f_2\|_{H^{p_2}}, \tag{7.4}$$

where $m_j^{(\epsilon)}$ is defined by (1.5) with m replaced by $m^{(\epsilon)}$.

To estimate the norm $\|m_j^{(\epsilon)}\|_{W^{(s_1,s_2)}}$, we choose the function $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$, which appeared in the definition of m_j, so that we have

$$\begin{aligned}
&\operatorname{supp} \Psi \subset \{\xi \in \mathbb{R}^{2n} : 2^{-1/2-\alpha} \le |\xi| \le 2^{1/2+\alpha}\},\\
&\Psi(\xi) = 1 \quad \text{if } 2^{-1/2+\alpha} \le |\xi| \le 2^{1/2-\alpha},
\end{aligned}$$

where $\alpha > 0$ is a sufficiently small number. If $\epsilon > 0$ is sufficiently small, then

$$\begin{aligned}\operatorname{supp} m^{(\epsilon)} &\subset \big\{(\xi_1, \xi_2) \in \mathbb{R}^{2n} : |\xi_1 - \zeta^\circ| \le \epsilon,\ 9/10 \le |\xi_2| \le 11/10\big\} \\ &\subset \big\{(\xi_1, \xi_2) \in \mathbb{R}^{2n} : 2^{-1/2+\alpha} \le |(\xi_1, \xi_2)| \le 2^{1/2-\alpha}\big\}.\end{aligned}$$

This implies

$$m_j^{(\epsilon)}(\xi) = m^{(\epsilon)}(2^j \xi)\, \Psi(\xi) = \begin{cases} m^{(\epsilon)}(\xi) & \text{if } j = 0, \\ 0 & \text{if } j \ne 0, \end{cases}$$

and consequently

$$\begin{aligned}\sup_{j\in\mathbb{Z}} \|m_j^{(\epsilon)}\|_{W^{(s_1,s_2)}} = \|m^{(\epsilon)}\|_{W^{(s_1,s_2)}} &= \|\widehat{\varphi}((\xi_1 - \zeta^\circ)/\epsilon)\, \widehat{\psi}(\xi_2)\|_{W^{(s_1,s_2)}} \\ &= \|\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\|_{W^{s_1}}\, \|\widehat{\psi}\|_{W^{s_2}}.\end{aligned}$$

Let $N > 0$ be large enough. Then

$$\begin{aligned}\|\widehat{\varphi}((\cdot{-}\zeta^\circ)/\epsilon)\|_{W^{s_1}} &= \|\epsilon^n \varphi(\epsilon x)\langle x\rangle^{s_1}\|_{L^2} \\ &\lesssim \epsilon^n \Big(\int_{\mathbb{R}^n} (1+|x|)^{2s_1}(1+\epsilon|x|)^{-2N}\, dx\Big)^{1/2} \\ &\approx \epsilon^n \Big(\int_{|x|\le 1} dx + \int_{1<|x|\le 1/\epsilon} |x|^{2s_1}\, dx + \int_{1/\epsilon<|x|<\infty} |x|^{2s_1}(\epsilon|x|)^{-2N}\, dx\Big)^{1/2} \\ &\approx \epsilon^{-s_1+n/2}.\end{aligned}$$

Hence, by (7.4),

$$\big\|\mathcal{F}^{-1}\big[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\, \widehat{f_1}\big]\, \mathcal{F}^{-1}\big[\widehat{\psi}\, \widehat{f_2}\big]\,\big\|_{L^p} \lesssim \epsilon^{-s_1+n/2}\, \|f_1\|_{H^{p_1}}\, \|f_2\|_{H^{p_2}}. \tag{7.5}$$

To obtain $s_1 \ge n/2$, we test (7.5) for

$$\widehat{f_1}(\xi_1) = \epsilon^{n/p_1 - n}\, \widehat{\varphi}((\xi_1 - \zeta^\circ)/\epsilon) \quad \text{and} \quad \widehat{f_2}(\xi_2) = \epsilon^{n/p_2 - n}\, \widehat{\varphi}((\xi_2 - e_1)/\epsilon),$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Since $\operatorname{supp} \widehat{f_1}$ and $\operatorname{supp} \widehat{f_2}$ are included in compact subsets of $\mathbb{R}^n \setminus \{0\}$ which are independent of ϵ, it follows from Remark 7.1 that

$$(\text{the right-hand side of (7.5)}) \approx \epsilon^{-s_1+n/2}\, \|f_1\|_{L^{p_1}}\, \|f_2\|_{L^{p_2}} = C\, \epsilon^{-s_1+n/2}.$$

On the other hand, since

$$\begin{aligned}\mathcal{F}^{-1}\big[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\, \widehat{f_1}\big](x)\, \mathcal{F}^{-1}\big[\widehat{\psi}\, \widehat{f_2}\big](x) &= \mathcal{F}^{-1}\big[\epsilon^{n/p_1 - n}\, \widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)^2\big](x)\, \mathcal{F}^{-1}\big[\widehat{f_2}\big](x) \\ &= \epsilon^{n/p_1}\, e^{i\zeta^\circ\cdot x}\, \varphi * \varphi(\epsilon x)\, \epsilon^{n/p_2} e^{ie_1\cdot x}\, \varphi(\epsilon x),\end{aligned}$$

we have

$$(\text{the left-hand side of (7.5)}) = \epsilon^{n/p_1 + n/p_2}\, \|\varphi * \varphi(\epsilon\cdot)\, \varphi(\epsilon\cdot)\|_{L^p} = C.$$

Hence, $1 \lesssim \epsilon^{-s_1+n/2}$ and $s_1 \ge n/2$.

To obtain $s_1 \geq n/p_1 - n/2$, we test (7.5) for

$$\widehat{f_1}(\xi_1) = \widehat{\psi}'(\xi_1) \quad \text{and} \quad \widehat{f_2}(\xi_2) = \epsilon^{n/p_2 - n}\,\widehat{\varphi}((\xi_2 - e_1)/\epsilon),$$

where $\psi' \in \mathcal{S}(\mathbb{R}^n)$ is chosen so that $\operatorname{supp}\widehat{\psi}'$ is a compact subset of $\mathbb{R}^n \setminus \{0\}$ and $\widehat{\psi}' = 1$ in a neighborhood of ζ°. It follows from Remark 7.1 that

$$(\text{the right-hand side of (7.5)}) \approx \epsilon^{-s_1+n/2}\,\|f_1\|_{L^{p_1}}\,\|f_2\|_{L^{p_2}} = C\,\epsilon^{-s_1+n/2}.$$

On the other hand, since

$$\begin{aligned}\mathcal{F}^{-1}\big[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\,\widehat{f_1}\big](x)\,\mathcal{F}^{-1}\big[\widehat{\psi}\,\widehat{f_2}\big](x) &= \mathcal{F}^{-1}\big[\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\big](x)\,\mathcal{F}^{-1}\big[\widehat{f_2}\big](x)\\ &= \epsilon^n\,e^{i\zeta^\circ\cdot x}\,\varphi(\epsilon x)\,\epsilon^{n/p_2}\,e^{ie_1\cdot x}\varphi(\epsilon x),\end{aligned}$$

we have

$$(\text{the left-hand side of (7.5)}) = \epsilon^{n+n/p_2}\,\|\varphi(\epsilon\cdot)^2\|_{L^p} = C\,\epsilon^{n-n/p_1}.$$

Therefore, $\epsilon^{n-n/p_1} \lesssim \epsilon^{-s_1+n/2}$ and $s_1 \geq n/p_1 - n/2$.

Since $|||f|||_{\mathrm{BMO}} \lesssim \|f\|_{\mathrm{BMO}}$ and $\|f(\epsilon\cdot)\|_{\mathrm{BMO}} = \|f\|_{\mathrm{BMO}}$, our argument above works for the case $p = \infty$ as well. □

Remark 7.2. For the multiplier $m^{(\epsilon)}$ of the above proof, we actually have

$$\|m^{(\epsilon)}\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} \approx \epsilon^{-s_1+n/2}.$$

The estimate $\|m^{(\epsilon)}\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} \lesssim \epsilon^{-s_1+n/2}$ has been proved above. To see the converse estimate, take a point $x_0 \in \mathbb{R}^n \setminus \{0\}$ and a number δ such that $0 < \delta < |x_0|/2$ and $|\varphi(x)| > \delta$ for $|x - x_0| < \delta$. Then, for sufficiently small $\epsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n}\big|\epsilon^n\varphi(\epsilon x)\langle x\rangle^{s_1}\big|^2 dx \geq \int_{|\epsilon x - x_0|\leq\delta}\big\{\epsilon^n\delta|x|^{s_1}\big\}^2 dx &\approx \int_{|x - x_0/\epsilon|\leq\delta/\epsilon}\Big\{\epsilon^n\delta\Big(\frac{|x_0|}{\epsilon}\Big)^{s_1}\Big\}^2 dx\\ &\approx \big\{\epsilon^{n-s_1}\big\}^2\epsilon^{-n} = \epsilon^{n-2s_1}\end{aligned}$$

and consequently

$$\|m^{(\epsilon)}\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})} \approx \|\widehat{\varphi}((\cdot - \zeta^\circ)/\epsilon)\|_{W^{s_1}(\mathbb{R}^n)} = \|\epsilon^n\,\varphi(\epsilon x)\,\langle x\rangle^{s_1}\|_{L^2} \gtrsim \epsilon^{-s_1+n/2}.$$

We next prove the necessity of the condition (7.3).

Proof of (7.3). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi(0) \neq 0, \quad \operatorname{supp}\widehat{\varphi} \subset \{|\xi| \leq 1/10\}, \quad \widehat{\varphi}(\xi) = 1 \quad \text{if } |\xi| \leq 1/20.$$

Take a point ζ° in $\mathbb{R}^n$ satisfying $|\zeta^\circ| = \sqrt{2}$, and set, for sufficiently small $\epsilon > 0$,

$$m^{(\epsilon)}(\xi_1, \xi_2) = \widehat{\varphi}\Big(\frac{\xi_1 + \xi_2 - \zeta^\circ}{\epsilon}\Big)\,\widehat{\varphi}(\xi_1 - \xi_2).$$

Note that

$$\begin{aligned}\operatorname{supp} m^{(\epsilon)} &\subset \Big\{|\xi_1+\xi_2-\zeta^\circ| \le \frac{\epsilon}{10},\ \ |\xi_1-\xi_2| \le \frac{1}{10}\Big\}\\ &\subset \Big\{\Big|\xi_1-\frac{\zeta^\circ}{2}\Big| \le \frac{\epsilon}{20}+\frac{1}{20},\ \ \Big|\xi_2-\frac{\zeta^\circ}{2}\Big| \le \frac{\epsilon}{20}+\frac{1}{20}\Big\}\\ &\subset \Big\{1-\frac{\epsilon}{10}-\frac{1}{10} \le |(\xi_1,\xi_2)| \le 1+\frac{\epsilon}{10}+\frac{1}{10}\Big\}\end{aligned}$$

and

$$\begin{aligned}&\mathcal{F}^{-1}(m^{(\epsilon)})(x_1,x_2)\\ &\quad= \frac{1}{(2\pi)^{2n}}\iint \widehat{\varphi}\Big(\frac{\xi_1+\xi_2-\zeta^\circ}{\epsilon}\Big)\,\widehat{\varphi}(\xi_1-\xi_2)\,\exp\{i(x_1\cdot\xi_1+x_2\cdot\xi_2)\}\,d\xi_1 d\xi_2\\ &\quad= c\iint \widehat{\varphi}\Big(\frac{\eta_1-\zeta^\circ}{\epsilon}\Big)\,\widehat{\varphi}(\eta_2)\,\exp\Big\{i\Big(x_1\cdot\frac{\eta_1+\eta_2}{2}+x_2\cdot\frac{\eta_1-\eta_2}{2}\Big)\Big\}\,d\eta_1 d\eta_2\\ &\quad= c\exp\Big(i\zeta^\circ\cdot\frac{x_1+x_2}{2}\Big)\,\epsilon^n\,\varphi\Big(\epsilon\frac{x_1+x_2}{2}\Big)\,\varphi\Big(\frac{x_1-x_2}{2}\Big).\end{aligned}$$

Since $\operatorname{supp} m^{(\epsilon)} \subset \{2^{-1/2+\alpha} < |(\xi_1,\xi_2)| < 2^{1/2-\alpha}\}$ for sufficiently small $\epsilon > 0$, it follows from the argument used in the proof of (7.2) that

$$\sup_{j\in\mathbb{Z}} \|m_j^{(\epsilon)}\|_{W^{(s_1,s_2)}} = \|m^{(\epsilon)}\|_{W^{(s_1,s_2)}}, \tag{7.6}$$

where $m_j^{(\epsilon)}$ is defined by (1.5) with m replaced by $m^{(\epsilon)}$. In order to obtain $s_1+s_2 \ge n/p_1+n/p_2-n/2$, we shall prove that

$$\|m^{(\epsilon)}\|_{W^{(s_1,s_2)}} = c\,\Big\|\epsilon^n\varphi\Big(\epsilon\frac{x_1+x_2}{2}\Big)\varphi\Big(\frac{x_1-x_2}{2}\Big)\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}\Big\|_{L^2} \lesssim \epsilon^{\frac{n}{2}-s_1-s_2} \tag{7.7}$$

for $s_1, s_2 > 0$.

Before proving (7.7), let us observe that this implies the desired result. Take a function $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}\widehat{f} \subset \Big\{\Big|\xi-\frac{\zeta^\circ}{2}\Big| \le \frac{2}{10}\Big\}, \quad \widehat{f}(\xi)=1 \quad \text{if } \Big|\xi-\frac{\zeta^\circ}{2}\Big| \le \frac{1}{10}.$$

Since $\widehat{f}(\xi_1)\,\widehat{f}(\xi_2)=1$ on $\operatorname{supp} m^{(\epsilon)}(\xi_1,\xi_2)$, we have

$$T_{m^{(\epsilon)}}(f,f)(x) = \mathcal{F}^{-1}(m^{(\epsilon)})(x,x) = c\,\exp(i\zeta^\circ\cdot x)\,\epsilon^n\,\varphi(\epsilon x)\,\varphi(0),$$

and hence

$$\|T_{m^{(\epsilon)}}(f,f)\|_{L^p} = c\,\|\epsilon^n\,\varphi(\epsilon x)\,\varphi(0)\|_{L^p} = C\,\epsilon^{n-n/p}. \tag{7.8}$$

On the other hand, since $\operatorname{supp}\widehat{f} \subset \mathbb{R}^n\setminus\{0\}$, we see that $f_i \in H^{p_i}(\mathbb{R}^n)$, $i=1,2$. Hence, it follows from (7.1) with $m=m^{(\epsilon)}$ and $f_1=f_2=f$ and from (7.6), (7.7) and (7.8) that

$$\epsilon^{n-\frac{n}{p}} \lesssim \epsilon^{\frac{n}{2}-s_1-s_2},$$

and consequently $s_1+s_2 \ge n/p-n/2 = n/p_1+n/p_2-n/2$.

We shall prove (7.7), that is,

$$\iint \Big| \epsilon^n \varphi\Big(\epsilon \frac{x_1+x_2}{2}\Big) \varphi\Big(\frac{x_1-x_2}{2}\Big) \langle x_1\rangle^{s_1} \langle x_2\rangle^{s_2} \Big|^2 dx_1 dx_2 \lesssim \epsilon^{n-2s_1-2s_2}. \tag{7.9}$$

Let $N>0$ be large enough. Then the left-hand side of (7.9) is majorized by

$$\begin{aligned}
&\iint \left\{\epsilon^n (1+\epsilon|x_1+x_2|)^{-N} (1+|x_1-x_2|)^{-N} \langle x_1\rangle^{s_1} \langle x_2\rangle^{s_2}\right\}^2 dx_1 dx_2 \\
&\qquad \approx \iint \left\{\epsilon^n (1+\epsilon|y_1|)^{-N} (1+|y_2|)^{-N} \langle y_1+y_2\rangle^{s_1} \langle y_1-y_2\rangle^{s_2}\right\}^2 dy_1 dy_2 \\
&\qquad \approx \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left\{\epsilon^n (1+2^j\epsilon)^{-N} (2^k)^{-N}\right\}^2 \\
&\qquad\quad \times \iint_{\substack{2^j<|y_1|<2^{j+1},\\ 2^k<|y_2|<2^{k+1}}} \langle y_1+y_2\rangle^{2s_1} \langle y_1-y_2\rangle^{2s_2}\, dy_1 dy_2 \\
&\qquad = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} I_{j,k},
\end{aligned}$$

where we replace $\int_{2^j<|y_1|<2^{j+1}}$ (respectively, $\int_{2^k<|y_2|<2^{k+1}}$) by $\int_{|y_1|<2}$ (respectively, $\int_{|y_2|<2}$) if $j=0$ (respectively, $k=0$). We assume ϵ is sufficiently small, say $4\epsilon<1$.

To estimate $I_{j,k}$, we divide (j,k) into six classes.

For (j,k) satisfying $j\ge k+2$ and $2^j\epsilon>1$, we have

$$\begin{aligned}
I_{j,k} &\approx \{\epsilon^n (2^j\epsilon)^{-N} (2^k)^{-N}\}^2\, 2^{j\cdot 2s_1}\, 2^{j\cdot 2s_2}\, 2^{jn}\, 2^{kn} \\
&= \epsilon^{2n-2N}\, 2^{j(-2N+n+2s_1+2s_2)}\, 2^{k(-2N+n)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \chi\{j\ge k+2,\ 2^j\epsilon>1\} I_{j,k} &\approx \sum_{j=0}^{\infty} \chi\{2^j\epsilon>1\} \epsilon^{2n-2N} 2^{j(-2N+n+2s_1+2s_2)} \\
&\approx \epsilon^{2n-2N} \epsilon^{-(-2N+n+2s_1+2s_2)} = \epsilon^{n-2s_1-2s_2}.
\end{aligned}$$

For (j,k) satisfying $j\le k-2$ and $2^j\epsilon>1$, we have

$$\begin{aligned}
I_{j,k} &\approx \{\epsilon^n (2^j\epsilon)^{-N}\, (2^k)^{-N}\}^2\, 2^{k\cdot 2s_1}\, 2^{k\cdot 2s_2}\, 2^{jn}\, 2^{kn} \\
&= \epsilon^{2n-2N}\, 2^{j(-2N+n)}\, 2^{k(-2N+n+2s_1+2s_2)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \chi\{j\le k-2,\ 2^j\epsilon>1\} I_{j,k} &\approx \sum_{j=0}^{\infty} \chi\{2^j\epsilon>1\} \epsilon^{2n-2N} 2^{j(-4N+2n+2s_1+2s_2)} \\
&\approx \epsilon^{2n-2N} \epsilon^{-(-4N+2n+2s_1+2s_2)} \\
&= \epsilon^{2N-2s_1-2s_2} < \epsilon^{n-2s_1-2s_2}.
\end{aligned}$$

For (j,k) satisfying $k-2<j<k+2$ and $2^j\epsilon>1$, we have

$$I_{j,k}\approx\{\epsilon^n(2^j\epsilon)^{-N}\,(2^j)^{-N}\}^2\,2^{j\cdot 2s_1}\,2^{j\cdot 2s_2}\,2^{j\cdot 2n}=\epsilon^{2n-2N}\,2^{j(-4N+2n+2s_1+2s_2)}.$$

Hence

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\chi\{k-2<j\ <k+2,2^j\epsilon>1\}I_{j,k}&\\ &\approx\sum_{j=0}^{\infty}\chi\{2^j\epsilon>1\}\,\epsilon^{2n-2N}\,2^{j(-4N+2n+2s_1+2s_2)}\\ &\approx\epsilon^{2n-2N}\,\epsilon^{-(-4N+2n+2s_1+2s_2)}\\ &=\epsilon^{2N-2s_1-2s_2}<\epsilon^{n-2s_1-2s_2}.\end{aligned}$$

For (j,k) satisfying $j\geq k+2$ and $2^j\epsilon\leq 1$, we have

$$I_{j,k}\approx\{\epsilon^n(2^k)^{-N}\}^2\,2^{j\cdot 2s_1}\,2^{j\cdot 2s_2}\,2^{jn}\,2^{kn}=\epsilon^{2n}\,2^{j(n+2s_1+2s_2)}\,2^{k(-2N+n)}.$$

Hence

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\chi\{j\geq k+2,\ 2^j\epsilon\leq 1\}I_{j,k}&\approx\sum_{j=0}^{\infty}\chi\{j\geq 2,\ 2^j\epsilon\leq 1\}\,\epsilon^{2n}\,2^{j(n+2s_1+2s_2)}\\ &\approx\epsilon^{2n}\,\epsilon^{-(n+2s_1+2s_2)}=\epsilon^{n-2s_1-2s_2}.\end{aligned}$$

For (j,k) satisfying $j\leq k-2$ and $2^j\epsilon\leq 1$, we have

$$I_{j,k}\approx\{\epsilon^n(2^k)^{-N}\}^2\,2^{k\cdot 2s_1}\,2^{k\cdot 2s_2}\,2^{jn}\,2^{kn}=\epsilon^{2n}\,2^{jn}\,2^{k(-2N+n+2s_1+2s_2)}.$$

Hence

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\chi\{j\leq k-2,\ 2^j\epsilon\leq 1\}I_{j,k}&\approx\sum_{j=0}^{\infty}\chi\{2^j\epsilon\leq 1\}\,\epsilon^{2n}\,2^{j(-2N+2n+2s_1+2s_2)}\\ &\approx\epsilon^{2n}<\epsilon^{n-2s_1-2s_2}.\end{aligned}$$

Finally, for (j,k) satisfying $k-2<j<k+2$ and $2^j\epsilon\leq 1$, we have

$$I_{j,k}\approx\{\epsilon^n(2^j)^{-N}\}^2\,2^{j\cdot 2s_1}\,2^{j\cdot 2s_2}\,2^{j\cdot 2n}=\epsilon^{2n}\,2^{j(-2N+2n+2s_1+2s_2)}.$$

Hence

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\chi\{k-2<j<k+2,\ 2^j\epsilon\leq 1\}I_{j,k}&\approx\sum_{j=0}^{\infty}\chi\{2^j\epsilon\leq 1\}\,\epsilon^{2n}\,2^{j(-2N+2n+2s_1+2s_2)}\\ &\approx\epsilon^{2n}<\epsilon^{n-2s_1-2s_2}.\end{aligned}$$

This completes the proof of Theorem 1.2. $\square$

Remark 7.3. In the estimate (7.9), $\lesssim$ can be replaced by $\approx$. In fact, taking $\delta > 0$ such that $|\varphi(x)| \geq |\varphi(0)|/2 > 0$ if $|x| \leq \delta$, we have

$$\begin{aligned}
&\iint \Big|\epsilon^n \varphi\Big(\epsilon\frac{x_1+x_2}{2}\Big)\varphi\Big(\frac{x_1-x_2}{2}\Big)\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}\Big|^2 dx_1dx_2 \\
&\qquad\gtrsim \iint \Big\{\epsilon^n \chi\{\epsilon|x_1+x_2|\leq\delta\}\chi\{|x_1-x_2|\leq\delta\}\langle x_1\rangle^{s_1}\langle x_2\rangle^{s_2}\Big\}^2 dx_1dx_2 \\
&\qquad\approx \iint \Big\{\epsilon^n \chi\{\epsilon|y_1|\leq\delta\}\chi\{|y_2|\leq\delta\}\langle y_1+y_2\rangle^{s_1}\langle y_1-y_2\rangle^{s_2}\Big\}^2 dy_1dy_2 \\
&\qquad\gtrsim \iint \Big\{\epsilon^n \chi\{\delta/2\leq\epsilon|y_1|\leq\delta\}\chi\{|y_2|\leq\delta\}\Big(\frac{\delta}{\epsilon}\Big)^{s_1}\Big(\frac{\delta}{\epsilon}\Big)^{s_2}\Big\}^2 dy_1dy_2 \\
&\qquad\approx \big\{\epsilon^{n-s_1-s_2}\big\}^2\epsilon^{-n} = \epsilon^{n-2s_1-2s_2}.
\end{aligned}$$

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