Rev. Mat. Iberoam. **29** (2013), no. 3, 749–764
DOI 10.4171/RMI/738

The twisting representation of the L-function of a curve

Francesc Fité and Joan-C. Lario

Abstract. Let C be a smooth projective curve defined over a number field and let C' be a twist of C. In this article we relate the ℓ-adic representations attached to the ℓ-adic Tate modules of the Jacobians of C and C' through an Artin representation. This representation induces *global* relations between the local factors of the respective Hasse–Weil L-functions. We make these relations explicit in a particularly illustrative situation. For all but a finite number of $\overline{\mathbb{Q}}$-isomorphism classes of genus 2 curves defined over $\mathbb{Q}$ with $\mathrm{Aut}(C) \simeq D_8$ or D_{12}, we find a representative curve $C/\mathbb{Q}$ such that, for every isomorphism $\phi\colon C' \to C$ satisfying some mild condition, we are able to determine either the local factor $L_p(C'/\mathbb{Q}, T)$ or the product $L_p(C'/\mathbb{Q}, T) \cdot L_p(C'/\mathbb{Q}, -T)$ from the local factor $L_p(C/\mathbb{Q}, T)$.

1. Introduction

Let C and C' be smooth projective curves of genus $g \geq 1$ defined over a number field k that become isomorphic over an algebraic closure of k (that is, they are *twists* of each other). The aim of this article is to relate the ℓ-adic representations attached to the $\mathbb{Q}_\ell$-vector spaces $V_\ell(C)$ and $V_\ell(C')$. Here, for a prime ℓ, $V_\ell(C)$ stands for $\mathbb{Q}_\ell \otimes T_\ell(C)$, where $T_\ell(C)$ denotes the ℓ-adic Tate module of the Jacobian variety $J(C)$ attached to C (and similarly for C').

The case of quadratic twists of elliptic curves is well known. If E and E' are elliptic curves defined over k that become isomorphic over a quadratic extension L/k, then there exists a character χ of $\mathrm{Gal}(L/k)$ such that

$$V_\ell(E') \simeq \chi \otimes V_\ell(E)\,. \tag{1.1}$$

This translates into a relation of local factors of the corresponding Hasse–Weil L-functions. Indeed, one has that, for every prime $\mathfrak{p}$ of k unramified in L,

$$L_\mathfrak{p}(E'/k, T) = L_\mathfrak{p}(E/k, \chi(\mathrm{Frob}_\mathfrak{p})T)\,. \tag{1.2}$$

Mathematics Subject Classification (2010): Primary 11G40; Secondary 14H45.
Keywords: Abelian varieties, genus 2 curves, L-functions, Artin representations.

Hence, from now on we will assume that the genus of C (and C') is $g \geq 2$, and we will focus on obtaining a generalization of relation (1.1).

Let us fix some notation. Hereafter, $\overline{\mathbb{Q}}$ denotes a fixed algebraic closure of $\mathbb{Q}$ that is assumed to contain k and all of its algebraic extensions. For any algebraic extension F/k, we will write $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. For abelian varieties A and B defined over k, denote by $\mathrm{Hom}_F(A,B)$ the $\mathbb{Z}$-module of homomorphisms from A to B defined over F, and by $\mathrm{End}_F(A)$ the ring of endomorphisms of A defined over F. Write $\mathrm{Hom}^0_F(A,B)$ for the $\mathbb{Q}$-vector space $\mathbb{Q}\otimes\mathrm{Hom}_F(A,B)$, and $\mathrm{End}^0_F(A)$ for the algebra $\mathbb{Q}\otimes\mathrm{End}_F(A)$. We write $A \sim_F B$ to denote that A and B are isogenous over F.

1.1. Relating ℓ-adic representations of twisted curves

Let $\mathrm{Aut}(C)$ be the group of automorphisms defined over $\overline{\mathbb{Q}}$ of C, and let $\mathrm{Isom}(C',C)$ be the set of all isomorphisms from C' to C. Throughout the paper, L/k (respectively K/k) will denote the minimal extension of k where all the elements in $\mathrm{Isom}(C',C)$ (respectively in $\mathrm{Aut}(C)$) are defined. By a theorem of Hurwitz, $\mathrm{Aut}(C)$ has order less than or equal to $84(g-1)$. Since the isomorphism ϕ induces a bijection between $\mathrm{Aut}(C)$ and $\mathrm{Isom}(C',C)$, we have, in particular, that these two sets are finite. Thus, the extensions K/k and L/k are finite. Since the curves C and C' are defined over k, the extensions K/k and L/k are Galois extensions. Clearly, K/k is a subextension of L/k. We can now state the principal result of Section 2.

Theorem 1.1. *The representation*

$$\theta_C\colon G_C := \mathrm{Aut}(C) \rtimes_{\lambda_C} \mathrm{Gal}(K/k) \to \mathrm{Aut}_{\mathbb{Q}}(\mathrm{End}^0_K(J(C)))\,,$$

defined by equation (2.2) and called the twisting representation of C, *satisfies that, for every θ_C-twist $\phi\colon C' \to C$, there is an inclusion of $\mathbb{Q}_\ell[G_k]$-modules*

$$V_\ell(C') \subseteq (\theta_C \circ \lambda_\phi) \otimes V_\ell(C)\,. \tag{1.3}$$

Here $\lambda_\phi\colon \mathrm{Gal}(L/k) \to G_C$ stands for the monomorphism defined by equation (2.1).

This result encompasses Remark 2.1, Proposition 2.3 and Theorem 2.1, and we refer to the remaining results of Section 2 for proofs that the objects involved in the statement are well defined. Requiring a twist C' of C to be a θ_C-twist is a mild condition that we make precise in Definition 2.1. In Proposition 2.4, we show that (1.3) indeed generalizes (1.1).

1.2. Applications

In the particular cases that we will consider, one can in fact compute the whole decomposition of $(\theta_C \circ \lambda_\phi) \otimes V_\ell(C)$. This leads to a relation between local factors of C and C' of the style of (1.2), that is, a relation written in terms of an Artin representation. Such *global* relations have proved to be most useful when one is interested in the study of the behaviour of the local factor at a varying prime (e.g., generalized Sato–Tate distributions; see Section 4 of [5] and especially [6]).

The essential feature of the cases considered in which one can perform the computation of the decomposition of $(\theta_C \circ \lambda_\phi) \otimes V_\ell(C)$ is the splitting of the Jacobian $J(C)$ over K as the power of an elliptic curve E/K (what we call the completely split Jacobian case). In this article we restrict to the case in which E does not have complex multiplication (CM), and we refer to [6] for a treatment of the case in which E has CM.

After some considerations of general type for the completely split Jacobian case of Section 3, we restrict our attention in Section 4 to the situation in which C is a genus 2 curve defined over $\mathbb{Q}$ with $\mathrm{Aut}(C) \simeq D_8$ (resp. D_{12}). Recall that every such a curve is $\overline{\mathbb{Q}}$-isomorphic to a curve C_u in the family of (4.3) (resp. in the family of (4.4)) for some u in $\mathbb{Q}^* \smallsetminus \{1/4, 9/100\}$ (resp. in $\mathbb{Q}^* \smallsetminus \{1/4, -1/50\}$). We then prove the following result:

Theorem 1.2. *Let $\phi : C' \to C$ be a twist of $C = C_u$ with $\mathrm{Aut}(C) \simeq D_8$ (respectively $\mathrm{Aut}(C) \simeq D_{12}$). Assume that u does not belong to the finite list (4.1) (respectively (4.2)). If $V_\ell(C')$ is a simple $\mathbb{Q}_\ell[G_K]$-module, then for every prime p unramified in $L/\mathbb{Q}$, we have*

$$L_p(C/\mathbb{Q}, \theta_C \circ \lambda_\phi, T) = \begin{cases} L_p(C'/\mathbb{Q}, T)^4 & \text{if } f = 1, \\ L_p(C'/\mathbb{Q}, T)^2 \, L_p(C'/\mathbb{Q}, -T)^2 & \text{if } f = 2, \end{cases}$$

where f denotes the residue class degree of p in K.

In the statement of the theorem, $L_p(C/\mathbb{Q}, \theta_C \circ \lambda_\phi, T)$ stands for the Rankin–Selberg polynomial whose roots are all the products of roots of $L_p(C/\mathbb{Q}, T)$ and roots of $\det(1 - \theta_C \circ \lambda_\phi(\mathrm{Frob}_p)T)$.

2. The twisting representation θ_C

For any twist C' of a smooth projective curve C defined over k of genus $g \geq 2$, let K/k and L/k be as in the introduction. We will write the natural action of the group $\mathrm{Gal}(L/k)$ on $\mathrm{Aut}(C)$, $\mathrm{Isom}(C', C)$, $\mathrm{End}^0_L(J(C))$, and $\mathrm{Hom}^0_L(J(C), J(C'))$ using left exponentiation and we will often avoid writing $\circ$ for the composition of maps. Then, we have the following monomorphism of groups:

$$\lambda_C \colon \mathrm{Gal}(K/k) \to \mathrm{Aut}(\mathrm{Aut}(C)), \quad \lambda_C(\sigma)(\alpha) = {}^\sigma\alpha\,.$$

Indeed, the minimality of K guarantees that if $\sigma \in \mathrm{Gal}(K/k)$ is such that $\alpha = {}^\sigma\alpha$ for every $\alpha \in \mathrm{Aut}(C)$, then σ is trivial. We define the twisting group of C as

$$G_C := \mathrm{Aut}(C) \rtimes_{\lambda_C} \mathrm{Gal}(K/k)\,,$$

where $\rtimes_{\lambda_C}$ denotes the semidirect product through the morphism λ_C. We next justify the name for G_C. First, we fix some notation. Suppose that F'/k is a Galois extension and that F/k is a Galois subextension of F'/k. Let $\pi_{F'/F} \colon \mathrm{Gal}(F'/k) \to$

Gal(F/k) stand for the canonical projection. For every isomorphism $\phi\colon C' \to C$, define the map

$$\lambda_\phi\colon \operatorname{Gal}(L/k) \to G_C, \quad \lambda_\phi(\sigma) = (\phi({}^\sigma\phi)^{-1}, \pi_{L/K}(\sigma)) . \tag{2.1}$$

Lemma 2.1. *The map λ_ϕ is a monomorphism of groups.*

Proof. Let σ and τ belong to Gal(L/k). Then we have

$$\begin{aligned}\lambda_\phi(\sigma\tau) &= (\phi({}^{\sigma\tau}\phi)^{-1}, \pi_{L/K}(\sigma\tau)) = (\phi({}^\sigma\phi)^{-1} \circ {}^\sigma(\phi({}^\tau\phi)^{-1}), \pi_{L/K}(\sigma\tau))\\ &= (\phi({}^\sigma\phi)^{-1}\lambda_C(\pi_{L/K}(\sigma))(\phi({}^\tau\phi)^{-1}), \pi_{L/K}(\sigma)\circ\pi_{L/K}(\tau))\\ &= (\phi({}^\sigma\phi)^{-1}, \pi_{L/K}(\sigma))(\phi({}^\tau\phi)^{-1}, \pi_{L/K}(\tau)) = \lambda_\phi(\sigma)\circ\lambda_\phi(\tau) .\end{aligned}$$

Let $\sigma \in \operatorname{Gal}(L/k)$ be such that $\phi({}^\sigma\phi)^{-1} = \mathrm{id}$ and $\pi_{L/K}(\sigma)$ is trivial, i.e., $\phi = {}^\sigma\phi$ and $\sigma \in \operatorname{Gal}(L/K)$. Let ψ be any element of Isom(C', C). Since $\psi\phi^{-1}$ is an element of Aut(C), it is fixed by σ. Then, one has

$${}^\sigma\psi = {}^\sigma(\psi\phi^{-1}\phi) = {}^\sigma(\psi\phi^{-1}){}^\sigma\phi = \psi\phi^{-1}\phi = \psi .$$

The minimality of L now guarantees that σ is trivial. □

Proposition 2.1. *There is a one-to-one correspondence between the elements of the following sets:*

i) *The set* Twist(C/k) *of twists of C up to k-isomorphism;*

ii) *The set of monomorphisms $\lambda\colon \operatorname{Gal}(F/k) \to G_C$ of the form $\lambda = \xi \rtimes_{\lambda_C} \pi_{F/K}$, with ξ a map from* Gal(F/k) *to* Aut(C)*, where we identify*

$$\lambda_1\colon \operatorname{Gal}(F_1/k) \to G_C \quad \textit{and} \quad \lambda_2\colon \operatorname{Gal}(F_2/k) \to G_C$$

if there exists $\alpha \in$ Aut(C) *such that, for every $\sigma \in$* Gal(F_1F_2/k)*, one has*

$$\lambda_1 \circ \pi_{F_1F_2/F_1}(\sigma)(\alpha, 1) = (\alpha, 1)\lambda_2 \circ \pi_{F_1F_2/F_2}(\sigma) ;$$

is given by associating to a twist C' of C the class of the monomorphism λ_ϕ, where ϕ is any isomorphism from C to C'.

Proof. There is a well-known bijection between the elements of Twist(C/k) and the elements of the cohomology set $H^1(G_k, \operatorname{Aut}(C))$, given by associating to a twist C' of C the class of the cocycle $\xi(\sigma) = \phi({}^\sigma\phi)^{-1}$ (see [11], chapter X). Now, associate to the cocycle ξ the morphism $\tilde\lambda\colon G_k \to G_C$ defined by $\tilde\lambda = \xi \rtimes_{\lambda_C} \pi_{\overline{k}/K}$. Observe that for σ and τ in G_k, one has that $\tilde\lambda(\sigma\tau) = \tilde\lambda(\sigma)\tilde\lambda(\tau)$ if and only if $\xi(\sigma\tau) = \xi(\sigma)\circ{}^\sigma\xi(\tau)$. Let G_F denote the kernel of $\tilde\lambda$ and let $\lambda\colon \operatorname{Gal}(F/k) \to G_C$ satisfy $\tilde\lambda = \lambda \circ \pi_{\overline{k}/F}$. Then λ is injective. Moreover, the cocycles ξ_1 and ξ_2 are cohomologous if and only if there exists α in Aut(C) such that for all σ in G_k there holds $\xi_1(\sigma)\circ{}^\sigma\alpha = \alpha\circ\xi_2(\sigma)$, which is equivalent to $\tilde\lambda_1(\sigma)(\alpha, 1) = (\alpha, 1)\tilde\lambda_2(\sigma)$. Finally, this amounts to requiring that $\lambda_1 \circ \pi_{F_1F_2/F_1}(\sigma)(\alpha, 1) = (\alpha, 1)\lambda_2 \circ \pi_{F_1F_2/F_2}(\sigma)$ for every $\sigma \in \operatorname{Gal}(F_1F_2/k)$. □

Proposition 2.2. *The monomorphism λ_ϕ is an isomorphism if and only if the action of* $\mathrm{Gal}(L/K)$ *on* $\mathrm{Isom}(C', C)$ *has a single orbit.*

Proof. One has that λ_ϕ is exhaustive if and only if $|\mathrm{Aut}(C)| = |\,\mathrm{Gal}(L/K)|$. This is equivalent to the fact that the injective morphism

$$\lambda\colon \mathrm{Gal}(L/K) \to \mathrm{Aut}(C), \quad \lambda(\sigma) = \phi({}^\sigma\phi)^{-1}$$

is an isomorphism. This happens if and only if for every $\alpha \in \mathrm{Aut}(C)$ there exists $\sigma \in \mathrm{Gal}(L/K)$ such that $\alpha\phi = {}^\sigma\phi$. That is, if and only if for every $\psi \in \mathrm{Isom}(C', C)$, there exists $\sigma \in \mathrm{Gal}(L/K)$ such that $\psi = {}^\sigma\phi$. □

Remark 2.1. For any twist C' of C, the abelian varieties $J(C)$ and $J(C')$ are defined over k and are isogenous over L. Let F/k be a subextension of L/k. Denote by $\theta(C, C'; L/F)$ the representation afforded by the $\mathbb{Q}[\mathrm{Gal}(L/F)]$-module $\mathrm{Hom}^0_L(J(C), J(C'))$. We will write $\theta(C, C') := \theta(C, C'; L/k)$. We recall that Theorem 3.1 of [5] asserts that

$$V_\ell(C') \subseteq \theta(C, C') \otimes V_\ell(C)$$

as $\mathbb{Q}_\ell[G_k]$-modules.

Every isomorphism ϕ from C' to C induces an isomorphism from $J(C')$ to $J(C)$, that we will also call ϕ. Consider the map

$$\theta_\phi\colon \mathrm{Gal}(L/k) \to \mathrm{Aut}_{\mathbb{Q}}(\mathrm{End}^0_L(J(C)))\,, \quad \theta_\phi(\sigma)(\psi) = \phi({}^\sigma\phi)^{-1} \circ {}^\sigma\psi\,,$$

where σ is in $\mathrm{Gal}(L/k)$ and ψ in $\mathrm{End}^0_L(J(C))$.

Proposition 2.3. *For every isomorphism $\phi\colon C' \to C$, the map θ_ϕ is a rational representation of* $\mathrm{Gal}(L/k)$ *isomorphic to* $\theta(C, C')$.

Proof. It is indeed a representation. For σ and τ in $\mathrm{Gal}(L/k)$, one has

$$\begin{aligned}\theta_\phi(\sigma\tau)(\psi) &= \phi({}^{\sigma\tau}\phi)^{-1} \circ {}^{\sigma\tau}\psi\\ &= \phi(\phi^\sigma)^{-1} \circ {}^\sigma(\phi({}^\tau\phi)^{-1} \circ {}^\tau\psi)\\ &= (\theta_\phi(\sigma) \circ \theta_\phi(\tau))(\psi)\,.\end{aligned}$$

The map $\tilde\phi\colon \mathrm{Hom}^0_L(J(C), J(C')) \to \mathrm{End}^0_L(J(C))$, defined by $\tilde\phi(\varphi) = \phi \circ \varphi$ for $\varphi \in \mathrm{Hom}^0_L(J(C), J(C'))$ is an isomorphism of $\mathbb{Q}$-vector spaces. Now, one deduces that $\theta(C, C')$ and θ_ϕ are isomorphic from the fact that, for every σ in $\mathrm{Gal}(L/k)$, the following diagram is commutative:

$$\begin{array}{ccc}\mathrm{Hom}^0_L(J(C), J(C')) & \xrightarrow{\theta(C,C')(\sigma)} & \mathrm{Hom}^0_L(J(C), J(C'))\\ \downarrow{\scriptstyle\tilde\phi} & & \downarrow{\scriptstyle\tilde\phi}\\ \mathrm{End}^0_L(J(C)) & \xrightarrow{\theta_\phi(\sigma)} & \mathrm{End}^0_L(J(C))\,.\end{array}$$

□

Denote also by α the endomorphism of $J(C)$ induced by an automorphism α in $\mathrm{Aut}(C)$. We define the twisting representation of the L-function of C as the map

$$\theta_C\colon G_C \to \mathrm{Aut}_{\mathbb{Q}}(\mathrm{End}^0_K(J(C)))\,, \quad \theta_C((\alpha,\sigma))(\psi) = \alpha \circ {}^{\sigma}\psi\,, \tag{2.2}$$

where σ in $\mathrm{Gal}(K/k)$ and ψ in $\mathrm{End}^0_K(J(C))$.

Definition 2.1. *We will say that a twist C' of C is a θ_C-*twist *of C if L is such that* $\mathrm{End}^0_K(J(C)) = \mathrm{End}^0_L(J(C))$.

Theorem 2.1. *The map θ_C is a faithful representation of G_C. Moreover, for every θ_C-twist C' of C and every isomorphism $\phi\colon C' \to C$, one has $\theta_C \circ \lambda_\phi = \theta_\phi$. That is, the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Gal}(L/k) & \overset{\lambda_\phi}{\hookrightarrow} & G_C \\ & \underset{\theta_\phi}{\searrow} & \downarrow \theta_C \\ & & \mathrm{Aut}_{\mathbb{Q}}(\mathrm{End}^0_K(J(C)))\,. \end{array}$$

Proof. For ψ_1, $\psi_2 \in \mathrm{Aut}(C)$ and σ_1, $\sigma_2 \in \mathrm{Gal}(K/k)$, one has

$$\begin{aligned} \theta_C((\alpha_1,\sigma_1)(\alpha_2,\sigma_2))(\psi) &= \theta_C((\alpha_1 \circ {}^{\sigma_1}\alpha_2, \sigma_1\sigma_2))(\psi) = \alpha_1 \circ {}^{\sigma_1}\alpha_2 \circ^{\sigma_1\sigma_2} \psi \\ &= \alpha_1 \circ {}^{\sigma_1}(\alpha_2 \circ^{\sigma_2} \psi) = (\theta_C((\alpha_1,\sigma_1)) \circ \theta_C((\alpha_2,\sigma_2)))(\psi)\,. \end{aligned}$$

Let α in $\mathrm{Aut}(C)$ and σ in $\mathrm{Gal}(K/k)$ be such that $\theta_C(\alpha,\sigma)(\psi) = \psi$ for every ψ in $\mathrm{End}^0_K(J(C))$. In particular, for $\psi = \alpha$, one obtains that ${}^{\sigma}\alpha = \mathrm{id}$, which implies $\alpha = \mathrm{id}$. Then $\psi = {}^{\sigma}\psi$ for all ψ in $\mathrm{End}^0_K(J(C))$ and the minimality of K implies that σ is trivial. Finally, there holds

$$(\theta_C \circ \lambda_\phi)(\sigma)(\psi) = \theta_C(\phi({}^{\sigma}\phi)^{-1}, \pi_{L/K}(\sigma))(\psi) = \phi({}^{\sigma}\phi)^{-1} \circ {}^{\sigma}\psi = \theta_\phi(\sigma)(\psi)\,,$$

for $\sigma \in \mathrm{Gal}(L/k)$ and $\psi \in \mathrm{End}^0_L(J(C))$. □

As a corollary of the previous results one obtains the desired inclusion

$$V_\ell(C') \subseteq (\theta_C \circ \lambda_\phi) \otimes V_\ell(C) \tag{2.3}$$

for every θ_C-twist C' of C. This inclusion is a generalization of the identity (1.1).

Proposition 2.4. *If C' is a nontrivial twist of C such that $\mathrm{End}^0_L(J(C)) \simeq \mathbb{Q}$, then the extension L/k is quadratic, the representation $\theta_C \circ \lambda_\phi$ is the quadratic character of* $\mathrm{Gal}(L/k)$*, and one has $V_\ell(C') \simeq (\theta_C \circ \lambda_\phi) \otimes V_\ell(C)$.*

Proof. By the inclusion (2.3), it is enough to prove that L/k is quadratic and that $\theta(C,C')$ is the quadratic character of L/k. Since $\mathrm{Aut}(C)$ injects in $\mathrm{End}^0_L(J(C)) = \mathrm{End}^0_k(J(C)) \simeq \mathbb{Q}$, we have that $\mathrm{Aut}(C)$ injects in C_2 and that $K = k$. Since C' is nontrivial, $\mathrm{Aut}(C)$ is nontrivial and, by Lemma 2.1, we deduce that L/k is a quadratic extension. Since the 1-dimensional representation $\theta(C,C')$ is faithful, it corresponds to the quadratic character of $\mathrm{Gal}(L/k)$. □

3. The completely split Jacobian case

In this section we explore the twisting representation θ_C when the Jacobian $J(C)$ splits over K as the power E^g of an elliptic curve E defined over K without complex multiplication (CM). Note that in this case $\dim \theta_C = g^2$. We will use the notation $H_C = \mathrm{Aut}(C)$ when we view $\mathrm{Aut}(C)$ as a subgroup of the twisting group G_C. We will be interested in the following cases:

(I) $[K\colon k] = g^2$, the elliptic curve E does not have CM, and θ_C is absolutely irreducible.

(II) $[K\colon k] = g^2/2$, the elliptic curve E does not have CM, and $\theta_C \simeq_{\overline{\mathbb{Q}}} \theta_1 \oplus \theta_2$ for θ_1 and θ_2 absolutely irreducible non-isomorphic representations such that $\mathrm{Res}^{G_C}_{H_C} \theta_1 = \mathrm{Res}^{G_C}_{H_C} \theta_2$.

Lemma 3.1. *Suppose that $J(C) \sim_K E^g$, for E an elliptic curve defined over K without CM. One has:*

$$\mathrm{Res}^{G_C}_{H_C} \theta_C \simeq g \cdot \varrho\,,$$

where ϱ is a rational representation of H_C of dimension g.

Proof. Consider the isomorphism

$$\Phi\colon\ \mathrm{End}^0_K(J(C)) \simeq \mathrm{End}^0_K(E^g) \to \bigoplus_{i=1}^{g} \mathrm{Hom}^0_K(E, E^g)\,,$$

defined by $\Phi(\varphi) = (\varphi\circ\iota_1, \dots, \varphi\circ\iota_g)$, where $\iota_i\colon E \to E^g$ is the inclusion of E as the i-th component of E^g. The action of $H_C = \mathrm{Aut}(C)$, which is by right composition, clearly restricts to each $\mathrm{Hom}^0_K(E, E^g)$. The rational representation ϱ afforded by $\mathrm{Hom}^0_K(E, E^g)$ satisfies $\mathrm{Res}^{G_C}_{H_C} \theta_C \simeq g \cdot \varrho$, and has dimension g provided that E has no CM. □

Proposition 3.1. *Suppose that $J(C) \sim_K E^g$, for E an elliptic curve defined over K. Suppose we are in either case* (I) *or* (II). *Let ϱ be as in Lemma 3.1. Then one has*

$$\mathrm{Ind}^{G_C}_{H_C} \varrho \simeq \frac{[K:k]}{g} \cdot \theta_C.$$

Proof. Let $(\cdot,\cdot)_{G_C}$ and $(\cdot,\cdot)_{H_C}$ denote the scalar products on complex-valued functions on G_C and H_C, respectively. For the case (I), by Frobenius reciprocity, the multiplicity of θ_C in $\mathrm{Ind}^{G_C}_{H_C} \varrho$ is

$$(\mathrm{Tr}\,\mathrm{Ind}^{G_C}_{H_C} \varrho, \mathrm{Tr}\,\theta_C)_{G_C} = (\mathrm{Tr}\,\varrho, \mathrm{Tr}\,\mathrm{Res}^{G_C}_{H_C} \theta_C)_{H_C} = g \cdot (\mathrm{Tr}\,\varrho, \mathrm{Tr}\,\varrho)_{H_C} \geq g\,.$$

Since $[K\colon k] = g^2$, the dimensions of $\mathrm{Ind}^{G_C}_{H_C} \varrho$ and $g \cdot \theta_C$ equal g^3, and the result follows.

For the case (II), observe that $\mathrm{Res}^{G_C}_{H_C} \theta_1 = \mathrm{Res}^{G_C}_{H_C} \theta_2$ implies that $\mathrm{Res}^{G_C}_{H_C} \theta_1 = g/2 \cdot \varrho$. Then, the multiplicity of θ_1 in $\mathrm{Ind}^{G_C}_{H_C} \varrho$ is

$$(\mathrm{Tr}\,\mathrm{Ind}^{G_C}_{H_C} \varrho, \mathrm{Tr}\,\theta_1)_{G_C} = (\mathrm{Tr}\,\varrho, \mathrm{Tr}\,\mathrm{Res}^{G_C}_{H_C} \theta_1)_{H_C} = \frac{g}{2} \cdot (\mathrm{Tr}\,\varrho, \mathrm{Tr}\,\varrho)_{H_C} \geq \frac{g}{2}\,,$$

from which one sees that $g/2 \cdot \theta_1$ is a subrepresentation of $\mathrm{Ind}_{H_C}^{G_C} \varrho$. Similarly, one proves that $g/2 \cdot \theta_2$ is a subrepresentation of $\mathrm{Ind}_{H_C}^{G_C} \varrho$. Therefore, $g/2 \cdot \theta_C$ is a subrepresentation of $\mathrm{Ind}_{H_C}^{G_C} \varrho$ and, since they both have dimension equal to $g^3/2$, they are isomorphic. □

Corollary 3.1. *Suppose that $J(C) \sim_K E^g$, for E an elliptic curve defined over K. Suppose we are in either case* (I) *or* (II). *Then one has*

$$\mathrm{Ind}_{H_C}^{G_C} \mathrm{Res}_{H_C}^{G_C} \theta_C \simeq [K:k] \cdot \theta_C \,.$$

In what follows we will be particularly interested in the structure of $V_\ell(C)$ as a $\mathbb{Q}_\ell[G_K]$-module. First, we define some notation. For an isomorphism $\phi\colon C' \to C$, denote by

$$\mathrm{Res}\, \lambda_\phi \colon \mathrm{Gal}(L/K) \to \mathrm{Aut}(C)$$

the restriction of the morphism λ_ϕ to the subgroup $\mathrm{Gal}(L/K)$. Observe that

$$\mathrm{Res}_{H_C}^{G_C} \theta_C \circ \mathrm{Res}\, \lambda_\phi \simeq \theta(C, C'; L/K) \,.$$

Theorem 3.1. *Suppose that $J(C) \sim_K E^g$, for E an elliptic curve defined over K. Let C' be a θ_C-twist of C. Suppose that $V_\ell(C')$ is a simple $\mathbb{Q}_\ell[G_K]$-module. Then, one has:*

$$\theta(C, C') \otimes V_\ell(C) \simeq \begin{cases} \mathbb{Q}[\mathrm{Gal}(K/k)] \otimes V_\ell(C') & \text{if (I)}, \\ 2 \cdot \mathbb{Q}[\mathrm{Gal}(K/k)] \otimes V_\ell(C') & \text{if (II)}. \end{cases}$$

Proof. For the case (I), recall that by Theorem 3.1 in [5] there is an inclusion of $\mathbb{Q}_\ell[G_K]$-modules

$$\begin{aligned} V_\ell(C') \subseteq\ & \theta(C, C'; L/K) \otimes V_\ell(C) \simeq\ (\mathrm{Res}_{H_C}^{G_C} \theta_C \circ \mathrm{Res}\, \lambda_\phi) \otimes V_\ell(C) \\ &\simeq\ g^2 \cdot (\varrho \circ \mathrm{Res}\, \lambda_\phi) \otimes V_\ell(E) \,. \end{aligned}$$

Since $V_\ell(C')$ is a simple $\mathbb{Q}_\ell[G_K]$-module, we obtain that

$$V_\ell(C') \simeq (\varrho \circ \mathrm{Res}\, \lambda_\phi) \otimes V_\ell(E) \,. \tag{3.1}$$

Tensoring both sides of the previous isomorphism with $g \cdot \mathbb{Q}[\mathrm{Gal}(K/k)]$ we get

$$\begin{aligned} g \cdot \mathbb{Q}[\mathrm{Gal}(K/k)] \otimes V_\ell(C') &\simeq\ g \cdot \mathrm{Ind}_K^k (\varrho \circ \mathrm{Res}\, \lambda_\phi) \otimes V_\ell(E) \\ &\simeq\ \mathrm{Ind}_K^k (\varrho \circ \mathrm{Res}\, \lambda_\phi) \otimes V_\ell(C) \ \simeq\ (\mathrm{Ind}_{H_C}^{G_C} \varrho \circ \lambda_\phi) \otimes V_\ell(C) \\ &\simeq\ g \cdot (\theta_C \circ \lambda_\phi) \otimes V_\ell(C) \ \simeq\ g \cdot \theta_\phi \otimes V_\ell(C) \ \simeq\ g \cdot \theta(C, C') \otimes V_\ell(C) \,, \end{aligned}$$

where we have used that $\mathrm{Ind}_{H_C}^{G_C} \varrho = g \cdot \theta_C$, as follows from Proposition 3.1. For the case (II), everything is as for case (I) until equation (3.1). Then, tensoring by $2g \cdot \mathbb{Q}[\mathrm{Gal}(K/k)]$, we get

$$\begin{aligned} 2g \cdot \mathbb{Q}[\mathrm{Gal}(K/k)] \otimes V_\ell(C') &\simeq\ 2g \cdot \mathrm{Ind}_K^k (\varrho \circ \mathrm{Res}\, \lambda_\phi) \otimes V_\ell(E) \\ &\simeq\ 2\,\mathrm{Ind}_K^k (\varrho \circ \mathrm{Res}\, \lambda_\phi) \otimes V_\ell(C) \ \simeq\ 2(\mathrm{Ind}_{H_C}^{G_C} \varrho \circ \lambda_\phi) \otimes V_\ell(C) \\ &\simeq\ g \cdot (\theta_C \circ \lambda_\phi) \otimes V_\ell(C) \ \simeq\ g \cdot \theta(C, C') \otimes V_\ell(C) \,. \end{aligned}$$

□

Corollary 3.2. *Assume the hypotheses of Theorem 3.1, and that one of the cases* (I) *or* (II) *holds. Let* $\mathfrak{p}$ *a prime of good reduction for both* C *and* C' *unramified in* L/k*. Write* $a_{\mathfrak{p}} = \operatorname{Tr} \varrho_C(\operatorname{Frob}_{\mathfrak{p}})$ *and* $a'_{\mathfrak{p}} = \operatorname{Tr} \varrho_{C'}(\operatorname{Frob}_{\mathfrak{p}})$*. Then:*

i) *If* $\operatorname{Frob}_{\mathfrak{p}} \in G_K$*, one has*

$$\operatorname{sgn}(a_{\mathfrak{p}} \cdot \operatorname{Tr}(\theta(C, C')(\operatorname{Frob}_{\mathfrak{p}}))) = \operatorname{sgn}(a'_{\mathfrak{p}}) .$$

ii) *If* $\operatorname{Frob}_{\mathfrak{p}} \notin G_K$*, one has*

$$\operatorname{Tr} \theta(C, C')(\operatorname{Frob}_{\mathfrak{p}}) = 0 .$$

Proof. Theorem 3.1 implies

$$\operatorname{Tr}(\theta(C, C')(\operatorname{Frob}_{\mathfrak{p}})) \cdot a_{\mathfrak{p}} = a'_{\mathfrak{p}} \cdot \operatorname{Tr}(\mathbb{Q}[\operatorname{Gal}(K/k)](\operatorname{Frob}_{\mathfrak{p}})) .$$

Part i) follows from the fact that if $\operatorname{Frob}_{\mathfrak{p}} \in G_K$, then

$$\operatorname{Tr}(\mathbb{Q}[\operatorname{Gal}(K/k)](\operatorname{Frob}_{\mathfrak{p}})) = |\operatorname{Gal}(K/k)| .$$

For part ii), suppose that $\operatorname{Frob}_{\mathfrak{p}} \notin G_K$. Corollary 3.1 implies that $\operatorname{Tr} \theta_C(\sigma) = 0$ for any $\sigma \notin H_C$. Then, $\operatorname{Tr} \theta(C, C')(\operatorname{Frob}_{\mathfrak{p}}) = \operatorname{Tr} \theta_C \circ \lambda_\phi(\operatorname{Frob}_{\mathfrak{p}}) = 0$. □

4. The genus 2 case

Throughout this section, C denotes a genus 2 curve defined over $\mathbb{Q}$. Let us recall some basic facts that may be found in [2]. It is well known that C admits an affine model given by a hyperelliptic equation $Y^2 = f(X)$, where $f(X) \in \mathbb{Q}[X]$. Any element $\alpha \in \operatorname{Aut}(C)$ can then be written in the form

$$\alpha(X, Y) = \left(\frac{mX + n}{pX + q}, \frac{mq - np}{(pX + q)^3} Y \right) ,$$

for unique $m, n, p, q \in K$. Moreover, the map

$$\operatorname{Aut}(C) \to \operatorname{GL}_2(K) , \quad \alpha \mapsto \begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

defines a 2-dimensional faithful representation of $\operatorname{Aut}(C)$. We will often identify an automorphism of C with its corresponding matrix. Note that $w(X, Y) = (X, -Y)$ is always an automorphism of C, called the hyperelliptic involution of C, which lies in the center $Z(\operatorname{Aut}(C))$ of $\operatorname{Aut}(C)$.

The group $\operatorname{Aut}(C)$ is isomorphic to one of the groups

$$C_2,\ C_2 \times C_2,\ D_8,\ D_{12},\ 2D_{12},\ \tilde{S}_4,\ C_2 \times C_5 ,$$

where $2D_{12}$ and $\tilde{S}_4$ denote certain double covers of the dihedral group of 12 elements D_{12} and the symmetric group on 4 letters S_4. Completing the study initiated by Clebsch and Bolza, Igusa [8] computed the 3-dimensional variety $\mathcal{M}_2$ of moduli of genus 2 curves defined over $\overline{\mathbb{Q}}$. Generically, the only nontrivial automorphism of a curve in $\mathcal{M}_2$ is the hyperelliptic involution and, thus, $\mathrm{Aut}(C) \simeq C_2$. The curves with $\mathrm{Aut}(C)$ containing $C_2 \times C_2$ constitute a surface in $\mathcal{M}_2$. The moduli points corresponding to curves such that $\mathrm{Aut}(C)$ contains D_8 or D_{12} describe two curves contained in this surface. The curves with $\mathrm{Aut}(C) \simeq 2D_{12}$, $\tilde{S}_4$, or $C_2 \times C_5$ correspond to three isolated points of $\mathcal{M}_2$.

In this section, we will explicitly compute the twisting representation θ_C of C and the decomposition of $\theta(C, C') \otimes V_\ell(C)$ when $\mathrm{Aut}(C) \simeq D_8$ or D_{12}. In both cases, the irreducible characters of G_C will be denoted χ_i, even though they refer to different groups (we will always refer the reader to the corresponding character table in Section 5). We will denote by ϱ_i a representation of character χ_i.

Lemma 4.1. *If* $\mathrm{Aut}(C)$ *is nonabelian, then* $J(C) \sim_K E^2$*, where* E *is an elliptic curve defined over* K*.*

Proof. It is straightforward to check that $\mathrm{Aut}(C)$ contains a nonhyperelliptic involution u. Then the quotient $E = C/\langle u \rangle$ is an elliptic curve defined over K (see Lemmas 2.1 and 2.2 in [2]). The injection $E \hookrightarrow J(C)$ is also defined over K and the Poincaré Decomposition Theorem ensures the existence of an elliptic curve E' defined over K such that $J(C) \sim_K E \times E'$. Since $\mathrm{End}_K(J(C))$ contains $\mathrm{Aut}(C)$, it is non-abelian and so $\mathrm{End}_K(J(C)) \simeq \mathbb{M}_2(\mathrm{End}_K(E))$, from which $E \sim_K E'$. □

Remark 4.1. Henceforth, for the cases $\mathrm{Aut}(C) \simeq D_8$ or D_{12}, we will make the assumption that the elliptic quotient E does not have complex multiplication, i.e., $\mathrm{End}^0_K(J(C)) \simeq \mathbb{M}_2(\mathbb{Q})$. This only excludes a finite number of $\overline{\mathbb{Q}}$-isomorphism classes. Indeed, curves with $\mathrm{Aut}(C) \simeq D_8$ or D_{12} defined over $\mathbb{Q}$ are parameterized by rational values of the absolute invariant u (see subsections 4.1 and 4.2 for details). According to Proposition 8.2.1 of [1], the j-invariant of the elliptic quotient E has two possible forms:

$$j(E) = \begin{cases} \dfrac{2^6(3 \mp 10\sqrt{u})^3}{(1 \mp 2\sqrt{u})(1 \pm 2\sqrt{u})^2} & \text{if } \mathrm{Aut}(C) \simeq D_8, \\[2ex] \dfrac{2^8 3^3(2 \mp 5\sqrt{u})^3(\pm\sqrt{u})}{(1 \mp 2\sqrt{u})(1 \pm 2\sqrt{u})^3} & \text{if } \mathrm{Aut}(C) \simeq D_{12}. \end{cases}$$

Since the degree of the extension $\mathbb{Q}(j(E))/\mathbb{Q}$ is 1 or 2 and the number of quadratic imaginary fields of class number 1 or 2 is finite, we deduce that there exists only a finite number of rational absolute invariants u for which E has CM. According to the table on page 112 of [1], for $\mathrm{Aut}(C) \simeq D_8$ these values of u are:

$$\frac{81}{196}, \frac{3969}{16900}, \frac{-81}{700}, \frac{1}{5}, \frac{9}{32}, \frac{12}{49}, \frac{81}{320}, \frac{81}{325}, \frac{2401}{9600}, \frac{9801}{39200}, \frac{6480}{25920}, \frac{194481}{777925}, \frac{96059601}{384238400}. \tag{4.1}$$

For $\mathrm{Aut}(C) \simeq D_{12}$ the values of u for which E has CM are:

$$\frac{4}{25}, \frac{-4}{11}, \frac{1}{20}, \frac{1}{2}, \frac{27}{100}, \frac{4}{17}, \frac{125}{484}, \frac{20}{81}, \frac{256}{1025}, \frac{756}{3025}, \frac{62500}{250001}. \tag{4.2}$$

Remark 4.2. By Lemma 4.1, if $\mathrm{Aut}(C) \simeq D_8$ or D_{12}, then for every twist C' of C, one has that

$$\mathrm{End}_L^0(J(C)) = \mathrm{End}_K^0(J(C)) \simeq \mathbb{M}_2(\mathrm{End}_K(E)) .$$

In other words, every twist C' of C is a θ_C-twist of C.

4.1. $\mathrm{Aut}(C) \simeq D_8$

Proposition 4.1 (Proposition 2.1 of [3]). *There is a bijection between the $\overline{\mathbb{Q}}$-isomorphism classes of genus 2 curves defined over $\mathbb{Q}$ with $\mathrm{Aut}(C) \simeq D_8$ and the open set of the affine line $\mathbb{Q}^* \setminus \{1/4, 9/100\}$, given by associating to each $u \in \mathbb{Q}^* \setminus \{1/4, 9/100\}$ the projective curve of equation*

$$Y^2Z^3 = X^5 + X^3Z^2 + uXZ^4 .$$

As follows from Proposition 4.4 of [3], the curve in the previous proposition is $\overline{\mathbb{Q}}$-isomorphic to

$$C = C_u \colon Y^2Z^4 = X^6 - 8X^5Z + \frac{3}{u}X^4Z^2 + \frac{3}{u^2}X^2Z^4 + \frac{8}{u^2}XZ^5 + \frac{1}{u^3}Z^6 . \tag{4.3}$$

where we have chosen parameters $z = 0$, $s = 1$ and $v = 1/u$. Its group of automorphisms is computed in Proposition 3.3 of [3], and it is generated by

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2u} \\ \sqrt{u/2} & -1/\sqrt{2} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1/\sqrt{u} \\ \sqrt{u} & 0 \end{pmatrix},$$

from which we see that $K = \mathbb{Q}(\sqrt{u}, \sqrt{2})$. Note that U and V satisfy the relations $U^2 = 1$, $V^4 = 1$ and $UV = V^3U$. For the character table of the group G_C, see in Section 5 Table 1 if u and $2u \notin \mathbb{Q}^{*2}$; Table 2 if $u \in \mathbb{Q}^{*2}$; and Table 3 if $2u \in \mathbb{Q}^{*2}$.

Proposition 4.2. *One has*

$$\mathrm{Tr}\,\theta_C = \begin{cases} \chi_{11} & \text{if } u \text{ and } 2u \notin \mathbb{Q}^{*2}, \\ \chi_9 + \chi_{10} & \text{if } u \in \mathbb{Q}^{*2}, \\ \chi_6 + \chi_7 & \text{if } 2u \in \mathbb{Q}^{*2}. \end{cases}$$

Moreover, $\mathrm{Res}^{G_C}_{H_C}\chi_9 = \mathrm{Res}^{G_C}_{H_C}\chi_{10}$ *in the second case, and* $\mathrm{Res}^{G_C}_{H_C}\chi_6 = \mathrm{Res}^{G_C}_{H_C}\chi_7$ *in the third case.*

Proof. The dimension of θ_C is 4. Suppose that u and $2u \notin \mathbb{Q}^{*2}$. By looking at the column of the conjugacy class $2A$ in Table 1, one sees that ϱ_{11} is the only faithful representation of dimension 4 of G_C.

One can also directly compute the representation θ_C. Denote by α^* the image of $\alpha \in \mathrm{Aut}(C)$ under the inclusion $\mathrm{Aut}(C) \hookrightarrow \mathrm{End}_K^0(J(C))$. We will prove that $\mathrm{End}_K^0(J(C)) = \langle 1^*, U^*, V^*, U^*V^* \rangle_{\mathbb{Q}}$. Indeed, it is enough to see that 1^*, U^*, V^* and U^*V^* are linearly independent. Suppose that for certain λ_i in $\mathbb{Q}$, one has $\lambda_1 1^* + \lambda_2 U^* + \lambda_3 V^* + \lambda_4 U^*V^* = 0$. Conjugating by V^* one obtains $\lambda_1 1^* - \lambda_2 U^* + \lambda_3 V^* - \lambda_4 U^*V^* = 0$, which implies $\lambda_1 1^* + \lambda_3 V^* = 0$ and thus $\lambda_1 = \lambda_3 = 0$.

Similarly, one has $\lambda_2 U^* + \lambda_4 U^* V^* = 0$, that is $\lambda_2 1^* + \lambda_4 V^* = 0$, which implies $\lambda_2 = \lambda_4 = 0$. Let σ, $\tau \in \mathrm{Gal}(K/\mathbb{Q})$ be such that $\sigma(\sqrt{u}) = -\sqrt{u}$ and $\tau(\sqrt{2}) = -\sqrt{2}$. Now, θ_C can be computed by observing that ${}^{\sigma}U = UV$, ${}^{\sigma}V = V^3$, ${}^{\tau}U = UV$, and ${}^{\tau}V = V$.

Suppose that $u \in \mathbb{Q}^{*2}$. By looking at the column of the conjugacy class $2A$ in Table 2, one sees that either ϱ_9 or ϱ_{10} is a constituent of θ_C, since otherwise θ_C would not be faithful. Since $\varrho_9 = \overline{\varrho}_{10}$, we deduce that $\theta_C = \varrho_9 + \varrho_{10}$. Moreover, by Lemma 3.1, $\mathrm{Res}^{G_C}_{H_C} \theta_C = 2 \cdot \varrho$, where ϱ is a representation of $H_C \simeq D_8$. Since the only faithful representation of D_8 is irreducible, it follows that $\mathrm{Res}^{G_C}_{H_C} \varrho_9 = \mathrm{Res}^{G_C}_{H_C} \varrho_{10} = \varrho$. The case $2u \in \mathbb{Q}^{*2}$ is analogous. □

As a consequence of the previous proposition and Theorem 3.1, we obtain the following result:

Corollary 4.1. *If C' is a twist of C such that $V_\ell(C')$ is a simple $\mathbb{Q}_\ell[G_K]$-module, then*

$$\theta(C, C') \otimes V_\ell(C) \simeq \begin{cases} \mathbb{Q}[\mathrm{Gal}(K/\mathbb{Q})] \otimes V_\ell(C') & \text{if } u \text{ and } 2u \notin \mathbb{Q}^{*2}. \\ 2 \cdot \mathbb{Q}[\mathrm{Gal}(K/\mathbb{Q})] \otimes V_\ell(C') & \text{if } u \text{ or } 2u \in \mathbb{Q}^{*2}. \end{cases}$$

Proof. If $u \in \mathbb{Q}^{*2}$, the fact that $\mathrm{Tr}\, \theta_C = \chi_9 + \chi_{10}$ together with $g^2/2 = [K\colon \mathbb{Q}] = 2$, guarantees that we are in case (II) of Theorem 3.1. The case $2u \in \mathbb{Q}^{*2}$ is analogous. If u and $2u \notin \mathbb{Q}^{*2}$, then we are in case (I). □

4.2. $\mathrm{Aut}(C) \simeq D_{12}$

Proposition 4.3 (Proposition 2.2 of [3]). *There is a bijection between the $\overline{\mathbb{Q}}$-isomorphism classes of genus 2 curves defined over $\mathbb{Q}$ with $\mathrm{Aut}(C) \simeq D_{12}$ and the open set of the affine line $\mathbb{Q}^* \smallsetminus \{1/4, -1/50\}$, given by associating to each $u \in \mathbb{Q}^* \smallsetminus \{1/4, -1/50\}$ the projective curve of equation*

$$Y^2 Z^4 = X^6 + X^3 Z^3 + u Z^6 \,.$$

As follows from Proposition 4.9 of [3], the curve of the previous proposition is $\overline{\mathbb{Q}}$-isomorphic to

$$C = C_u \colon Y^2 Z^4 = \begin{array}{l} 27\, u X^6 - 2916\, u^2 X^5 Z + 243\, u^2 X^4 Z^2 + 29160\, u^3 X^3 Z^3 \\ +729\, u^3 X^2 Z^4 - 26244\, u^4 X Z^5 + 729\, u^4 Z^6 \,. \end{array} \tag{4.4}$$

This curve corresponds to the curve appearing in Proposition 4.9 of [3], with the choice of parameters $z = 0$, $s = u$ and $v = u/3$. Its group of automorphisms is computed in Proposition 3.5 of [3], and is generated by

$$U = \begin{pmatrix} 0 & \sqrt{u}/3 \\ 3/\sqrt{u} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1/2 & -\sqrt{u}/\sqrt{12} \\ 3\sqrt{3}/\sqrt{4u} & 1/2 \end{pmatrix},$$

from which we see that $K = \mathbb{Q}(\sqrt{u}, \sqrt{3})$ (observe the change of two signs in the matrix V with respect [3]). Note that U and V satisfy the relations $U^2 = 1$, $V^6 = 1$ and $UV = V^5 U$. For the character table of the group G_C, see in Section 5 Table 4 if u and $3u \notin \mathbb{Q}^{*2}$; Table 5 if $u \in \mathbb{Q}^{*2}$; and Table 6 if $3u \in \mathbb{Q}^{*2}$.

Proposition 4.4. *One has*

$$\operatorname{Tr}\theta_C = \begin{cases} \chi_{15} & \text{if } u \text{ and } 3u \notin \mathbb{Q}^{*2}, \\ \chi_i + \chi_j, \text{ for } i \neq j \in \{10, 11, 12\} & \text{if } u \in \mathbb{Q}^{*2}, \\ \chi_8 + \chi_9 & \text{if } 3u \in \mathbb{Q}^{*2}. \end{cases}$$

Moreover, $\operatorname{Res}^{G_C}_{H_C} \chi_i = \operatorname{Res}^{G_C}_{H_C} \chi_j$ *in the second case, and* $\operatorname{Res}^{G_C}_{H_C} \chi_8 = \operatorname{Res}^{G_C}_{H_C} \chi_9$ *in the third case.*

Proof. The dimension of θ_C is 4. Suppose that u and $3u \notin \mathbb{Q}^{*2}$. By Lemma 4.2, and by looking at the column of the conjugacy class $2A$ in Table 4, one sees that ϱ_{13}, ϱ_{14} and ϱ_{15} are the only possible constituents of θ_C. We deduce that $\theta_C \simeq \varrho_{15}$ from the fact that none of the representations $2\cdot\varrho_{13}$, $2\cdot\varrho_{14}$ and $\varrho_{13}\oplus\varrho_{14}$ is faithful.

One can also directly compute the representation θ_C. Analogously to the case $\operatorname{Aut}(C) \simeq D_8$ one has $\operatorname{End}^0_K(J(C)) = \langle 1^*, U^*, V^*, U^*V^*\rangle_{\mathbb{Q}}$. Moreover, since the algebra $\langle 1^*, V^*\rangle$ has no zero divisors, one deduces that $V^{*2} = V^* - 1$. Let $\sigma, \tau \in \operatorname{Gal}(K/\mathbb{Q})$ be such that $\sigma(\sqrt{u}) = -\sqrt{u}$ and $\tau(\sqrt{3}) = -\sqrt{3}$. Then ${}^{\sigma}U = UV^3$, ${}^{\sigma}V = V^5$, ${}^{\tau}U = U$, and ${}^{\tau}V = V^5$.

Suppose that $u \in \mathbb{Q}^{*2}$. By Lemma 3.1, $\operatorname{Res}^{G_C}_{H_C} \theta_C = 2\cdot\varrho$. The only faithful representation of $H_C \simeq D_{12}$ is irreducible. This, together with the fact that the dimension of an irreducible representation of G_C is at most 2 (see Table 5), implies that θ_C is the sum of two irreducible representations of dimension 2. The only sums of two irreducible representations of dimension 2 of G_C which are faithful are $\chi_{10} + \chi_{11}$, $\chi_{11} + \chi_{12}$, or $\chi_{10} + \chi_{12}$. The case $3u \in \mathbb{Q}^{*2}$ is analogous. □

Lemma 4.2. *Let C be a smooth projective hyperelliptic curve. Let w be the hyperelliptic involution of C. Then, one has*

$$\operatorname{Tr}\theta_C((w, \mathrm{id})) = -\dim \operatorname{End}^0_K(J(C)).$$

Proof. Observe that for $\psi \in \operatorname{End}^0_K(J(C))$, one has $\theta_C((w, \mathrm{id}))(\psi) = -\psi$. □

As a consequence of the previous proposition and Theorem 3.1, we obtain the following result:

Corollary 4.2. *If C' is a twist of C such that $V_\ell(C')$ is a simple $\mathbb{Q}_\ell[G_K]$-module, then*

$$\theta(C, C') \otimes V_\ell(C) \simeq \begin{cases} \mathbb{Q}[\operatorname{Gal}(K/\mathbb{Q})] \otimes V_\ell(C') & \text{if } u \text{ and } 3u \notin \mathbb{Q}^{*2}. \\ 2\cdot\mathbb{Q}[\operatorname{Gal}(K/\mathbb{Q})] \otimes V_\ell(C') & \text{if } u \text{ or } 3u \in \mathbb{Q}^{*2}. \end{cases}$$

Proof. If u and $3u \notin \mathbb{Q}^{*2}$, the fact that $\operatorname{Tr}\theta_C = \chi_{15}$ together with $g^2 = [K\colon \mathbb{Q}] = 4$, guarantees that we are in case (I) of Theorem 3.1. If u or $3u \in \mathbb{Q}^{*2}$, then we are in case (II). □

4.3. *L*-functions of twisted genus 2 curves

Now the proof of Theorem 1.2 is immediate. If p is an unramified prime in $L/\mathbb{Q}$, then the reciprocal of the characteristic polynomial of Frob_p acting on the $\mathbb{Q}_\ell[G_{\mathbb{Q}}]$-module on the left-hand side of the isomorphism of Corollary 4.1 or Corollary 4.2

is $L_p(C/\mathbb{Q}, \theta_C \circ \lambda_\phi, T)$. Recall that f denotes the residue class degree of p in $K/\mathbb{Q}$. The result follows from the fact that the right-hand side of the isomorphism of Corollary 4.1 or Corollary 4.2 is of the form $\varrho \otimes V_\ell(C')$, where ϱ is a 4-dimensional representation of $\mathrm{Gal}(K/\mathbb{Q})$ such that $\varrho(\mathrm{Frob}_p)$ has four eigenvalues equal to 1 if $f = 1$, and two eigenvalues equal to 1, and two equal to -1 if $f = 2$.

Observe that thanks to Theorem 1.2, from the local factor $L_p(C/\mathbb{Q}, T)$ and the representation $\theta(C, C') \simeq \theta_C \circ \lambda_\phi$, either the polynomial $L_p(C'/\mathbb{Q}, T)$ or the product $L_p(C'/\mathbb{Q}, T) \cdot L_p(C'/\mathbb{Q}, -T)$ can be determined. The indeterminacy of the sign of a'_p which follows from the product $L_p(C'/\mathbb{Q}, T) \cdot L_p(C'/\mathbb{Q}, -T)$, can not be handled with the relation

$$\mathrm{sgn}(\mathrm{Tr}(\theta(C, C')(\mathrm{Frob}_p)) = \mathrm{sgn}(a_p \cdot a'_p)$$

from Proposition 3.2, since this relation only holds for $f = 1$.

5. Appendix: Character tables of twisting groups

In the following tables, the notation $\mathrm{GAP}(n, m)$ indicates the m-th group of order n in the ordered list of finite groups of [7].

Class	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$4A$	$4B$	$4C$	$8A$	$8B$
Size	1	1	2	4	4	4	2	2	4	4	4
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	-1	1	1	-1	-1	-1	1
χ_3	1	1	1	1	-1	-1	1	1	1	-1	-1
χ_4	1	1	-1	1	1	-1	1	-1	-1	1	-1
χ_5	1	1	-1	-1	1	-1	1	-1	1	-1	1
χ_6	1	1	1	-1	-1	-1	1	1	-1	1	1
χ_7	1	1	-1	-1	-1	1	1	-1	1	1	-1
χ_8	1	1	1	-1	1	1	1	1	-1	-1	-1
χ_9	2	2	2	0	0	0	-2	-2	0	0	0
χ_{10}	2	2	-2	0	0	0	-2	2	0	0	0
χ_{11}	4	-4	0	0	0	0	0	0	0	0	0

TABLE 1. Character table of $D_8 \rtimes (C_2 \times C_2) \simeq \mathrm{GAP}(32, 43)$.

Class	$1A$	$2A$	$2B$	$2C$	$2D$	$4A$	$4B$	$4C$	$4D$	$4E$
Size	1	1	2	2	2	1	1	2	2	2
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	1	-1	-1	1	-1	-1
χ_3	1	1	-1	-1	-1	-1	-1	1	1	1
χ_4	1	1	1	-1	-1	1	1	1	-1	-1
χ_5	1	1	1	-1	1	-1	-1	-1	1	-1
χ_6	1	1	1	1	-1	-1	-1	-1	-1	1
χ_7	1	1	-1	-1	1	1	1	-1	-1	1
χ_8	1	1	-1	1	-1	1	1	-1	1	-1
χ_9	2	-2	0	0	0	$2i$	$-2i$	0	0	0
χ_{10}	2	-2	0	0	0	$-2i$	$2i$	0	0	0

TABLE 2. Character table of $D_8 \rtimes C_2 \simeq \mathrm{GAP}(16, 13)$

Class	$1A$	$2A$	$2B$	$2C$	$4A$	$8A$	$8B$
Size	1	1	4	4	2	2	2
χ_1	1	1	1	1	1	1	1
χ_2	1	1	-1	-1	1	1	1
χ_3	1	1	-1	1	1	-1	-1
χ_4	1	1	1	-1	1	-1	-1
χ_5	2	2	0	0	-2	0	0
χ_6	2	-2	0	0	0	ζ_8	$-\zeta_8$
χ_7	2	-2	0	0	0	$-\zeta_8$	ζ_8

Table 3. Character table of $D_8 \rtimes C_2 \simeq \mathrm{GAP}(16,7)$

Class	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$2F$	$2G$	$3A$	$4A$	$4B$	$6A$	$6B$	$6C$	$12A$
Size	1	1	2	2	3	3	6	6	2	2	6	2	4	4	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	1	1	1	-1	-1	-1	-1	1	1	-1	1	1	1	-1	-1
χ_1	1	1	-1	1	-1	-1	1	-1	1	-1	1	1	-1	1	-1
χ_1	1	1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1
χ_1	1	1	1	1	-1	-1	-1	-1	1	1	-1	1	1	1	1
χ_1	1	1	1	-1	1	1	1	-1	1	-1	-1	1	1	-1	-1
χ_1	1	1	-1	1	1	1	-1	1	1	-1	-1	1	-1	1	-1
χ_1	1	1	-1	-1	-1	-1	1	1	1	1	-1	1	-1	-1	1
χ_1	2	2	2	2	0	0	0	0	-1	2	0	-1	-1	-1	-1
χ_{10}	2	2	-2	-2	0	0	0	0	-1	2	0	-1	1	1	-1
χ_{11}	2	2	2	-2	0	0	0	0	-1	-2	0	-1	-1	1	1
χ_{12}	2	2	-2	2	0	0	0	0	-1	-2	0	-1	1	-1	1
χ_{13}	2	-2	0	0	-2	2	0	0	2	0	0	-2	0	0	0
χ_{14}	2	-2	0	0	2	-2	0	0	2	0	0	-2	0	0	0
χ_{15}	4	-4	0	0	0	0	0	0	-2	0	0	2	0	0	0

Table 4. Character table of $D_{12} \rtimes (C_2 \times C_2) \simeq \mathrm{GAP}(48,38)$

Size	1	1	1	1	3	3	3	3	2	2	2	2
Class	$1A$	$2A$	$2B$	$2C$	$2D$	$2E$	$2F$	$2G$	$3A$	$6A$	$6B$	$6C$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	-1	1	1	-1	-1	1	1	-1	1	-1
χ_3	1	-1	1	-1	-1	-1	1	1	1	-1	-1	1
χ_4	1	1	-1	-1	-1	1	-1	1	1	1	-1	-1
χ_5	1	1	1	1	-1	-1	-1	-1	1	1	1	1
χ_6	1	-1	-1	1	-1	1	1	-1	1	-1	1	-1
χ_7	1	-1	1	-1	1	1	-1	-1	1	-1	-1	1
χ_8	1	1	-1	-1	1	-1	1	-1	1	1	-1	-1
χ_9	2	2	2	2	0	0	0	0	-1	-1	-1	-1
χ_{10}	2	-2	-2	2	0	0	0	0	-1	1	-1	1
χ_{11}	2	2	-2	-2	0	0	0	0	-1	-1	1	1
χ_{12}	2	-2	2	-2	0	0	0	0	-1	1	1	-1

Table 5. Character table of $D_{12} \rtimes C_2 \simeq \mathrm{GAP}(24,14)$

Class	1A	2A	2B	2C	3A	4A	6A	6B	6C
Size	1	1	2	6	2	6	2	2	2
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	-1	1	1	1
χ_3	1	1	-1	-1	1	1	-1	-1	1
χ_4	1	1	-1	1	1	-1	-1	-1	1
χ_5	2	2	-2	0	-1	0	1	1	-1
χ_6	2	-2	0	0	2	0	0	0	-2
χ_7	2	2	2	0	-1	0	-1	-1	-1
χ_8	2	-2	0	0	-1	0	$-\sqrt{-3}$	$\sqrt{-3}$	1
χ_9	2	-2	0	0	-1	0	$\sqrt{-3}$	$-\sqrt{-3}$	1

Table 6. Character table of $D_{12} \rtimes C_2 \simeq \mathrm{GAP}(24,8)$

References

[1] Cardona, G.: *Models racionals de corbes de gènere* 2. Doctoral thesis, Universitat Politècnica de Catalunya, Barcelona, 2001.

[2] Cardona, G., González, J., Lario, J-C. and Rio, A.: On curves of genus 2 with Jacobian of GL_2-type. *Manuscripta Math.* **98** (1999), no. 1, 37–54.

[3] Cardona, G. and Quer, J.: Curves of genus 2 with group of automorphisms isomorphic to D_8 or D_{12}. *Trans. Amer. Math. Soc.* **359** (2007), no. 6, 2831–2849.

[4] Cardona, G.: Representations of G_k-groups and twists of the genus two curve $y^2 = x^5 - x$. *J. Algebra* **303** (2006), no. 2, 707–721.

[5] Fité, F.: Artin representations attached to pairs of isogenous abelian varieties. *J. Number Theory* **133** (2013), no. 4, 1331–1345.

[6] Fité, F. and Sutherland, A.V.: Sato–Tate distributions of twists of $y^2 = x^5 - x$ and $y^2 = x^6 + 1$. ArXiv: `1203.1476v2`, 2012.

[7] Gap: Gap system for computational discrete algebra. `http://www.gap-system.org`.

[8] Igusa, J.-I.: Arithmetic variety of moduli for genus two. *Ann. of Math. (2)* **72** (1960), no. 3, 612–649.

[9] Magma: The magma computational algebra system. Available from: `http://magma.maths.usyd.edu.au/magma/`.

[10] Serre, J.-P.: *Linear representations of finite groups.* Graduate Texts in Mathematics 42, Springer-Verlag, New York, 1977.

[11] Silverman, J. H.: *The arithmetic of elliptic curves.* Graduate Texts in Mathematics 106, Springer-Verlag, New York, 1986.

Received September 12, 2011.

Joan-C. Lario: Departament de Matemàtica Aplicada 2, Universitat Politècnica de Catalunya, C/ Jordi Girona 1-3, 08034-Barcelona, Spain.
E-mail: `joan.carles.lario@upc.edu`

Francesc Fité: Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501-Bielefeld, Germany.
E-mail: `francesc.fite@gmail.com`

The authors were supported by the grants 2009 SGR 1220 and MTM2009-13060-C02-01.