



The automorphism group of Thompson's group F : subgroups and metric properties

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Abstract. We describe some of the geometric properties of the automorphism group $\text{Aut}(F)$ of Thompson's group F . We give realizations of $\text{Aut}(F)$ geometrically via periodic tree pair diagrams, which lead to natural presentations and give effective methods for estimating the word length of elements. We study some natural subgroups of $\text{Aut}(F)$ and their metric properties. In particular, we show that the subgroup of inner automorphisms of F is at least quadratically distorted in $\text{Aut}(F)$, whereas other subgroups of $\text{Aut}(F)$ isomorphic to F are undistorted.

Introduction

Thompson's group F has a number of perplexing properties. These include a wide range of unusual geometric properties, both on the large scale and the local scale. These have been studied by a number of authors. Here, we develop a geometric understanding of the automorphism group of F . The algebraic structure of $\text{Aut}(F)$ was described by Brin [1] via a short exact sequence involving the subgroup of inner automorphisms of F together with the product of two copies of Thompson's group T . The geometric structure of $\text{Aut}(F)$ is related to the geometric description of the commensurator of F , developed in Burillo, Cleary and Röver [5]. Here, we describe $\text{Aut}(F)$ in terms of periodic infinite tree pair diagrams. These descriptions lead to a natural presentation for $\text{Aut}(F)$ as well as some effective estimates for word length in $\text{Aut}(F)$ in terms of the complexity of their eventually periodic tree pair diagrams. We estimate the word length as a function of the size of the periodic part of the diagrams, together with the size of a remaining non-periodic part.

There are several natural subgroups of $\text{Aut}(F)$ arising naturally via Brin's algebraic description as a short exact sequence. We consider several such subgroups, giving geometric descriptions of them and their presentations. We estimate the word length in these subgroups and compute the exact distortion to be quadratic in $\text{Aut}(F)$ for the simplest of these subgroups. The construction shows that F

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itself is quadratically distorted in its most natural realization as a subgroup of $\text{Aut}(F)$, as the subgroup of inner automorphisms. In contrast, there are many other realizations of F as a subgroup in $\text{Aut}(F)$, and for some of them, we show that it is undistorted.

1. Background on F and its automorphism group

Thompson's group F is typically defined as a group of homeomorphisms of the unit interval. Elements of F are maps

$$f : [0, 1] \longrightarrow [0, 1]$$

which satisfy

1. they are orientation-preserving homeomorphisms,
2. they are piecewise linear, with finitely many breakpoints (points where the linear slope changes),
3. all breakpoints are dyadic integers (lying in $[0, 1] \cap \mathbb{Z}[\frac{1}{2}]$),
4. all slopes of the linear parts are powers of 2.

For an introduction to F , and proofs of its most important properties, see Cannon, Floyd and Parry [8].

Thompson's group T is defined in a manner completely analogous to F , but the homeomorphisms are defined on the circle S^1 instead of the unit interval. We usually interpret elements of T also as maps of the interval but with the endpoints identified, and then we can also think of them as piecewise linear, having dyadic breakpoints, etc. Hence, F is a subgroup of T . Again, see [8] for details.

To study the group of automorphisms of F , it is convenient to introduce the group $PL_2(\mathbb{R})$. This group is the group of piecewise linear homeomorphisms of the real line which have also dyadic breakpoints and power-of-two slopes, but whose set of breakpoints can be infinite, although discrete. Thompson's group F can be seen as a group of homeomorphisms of the real line, rather than $[0, 1]$, by conjugating elements with a suitable map (see Figure 1).

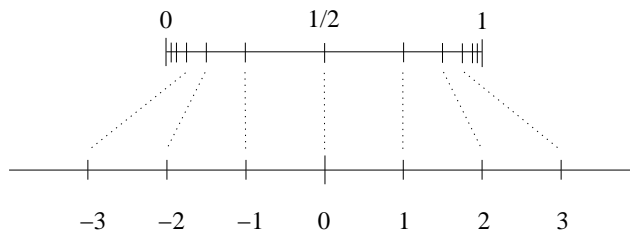


FIGURE 1: The map used to transport F from $[0, 1]$ to the real line.

Even though it seems that this conjugation could introduce an infinite number of breakpoints, it is easy to see that it does not. The slope 2^k near 0, for instance,

results in translation by k near $-\infty$ when conjugated into the real line. Hence, F is seen as the subgroup of $PL_2(\mathbb{R})$ comprising those elements that satisfy the following property: there exist a real number $M > 0$ and integers k and l such that

- 1) for all $x > M$, we have $f(x) = x + k$;
- 2) for all $x < -M$, we have $f(x) = x + l$.

In particular, this implies that all breakpoints are within the interval $[-M, M]$ and hence the set of breakpoints is finite.

The group $PL_2(\mathbb{R})$ is the group where all the action will be contained. Brin [1] shows that the group $PL_2(\mathbb{R})$ acts as a natural setting for these studies, showing that the automorphism groups of both Thompson's group F and its commutator subgroup are subgroups of $PL_2(\mathbb{R})$.

Theorem 1.1 (Brin [1]). *For a subgroup $G \subset PL_2(\mathbb{R})$, let $N(G)$ be its normalizer in $PL_2(\mathbb{R})$, that is, all those elements in $PL_2(\mathbb{R})$ which conjugate G to itself. Then we have:*

- 1) $\text{Aut } F \cong N(F)$,
- 2) $\text{Aut } F' \cong N(F') = PL_2(\mathbb{R})$.

This theorem is what allows us to investigate the automorphism group of F as a subgroup of $PL_2(\mathbb{R})$, where a piecewise linear map acts on F by conjugation. That is, the group of automorphisms of F is isomorphic to those elements in $PL_2(\mathbb{R})$ which conjugate F to itself.

We note that $PL_2(\mathbb{R})$ has an index two subgroup, which we will denote $PL_2^+(\mathbb{R})$, comprising those maps in $PL_2(\mathbb{R})$ which preserve the orientation. A natural representative of the reversing coset is the map $x \mapsto -x$. In the same way, the group of automorphisms of F has also a subgroup $\text{Aut}^+ F$, the automorphisms preserving orientation. Since in both cases the subgroup has index two, we will restrict our study to the orientation-preserving automorphisms, noting that F is actually included in $PL_2^+(\mathbb{R})$.

The key characteristic of the elements of F inside $PL_2^+(\mathbb{R})$ is the fact that near infinity, they are translations by integers. The two integers k and l which exist for each element of F give precisely the abelianization map for F (see Cannon, Floyd and Parry [8]). Then, the commutator F' is given by the elements which are the identity near infinity. This shows that F' is exactly the subgroup of $PL_2^+(\mathbb{R})$ of those elements which have bounded support; that is, those which are the identity outside a bounded interval. This gives an idea of why the automorphism group of F' is the whole $PL_2(\mathbb{R})$. See Brin's paper [1] for details.

To see which elements of $PL_2^+(\mathbb{R})$ conjugate F to itself, we note that the condition that $f(x) = x + k$, for $x > M$, and the fact that all integers appear somewhere, forces a conjugating element α to satisfy $\alpha(x+1) = \alpha(x) + 1$ for sufficiently large x . The corresponding behaviour at $-\infty$ means that this same condition is satisfied on the other side. Hence an element $\alpha \in \text{Aut}^+ F$ must satisfy $\alpha(x+1) = \alpha(x) + 1$ outside of some bounded interval.

Finally, if we consider a map α on \mathbb{R} which satisfies $\alpha(x+1) = \alpha(x) + 1$, it descends to S^1 , viewed as \mathbb{R}/\mathbb{Z} , so it corresponds to a map of S^1 . This means that to an element of Aut^+F we can assign two elements of T , one given by the behavior near ∞ and another by the behavior near $-\infty$ (which do not interact, and hence, commute). This fact is summarized in the following structure theorem for Aut^+F , proved by Brin in [1].

Theorem 1.2 ([1]). 1) *The group Aut^+F is exactly the subgroup of $PL_2^+(\mathbb{R})$ given by those elements α for which there exists $M > 0$ such that $\alpha(x+1) = \alpha(x) + 1$, for all $x \notin [-M, M]$.*

2) *We have a short exact sequence*

$$1 \longrightarrow F \longrightarrow \text{Aut}^+F \longrightarrow T \times T \longrightarrow 1.$$

In this paper, we introduce an interpretation for Aut^+F given by binary trees analogous to the one for F , use it to construct a presentation for Aut^+F , describe some interesting subgroups of Aut^+F , and study the large-scale metric properties of some of these subgroups.

2. Binary trees

Elements of Thompson's group F can be represented using pairs of binary trees to encode in a standard way the subdivisions of the unit interval needed to construct the piecewise linear maps. By the "caret" of a rooted binary tree, we mean an internal node together with its two downward directed edges. Each caret represents a subdivision of an appropriate subinterval into two halves, which are its children. In this way, a binary tree represents a subdivision of the interval. Two trees with the same number of leaves represent an element of F , by mapping the leaf subintervals linearly in an order-preserving way. The carets are naturally ordered from left to right, according to the order of the subintervals they represent in $[0, 1]$. See Figure 2 for an example.

Our goal is to extend this representation to elements of $\text{Aut}F$, and even to all of $PL_2(\mathbb{R})$. Since according to the previous section, elements of $\text{Aut}F$ are represented by piecewise linear maps on the line instead of $[0, 1]$, we need to clarify how pairs of binary trees will correspond to maps of the real line. If we consider the conjugation map which relates the interval to the line, we see how to construct this representation. The carets located on the right and left branches of a tree represent the main intervals which appear in the conjugating map, see Figure 1. Hence, these carets will represent integer intervals of length one in the real line representation, with the exception that the leftmost and rightmost carets represent intervals extending to $-\infty$ and $+\infty$ respectively. See Figure 3.

So we are now considering infinite binary trees (with infinitely many leaves). We need a mark on the two trees to signal a reference starting point. In this way, each element of $PL_2(\mathbb{R})$ can be represented by a pair of (possibly infinite) binary trees with a marking. Elements of F correspond to those for which only finitely

many of the unit length intervals are subdivided, and hence the trees are finite with the exception of the right and left "tails." See Figure 4 for an example. The choice of marking is arbitrary; there are many marked diagrams representing a given element. We note that it is possible to represent all of $\text{Aut } F$ by indicating orientations with the markings. However, here, for simplicity, we will restrict our consideration to the index two orientation-preserving subgroup $\text{Aut}^+ F$.

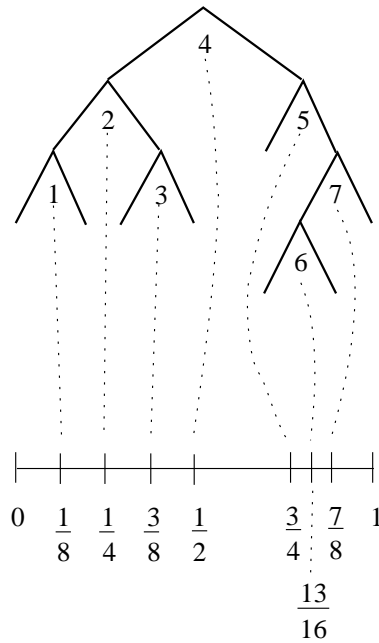


FIGURE 2: An example of a binary tree and the subdivision of the unit interval it represents.

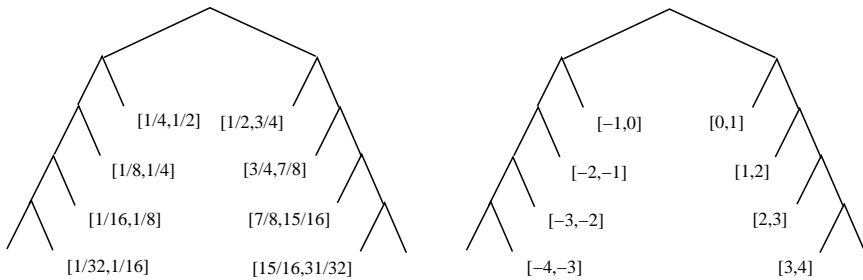


FIGURE 3: The conjugation between the unit interval and the real line, seen with trees representing elementary subdivisions.

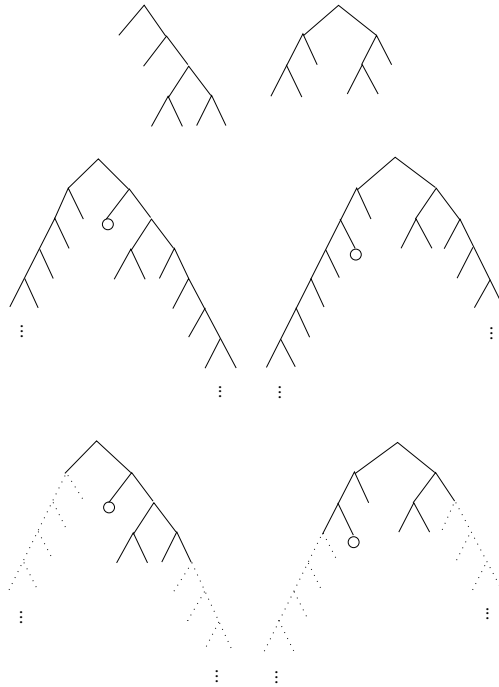


FIGURE 4: An element of F seen as a pair of trees encoding a map of the real line. Here we see: the element in $[0, 1]$, then the same element in \mathbb{R} , and in the third picture dotted carets representing the tails that have been added.

In the particular case of $\text{Aut } F$, the pairs of trees have to be periodic. The condition $\alpha(x + 1) = \alpha(x) + 1$ means that, for $k \in \mathbb{Z}$ sufficiently large, whatever subdivision we have of the interval $[k, k + 1]$, it must repeat in the subsequent intervals. Hence, the infinite trees representing elements of $\text{Aut } F$ can be coded in a finite way, because from some point on, the trees are periodic. This leads to the following definition.

Definition 2.1. An *eventually periodic binary tree* is a subtree of the infinite binary tree, for which there exists an integer $N > 0$ such that, for $k \geq N$, all intervals $[k, k + 1]$ have the same subdivision, and also for $k \geq N$, all intervals $[-k, -k + 1]$ have the same subdivision. The subdivision of the intervals near $+\infty$, although the same for all these intervals, is not necessarily the same as that used for the intervals near $-\infty$.

See Figure 5 for an example of an eventually periodic element. The trees are infinite, but only at the two extremes, and that each integer interval is subdivided only finitely many times. Hence the breakpoints form a discrete subset. The fact that the trees are eventually periodic means that for each tree, the right and left tails are periodic. Since now the trees are infinite, we need a marking to indicate how the leaves are mapped to each other. The marking (unoriented in

the case of $\text{Aut}^+ F$) is given by a little circle in a leaf of each tree. These two leaves are mapped to each other, and the corresponding leaves are mapped in an order-preserving way.

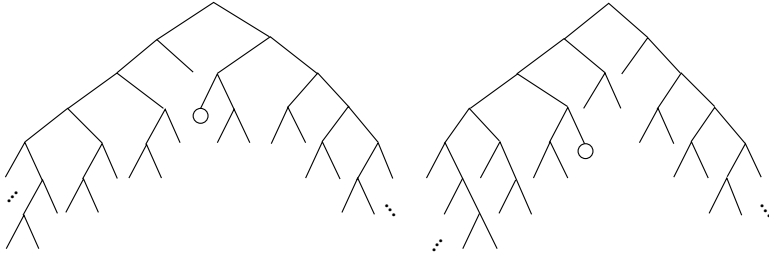


FIGURE 5: An example of an eventually periodic map represented by infinite, also eventually periodic trees.

It is straightforward to verify that conjugating an element of F by an eventually periodic tree pair results in another element of F . Both tails get cancelled because they are identical and give redundant subdivisions, capable of being reduced, using a process which is analogous to that for finite tree pairs.

Each orientation-preserving automorphism of F can be represented by an eventually periodic map of \mathbb{R} , and hence by a pair of eventually periodic trees. The map from $\text{Aut}^+ F$ to $T \times T$ can be read off directly from the infinite tree. The map is given by the trees which subdivide in both tails. See Figure 6 for the element of T corresponding to the left tail of the element of Figure 5.

We note that we are restricting to the binary case; the generalizations $F(p)$ of Thompson's group F have much more complicated automorphism groups; see Brin and Guzmán [2].

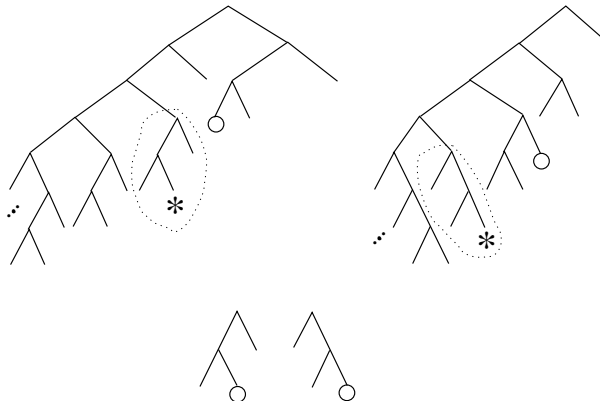


FIGURE 6: The left tail of the element in Figure 5. The two trees enclosed with a dotted line are the ones repeating indefinitely down the tail, and the two leaves marked with an asterisk map to each other (they are both at distance three from the marking). The resulting element of T is constructed with these two trees, and is represented on the bottom.

3. Presentation

The short exact sequence

$$1 \longrightarrow F \longrightarrow \text{Aut}^+ F \longrightarrow T \times T \longrightarrow 1.$$

can be used to compute a presentation for $\text{Aut}^+ F$. We use the standard presentation for F given by

$$\langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

A standard finite presentation for T , given in Cannon, Floyd and Parry [8], supplements the two generators x_0 and x_1 for F with a new generator c , which is an element of order 3. In our case, we will choose a different torsion element, denoted by t , which is diagrammatically smaller and easier to lift to $\text{Aut}^+ F$ and is also used in Burillo, Cleary, Stein and Taback [7]. The two elements c and t are depicted in Figure 7, and note that they satisfy the relation $c = tx_0$.

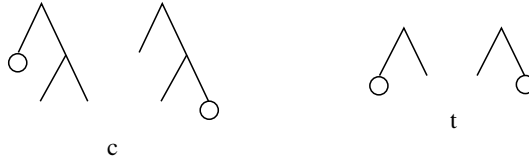


FIGURE 7: Two torsion elements of T . The element c is used as a generator in [8], and the element t is used here.

Using this relation and the presentation for T given in [8], it is easy to see that T is generated by x_0, x_1 and t , with the following relators, which are labelled for future reference:

- (f1) $[x_0 x_1^{-1}, x_0^{-1} x_1 x_0]$
- (f2) $[x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2]$
- (t1) $x_1^{-1} t x_0^2 x_1^{-1} x_0^{-1} t$
- (t2) $x_1^{-1} t x_0^2 x_1^{-1} t x_0 t x_0$
- (t3) $x_1^{-2} t x_0^2 x_1^{-1} x_0^{-1} t$
- (t4) t^2 .

Since we are using x_0 and x_1 for the embedding of F in $\text{Aut}^+ F$, the two lifts of T will be generated by $\{y_0, y_1, s\}$ and $\{z_0, z_1, t\}$, respectively. These lifts will be such that y_0, y_1 and s have support in $(-\infty, 0]$ whereas z_0, z_1 and t have support in $[0, \infty)$. The lifts of y_0, y_1, z_0 and z_1 are straightforward, and amount to attaching a $[0, 1]$ version of the F generators into the corresponding box (from 0 to 1 in the case of those supported on the positive half-line, or from -1 to 0 in the case of those supported on the negative half-line), and repeating it indefinitely in the whole support. The lifts of s and t to $\text{Aut}^+ F$ are trickier, and some choice is involved. Our choices are shown in Figures 8 and 9, but these should by no means be considered canonical.

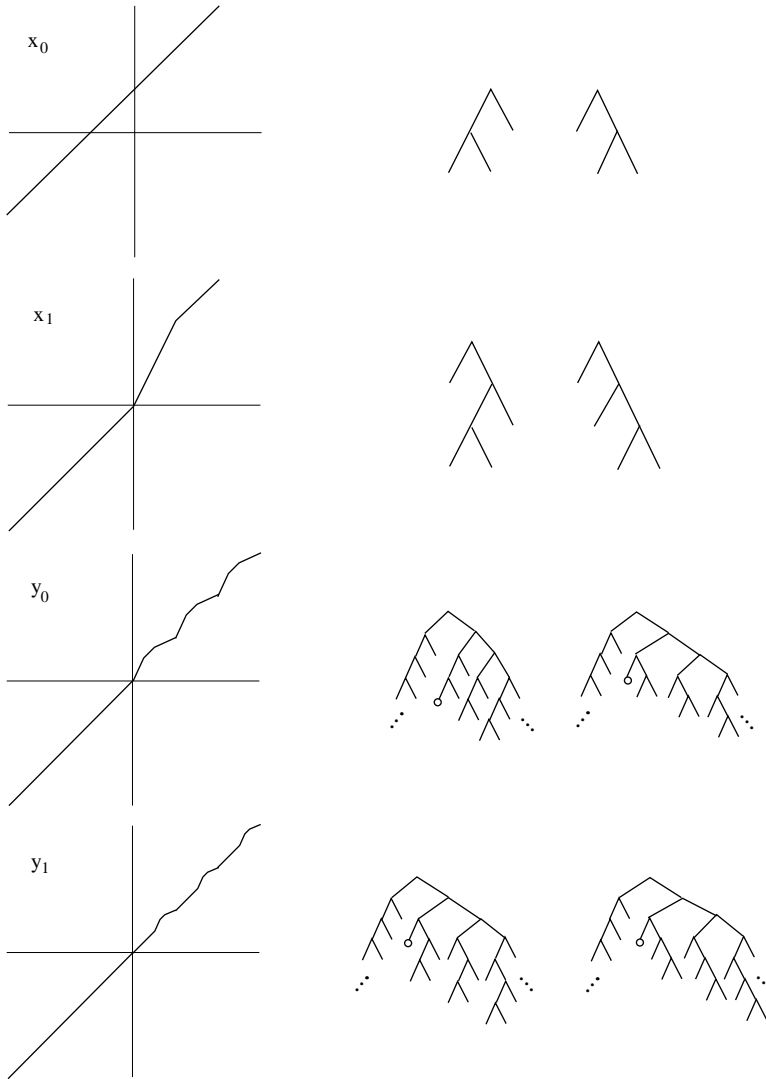


FIGURE 8: The first four generators of $\text{Aut}^+ F$, both as maps of \mathbb{R} and as pairs of periodic trees. For x_0 and x_1 , only the finite trees are shown as if they were in F , it is understood that for the periodic form, there are infinite tails at both sides of each tree (see Figure 4).

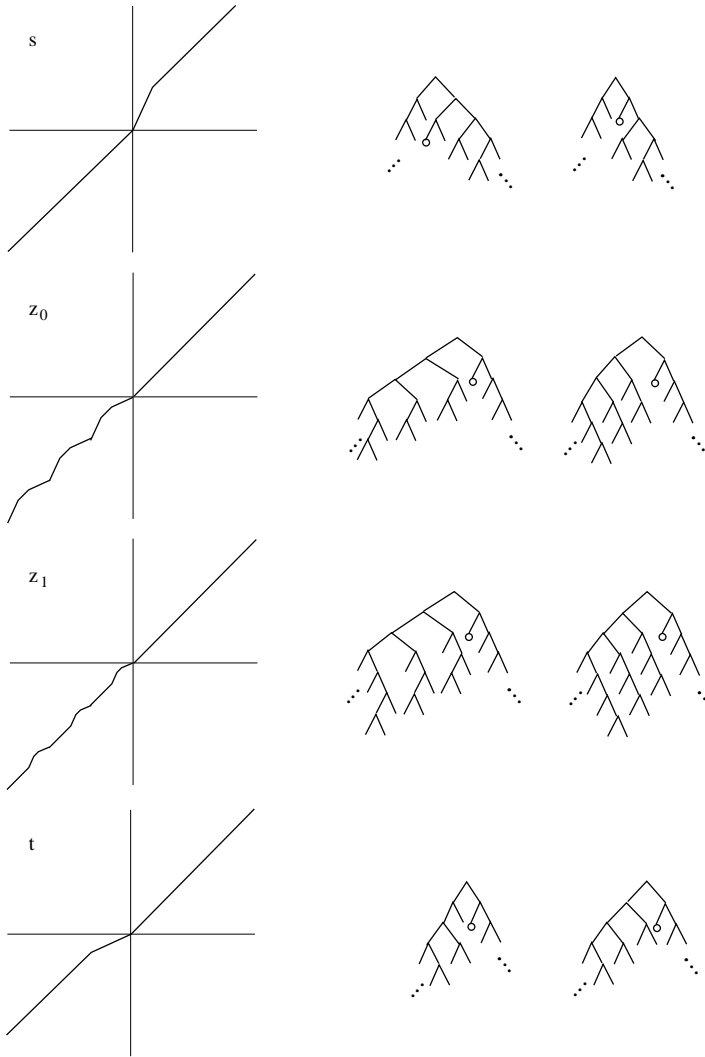


FIGURE 9: The last four generators of $\text{Aut}^+ F$.

Proposition 3.1. *The group $\text{Aut}^+ F$ is presented by the generators $\{x_0, x_1, y_0, y_1, s, z_0, z_1, t\}$, with a set of 35 relators.*

The relators for $\text{Aut}^+ F$ can be worked out following the standard procedure for short exact sequences. These relators can be classified in three types:

- 1) The two relators for the kernel F , namely, $[x_0 x_1^{-1}, x_0^{-1} x_1 x_0]$ and $[x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2]$ (2 relators).
- 2) The lifts of relators for $T \times T$. These are furthermore divided into two types:
 - (a) Commuting relations: the generators y_0, y_1 , and s commute with z_0, z_1 , and t (9 relators in total).

- (b) Lifts of the relators of the two copies of T to Aut^+F : lifting the relators (f1) and (f2) is direct, as they lift to the same relators but spelled out with y_0, y_1 and z_0, z_1 . Lifting the relators (t1) to (t4) requires an extra computation and it actually depends on the choice of lifts. For instance, we will compute two examples, based on the lifts given in Figures 8 and 9. The relator s^2 in the first copy of T lifts, with our choices, to $s^2 = y_1^2 y_0^{-1} y_1^{-1} y_0$; and the relator (t1), in its y_0, y_1, s version, lifts to

$$y_1^{-1} s y_0^2 y_1^{-1} y_0^{-1} s = y_1^4 y_0^{-3} y_1 y_0^2 y_1 y_0 y_1^{-1}.$$

There are a total of 12 relators in this class. Their tedious calculation is left to the reader.

- 3) The third type of relators appearing in a short exact sequence is given by the action relators: the lifts of the generators of $T \times T$ act on the kernel F , since this is a normal subgroup. This gives a total of 12 relators more, each given by the conjugation action of one of the generators y_0, y_1, s, z_0, z_1 or t on x_0 or x_1 . Again, the computation of these actions is somewhat tedious, and an example is

$$s x_0 s^{-1} = x_1^2 x_0^{-1} x_1^{-1} x_0 x_1^{-1}.$$

Some of these relators will be written explicitly in the following section, when we consider subgroups of Aut^+F generated by some of the generators.

The presentation of Aut^+F leads immediately to a presentation for the whole group $\text{Aut} F$, again using the corresponding short exact sequence. It is worth noticing that the orientation-reversing automorphism given by $\phi(x) = -x$ helps to reduce the number of generators and relators, since the generators z_0, z_1, t can be easily obtained from y_0, y_1, t by conjugating by ϕ . For brevity, we leave these computations to the reader as well. We have preferred to spell out the longer presentation for Aut^+F because it will be useful to understand the subgroups studied in the next section.

4. Some interesting subgroups

The structure as a short exact sequence obtained for Aut^+F shows that some interesting subgroups of it can be studied. Let

$$\pi : \text{Aut}^+F \longrightarrow T \times T$$

be the quotient map in the short exact sequence. If we consider F as a subgroup of T , we see that we have a subgroup $F \times F$ inside $T \times T$. Clearly, these two copies of F are generated by y_0, y_1 and z_0, z_1 , respectively. Since we will use them often, and to clarify to which we refer, we will rename them F_y and F_z . In the same way the kernel copy of F will be denoted by F_x .

The first subgroup we will study is the group $A = \pi^{-1}(F_y \times F_z)$. We have now a short exact sequence, similar to the previous one:

$$1 \longrightarrow F_x \longrightarrow A \longrightarrow F_y \times F_z \longrightarrow 1.$$

This short exact sequence is fundamentally different than the one for Aut^+F . Each element of $F_y \times F_z$ admits a canonical lifting into A , since elements in F , seen as maps of $[0, 1]$, fix 0 and 1. The elements y_0 and y_1 are lifts of the main generators of F to the first component, and z_0 and z_1 to the second component. Since these elements fix the points (k, k) for any $k \in \mathbb{Z}$, we see that these lifts define homomorphisms. Any element of F can be lifted to the first or second component, by introducing its graph (in $[0,1]$) in the corresponding one-by-one boxes of the plane. Note that any element obtained as the lift of an element of $F_y \times F_z$ has its graph included in the union of the boxes with sides $[k, k + 1]$, for all integers k .

Hence, we have proved the following result:

Theorem 4.1. *The short exact sequence*

$$1 \longrightarrow F_x \longrightarrow A \longrightarrow F_y \times F_z \longrightarrow 1$$

splits. This gives the group A the structure of A semidirect product $F_x \rtimes (F_y \times F_z)$ where the action of $F_y \times F_z$ on F_x is given by conjugation of the elements of F_x by the elements y_0, y_1 and z_0, z_1 .

The group A is the subgroup of Aut^+F generated by $\{x_0, x_1, y_0, y_1, z_0, z_1\}$. Its 18 relators can also be classified in three types, as before:

- 1) the F relators in F_x ;
- 2) the F relators in F_y and in F_z , which lift unmodified to A due to the splitting, and also the commuting relations: each of y_0 and y_1 commutes with z_0 and z_1 ;
- 3) the eight relators of the third type, which specify how the generators y_0, y_1 and z_0, z_1 act in the x_0, x_1 . The first two examples are

$$y_0x_0y_0^{-1} = x_0x_1x_0^{-1}x_1x_0x_1^{-2}, \quad y_0x_1y_0^{-1} = x_1^3x_0^{-1}x_1x_0^{-3}x_1x_0x_1x_0^3x_1^{-2},$$

and the rest are left to the reader. These eight relators also belong to the presentation for Aut^+F above. As an easy example, z_0 and z_1 commute with x_1 because of disjoint supports.

Inside this subgroup A we can obtain two versions of $F \rtimes F$ by taking the first component or the second component only, giving two subgroups $B_1 = F_x \rtimes F_y$ and $B_2 = F_x \rtimes F_z$, both isomorphic to $F \rtimes F$ but with different actions, and whose generators are $\{x_0, x_1, y_0, y_1\}$ and $\{x_0, x_1, z_0, z_1\}$, respectively. For instance, many relators of a presentation for B_1 are already computed above, and we have the actual presentation as

$$\begin{aligned} \langle x_0, x_1, y_0, y_1 \mid & [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2], \\ & [y_0y_1^{-1}, y_0^{-1}y_1y_0], [y_0y_1^{-1}, y_0^{-2}y_1y_0^2], \\ & y_0x_0y_0^{-1} = x_0x_1x_0^{-1}x_1x_0x_1^{-2}, \\ & y_0x_1y_0^{-1} = x_1^3x_0^{-1}x_1x_0^{-3}x_1x_0x_1x_0^3x_1^{-2} \\ & y_1x_0y_1^{-1} = x_0x_1x_0^{-1}x_1x_0^{-1}x_1x_0x_1^{-2}x_0x_1^{-1} \\ & y_1x_1y_1^{-1} = x_1^2x_0^{-1}x_1x_0^{-1}x_1x_0^{-2}x_1^2x_0^{-2}x_1x_0x_1^{-2}x_0x_1^{-1}x_0^3x_1^{-2}x_0x_1^{-1} \rangle \end{aligned}$$

However, the most interesting subgroup of Aut^+F is obtained by taking the diagonal copy of F inside $F_y \times F_z$. Let D be this diagonal:

$$D = \{(g, g) \in F_y \times F_z \mid g \in F\},$$

and consider $C = \pi^{-1}(D)$ as a subgroup of A , and hence, of Aut^+F . We intend to study the group C extensively, since its properties are quite interesting.

The group D is isomorphic to F , and the isomorphism is easy to understand. Regard an element of F as a map of $[0, 1]$, and construct a map from \mathbb{R} to \mathbb{R} by introducing the map in each one-by-one box of the form $[k, k + 1] \times [k, k + 1]$, for each $k \in \mathbb{Z}$. This map is the corresponding element in D . Notice too that these elements act on F_x , seen now as maps of the real line. Hence, D acts on F_x by conjugation, and the result is the group C , a copy of $F \rtimes F$.

Since D is isomorphic to F , it is generated by the images of the generators for F . Hence, D is generated by the two elements $w_0 = y_0z_0$ and $w_1 = y_1z_1$. For consistency, we will denote this copy of F by F_w as well. So the elements of F_w are characterized by having the same maps in each box $[k, k + 1] \times [k, k + 1]$, and when seen as a pair of eventually periodic trees, an element of F_w has actually completely periodic trees: each one of the integral leaves (see Figure 3) has the same tree attached. See Figure 10 for a picture of the generators of C , including the two generators for D .

A presentation for C can be given as usual, by giving the action of w_0 and w_1 on x_0 and x_1 . Since the maps in F_w are periodic, and x_0 is just the map $t \mapsto t + 1$ in \mathbb{R} , then the elements of F_w commute with x_0 , something that can be seen easily with the trees as well. Hence we only need to compute the two actions of w_0 and w_1 on x_1 . The presentation for C is:

$$\begin{aligned} \langle x_0, x_1, w_0, w_1 \mid & [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2], \\ & [w_0w_1^{-1}, w_0^{-1}w_1w_0], [w_0w_1^{-1}, w_0^{-2}w_1w_0^2], \\ & [x_0, w_0], [x_0, w_1] \\ & w_0x_1w_0^{-1} = x_1^3x_0^{-1}x_1x_0^{-3}x_1x_0x_1x_0^3x_1^{-2} \\ & w_1x_1w_1^{-1} = x_1^2x_0^{-1}x_1x_0^{-1}x_1x_0^{-2}x_1^2x_0^{-2}x_1x_0x_1^{-2}x_0x_1^{-1}x_0^3x_1^{-2}x_0x_1^{-1} \rangle \end{aligned}$$

Note also that $w_0x_1w_0^{-1} = y_0x_1y_0^{-1}$, and $w_1x_1w_1^{-1} = y_1x_1y_1^{-1}$ (compare the presentations for B_1 and C). This is because the support of x_1 is $[0, \infty)$, and inside this interval we have $w_0 = y_0$ and $w_1 = y_1$.

The reader is encouraged to verify these relators using periodic trees. The fact that the conjugates of elements of F_x by elements of F_w stay in F_x remains quite apparent with the trees, since in the final element, the two tails at both ends can always be cancelled and the resulting trees represent an element of F_x .

These subgroups give new groups, all of them obtained as semidirect products of copies of F , and interesting actions of F on itself, mixing the two interpretations of F as maps of $[0, 1]$ and maps of \mathbb{R} . In the next section we will study the metric properties of C , and the distortions of the two canonical embeddings of F in C .

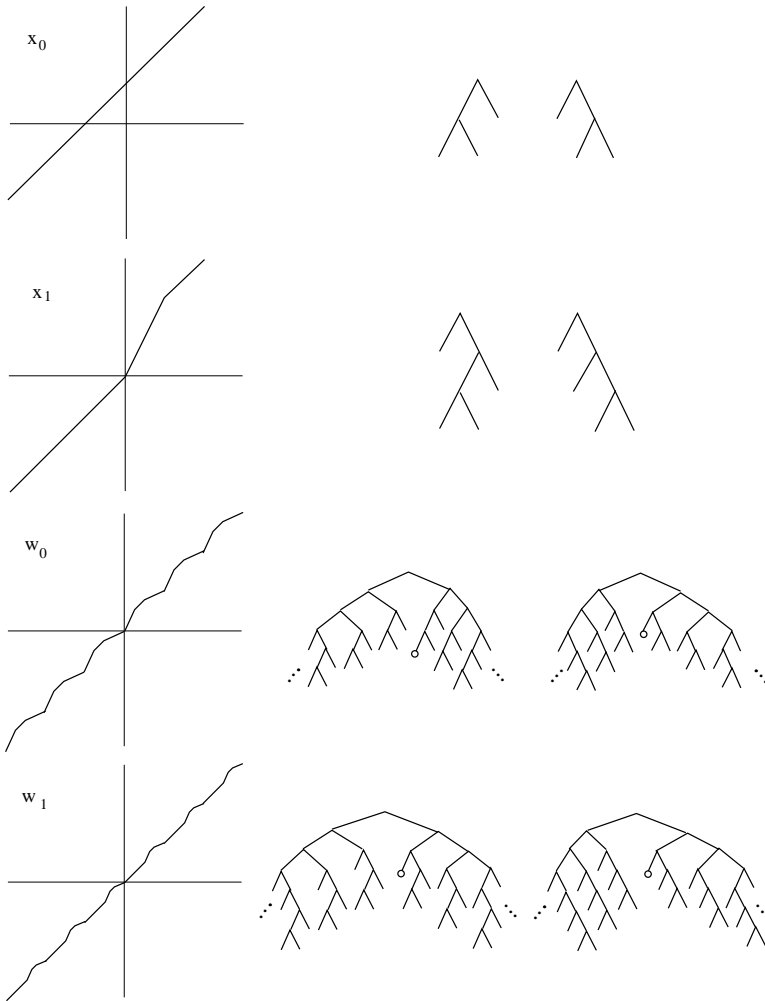


FIGURE 10: The generators for C .

5. Distortion of F in C

The two embeddings of F in C behave in different ways with respect to the metric properties. Recall that in F , the norm of an element (the length of the shortest word on x_0, x_1 representing it) is equivalent (up to a multiplicative constant) to the number of carets of the minimal reduced diagram representing it. We will use this fact to estimate the norm of an element of F , and to compare it with its norm inside C . The result is the following:

Theorem 5.1. *The subgroup F_x is quadratically distorted in C , whereas F_w is undistorted.*

Recall that the distortion function of a subgroup is obtained when elements shorten their length when embedded in the large group. If $H < G$, and both are finitely generated, we have the distortion function

$$\delta(n) = \max\{|x|_H, \text{ for all } x \in G, |x|_G \leq n\}.$$

Theorem 5.1, stating that a copy of F inside C is quadratically distorted, represents an interesting change, since many of the standard subgroups of F are undistorted (see Burillo [3], Burillo, Cleary and Stein [6], Cleary [9], Cleary and Taback [10], and Guba and Sapir [12]), as well as the embeddings $F \subset T \subset V$ (Burillo, Cleary, Stein and Taback [7]). However, F does admit subgroups which are at least polynomially distorted, see Guba and Sapir [12]. Recently, some exponentially distorted embeddings of generalizations of Thompson's groups have appeared (see Wladis [13], and Burillo and Cleary [4]) but they always involve more complicated groups whose elements are represented by two different types of carets. This is an example of a quadratically distorted embedding different from previously studied phenomena.

The proof of Theorem 5.1 occupy the rest of the article. We start by proving a lower bound for the quadratic distortion of F_x inside C with examples. We consider the family of words $r_n = w_0^{-n} x_1^n w_0^n$. Clearly these words have lengths at most $3n$ in C . We will prove that each r_n represents an element of F_x with a number of carets of the order of n^2 . This is enough to prove the distortion is at least quadratic.

For simplicity, we will draw pictures of a word of this type with $n = 4$, but the pattern will be quite clear. As shown in Figure 11, that the trees for w_0^{-4} have a repeated tree with 4 carets in each integral leaf. Multiplying by x_1^4 has the effect of shifting four of these trees further down the right-hand side of the tree, opening up a sequence of 4 empty carets on the right arm of the tree, shown in Figure 12. When bringing w_0^4 back again, these formerly empty spots are filled with trees of four carets each (see Figure 13), hence having at least 16 carets (actually 27). This process will give for general n a quadratically distorted word, with more than n^2 carets.

To prove that the distortion is no worse than quadratic, and also to find the linear distortion of the subgroup $\langle w_0, w_1 \rangle$, we will give a lower bound on the metric for an element of C in terms of the different features of its eventually periodic tree diagram, described in the next section.

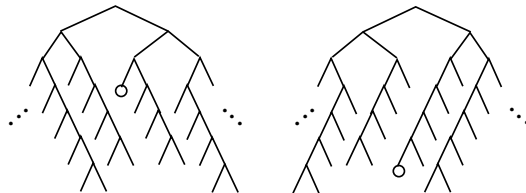


FIGURE 11: The word w_0^4 to start the construction of r_4 . Notice the four-caret tree repeated throughout.

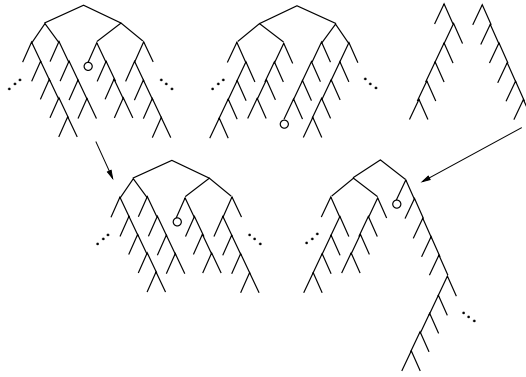


FIGURE 12: Multiplication of the word w_0^4 with x_1^4 to shift trees down the right-hand side. Observe the empty area created, which will be filled by quadratically many carets.

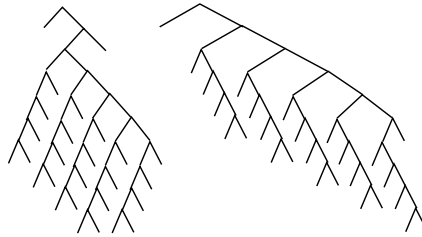


FIGURE 13: The element represented by the word r_4 in F , obtained by multiplication with w_0^4 , which fills the five empty spaces with four-caret trees.

6. Metric properties of C

In this section we will relate the metric of the group C with features of its elements, as happens with F . Given an element of F , its norm with respect to the generating set $\{x_0, x_1\}$ is equivalent to the number of carets of its reduced tree diagram (see [6] and Fordham [11]). Since C is a semidirect product of two copies of F , and the elements also admit tree pair diagrams which are related to those of the copies of F , it makes sense to expect similar results for the elements of F . We have already seen that the inclusion is distorted (at least quadratically), so a completely analogous result is not possible, but we can find a bound for the metric which guarantees that quadratic distortion is the worst possible.

We consider the short exact sequence defining C

$$1 \longrightarrow F_x \longrightarrow C \xrightarrow{\pi} F_w \longrightarrow 1,$$

and let

$$\sigma : F_w \longrightarrow C$$

be the section of the map for π obtained by repeating the element of F_w thought of as lying in $[0, 1]$ indefinitely in both directions to lie in \mathbb{R} , satisfying $\pi \circ \sigma = \text{id}$.

Given an element $g \in C$, its periodic tree diagram is periodic with the same periodic map at both ends, and the single repeating tree gives an element of F_w , which is exactly $\pi(g)$. The number of carets of the trees of $\pi(g)$ is the first feature we will consider. It will be called $a(g)$ in Definition 6.2. This number of carets relates to the F_w part of g . To consider the F_x part of g , we need to see which part of g is not related to the periodic parts. So we take $\pi(g)$, which is the F_w part, lift it back to C , and use it to cancel the periodic part. This leads to the following definition.

Definition 6.1. Let g be an element of C . Then, the element of C given by $d_g = (\sigma \circ \pi(g))^{-1}g$ is actually an element of F_x , which we will call the *debris* of g .

Clearly g and $\sigma \circ \pi(g)$ are both mapped into $\pi(g)$ by π , so d_g is in the kernel of π , which is F_x . Related to this element we will consider two other quantities b and c which estimate the complexity and help us to understand the metric in C .

Definition 6.2. Given an element $g \in C$, we will define the following three numbers associated to it:

- 1) The number $a(g)$ is the number of carets of the reduced diagram of the element $\pi(g)$ in F_w .
- 2) The number $b(g)$ is the same except for d_g . That is, it is the number of carets of the reduced diagram of $d_g \in F_x$.
- 3) The number $c(g)$ is also computed with the reduced diagram of the element d_g . It is given by the sum of the number of left and right carets (not counting the root) in both trees of the reduced tree diagram for d_g .

See Figure 14 to clarify these notions. We would like to see what is the effect on these three quantities for g when it is multiplied by each one of the four generators. Studying these effects carefully we will be able to see which is the worst possible scenario for a, b, c in an element of length L , and use it to find a lower bound for the metric.

1. **Multiplication by x_0 .** We see that $\pi(g)$ and $\pi(gx_0)$ are equal, corresponding to the periodic part of these elements, which is unaffected by x_0 . From this we see that

$$d_{gx_0} = (\sigma \circ \pi(gx_0))^{-1}gx_0 = (\sigma \circ \pi(g))^{-1}gx_0 = d_gx_0.$$

So the effect of multiplication by x_0 is apparent only on d_g , and we can consider the multiplication d_gx_0 as if it took place inside F_x . The quantity a itself will remain unchanged, whereas b and c may grow, but by at most two carets. So we have:

$$a(gx_0) = a(g), \quad b(gx_0) \leq b(g) + 2, \quad c(gx_0) \leq c(g) + 2.$$

2. **Multiplication by x_1 .** The situation is quite similar to the previous case. Multiplying by x_1 only affects the debris, so a stays the same, b and c can grow by at most 3:

$$a(gx_1) = a(g), \quad b(gx_1) \leq b(g) + 3, \quad c(gx_1) \leq c(g) + 3.$$

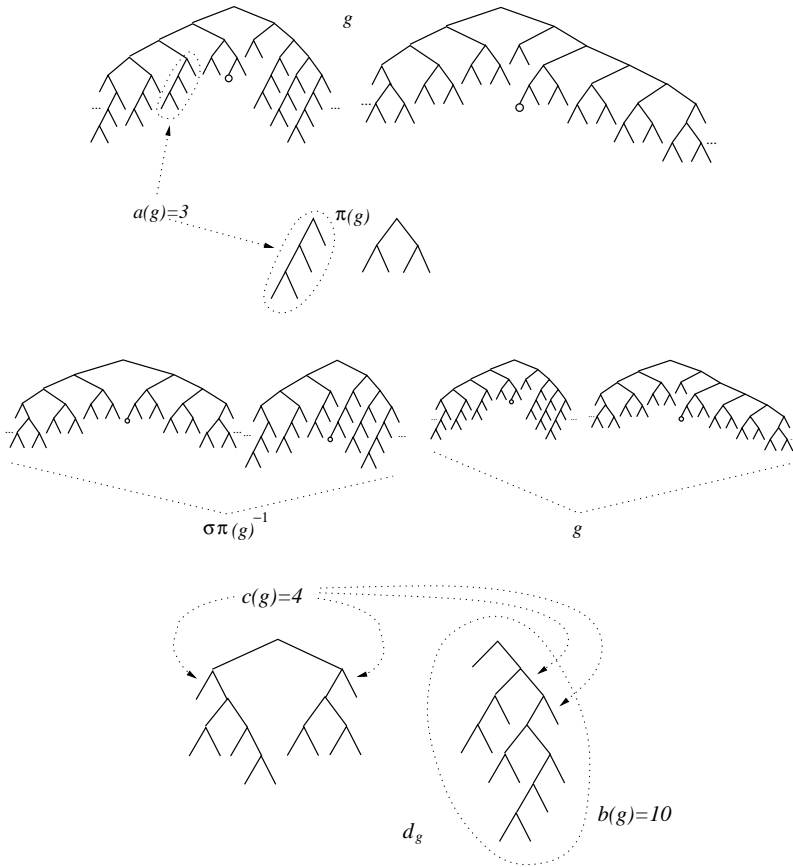


FIGURE 14: An example of an element in C and its values for a , b and c . From top to bottom, we see the element with a , the process of computing d_g and the values of b and c .

3. **Multiplication by w_0 .** When we multiply g by w_0 , we notice that a can grow by two carets, in case that there is no periodic part in g . If there is already a periodic part, it grows by at most one. We see that c is unchanged from g to gw_0 . The integral leaves that will get canceled when multiplying $(\sigma \circ \pi(g))^{-1}$ by g will get modified by w_0 , but they will still be the same for the two elements $(\sigma \circ \pi(gw_0))^{-1}$ and gw_0 , and they will get canceled the same way. So the left and right carets for d_{gw_0} will be the same as those for d_g .

Another way of seeing that c is unchanged is by appealing to the maps in \mathbb{R} . If g is periodic outside of a box $[p, q] \times [r, s]$, for integers p, q, r, s , then gw_0 is also periodic outside this box. The element w_0 sends the integer interval $[r, s]$ to itself.

Finally, if c is unchanged, we see that the worst that can happen to b is that it increases by two carets in every integral leaf that survives in the debris.

We note that when premultiplying by $(\sigma \circ \pi(gw_0))^{-1}$, in each integral leaf there may be two more carets than before, and in the debris, there could be 2 carets more surviving in each integral leaf. Hence, b grows by at most $2c$. Summarizing:

$$a(gw_0) \leq a(g) + 2, \quad b(gw_0) \leq b(g) + 2c(g), \quad c(gw_0) \leq c(g).$$

4. **Multiplication by w_1 .** The situation is again similar to the one for w_0 . The number a can grow by at most 3, c stays the same, and b can grow in the worst case by 3 times c . So we have

$$a(gw_1) \leq a(g) + 3, \quad b(gw_1) \leq b(g) + 3c(g), \quad c(gw_1) \leq c(g).$$

We note that multiplying by the inverses of the generators will give the same bounds for a , b , and c .

The inequalities for $b(gw_0)$ and $b(gw_1)$ where the number of additional carets can grow by a multiple of $c(g)$ is the reason for the quadratic distortion. In the example shown in Section 5 each surviving integral leaf in r_n (of which there are n), gets n carets, giving n^2 carets in the element.

Armed with these inequalities, we now consider any element g in C , and assume it has length $L = |g|_C$. Therefore it has been constructed by multiplication of L generators, and hence in each one of the L multiplications, the quantities a , b and c may have suffered the maximal increases detailed above. This gives the following inequalities:

- 1) $a(g)$ and $c(g)$ may have grown by at most 3 in each step, so $a(g) \leq 3L$ and $c(g) \leq 3L$.
- 2) $b(g)$ grows, in the worst case, by $3c$. Since c is at most $3L$ at the end and thus also in each step, clearly we have that $b(g) \leq 9L^2$, or, in a formulation better for our purposes, $\sqrt{b(g)} \leq 3L$.

The conclusion is the following: there exist a constant $K > 0$ such that

$$a(g) + \sqrt{b(g)} + c(g) \leq K|g|_C.$$

This inequality is all we need to finish the proof of Theorem 5.1. Let w be an element of F_w with $N(w)$ carets, and recall that $N(w)$ is equivalent to $|w|_{F_w}$. When lifted to C , the element $\sigma(w)$ is everywhere periodic, so it has $a(\sigma(w)) = N(w)$, and $d_w = 1$, so $b(\sigma(w)) = 0$ and $c(\sigma(w)) = 0$. Hence, $N(w)$ is bounded above by $|w|_C$ (up to a multiplicative constant), and so is $|w|_{F_w}$. Thus the periodic subgroup F_w is undistorted.

For the embedding of the nonperiodic subgroup F_x , we take a general element x of F_x and consider it also as an element of C . When seen in F_x , it has some number of carets $N(x)$, equivalent to $|x|_{F_x}$. However, naturally $d_x = x$ if $x \in F_x$, so we have $b(x) = N(x)$. Our inequalities above say that there exists a constant K such that

$$|x|_{F_x} \leq K|x|_C^2,$$

which proves that the distortion of F_x in C is at most quadratic. From the examples explicitly constructed in Section 5, we see that the distortion is exactly quadratic.

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