Rev. Mat. Iberoam. **29** (2013), no. 3, 997–1020
DOI 10.4171/RMI/747

Infinitely many nonradial solutions for the Hénon equation with critical growth

Juncheng Wei and Shusen Yan

Abstract. We consider the following Hénon equation with critical growth:

$$(*)\begin{cases} -\Delta u = |y|^{\alpha}\, u^{\frac{N+2}{N-2}},\ u>0 & y\in B_1(0),\\ u=0, & \text{on } \partial B_1(0),\end{cases}$$

where $\alpha > 0$ is a positive constant, $B_1(0)$ is the unit ball in $\mathbb{R}^N$, and $N \geq 4$. Ni [9] proved the existence of a radial solution and Serra [12] proved the existence of a nonradial solution for α *large* and $N \geq 4$. In this paper, we show the existence of a nonradial solution for *any* $\alpha > 0$ and $N \geq 4$. Furthermore, we prove that equation (*) has *infinitely many nonradial* solutions, whose energy can be made arbitrarily large.

1. Introduction

Of concern is the following Hénon equation with critical growth:

$$\begin{cases} -\Delta u = |y|^{\alpha}\, u^{\frac{N+2}{N-2}},\ u>0, & y\in B_1(0),\\ u=0, & \text{on } \partial B_1(0),\end{cases} \tag{1.1}$$

where $\alpha > 0$ is a positive constant, $B_1(0)$ is the unit ball in $\mathbb{R}^N$, and $N \geq 3$.

Equation (1.1) arises in the study of astrophysics, see [7]. If the exponent $(N+2)/(N-2)$ is replaced by p, where $p < (N+2)/(N-2)$, a solution can be obtained easily by variational methods. When $p = (N+2)/(N-2)$, the loss of compactness from $H_0^1(B_1(0))$ to $L^{\frac{2N}{N-2}}(B_1(0))$ makes the problem (1.1) very difficult to study. Ni [9] first proved the existence of a *radial solution* for *any* $\alpha > 0$. On the other hand, it is easy to check that the mountain pass value c corresponding to (1.1) is

$$c = \frac{1}{N}\, S^{N/2},$$

Mathematics Subject Classification (2010): Primary 35J65; Secondary 35B38, 47J30.
Keywords: Henon's equation, infinitely many solutions, critical Sobolev exponent, reduction method.

where S is best Sobolev constant of the embedding from $D^{1,2}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, from which we can deduce that c is not a critical value of the functional corresponding to (1.1). When $N = 2$, Smets–Su–Willem [13] showed that the mountain pass solution is nonradial when α is large. When $N \geq 3$, for the Hénon equations with nearly critical growth (replacing $(N+2)/(N-2)$ in (1.1) by $(N+2)/(N-2)-\varepsilon$ with $\varepsilon > 0$ small), Cao–Peng [3] proved that the mountain pass solution is nonradial and blows up as $\varepsilon \to 0$. Thus, it is natural to ask whether (1.1) has a nonradial solution. Using a variational method, Serra [12] proved that (1.1) has a nonradial solution when $N \geq 4$ and α *is large*. As far as we know, up to now, there is no result showing the existence of *nonradial solution* of (1.1), nor is there a *multiplicity* result, with arbitrary $\alpha > 0$, for (1.1).

The aim of this paper is to prove that (1.1) has infinitely many nonradial solutions if $N \geq 4$. In fact, we will study a more general problem:

$$\begin{cases} -\Delta u = K(|y|)\, u^{\frac{N+2}{N-2}}, \ u > 0, & y \in B_1(0), \\ u = 0, & \text{on } \partial B_1(0), \end{cases} \tag{1.2}$$

where $K(r)$ is a bounded function defined in $[0,1]$. It is easy to see that a necessary condition for the existence of a solution of (1.2) is that $K(r)$ is positive somewhere. On the other hand, Pohozaev identity implies (1.2) has no solution if $K'(r) \leq 0$ in $[0,1]$. Concerning the existence of solutions for (1.2), using the same method as in [15], we can prove the following existence result:

Theorem A. *Suppose that there is a $r_0 \in (0,1)$, such that $K(r_0) > 0$, and*

$$K(r) = K(r_0) - K_0|r - r_0|^m + O(|r - r_0|^{m+\theta}), \quad \text{as } r \to r_0, \tag{1.3}$$

where $m \in [2, N-2)$, $K_0 > 0$, and $\theta > 0$ are some constants. Then, for $N \geq 5$, the problem (1.2) *has infinitely many nonradial solutions.*

Note that for the Hénon equation, $K(r) = r^\alpha$, which has no critical point in $(0,1)$. So, Theorem A does not apply to the Henon equation (1.1).

Condition (1.3) implies that r_0 is a local maximum point of $K(r)$, and thus a critical point of $K(r)$. The function r^α attains its maximum on $[0,1]$ at $r_0 = 1$, but $r_0 = 1$ is not a critical point of r^α.

The aim of this paper is to show that if $K(r)$ is increasing near $r_0 = 1$ (so it is a maximum point of $K(r)$ on $[1-\delta, 1]$ for some small $\delta > 0$), the zero Dirichlet boundary condition makes it possible to construct infinitely many solutions of (1.2), although $r_0 = 1$ is not a critical point of $K(r)$. Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that $N \geq 4$. If $K(r)$ satisfies $K(1) > 0$ and $K'(1) > 0$, then problem* (1.2)*)has infinitely many nonradial solutions. In particular, the Hénon equation* (1.1) *has infinitely many nonradial solutions.*

Recall that a necessary condition for the existence of at least one solution of (1.2) is that $K'(r)$ is positive somewhere on $[0,1]$. If $K(r) \geq 0$ and $N \geq 5$, Theorems A and 1.1 show that under a condition which is slightly stronger than this necessary condition, (1.2) has infinitely many solutions.

We think that the condition that $N \geq 4$ is just technical. The reason is that the reduced energy does have a critical point when $N = 3$. The problem lies in the reduction part which should be only technical. (Some partial (negative) results are obtained by O. Druet and Laurain [6].).

The reader can refer to [1], [2], [4], [8], [10], [11], and [14] for results on Hénon equations involving subcritical and near critical exponents.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

Let us fix a positive integer $k \geq k_0$, where k_0 is large, which is to be determined later.

Set

$$\mu = k^{\frac{N-1}{N-2}}, \quad N \geq 4$$

to be the scaling parameter.

Let $2^* = 2N/(N-2)$. Using the transformation $u(y) \mapsto \mu^{-(N-2)/2} u(y/\mu)$, we find that (1.2) becomes

$$\begin{cases} -\Delta u = K\left(\frac{|y|}{\mu}\right) u^{2^*-1}, u > 0, & y \in B_\mu(0), \\ u = 0, & \text{on } \partial B_\mu(0). \end{cases} \tag{1.4}$$

It is well known that the functions

$$U_{x,\Lambda}(y) = \big(N(N-2)\big)^{(N-2)/4} \Big(\frac{\Lambda}{1 + \Lambda^2 |y-x|^2}\Big)^{(N-2)/2}, \quad \mu > 0, \quad x \in \mathbb{R}^N$$

are the only solutions to the following problem

$$-\Delta u = u^{\frac{N+2}{N-2}}, \qquad u > 0 \ \text{ in } \mathbb{R}^N.$$

As the scaling parameter $\Lambda \to +\infty$, $U_{x,\Lambda}$ is called a *single-bubble* centered at the point x. Since there is no *small* parameter in (1.1) (here μ is fixed), we use the scaling parameter Λ as the blow-up parameter. Our main idea is to *place* a large number of bubbles inside Ω. Then the scaling parameter will be determined by the *number of bubbles*. We put many bubbles *along a $k-$polygon inside the domain $B_1(0)$ but near the boundary*. See Figure 1. (The idea of using the number of bubbles as parameter was first introduced in [15].)

Let us remark that the variational method of Serra [12] also uses the dihedral symmetry of $k-$polygons. By using the $D_k \times O(N-2)$ symmetry, the problem (1.1) can be reduced to the one in a sector. He then showed that under dihedral symmetry, the loss of compactness can be recovered if the critical value is below some constant, which holds true when $N \geq 4$. To show that the solution is nonradial, he needed to compare with the energy level of a radial solution. There the condition that α is large is needed. Our method of construction is direct and gives more information.

We continue our construction. Since $U_{x,\Lambda}$ is not zero on $\partial B_\mu(0)$, we define $PU_{x,\Lambda}$ as the solution of the following problem:

$$\Delta PU_{x,\Lambda} = \Delta U_{x,\Lambda}, \ \text{ in } B_\mu(0), \quad \Delta PU_{x,\Lambda} = 0 \ \text{ on } \partial B_\mu(0). \tag{1.5}$$

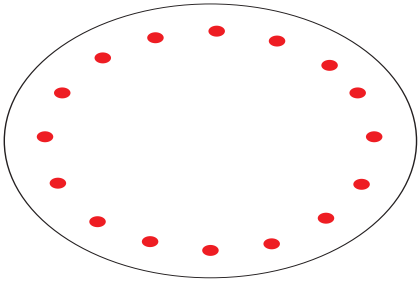

FIGURE 1. The location of the bubbles.

Let $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$. Define

$$H_s = \big\{u : u \in H_0^1(B_\mu(0)), u \text{ is even in } y_h, h = 2, \dots, N,$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \tfrac{2\pi j}{k}), r\sin(\theta + \tfrac{2\pi j}{k}), y'')\big\}.$$

Let

$$x_j = \big(r\cos\tfrac{2(j-1)\pi}{k}, r\sin\tfrac{2(j-1)\pi}{k}, 0\big), \quad j = 1, \dots, k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and let

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}.$$

In this paper, we always assume that

$$r \in \Big[\mu\,(1 - \frac{r_0}{k}), \mu\,(1 - \frac{r_1}{k})\Big], \quad \text{for some constants } r_1 > r_0 > 0,$$

and

$$L_0 \le \Lambda \le L_1, \quad \text{for some constants } L_1 > L_0 > 0.$$

Theorem 1.1 is a direct consequence of the following result:

Theorem 1.2. *Suppose that $N \ge 4$. If $K(1) > 0$ and $K'(1) > 0$, then there is an integer $k_0 > 0$, such that for any integer $k \ge k_0$, (1.4) has a solution u_k of the form*

$$u_k = W_{r_k,\Lambda_k}(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\|\omega_k\|_{L^\infty} \to 0$, $L_0 \le \Lambda_k \le L_1$, and $r_k \in \big(\mu\,(1 - r_0/k), \mu\,(1 - r_1/k)\big)$.

Unlike Theorem A, where the result was proved by constructing solutions with many bubbles near the local maximum point $r_0 \in (0, 1)$, the solutions constructing in Theorem 1.1 have many bubbles near the boundary of the unit ball $B_1(0)$. In Theorem 1.1, $r_0 = 1$ is not a critical point of $K(r)$ anymore. It is the zero boundary condition that plays a very important role in the construction of solutions with many bubbles near $|y| = 1$.

2. Finite-dimensional reduction

In this section, we perform a finite-dimensional reduction. Let

$$\|u\|_* = \sup_{y\in B_\mu(0)} \Big(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1} |u(y)|, \tag{2.1}$$

and

$$\|f\|_{**} = \sup_{y\in B_\mu(0)} \Big(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1} |f(y)|, \tag{2.2}$$

where $\tau = (N-2)/(N-1)$ if $N \geq 4$. For this choice of τ, we find that

$$\sum_{j=2}^{k} \frac{1}{|x_j - x_1|^\tau} \leq \frac{Ck^\tau}{\mu^\tau} \sum_{j=2}^{k} \frac{1}{j^\tau} \leq \frac{Ck}{\mu^\tau} \leq C'.$$

Let

$$Z_{i,1} = \frac{\partial PU_{x_i,\Lambda}}{\partial r}, \quad Z_{i,2} = \frac{\partial PU_{x_i,\Lambda}}{\partial \Lambda}.$$

Consider

$$\begin{cases} -\Delta\phi_k - (2^*-1)K\big(\frac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi_k = h + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^*-2} Z_{i,j}, & \text{in } B_\mu(0), \\ \phi_k \in H_s, \\ < U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \phi_k >= 0, \quad i = 1,\dots,k, \ l = 1,2 \end{cases} \tag{2.3}$$

for some numbers c_i, where $< u, v >= \int_{B_\mu(0)} uv$.

Lemma 2.1. *Assume that ϕ_k solves (2.3) for $h = h_k$. If $\|h_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. The proof of this lemma is similar to the proof of Lemma 2.1 in [15]. Therefore, we only sketch it.

We argue by contradiction. Suppose that there are $k \to +\infty$, $h = h_k$, $\Lambda_k \in [L_1, L_2]$, $r_k \in \big[\mu(1-r_0/k), \mu(1-r_1/k)\big]$, and ϕ_k solving (2.3) for $h = h_k$, $\Lambda = \Lambda_k$, and $r = r_k$, with $\|h_k\|_{**} \to 0$, and $\|\phi_k\|_* \geq c' > 0$. We may assume that $\|\phi_k\|_* = 1$. For simplicity, we drop the subscript k.

We rewrite (2.3) as

$$\begin{aligned} \phi(y) &= (2^*-1)\int_{B_\mu(0)} \frac{1}{|z-y|^{N-2}} K\big(\tfrac{|z|}{\mu}\big) W_{r,\Lambda}^{2^*-2}\phi(z)\,dz \\ &\quad + \int_{B_\mu(0)} \frac{1}{|z-y|^{N-2}} \Big(h(z) + \sum_{j=1}^{2} c_j \sum_{i=1}^{k} Z_{i,j}(z)\, U_{x_i,\Lambda}^{2^*-2}(z)\Big)\,dz. \end{aligned} \tag{2.4}$$

Using Lemma B.3, we have

$$\begin{aligned}&\Big|(2^*-1)\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}\,K\big(\tfrac{|z|}{\mu}\big)\,W_{r,\Lambda}^{2^*-2}\,\phi(z)\,dz\Big|\\ &\quad\le C\,\|\phi\|_*\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}\,W_{r,\Lambda}^{2^*-2}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz\\ &\quad\le C\,\|\phi\|_*\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.\end{aligned}\tag{2.5}$$

It follows from Lemma B.2 that

$$\Big|\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}\,h(z)\,dz\Big|\le C\,\|h\|_{**}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}.\tag{2.6}$$

and

$$\Big|\int_{B_\mu(0)}\frac{1}{|z-y|^{N-2}}\sum_{i=1}^{k}Z_{i,l}(z)\,U_{x_i,\Lambda}^{2^*-2}(z)\,dz\Big|\le C\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.\tag{2.7}$$

Next, we estimate c_l, $l=1,2$. Multiplying (2.3) by $Z_{1,l}$ and integrating, we see that c_t satisfies

$$\sum_{t=1}^{2}\sum_{i=1}^{k}\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,t},Z_{1,l}\rangle c_t=\big\langle-\Delta\phi-(2^*-1)K\big(\tfrac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi,Z_{1,l}\big\rangle-\big\langle h,Z_{1,l}\big\rangle.\tag{2.8}$$

It follows from Lemma B.1 that

$$\big|\langle h,Z_{1,l}\rangle\big|\le C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{(1+|z-x_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz\le C\|h\|_{**}.$$

On the other hand, using Lemma B.3, we can prove

$$\begin{aligned}&\big\langle-\Delta\phi-(2^*-1)K\big(\tfrac{|z|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\phi,Z_{1,l}\big\rangle\\ &\qquad=(2^*-1)\big\langle\big(1-K\big(\tfrac{|z|}{\mu}\big)W_{r,\Lambda}^{2^*-2}Z_{1,l},\phi\big\rangle=o(\|\phi\|_*).\end{aligned}\tag{2.9}$$

However, there is a constant $\bar c>0$,

$$\sum_{i=1}^{k}\big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,t}\,,\,Z_{1,l}\big\rangle=(\bar c+o(1))\,\delta_{tl}.$$

Thus we obtain from (2.8) that

$$c_l=o(\|\phi\|_*)+O(\|h\|_{**}).\tag{2.10}$$

So,

$$\|\phi\|_* \le \left(o(1) + \|h_k\|_{**} + \frac{\sum_{j=1}^k (1+|y-x_j|)^{-\frac{N-2}{2}-\tau-\theta}}{\sum_{j=1}^k (1+|y-x_j|)^{-\frac{N-2}{2}-\tau}} \right). \tag{2.11}$$

Since $\|\phi\|_* = 1$, we obtain from (2.11) that there is $R > 0$, such that

$$\|\phi(y)\|_{B_R(x_i)} \ge a > 0, \tag{2.12}$$

for some i. However, $\bar{\phi}(y) = \phi(y - x_i)$ converges uniformly in any compact set to a solution u of

$$-\Delta u - (2^* - 1)\, U_{0,\Lambda}^{2^*-2}\, u = 0, \quad \text{in } \mathbb{R}^N, \tag{2.13}$$

for some $\Lambda \in [L_1, L_2]$, and u is perpendicular to the kernel of (2.13). Hence, $u = 0$. This is a contradiction to (2.12). □

From Lemma 2.1, using the same argument as in the proof of Proposition 4.1 in [5], we can prove the following result :

Proposition 2.2. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \ge k_0$ and all $h \in L^\infty(\mathbb{R}^N)$, problem (2.3) has a unique solution $\phi \equiv L_k(h)$. Moreover,*

$$\|L_k(h)\|_* \le C\, \|h\|_{**}, \quad |c_l| \le C\, \|h\|_{**}. \tag{2.14}$$

Now, we consider

$$\begin{cases} -\Delta\big(W_{r,\Lambda} + \phi\big) = K\big(\frac{y}{\mu}\big)\big(W_{r,\Lambda} + \phi\big)^{2^*-1} + \displaystyle\sum_{t=1}^{2} c_t \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^*-2} Z_{i,t}, & \text{in } B_\mu(0), \\ \phi_k \in H_s, \\ < U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \phi_k >= 0, \quad i = 1,\dots,k,\ l = 1,2. \end{cases} \tag{2.15}$$

We have

Proposition 2.3. *There is an integer $k_0 > 0$, such that for each $k \ge k_0$, $L_0 \le \Lambda \le L_1$, $r \in \big[\mu(1 - r_0/k), \mu(1 - r_1/k)\big]$, (2.15) has a unique solution $\phi = \phi(r, \Lambda)$, satisfying*

$$\|\phi\|_* \le C\Big(\frac{1}{\mu}\Big)^{1/2+\sigma}, \quad |c_t| \le C\Big(\frac{1}{\mu}\Big)^{1/2+\sigma},$$

if $N \ge 4$, where $\sigma > 0$ is a small constant, and $\mu = k^{\frac{N-1}{N-2}}$.

Rewrite (2.15) as

$$\begin{cases} -\Delta\phi - (2^*-1) K\big(\frac{|y|}{\mu}\big) W_{r,\Lambda}^{2^*-2}\phi = N(\phi) + l_k + \displaystyle\sum_{t=1}^{2} c_i \sum_{i=1}^{k} U_{x_i,\Lambda}^{2^*-2} Z_{i,t}, & \text{in } B_\mu(0), \\ \phi \in H_s, \\ < U_{x_i,\Lambda}^{2^*-2} Z_{i,l}, \phi >= 0, \quad i = 1,\dots,k,\ l = 1,2, \end{cases} \tag{2.16}$$

where

$$N(\phi) = K\big(\tfrac{|y|}{\mu}\big)\,\big((W_{r,\Lambda}+\phi)^{2^*-1} - W_{r,\Lambda}^{2^*-1} - (2^*-1)W_r^{2^*-2}\phi\big),$$

and

$$l_k = K\big(\tfrac{|y|}{\mu}\big)\,W_{r,\Lambda}^{2^*-1} - \sum_{j=1}^k U_{x_j,\Lambda}^{2^*-1}.$$

In order to use the contraction mapping theorem to prove that (2.16) is uniquely solvable in the set where $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

Lemma 2.4. *If $N \ge 4$, then*

$$\|N(\phi)\|_{**} \le C\,\|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|N(\phi)| \le \begin{cases} C\,|\phi|^{2^*-1}, & N \ge 6;\\ C\,\big(W_{r,\Lambda}^{\frac{6-N}{N-2}}\,\phi^2 + |\phi|^{2^*-1}\big), & N = 4,5.\end{cases}$$

First, we consider $N \ge 6$. Using

$$\sum_{j=1}^k a_j b_j \le \big(\sum_{j=1}^k a_j^p\big)^{1/p}\,\big(\sum_{j=1}^k b_j^q\big)^{1/q}, \quad \frac{1}{p}+\frac{1}{q} = 1,\ a_j,\, b_j \ge 0,$$

we obtain

$$\begin{aligned} |N(\phi)| &\le C\,\|\phi\|_*^{2^*-1}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\ &\le C\,\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\tau}}\Big)^{\frac{4}{N-2}}\\ &\le C\,\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}. \end{aligned} \tag{2.17}$$

Thus, the result follows.

Suppose that $N = 4$ or 5. Noting that $N-2 \ge (N-2)/2+\tau$, we find

$$\begin{aligned} |N(\phi)| &\le C\,\|\phi\|_*^2\,\Big(\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{N-2}}\Big)^{\frac{6-N}{N-2}}\,\Big(\sum_{j=1}^k \frac{1}{1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2\\ &\quad + C\,\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\ &\le C\,\|\phi\|_*^2\Big(\sum_{j=1}^k \frac{1}{1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} + C\,\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\ &= C\,\|\phi\|_*^2\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}. \end{aligned}$$

So, we have proved that, for $N \geq 4$,

$$\|N(\phi)\|_{**} \leq C \|\phi\|_*^{\min(2,2^*-1)}. \qquad \square$$

Next, we estimate l_k.

Lemma 2.5. *Assume that* $r \in \big[\mu(1-r_0/k), \mu(1-r_1/k)\big]$. *If* $N \geq 4$, *then*

$$\|l_k\|_{**} \leq C\Big(\frac{1}{\mu}\Big)^{1/2+\sigma}.$$

Proof. Define

$$\Omega_j = \Big\{y: \ y=(y',y'') \in B_\mu(0), \langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\rangle \geq \cos\frac{\pi}{k}\Big\}.$$

We have

$$\begin{aligned} l_k &= K\big(\tfrac{|y|}{\mu}\big)\Big(W_{r,\Lambda}^{2^*-1} - \sum_{j=1}^k \big(PU_{x_j,\Lambda}\big)^{2^*-1}\Big) \\ &\quad + K\big(\tfrac{|y|}{\mu}\big)\Big(\sum_{j=1}^k \big(PU_{x_j,\Lambda}\big)^{2^*-1} - \sum_{j=1}^k U_{x_j,\Lambda}^{2^*-1}\Big) + \sum_{j=1}^k U_{x_j,\Lambda}^{2^*-1}\Big(K\big(\tfrac{|y|}{\mu}\big)-1\Big) \\ &=: J_0 + J_1 + J_2. \end{aligned}$$

Using the assumed symmetry, we can suppose that $y \in \Omega_1$. Then,

$$|y-x_j| \geq |y-x_1|, \quad \forall\, y \in \Omega_1.$$

First, we claim

$$\frac{1}{1+|y-x_j|} \leq \frac{C}{|x_j-x_1|}, \quad \forall\, y \in \Omega_1, \ j \neq 1. \tag{2.18}$$

In fact, if $|y-x_1| \leq \frac{1}{2}|x_1-x_j|$, then $|y-x_j| \geq \frac{1}{2}|x_1-x_j|$. If $|y-x_1| \geq \frac{1}{2}|x_1-x_j|$, then $|y-x_j| \geq |y-x_1| \geq \frac{1}{2}|x_1-x_j|$, since $y \in \Omega_1$.

For the estimate of J_0, we have

$$\begin{aligned} |J_0| \leq{}& C\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{N-2}} \\ &+ C\Big(\sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}. \end{aligned} \tag{2.19}$$

Using (2.18), and taking $1 < \alpha \leq N-2$, we obtain that, for any $y \in \Omega_1$,

$$\begin{aligned} &\frac{1}{(1+|y-x_1|)^4}\,\frac{1}{(1+|y-x_j|)^{N-2}} \\ &\leq C\frac{1}{(1+|y-x_1|)^{N+2-\alpha}}\,\frac{1}{|x_j-x_1|^\alpha}, \quad j>1. \end{aligned} \tag{2.20}$$

Take $\alpha > \max((N-1)/2, 1)$ satisfying $N+2-\alpha \geq (N+2)/2+\tau$. Then

$$\frac{1}{(1+|y-x_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}} \leq \frac{C}{(1+|y-x_1|)^{N+2-\alpha}}\Big(\frac{k}{\mu}\Big)^{\alpha}$$

$$= \frac{C}{(1+|y-x_1|)^{N+2-\alpha}}\mu^{-\alpha/(N-1)} \leq C\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{1}{\mu}\Big)^{1/2+\sigma}. \tag{2.21}$$

Using the Hölder inequality, we obtain

$$\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}$$
$$\leq \sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{4/(N-2)}.$$

Noting that $\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2}) > 1$ if $N \geq 4$, we obtain

$$\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2}}\Big)^{2^*-1}$$
$$\leq C\Big(\sum_{j=2}^{k}\frac{1}{|x_1-x_j|^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{4/(N-2)}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}$$
$$\leq C\Big(\frac{k}{\mu}\Big)^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})\frac{4}{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \tag{2.22}$$
$$= C\Big(\frac{1}{\mu}\Big)^{\frac{N+2}{N-1}(\frac{1}{2}-\frac{\tau}{N+2})}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}$$
$$= C\Big(\frac{1}{\mu}\Big)^{1/2+\sigma}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}},$$

since $\frac{N+2}{N-1}(\frac{1}{2}-\frac{\tau}{N+2}) > \frac{1}{2}$. Thus, we have proved that if $N \geq 4$,

$$\|J_0\|_{**} \leq C\Big(\frac{1}{\mu}\Big)^{1/2+\sigma}.$$

Now, we estimate J_1. Let $H(y,x)$ be the regular part of the Green function for $-\Delta$ in $B_1(0)$ with the zero boundary condition. Let $\bar{x}_j^*$ be the reflection point of $\bar{x}_j$ with respect to $\partial B_1(0)$. Then

$$\frac{H(\bar{y},\bar{x}_j)}{\mu^{N-2}} = \frac{C}{\mu^{N-2}|\bar{y}-\bar{x}_j^*|^{N-2}} \leq \frac{C}{(1+|y-x_j|)^{N-2}}.$$

Take $t = 1 - \theta$ with $\theta > 0$ small. Then using (A.1), we find

$$
\begin{aligned}
|J_1| &\le \sum_{j=1}^{k} \frac{C}{(1+|y-x_j|)^4} \frac{H(\bar{y},\bar{x}_j)}{\mu^{N-2}} \\
&\le \sum_{j=1}^{k} \frac{C}{(1+|y-x_j|)^{4+t(N-2)}} \Big(\frac{H(\bar{y},\bar{x}_j)}{\mu^{N-2}}\Big)^t \\
&\le C\Big(\frac{1}{\mu d}\Big)^{t(N-2)} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4+t(N-2)}} \\
&\le C\Big(\frac{1}{\mu}\Big)^{t\frac{N-2}{N-1}} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{4+t(N-2)}} \\
&\le C\Big(\frac{1}{\mu}\Big)^{1/2+\sigma} \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}},
\end{aligned}
\tag{2.23}
$$

since $t\frac{N-2}{N-1} > 1/2$ for $N \ge 4$, $4 + t(N-2) \ge (N+2)/2 + \tau$, and $d \ge r_0/k$.

Finally, we estimate J_2. For $y \in \Omega_1$, and $j > 1$, using (2.18), we have

$$
U_{x_j,\Lambda}^{2^*-1}(y) \le C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{|x_1-x_j|^{\frac{N+2}{2}-\tau}},
$$

which implies

$$
\begin{aligned}
\Big|\sum_{j=2}^{k}\Big(K\big(\frac{|y|}{\mu}\big)-1\Big)U_{x_j,\Lambda}^{2^*-1}\Big| &\le C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \sum_{j=2}^{k} \frac{1}{|x_1-x_j|^{\frac{N+2}{2}-\tau}} \\
\le C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \Big(\frac{k}{\mu}\Big)^{\frac{N+2}{2}-\tau} &\le C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \Big(\frac{1}{\mu}\Big)^{1/2+\sigma}.
\end{aligned}
\tag{2.24}
$$

For $y \in \Omega_1$ and $||y| - \mu| \ge \delta\mu$, where $\delta > 0$ is a fixed constant, then

$$
\big||y| - |x_1|\big| \ge \big||y| - \mu\big| - \big||x_1| - \mu\big| \ge \frac{1}{2}\delta\mu.
$$

As a result,

$$
\Big|U_{x_1,\Lambda}^{2^*-1}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big)\Big| \le C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{\mu^{\frac{N+2}{2}-\tau}}.
\tag{2.25}
$$

If $y \in \Omega_1$ and $||y| - \mu| \le \delta\mu$, then

$$
\begin{aligned}
\big|K\big(\tfrac{|y|}{\mu}\big) - 1\big| &\le C\,\big|\tfrac{|y|}{\mu} - 1\big| \le \frac{C}{\mu}\big((||y|-|x_1||) + ||x_1|-\mu|)\big) \\
&\le \frac{C}{\mu}\big||y|-|x_1|\big| + \frac{C}{k} = \frac{C}{\mu}\big||y|-|x_1|\big| + \frac{C}{\mu^{\frac{N-2}{N-1}}} \le \frac{C}{\mu}\big||y|-|x_1|\big| + \frac{C}{\mu^{1/2+\sigma}},
\end{aligned}
$$

and

$$
\big||y| - |x_1|\big| \le \big||y| - \mu\big| + \big|\mu - |x_1|\big| \le 2\delta\mu.
$$

However,

$$\frac{||y|-|x_1||}{\mu}\frac{1}{(1+|y-x_1|)^{N+2}}=\frac{C}{\mu^{1/2+\sigma}}\frac{||y|-|x_1||^{1/2-\sigma}}{(1+|y-x_1|)^{N+2}}$$
$$\le\frac{C}{\mu^{1/2+\sigma}}\frac{1}{(1+|y-x_1|)^{N+2-1/2+\sigma}}\le\frac{C}{\mu^{1/2+\sigma}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.$$

Thus, we obtain

$$\Big|U_{x_1,\Lambda}^{2^*-1}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big)\Big|\le\frac{C}{\mu^{1/2+\sigma}}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}},\quad ||y|-\mu|\le\delta\mu. \tag{2.26}$$

Combining (2.24), (2.25) and (2.26), we obtain

$$\|J_2\|_{**}\le C\Big(\frac{1}{\mu}\Big)^{1/2+\sigma}.$$ □

Now, we are ready to prove Proposition 2.3.

Proof of Proposition 2.3. Let us recall that

$$\mu=k^{\frac{N-1}{N-2}},\quad N\ge4.$$

Let

$$E=\Big\{u:u\in C(B_\mu(0))\cap H_s,\ \|u\|_*\le\big(\tfrac1k\big)^{1/2},$$
$$\int_{B_\mu(0)}U_{x_i,\Lambda}^{2*-2}Z_{i,l}\phi=0\,,\ i=1,\dots,k,\ l=1,2\Big\}.$$

Then, (2.16) is equivalent to

$$\phi=A(\phi)=:L_k(N(\phi))+L_k(l_k),$$

where L_k is defined in Proposition 2.2. We will prove that A is a contraction map from E to E.

We have

$$\begin{aligned}\|A(\phi)\|_*&\le C\|N(\phi)\|_{**}+C\|l_k\|_{**}\\&\le C\|\phi\|_*^{\min(2^*-1,2)}+C\|l_k\|_{**}\le\frac{C}{k^{1/2+\sigma}}\le\frac{1}{k^{1/2}}.\end{aligned} \tag{2.27}$$

Thus, A maps E to E.

On the other hand,

$$\|A(\phi_1)-A(\phi_2)\|_*=\|L_k(N(\phi_1))-L_k(N(\phi_2))\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}.$$

If $N\ge6$, then

$$|N'(t)|\le C\,|t|^{2^*-2}.$$

As a result,

$$|N(\phi_1)-N(\phi_2)| \le C\left(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\right)|\phi_1-\phi_2|$$
$$\le C\left(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\right)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}.$$

As before, we have

$$\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\le C\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

So,

$$\|A(\phi_1)-A(\phi_2)\|_*\le C\|N(\phi_1)-N(\phi_2)\|_{**}$$
$$\le C\big(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\big)\|\phi_1-\phi_2\|_*\le \frac12\|\phi_1-\phi_2\|_*.$$

Thus, A is a contraction map.

For $N=4$ or 5,

$$|N'(t)|\le C\,W_{r,\Lambda}^{\frac{6-N}{N-2}}\,|t|+C\,|t|^{2^*-2}.$$

So,

$$|N(\phi_1)-N(\phi_2)|$$
$$\le C\big(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2}\big)|\phi_1-\phi_2|+C\big(|\phi_1|+|\phi_2|\big)W_{r,\Lambda}^{\frac{6-N}{N-2}}|\phi_1-\phi_2|$$
$$\le C\left(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2}\right)\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}$$
$$\tag{2.28} +C\left(\|\phi_1\|_*+\|\phi_2\|_*\right)\|\phi_1-\phi_2\|_*W_{r,\Lambda}^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2}$$
$$\le C\left(\|\phi_1\|_*+\|\phi_2\|_*\right)\|\phi_1-\phi_2\|_*\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

Thus, A is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi\in E$, such that

$$\phi=A(\phi).$$

Moreover, it follows from Proposition 2.2 that

$$\|\phi\|_*\le C\,\|l_k\|_{**}+C\,\|N(\phi)\|_{**}\le C\,\|l_k\|_{**}+C\,\|\phi\|_*^{\min(2^*-1,2)},$$

which gives, if $N\ge 4$,

$$\|\phi\|_*\le C\left(\frac{1}{\mu}\right)^{1/2+\sigma},$$

Finally, the estimate of c_t comes from (2.14). See also (2.10). □

3. Proof of Theorem 1.2

Let

$$F(d,\Lambda)=I\big(W_{r,\Lambda}+\phi\big),$$

where $r=|x_1|$, $d=1-r/\mu$, ϕ is the function obtained in Proposition 2.3, and

$$I(u)=\frac{1}{2}\int_{B_\mu(0)}|Du|^2-\frac{1}{2^*}\int_{B_\mu(0)}K\Big(\frac{|y|}{\mu}\Big)\,|u|^{2^*}.$$

Proposition 3.1. *If $N\ge 4$, then*

$$\begin{aligned}F(d,\Lambda)&=I(W_{r,\Lambda})+O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\\&=k\Big(A+\frac{B_1H(\bar x_1,\bar x_1)}{\Lambda^{N-2}\mu^{N-2}}+B_2K'(1)d-\sum_{i=2}^{k}\frac{B_1G(\bar x_i,\bar x_1)}{\Lambda^{N-2}\mu^{N-2}}+O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big),\end{aligned}$$

where A, B_1 and B_2 are positive constants, and $\sigma>0$ is a small constant.

Proof. Since

$$\big\langle I'\big(W_{r,\Lambda}+\phi\big),\phi\big\rangle=0,\quad\forall\,\phi\in E,$$

there is $t\in(0,1)$ such that

$$\begin{aligned}F(d,\Lambda)&=I(W_{r,\Lambda})-\frac{1}{2}D^2I\big(W_{r,\Lambda}+t\phi\big)(\phi,\phi)\\&=I(W_{r,\Lambda})-\frac{1}{2}\int_{B_\mu(0)}\Big(|D\phi|^2-(2^*-1)K\Big(\frac{|y|}{\mu}\Big)\big(W_{r,\Lambda}+t\phi\big)^{2^*-2}\phi^2\Big)\\&=I(W_{r,\Lambda})+\frac{2^*-1}{2}\int_{B_\mu(0)}K\Big(\frac{|y|}{\mu}\Big)\Big(\big(W_{r,\Lambda}+t\phi\big)^{2^*-2}-W_{r,\Lambda}^{2^*-2}\Big)\phi^2\\&\quad-\frac{1}{2}\int_{B_\mu(0)}\big(N(\phi)+l_k\big)\phi\\&=I(W_{r,\Lambda})+O\Big(\int_{B_\mu(0)}\big(|\phi|^{2^*}+|N(\phi)||\phi|+|l_k||\phi|\big)\Big).\end{aligned}$$

However,

$$\begin{aligned}\int_{B_\mu(0)}\big(|N(\phi)||\phi|+|l_k||\phi|\big)&\le C\left(\|N(\phi)\|_{**}+\|l_k\|_{**}\right)\|\phi\|_*\\&\quad\times\int_{B_\mu(0)}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.\end{aligned}$$

Using Lemma B.1, for $N \geq 4$,

$$\begin{aligned}
&\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
&\quad = \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+2\tau}} + \sum_{j=1}^{k} \sum_{i\neq j} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
&\quad \leq \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+2\tau}} + C \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+\tau}} \sum_{j=2}^{k} \frac{1}{|x_j-x_1|^{\tau}} \\
&\quad \leq C \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{N+\tau}}.
\end{aligned}$$

Thus, we obtain

$$\int_{B_\mu(0)} \big(|N(\phi)||\phi| + |l_k||\phi|\big) \leq C\,k\,\big(\|N(\phi)\|_{**} + \|l_k\|_{**}\big)\,\|\phi\|_* \leq C\,k \left(\frac{1}{\mu}\right)^{1+\sigma}, \quad N \geq 4.$$

On the other hand,

$$\int_{B_\mu(0)} |\phi|^{2^*} \leq C\,\|\phi\|_*^{2^*} \int_{B_\mu(0)} \Big(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big)^{2^*}.$$

However, using (2.18), if $y \in \Omega_1$, and $N \geq 4$,

$$\begin{aligned}
&\sum_{j=2}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \\
&\qquad \leq C \frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}}} \sum_{j=2}^{k} \frac{1}{|x_j-x_1|^{\tau}} \leq C \frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}}},
\end{aligned}$$

Thus,

$$\Big(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big)^{2^*} \leq \frac{C}{(1+|y-x_1|)^{N}}, \quad y \in \Omega_1,$$

which gives

$$\int_{B_\mu(0)} \Big(\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big)^{2^*} \leq Ck \ln k.$$

So, we have proved

$$\int_{B_\mu(0)} |\phi|^{2^*} \leq C\,k \ln k\, \|\phi\|_*^{2^*} \leq C\,k \ln k \left(\frac{1}{\mu}\right)^{2^*(1/2+\sigma)}, \quad N \geq 4. \qquad \square$$

Proposition 3.2. *We have*

$$\frac{\partial F(d,\Lambda)}{\partial \Lambda}=kB_1(N-2)\Big(-\frac{H(\bar{x}_1,\bar{x}_1)}{\Lambda^{N-1}\mu^{N-2}}+\sum_{i=2}^{k}\frac{G(\bar{x}_i,\bar{x}_1)}{\Lambda^{N-1}\mu^{N-2}}+O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big),$$

and

$$\frac{\partial F(d,\Lambda)}{\partial d}=k\Big(\frac{B_1\frac{\partial H(\bar{x}_1,\bar{x}_1)}{\partial d}}{\Lambda^{N-2}\mu^{N-2}}+B_2K'(1)-\sum_{i=2}^{k}\frac{B_1\frac{\partial G(\bar{x}_i,\bar{x}_1)}{\partial d}}{\Lambda^{N-1}\mu^{N-2}}+O\Big(\frac{1}{\mu^{\sigma}}\Big)\Big),$$

if $N\ge 4$, where B_1 and B_2 are the same constants as in Proposition 3.1, and $\sigma>0$ is a small constant.

Proof. We estimate $\partial F(d,\Lambda)/\partial\Lambda$ first. We have

$$\begin{aligned}\frac{\partial F(d,\Lambda)}{\partial \Lambda}&=\Big\langle I'(W_{r,\Lambda}+\phi),\frac{\partial W_{r,\Lambda}}{\partial\Lambda}+\frac{\partial\phi}{\partial\Lambda}\Big\rangle\\&=\Big\langle I'(W_{r,\Lambda}+\phi),\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\Big\rangle+\sum_{l=1}^{2}\sum_{i=1}^{k}c_l\Big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\frac{\partial\phi}{\partial\Lambda}\Big\rangle.\end{aligned}$$

However,

$$\Big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\frac{\partial\phi}{\partial\Lambda}\Big\rangle=-\Big\langle\frac{\partial(U_{x_i,\Lambda}^{2^*-2}Z_{i,l})}{\partial\Lambda},\phi\Big\rangle$$

Thus, using Proposition 2.3,

$$\begin{aligned}&\Big|\sum_{i=1}^{k}c_l\Big\langle U_{x_i,\Lambda}^{2^*-2}Z_{i,l},\frac{\partial\phi}{\partial\Lambda}\Big\rangle\Big|\\&\qquad\le C\,|c_l|\,\|\phi\|_*\int_{\mathbb{R}^N}\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{N+2}}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\le\frac{C}{\mu^{1+\sigma}}.\end{aligned}$$

On the other hand,

$$\int_{\mathbb{R}^N}D(W_{r,\Lambda}+\phi)D\frac{\partial W_{r,\Lambda}}{\partial\Lambda}=\int_{\mathbb{R}^N}DW_{r,\Lambda}D\frac{\partial W_{r,\Lambda}}{\partial\Lambda},$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}K\big(\tfrac{|y|}{\mu}\big)(W_{r,\Lambda}+\phi)^{2^*-1}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\\&=\int_{\mathbb{R}^N}K\big(\tfrac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-1}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}+(2^*-1)\int_{\mathbb{R}^N}K\big(\tfrac{|y|}{\mu}\big)W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial\Lambda}\phi+O\Big(\int_{\mathbb{R}^N}|\phi|^{2^*}\Big).\end{aligned}$$

Moreover, from $\phi \in E$,

$$\begin{aligned}\int_{\mathbb{R}^N} K\big(\tfrac{|y|}{\mu}\big) W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial \Lambda}\phi &= \int_{\mathbb{R}^N} K\big(\tfrac{|y|}{\mu}\big)\Big(W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial \Lambda} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\Big)\phi \\ &\quad + \sum_{j=1}^{k}\int_{\mathbb{R}^N}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big) U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\phi \\ &= k\int_{\Omega_1} K\big(\tfrac{|y|}{\mu}\big)\Big(W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial \Lambda} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\Big)\phi \\ &\quad + k\int_{\mathbb{R}^N}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big) U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi,\end{aligned}$$

$$\begin{aligned}&\Big|\int_{\Omega_1} K\big(\tfrac{|y|}{\mu}\big)\Big(W_{r,\Lambda}^{2^*-2}\frac{\partial W_{r,\Lambda}}{\partial \Lambda} - \sum_{j=1}^{k} U_{x_j,\Lambda}^{2^*-2}\frac{\partial U_{x_j,\Lambda}}{\partial \Lambda}\Big)\phi\Big| \\ &\quad \le C\int_{\Omega_1}\Big(U_{x_1,\Lambda}^{2^*-2}\big(U_{x_1,\Lambda}-PU_{x_1,\Lambda}\big) + U_{x_1,\Lambda}^{2^*-2}\sum_{j=2}^{k}U_{x_j,\Lambda} + \sum_{j=2}^{k}U_{x_j,\Lambda}^{2^*-1}\Big)|\phi| \le \frac{C}{\mu^{1+\sigma}},\end{aligned}$$

and

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big) U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi\Big| &\le \Big|\int_{||y|-\mu|\le\sqrt{\mu}}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big) U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi\Big| \\ &\quad + \Big|\int_{||y|-\mu|\ge\sqrt{\mu}}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big) U_{x_1,\Lambda}^{2^*-2}\frac{\partial U_{x_1,\Lambda}}{\partial \Lambda}\phi\Big| \\ &\le \frac{C}{\mu^{1+\sigma}}.\end{aligned}$$

Thus, we have proved

$$\frac{\partial F(d,\Lambda)}{\partial \Lambda} = \frac{\partial I(W_{r,\Lambda})}{\partial \Lambda} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big),$$

and the result follows from Proposition A.2.

Finally, noting that $\partial/\partial d = -\mu\partial/\partial r$, we can estimate $\partial F(d,\Lambda)/\partial d$ in a similar way. □

Now, we estimate $H(\bar{x}_1,\bar{x}_1)$ and $G(\bar{x}_i,\bar{x}_1)$, $i \ge 2$. Let $\bar{x}_1^* = (1/(1-d), 0, \dots, 0)$ be the reflection of $\bar{x}_1$ with respect to the unit sphere. Then

$$H(y,\bar{x}_1) = \frac{1}{|y-\bar{x}_1^*|^{N-2}}\big(1+O(d)\big).$$

So, we obtain

$$H(\bar{x}_1, \bar{x}_1) = \frac{1}{2^{N-2}\, d^{N-2}} \big(1 + O(d)\big).$$

On the other hand,

$$|\bar{x}_i - \bar{x}_1^*| = \sqrt{|\bar{x}_i - \bar{x}_1|^2 + 4d^2 - 4d|\bar{x}_i - \bar{x}_1| \cos\theta_i},$$

where θ_i is the angle between $\bar{x}_i - \bar{x}_1$ and $(1, 0, \dots, 0)$. Thus, $\theta_i = \pi/2 + (i-1)\pi/2$.

$$\begin{aligned} G(\bar{x}_i, \bar{x}_1) &= \frac{1}{|\bar{x}_i - \bar{x}_1|^{N-2}} - \frac{1}{|\bar{x}_i - \bar{x}_1^*|^{N-2}} \big(1 + O(d)\big) \\ &= \frac{1}{|\bar{x}_i - \bar{x}_1|^{N-2}} \left(1 - \frac{1 + O(d)}{\big(1 + \frac{4d^2 + 4d|\bar{x}_i - \bar{x}_1| \sin((i-1)\pi/2)}{|\bar{x}_i - \bar{x}_1|^2}\big)^{(N-2)/2}}\right). \end{aligned}$$

Since

$$|\bar{x}_i - x_1| = 2\,|x_1|\, \sin \frac{(i-1)\pi}{k}, \quad i = 2, \dots, k,$$

using $dk \to c > 0$ and

$$0 < c' \le \frac{\sin((j-1)\pi/k)}{(j-1)\pi/k} \le c'', \quad j = 2, \dots, [k/2],$$

we obtain

$$\frac{a_0}{j^2} \le \frac{4\,d^2 + 4\,d\,|\bar{x}_i - \bar{x}_1| \sin((i-1)\pi/2)}{|\bar{x}_i - \bar{x}_1|^2} \le \frac{a_1}{j^2}$$

for some constant $a_1 \ge a_0 > 0$, which implies

$$\frac{a_0'}{j^N} + O\big(\frac{d}{j^{N-2}}\big) \le \frac{1}{k^{N-2}} G(\bar{x}_j, \bar{x}_1) \le \frac{a_1'}{j^N} + O\big(\frac{d}{j^{N-2}}\big)$$

for some constant $a_1' \ge a_0' > 0$. Hence, there is a constant $B_4 > 0$, such that

$$\sum_{j=2}^{k} G(\bar{x}_j, \bar{x}_1) = k^{N-2}\Big(\frac{B_4}{|\bar{x}_1|^{N-2}} + O\big(\frac{1}{k^{N-1}}\big) + O(d)\Big) = B_4 k^{N-2} + O(k^{N-2} d).$$

Thus, we obtain that there are positive constants A_1, A_2 and A_3, such that

$$F(d, \Lambda) = k\Big(A + \frac{A_1}{\Lambda^{N-2}\mu^{N-2}d^{N-2}} + A_2 d - \frac{A_3 k^{N-2}}{\Lambda^{N-2}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big), \tag{3.1}$$

$$\frac{\partial F(d, \Lambda)}{\partial \Lambda} = k\Big(- \frac{A_1(N-2)}{\Lambda^{N-1}\mu^{N-2}d^{N-2}} + \frac{A_3(N-2)k^{N-2}}{\Lambda^{N-1}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big), \tag{3.2}$$

$$\frac{\partial F(d, \Lambda)}{\partial d} = k\Big(- \frac{A_1(N-2)}{\Lambda^{N-2}\mu^{N-2}d^{N-1}} + A_2 + O\Big(\frac{1}{\mu^{\sigma}}\Big)\Big), \tag{3.3}$$

Note that $d = 1 - r/\mu$, and $\mu = k^{\frac{N-1}{N-2}}$. Define $D = d/k$. Then, from (3.2) and (3.3), $\partial F(d, \Lambda)/\partial \Lambda = 0$ and $\partial F(d, \Lambda)/\partial d = 0$ are equivalent to

$$- \frac{A_1(N-2)}{\Lambda^{N-1}D^{N-2}} + \frac{A_3(N-2)}{\Lambda^{N-1}} + O\Big(\frac{1}{\mu^{\sigma}}\Big) = 0, \tag{3.4}$$

and

$$-\frac{A_1(N-2)}{\Lambda^{N-2}D^{N-1}} + A_2 + O\Big(\frac{1}{\mu^{\sigma}}\Big) = 0, \tag{3.5}$$

respectively.

Proof of Theorem 1.2. Let

$$f_1(D,\Lambda) = -\frac{A_1(N-2)}{\Lambda^{N-1}D^{N-2}} + \frac{A_3(N-2)}{\Lambda^{N-1}} \quad \text{and} \quad f_2(D,\Lambda) = -\frac{A_1(N-2)}{\Lambda^{N-2}D^{N-1}} + A_2.$$

Then, $f_1 = 0$ and $f_2 = 0$ have a unique solution

$$D_0 = \Big(\frac{A_1}{A_3}\Big)^{1/(N-2)}, \quad \Lambda_0 = \Big(\frac{A_1(N-2)}{A_2 D_0^{N-1}}\Big)^{1/(N-2)}.$$

On the other hand, it is easy to see that

$$\frac{\partial f_1(D_0,\Lambda_0)}{\partial \Lambda} = 0, \quad \frac{\partial f_2(D_0,\Lambda_0)}{\partial D} > 0, \quad \text{and} \quad \frac{\partial f_1(D_0,\Lambda_0)}{\partial D} = \frac{\partial f_2(D_0,\Lambda_0)}{\partial \Lambda} > 0.$$

Thus the linear operator of $f_1 = 0$ and $f_2 = 0$ at (D_0,Λ_0) is invertible. As a result, (3.4) and (3.5) have a solution near (D_0,Λ_0). □

A. Energy expansion

In both appendices, we always assume that

$$x_j = \Big(r\cos\frac{2(j-1)\pi}{k}, r\sin\frac{2(j-1)\pi}{k}, 0\Big), \quad j = 1,\dots,k,$$

where 0 is the zero vector in $\mathbb{R}^{N-2}$, and $r \in \big[\mu(1-r_0/k), \mu(1-r_1/k)\big]$. Let

$$\bar{x}_j = \frac{1}{\mu}x_j.$$

Let $G(y,z)$ be the Green function of $-\Delta$ in $B_1(0)$ with the Dirichlet boundary condition. Let $H(y,z)$ be the regular part of the Green function.

Recall that

$$\mu = k^{\frac{N-1}{N-2}},$$

$$I(u) = \frac{1}{2}\int_{B_\mu(0)} |Du|^2 - \frac{1}{2^*}\int_{B_\mu(0)} K\big(\tfrac{|y|}{\mu}\big)\,|u|^{2^*},$$

$$U_{x_j,\Lambda}(y) = \big(N(N-2)\big)^{(N-2)/4}\frac{\Lambda^{(N-2)/2}}{(1+\Lambda^2|y-x_j|^2)^{(N-2)/2}},$$

and

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}(y),$$

where $PU_{x,\Lambda}$ is the solution of (1.5). It is well known that

$$U_{x_j,\Lambda}(y) - PU_{x_j,\Lambda}(y) = \frac{H(\bar{y},\bar{x})}{\mu^{N-2}} + O\Big(\frac{1}{d^N\mu^N}\Big), \tag{A.1}$$

where $d = 1 - |\bar{x}| = 1 - |x|/\mu$.

In this section, we will calculate $I(W_{r,\Lambda})$.

Proposition A.1. *We have*

$$I(W_{r,\Lambda}) = k\Big(A + \frac{B_1 H(\bar{x}_1,\bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + B_2K'(1)d - \sum_{i=2}^{k}\frac{B_1G(\bar{x}_i,\bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + O\big(\frac{1}{\mu^{1+\sigma}}\big)\Big),$$

where A, B_1 and B_2 are positive constants.

Proof. By using the assumed symmetry, we have

$$\begin{aligned}
\int_{B_\mu(0)}|DW_{r,\Lambda}|^2 &= \sum_{j=1}^{k}\sum_{i=1}^{k}\int_{B_\mu(0)}U_{x_j,\Lambda}^{2^*-1}PU_{x_i,\Lambda}\\
&= k\Big(\int_{B_\mu(0)}U_{0,1}^{2^*} - \int_{B_\mu(0)}U_{x_1,\Lambda}^{2^*-1}(U_{x_1,\Lambda}-PU_{x_1,\Lambda}) + \sum_{i=2}^{k}\int_{B_\mu(0)}U_{x_1,\Lambda}^{2^*-1}PU_{x_i,\Lambda}\Big)\\
&= k\Big(\int_{\mathbb{R}^N}U^{2^*} - \frac{\bar{B}_1H(\bar{x}_1,\bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + \frac{\bar{B}_1\sum_{i=2}^{k}G(\bar{x}_i,\bar{x}_1)}{\Lambda^{N-2}\mu^{N-2}} + O\big(\frac{1}{\mu^{N/(N-1)}}\big)\Big),
\end{aligned}$$

where

$$\bar{B}_1 = \int_{\mathbb{R}^N}U^{2^*-1}.$$

Let

$$\Omega_j = \Big\{y:\ y=(y',y'')\in B_\mu(0),\ \big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\big\rangle \ge \cos\frac{\pi}{k}\Big\}.$$

Then,

$$\begin{aligned}
\int_{B_\mu(0)}K\big(\tfrac{|y|}{\mu}\big)|W_{r,\Lambda}|^{2^*} &= k\int_{\Omega_1}K\big(\tfrac{|y|}{\mu}\big)|W_{r,\Lambda}|^{2^*}\\
&= k\Big(\int_{\Omega_1}K\big(\tfrac{|y|}{\mu}\big)(PU_{x_1,\Lambda})^{2^*} - 2^*\int_{\Omega_1}\sum_{i=2}^{k}(PU_{x_1,\Lambda})^{2^*-1}PU_{x_i,\Lambda}\\
&\quad + O\Big(\int_{\Omega_1}\big|K\big(\tfrac{|y|}{\mu}\big)-1\big|\sum_{i=2}^{k}U_{x_1,\Lambda}^{2^*-1}U_{x_i,\Lambda} + \int_{\Omega_1}U_{x_1,\Lambda}^{2^*/2}\big(\sum_{i=2}^{k}U_{x_i,\Lambda}\big)^{2^*/2}\Big)\Big).
\end{aligned}$$

Note that for $y\in\Omega_1$, $|y-x_i|\ge|y-x_1|$. Using (2.18), we find that for any $t\in(1,N-2)$,

$$\sum_{i=2}^{k}U_{x_i,\Lambda} \le \frac{C}{(1+|y-x_1|)^{N-2-t}}\sum_{i=2}^{k}\frac{1}{|x_i-x_1|^t}.$$

If we take the constant t close to $N-2$, then

$$\int_{\Omega_1} U_{x_1,\Lambda}^{2^*/2}\Big(\sum_{i=2}^{k} U_{x_i,\Lambda}\Big)^{2^*/2} = O\Big(\Big(\frac{k}{\mu}\Big)^{t\frac{N}{N-2}}\Big) = O\Big(\frac{1}{\mu^{1+\sigma}}\Big).$$

On the other hand, it is easy to show that

$$\begin{aligned}\int_{\Omega_1} \sum_{i=2}^{k} (PU_{x_1,\Lambda})^{2^*-1} PU_{x_i,\Lambda} &= \frac{\bar B_2\, G(\bar x_i,\bar x_1)}{\Lambda^{N-2}\,\mu^{N-2}} + O\Big(\frac{k^N}{\mu^N}\Big)\\ &= \frac{\bar B_2\, G(\bar x_i,\bar x_1)}{\Lambda^{N-2}\,\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big),\end{aligned}$$

and

$$\int_{\Omega_1} \big|K\big(\tfrac{|y|}{\mu}\big)-1\big| \,\sum_{i=2}^{k} U_{x_1,\Lambda}^{2^*-1} U_{x_i,\Lambda} = O\Big(\frac{1}{\mu^{1+\sigma}}\Big).$$

Moreover,

$$\begin{aligned}&\int_{\Omega_1} K\big(\tfrac{|y|}{\mu}\big)(PU_{x_1,\Lambda})^{2^*}\\ &= \int_{\Omega_1}(PU_{x_1,\Lambda})^{2^*} + \int_{\Omega_1}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big)U_{x_1,\Lambda}^{2^*} + O\Big(\int_{\Omega_1}\big|K(\tfrac{|y|}{\mu})-1\big|U_{x_1,\Lambda}^{2^*-1}\frac{H(y,x_1)}{\mu^{N-2}}\Big)\\ &= \int_{\mathbb{R}^N} U^{2^*} - 2^*\frac{\bar B_1 H(\bar x_1,\bar x_1)}{\Lambda^{N-2}\mu^{N-2}} + \int_{\Omega_1}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big)\,U_{x_1,\Lambda}^{2^*} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big).\end{aligned}$$

However,

$$\begin{aligned}\int_{\Omega_1}\big(K\big(\tfrac{|y|}{\mu}\big)-1\big)\,U_{x_1,\Lambda}^{2^*} &= \big(K(|\bar x_1|)-1\big)\int_{\mathbb{R}^N} U^{2^*} + O\Big(\frac{1}{\mu^2}\Big)\\ &= -K'(1)\,d\int_{\mathbb{R}^N} U^{2^*} + O(d^2) = -K'(1)\,d\int_{\mathbb{R}^N} U^{2^*} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big).\end{aligned}$$

Thus, we have proved

$$\begin{aligned}\int_{\mathbb{R}^N} K\big(\tfrac{|y|}{\mu}\big)|W_{r,\Lambda}|^{2^*} = k\Big(\int_{\mathbb{R}^N} U^{2^*} - K'(1)d\int_{\mathbb{R}^N} U^{2^*} - 2^*\frac{\bar B_1 H(\bar x_1,\bar x_1)}{\Lambda^{N-2}\mu^{N-2}}\\ + 2^*\sum_{i=2}^{k}\frac{\bar B_1 G(\bar x_i,\bar x_1)}{\Lambda^{N-2}\mu^{N-2}} + O(\frac{1}{\mu^{1+\sigma}})\Big).\end{aligned}$$

□

We also need to calculate

$$\frac{\partial I(W_{r,\Lambda})}{\partial\Lambda} \quad\text{and}\quad \frac{\partial I(W_{r,\Lambda})}{\partial r}.$$

Proposition A.2. *We have*

$$\frac{\partial I(W_{r,\Lambda})}{\partial \Lambda} = k(N-2)B_1\Big(-\frac{H(\bar{x}_1,\bar{x}_1)}{\Lambda^{N-1}\mu^{N-2}} + \sum_{i=2}^{k}\frac{G(\bar{x}_1,\bar{x}_i)}{\Lambda^{N-1}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big),$$

and

$$\frac{\partial I(W_{r,\Lambda})}{\partial r} = k\Big(B_1\frac{\frac{\partial H(\bar{x}_1,\bar{x}_1)}{\partial r}}{\Lambda^{N-2}\mu^{N-2}} - B_2K'(1)\frac{1}{\mu} - \sum_{i=2}^{k}\frac{B_1\frac{\partial G(\bar{x}_1,\bar{x}_i)}{\partial r}}{\Lambda^{N-1}\mu^{N-2}} + O\Big(\frac{1}{\mu^{1+\sigma}}\Big)\Big),$$

where B_1 is the same positive constant as in Proposition A.1.

Proof. We use ∂ to denote either $\partial/\partial\Lambda$ or $\partial/\partial r$. Using the symmetry, we have

$$\partial I(W_{r,\Lambda}) = k\,\Big((2^*-1)\sum_{i=2}^{k}\int_{\mathbb{R}^N} U_{x_1,\Lambda}^{2^*-2}\,\partial(U_{x_1,\Lambda})\,PU_{x_i,\Lambda} - \int_{\Omega_1} K\big(\tfrac{|y|}{\mu}\big)\,W_{r,\Lambda}^{2^*-1}\,\partial W_{r,\Lambda}\Big).$$

Then the proof of this proposition is similar to the proof of Proposition A.1, so we omit it. □

B. Basic estimates

In this section, we list some lemmas, whose proofs can be found in [15].

For each fixed i and j, $i \neq j$, consider the function

$$g_{ij}(y) = \frac{1}{(1+|y-x_j|)^{\alpha}}\,\frac{1}{(1+|y-x_i|)^{\beta}}, \tag{B.1}$$

where $\alpha \geq 1$ and $\beta \geq 1$ are constants.

Lemma B.1. *For any constant $0 < \sigma \leq \min(\alpha,\beta)$, there is a constant $C > 0$ such that*

$$g_{ij}(y) \leq \frac{C}{|x_i-x_j|^{\sigma}}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big).$$

Lemma B.2. *For any constant $0 < \sigma < N-2$, there is a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}\,\frac{1}{(1+|z|)^{2+\sigma}}\,dz \leq \frac{C}{(1+|y|)^{\sigma}}.$$

Recall that

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} PU_{x_j,\Lambda}.$$

Lemma B.3. *Suppose that $N \geq 4$. Then there is a small $\theta > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{4/(N-2)}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\, dz \leq C \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.$$

Proof. The proof can be found in [15]. We only need to use

$$W_{r,\Lambda}(y) \leq C \sum_{i=1}^{k} \frac{1}{(1+|y-x_i|)^{N-2}}.$$

□

References

[1] BYEON, J. AND WANG, Z.-Q.: On the Hénon equation: asymptotic profile of ground states. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23** (2006), 803–828.

[2] BYEON, J. AND WANG, Z.-Q.: On the Hénon equation: asymptotic profile of ground states. II. *J. Differential Equations* **216** (2005), 78–108.

[3] CAO, D. AND PENG, S.: The asymptotic behaviour of the ground state solutions for Hénon equation. *J. Math. Anal. Appl.* **278** (2003), 1–17.

[4] CAO, D., PENG, S. AND YAN, S.: Asymptotic behaviour of ground state solutions for the Hénon equation. *IMA J. Appl. Math.* **74** (2009), 468–480.

[5] DEL PINO, M., FELMER, P. AND MUSSO, M.: Two-bubble solutions in the super-critical Bahri–Coron's problem. *Calc. Var. Partial Differential Equations* **16** (2003), 113–145.

[6] DRUET, O. AND LAURAIN, P.: Private communication.

[7] HÉNON, M.: Numerical experiments on the stability of spherical stellar systems. *Astronom. and Astrophys.* **24** (1973), 229–238.

[8] HIRANO, N.: Existence of positive solutions for the Hénon equation involving critical Sobolev terms. *J. Differential Equations* **247** (2009), 1311–1333.

[9] NI, W.-M.: A nonlinear Dirichlet problem on the unit ball and its applications. *Indiana Univ. Math. J.* **31** (1982), 801–807.

[10] PENG, S.: Multiple boundary concentrating solutions to Dirichlet problem of Hénon equation. *Acta Math. Appl. Sin. Engl. Ser.* **22** (2006), 137–162.

[11] PISTOIA, A. AND SERRA, E.: Multi-peak solutions for the Hénon equation with slightly subcritical growth. *Math. Z.* **256** (2007), 75–97.

[12] SERRA, E.: Non-radial positive solutions for the Hénon equation with critical growth. *Cal. Var. Partial Differential Equations* **23** (2005), no. 3, 301–326.

[13] SMETS, D., SU, J. AND WILLEM, M.: Non-radial ground states for the Hénon equation. *Commun. Contemp. Math.* **4** (2002), no. 3, 467–480.

[14] Smets, D. and Willem, M.: Partial symmetry and asymptotic behavior for some elliptic variational problems. *Calc. Var. Partial Differential Equations* **18** (2003), 57–75.

[15] Wei, J. and Yan, S.: Infinitely many solutions for the prescribed scalar curvature problem on $\mathbb{S}^N$. *J. Funct. Anal.* **258** (2010), no. 9, 3048–3081.

Received December 1, 2011.

Juncheng Wei: Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.
E-mail: wei@math.cuhk.edu.hk

Shusen Yan: Department of Mathematics, The University of New England, Armidale, NSW 2351, Australia.
E-mail: syan@turing.une.edu.au

The first author is supported by an Earmarked Grant from the RGC of Hong Kong, a direct grant from the CUHK and an Oversea Joint Research Grant of the NSFC. The second author is partially supported by the ARC. The first author thanks the Institute EHESS for its hospitality during his stay in June, 2009.