Rev. Mat. Iberoam. **29** (2013), no. 3, 1091–1126
DOI 10.4171/RMI/750

Lewy–Stampacchia type estimates for variational inequalities driven by (non)local operators

Raffaella Servadei and Enrico Valdinoci

Abstract. The purpose of this paper is to derive some Lewy–Stampacchia estimates in some cases of interest, such as the ones driven by non-local operators. Since we will perform an abstract approach to the problem, this will provide, as a byproduct, Lewy–Stampacchia estimates in more classical cases as well. In particular, we can recover the known estimates for the standard Laplacian, the p-Laplacian, and the Laplacian in the Heisenberg group. In the non-local framework we prove a Lewy–Stampacchia estimate for a general integrodifferential operator and, as a particular case, for the fractional Laplacian. As far as we know, the abstract framework and the results in the non-local setting are new.

1. Introduction

1.1. The classical obstacle problem and its modifications

The simplest, classical example of obstacle problem consists of an elastic membrane, with vertical displacement u on a domain Ω, which is constrained at its boundary (say $u = u_0$ along $\partial\Omega$) and it is forced to lie below some obstacle (say, $u \leqslant \psi$). Then, at the equilibrium, whenever the membrane does not touch the obstacle, the elasticity provides a balance of the tension of the membrane, that, geometrically, reflects into a balance of the principal curvatures of the surface described by u. On the other hand, when the membrane sticks to the obstacle, its principal curvatures are expected to adapt to those of ψ. These physical considerations lead to the classical variational inequality

$$\int_\Omega \nabla u(x)(\nabla v(x) - \nabla u(x))\, dx \geqslant 0 \tag{1.1}$$

say, for any test function v, with $v \leqslant \psi$ and $v = u_0$ along $\partial\Omega$ (more formal details on the function spaces will be discussed in the sequel).

Mathematics Subject Classification (2010): Primary 35J86, 35J87, 49J40; Secondary 35R11, 35R35.
Keywords: Variational inequalities, integrodifferential operators, fractional Laplacian.

If an external force $-f$ is switched on, the rest configuration of the membrane will be such that the elastic tension of the membrane equilibrates the force, so that (1.1) becomes

$$\int_\Omega \nabla u(x)(\nabla v(x) - \nabla u(x))\,dx \geqslant \int_\Omega f(x)\,(v(x) - u(x))\,dx\,. \tag{1.2}$$

Many extensions of this problem has been considered in the literature, particularly for taking into account nonlinear elastic reactions of the membrane, non commutative effects, and non-local interactions. For instance, one may replace the linear elasticity (say, Hook's law) with a power-like one: this would change (1.2) into the following variational inequality of p-Laplace type:

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)(\nabla v(x) - \nabla u(x))\,dx \geqslant \int_\Omega f(x)\,(v(x) - u(x))\,dx\,, \tag{1.3}$$

for some $p \in (1, \infty)$. These types of obstacle problems have been recently considered in [4] and [23].

Similarly, one might replace the commutative Euclidean vector fields with some non-commutative ones, such as those of the Heisenberg group $\mathbb{H}^n$ (see, e.g., [28]). In this case, (1.2) is replaced by

$$\int_\Omega \nabla_{\mathbb{H}^n} u(\xi)(\nabla_{\mathbb{H}^n} v(\xi) - \nabla_{\mathbb{H}^n} u(\xi))\,d\xi \geqslant \int_\Omega f(\xi)\,(v(\xi) - u(\xi))\,d\xi\,. \tag{1.4}$$

This type of variational inequalities has been recently dealt with in [22].

Analogously, one might replace the local elastic reaction in (1.2) with a non-local one, with the purpose of taking into account the long-range interactions of particles. For instance, one might replace the standard Laplacian Δ with the so-called fractional Laplacian $-(-\Delta)^s$, with $s \in (0,1)$. In this case, (1.2) is replaced by the non-local variational inequality

$$\begin{aligned} \int_{\mathbb{R}^{2n}\setminus((\mathbb{R}^n\setminus\Omega)\times(\mathbb{R}^n\setminus\Omega))} & \frac{(u(x)-u(y))(v(x)-v(y)-u(x)+u(y))}{|x-y|^{n+2s}}\,dx\,dy \\ & \geqslant \int_\Omega f(x)\,(v(x)-u(x))\,dx\,. \end{aligned} \tag{1.5}$$

These kind of obstacle problems have been extensively studied in [3], [17] and [24] (see also [11], [26] and [27] for the basic definitions and properties of the fractional Laplacian). Obstacle problems for other integrodifferential kernels have been also studied in [15] and [16].

In Section 2, we will provide a unified framework which will comprise simultaneously, as particular cases, all the variational inequalities in (1.2)–(1.5).

1.2. Lewy–Stampacchia type estimates

Solutions of the variational inequality (1.1) have a bounded Laplacian. This fact may be heuristically guessed via the following argument. When u lies below the obstacle ψ, it is harmonic, $\Delta u = 0$. On the other hand, at the points where u sticks to the obstacle, one expects Δu to somewhat match with $\Delta\psi$, and the obstacle has to "bend up" at those contact points, that is $\Delta\psi \geqslant 0$. Therefore, though this argument is not rigorous (since it does not really take into account possible singularities that may occur at the free boundary $\partial\{u < \psi\}$, which, in fact, could be in principle quite a wild set), one can expect that solutions of the variational inequality (1.1) have Δu comprised between 0 and the positive part of $\Delta\psi$, namely

$$0 \leqslant \Delta u \leqslant (\Delta\psi)^+ . \tag{1.6}$$

The content of the so-called Lewy–Stampacchia estimates (named after [12]) is exactly a rigorous derivation of (1.6), and possible generalizations. Such kind of bounds are also called "dual estimates", since they are usually understood as an integral bound in the dual of $L^\infty(\Omega)$, that is (1.6) is derived in the distributional form

$$0 \leqslant \int_\Omega \Delta u(x)\varphi(x)\,dx \leqslant \int_\Omega (\Delta\psi)^+(x)\varphi(x)\,dx$$

for any nonnegative test function φ. Also, when an external force comes into play as in (1.2), then (1.6) gets modified as

$$0 \leqslant \Delta u + f \leqslant (\Delta\psi + f)^+ .$$

The purpose of this paper is to derive some Lewy–Stampacchia estimates in some cases of interest, such as the ones driven by non-local operators. Since we will perform an abstract approach to the problem, this will provide, as a byproduct, Lewy–Stampacchia estimates in more classical cases as well (in particular, we can recover the known estimates for the standard Laplacian, the p-Laplacian, and the Laplacian in the Heisenberg group).

In the non-local framework, the simplest example we can deal with is given by the fractional[1] Laplacian, according to the following result:

Theorem 1.1 (Lewy–Stampacchia type estimate for the fractional Laplacian). *Let $s \in (0,1)$ and $n > 2s$, and let Ω be an open bounded set of $\mathbb{R}^n$. Let $f \in L^\infty(\Omega)$ and let u_0 and $\psi : \mathbb{R}^n \to \mathbb{R}$ be two functions such that $u_0 \in H^s(\Omega) \cap L^\infty(\mathbb{R}^n \setminus \Omega)$ and $\psi \in H^s(\Omega)$ with $u_0 \leqslant \psi$ a.e. in $\mathbb{R}^n$ and $(-\Delta)^s\psi \in L^\infty(\Omega)$.*

If $u : \mathbb{R}^n \to \mathbb{R}$ is a solution of the variational inequality

$$\begin{cases} \displaystyle\int_{\mathbb{R}^{2n}\setminus((\mathbb{R}^n\setminus\Omega)\times(\mathbb{R}^n\setminus\Omega))} \frac{(u(x)-u(y))(v(x)-v(y)-u(x)+u(y))}{|x-y|^{n+2s}}\,dx\,dy \\ \qquad\qquad \geqslant \displaystyle\int_\Omega f(x)\,(v(x)-u(x))\,dx \\ \qquad\qquad \forall\, v \in H^s(\Omega),\ v = u_0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega,\ v \leqslant \psi \text{ a.e. in } \Omega \\ u \in H^s(\Omega),\ u = u_0 \text{ a.e. in } \mathbb{R}^n\setminus\Omega,\ u \leqslant \psi \text{ a.e. in } \Omega , \end{cases}$$

[1] As customary, we denoted by $H^s(\Omega)$ the fractional Sobolev space, endowed by the so-called Gagliardo norm, and by $H^s_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$ (for further details, see (5.5) in the sequel, and [6] for a crash introduction to fractional Sobolev spaces).

then

$$0 \leqslant -\int_{\mathbb{R}^{2n}\setminus((\mathbb{R}^n\setminus\Omega)\times(\mathbb{R}^n\setminus\Omega))} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_\Omega f(x)\varphi(x)\,dx$$

$$\leqslant \int_\Omega \big(-(-\Delta)^s\,\psi + f\big)^+(x)\,\varphi(x)\,dx \tag{1.7}$$

for any $\varphi \in H_0^s(\Omega)$ *with* $\varphi \geqslant 0$ *a.e. in* Ω *and* $\varphi = 0$ *a.e. in* $\mathbb{R}^n \setminus \Omega$.

Notice that (1.7) is a dual estimate, in the sense that it can be interpreted, in the sense of distributions, as a bound on the operator $-(-\Delta)^s u$, by writing, concisely

$$0 \leqslant -(-\Delta)^s u + f \leqslant \big(-(-\Delta)^s\psi + f\big)^+ .$$

This interpretation in the sense of distribution is a general property.

In fact, Theorem 1.1 is a particular case of more general result for integrodifferential operators of non-local type, as stated in the next theorem, where we consider the integral operator $\mathcal{L}_K$ defined as follows:

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} \big(u(x+y) + u(x-y) - 2u(x)\big)K(y)\,dy\,. \tag{1.8}$$

Note that when $K(y) = |y|^{-(n+2s)}$, then $\mathcal{L}_K = -(-\Delta)^s$, $s \in (0,1)\,$.

Theorem 1.2 (Lewy–Stampacchia type estimate for general integrodifferential operators). *Let* $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ *be a function with the following properties:*

$$mK \in L^1(\mathbb{R}^n),\ \textit{where}\ m(x) = \min\{|x|^2, 1\}\,; \tag{1.9}$$

$$\textit{there exists}\ \lambda > 0\ \textit{such that for any}\ x \in \mathbb{R}^n \setminus \{0\},\quad K(x) \geqslant \lambda |x|^{-(n+2s)}\,; \tag{1.10}$$

$$K(x) = K(-x)\ \textit{for any}\ x \in \mathbb{R}^n \setminus \{0\}\,. \tag{1.11}$$

Let Ω *be an open bounded set of* $\mathbb{R}^n$, *with* $n > 2s$. *Moreover, let* $f \in L^\infty(\Omega)$ *and let* $u_0 \in X \cap L^\infty(\mathbb{R}^n \setminus \Omega)$ *and* $\psi \in X$ *with* $u_0 \leqslant \psi$ *a.e. in* $\mathbb{R}^n$ *and* $\mathcal{L}_K\,\psi \in L^\infty(\Omega)\,$, *where* X *is the linear space of Lebesgue measurable functions from* $\mathbb{R}^n$ *to* $\mathbb{R}$ *such that the restriction to* Ω *of any function in* X *belongs to* $L^2(\Omega)$ *and the map* $(x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)}$ *is in* $L^2(Q, dxdy)$, *with* $Q = \mathbb{R}^{2n} \setminus ((\mathbb{R}^n \setminus \Omega) \times (\mathbb{R}^n \setminus \Omega))\,$.

If $u : \mathbb{R}^n \to \mathbb{R}$ *is a solution of the variational inequality*

$$\begin{cases} \displaystyle\int_Q (u(x)-u(y))\,(v(x)-v(y)-u(x)+u(y))\,K(x-y)\,dx\,dy \\ \displaystyle\qquad\qquad\qquad \geqslant \int_\Omega f(x)(v(x)-u(x))\,dx \\ \qquad\qquad\qquad \forall\, v \in X,\ v = u_0\ \textit{a.e. in}\ \mathbb{R}^n \setminus \Omega,\ v \leqslant \psi\ \textit{a.e. in}\ \Omega \\ u \in X,\ u = u_0\ \textit{a.e. in}\ \mathbb{R}^n \setminus \Omega,\ u \leqslant \psi\ \textit{a.e. in}\ \Omega\,, \end{cases} \tag{1.12}$$

then, for any $\varphi \in C_0^\infty(\Omega)$ *with* $\varphi \geqslant 0$ *in* Ω *and* $\varphi = 0$ *in* $\mathbb{R}^n \setminus \Omega$,

$$\begin{aligned} 0 &\leqslant -\int_Q (u(x)-u(y))(\varphi(x)-\varphi(y))\,dx\,dy + \int_\Omega f(x)\varphi(x)\,dx \\ &\leqslant \int_\Omega \big(\mathcal{L}_K\,\psi + f\big)^+(x)\,\varphi(x)\,dx \end{aligned} \tag{1.13}$$

We observe that (1.13) may be stated, in the distributional sense, as

$$0 \leqslant \mathcal{L}_K u + f \leqslant (\mathcal{L}_K\psi + f)^+ .$$

As far as we know, Theorems 1.1 and 1.2 are new. On the other hand, as already mentioned, since we will perform a functional analytic approach, these results will be the consequence of an abstract framework, which comprises some known results as particular cases. For instance, we list some classical and recent results that can be recovered by our approach.

Theorem 1.3 (Lewy–Stampacchia type estimate for the Laplacian). *Let* $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, *be an open bounded set of class* C^1. *Let* $f \in L^\infty(\Omega)$ *and let* u_0, $\psi : \mathbb{R}^n \to \mathbb{R}$ *be two functions such that* $u_0 \in C(\overline{\Omega})$ *and* $\psi \in C^2(\overline{\Omega})$ *with* $u_0 \leqslant \psi$ *a.e. in* $\mathbb{R}^n$.

If $u : \Omega \to \mathbb{R}$ *is a solution of the variational inequality*

$$\begin{cases} \displaystyle\int_\Omega \nabla u(x)(\nabla v(x) - \nabla u(x))\,dx \geqslant \int_\Omega f(x)(v(x)-u(x))\,dx \\ \qquad\qquad \forall\, v \in H^1(\Omega),\ v - u_0 \in H_0^1(\Omega),\ v \leqslant \psi \text{ a.e. in } \Omega \\ u \in H^1(\Omega),\ u - u_0 \in H_0^1(\Omega),\ u \leqslant \psi \text{ a.e. in } \Omega\,, \end{cases} \tag{1.14}$$

then, for any $\varphi \in H_0^1(\Omega)$ *with* $\varphi \geqslant 0$ *a.e. in* Ω,

$$0 \leqslant -\int_\Omega \nabla u(x)\nabla\varphi(x)\,dx + \int_\Omega f(x)\varphi(x)\,dx \leqslant \int_\Omega \big(\Delta\psi + f\big)^+(x)\,\varphi(x)\,dx. \tag{1.15}$$

Theorem 1.4 (Lewy–Stampacchia type estimate for the p-Laplacian). *Let* $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, *be an open bounded set of class* $C^{1,\alpha}$, *with* $\alpha \in (0,1]$. *Let* $f \in L^\infty(\Omega)$ *and let* u_0, $\psi : \mathbb{R}^n \to \mathbb{R}$ *be two functions such that* $u_0 \in C^{1,\alpha}(\overline{\Omega})$ *and* $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$ *with* $u_0 \leqslant \psi$ *a.e. in* $\mathbb{R}^n$ *and* $\Delta_p\,\psi \in L^\infty(\Omega)$, $p > 1$.

If $u : \Omega \to \mathbb{R}$ *is a solution of the variational inequality*

$$\begin{cases} \displaystyle\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)(\nabla v(x) - \nabla u(x))\,dx \geqslant \int_\Omega f(x)(v(x)-u(x))\,dx \\ \qquad\qquad \forall\, v \in W^{1,p}(\Omega),\ v - u_0 \in W_0^{1,p}(\Omega),\ v \leqslant \psi \text{ a.e. in } \Omega \\ u \in W^{1,p}(\Omega),\ u - u_0 \in W_0^{1,p}(\Omega),\ u \leqslant \psi \text{ a.e. in } \Omega\,, \end{cases} \tag{1.16}$$

then, for any $\varphi \in W_0^{1,p}(\Omega)$ *with* $\varphi \geqslant 0$ *a.e. in* Ω,

$$\begin{aligned} 0 &\leqslant -\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla\varphi(x)\,dx + \int_\Omega f(x)\,\varphi(x)\,dx \\ &\leqslant \int_\Omega \big(\Delta_p\,\psi + f\big)^+(x)\,\varphi(x)\,dx. \end{aligned} \tag{1.17}$$

Theorem 1.5 (Lewy–Stampacchia type estimate in the Heisenberg group). *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be an open bounded set with smooth boundary. Let $f \in L^\infty(\Omega)$ and let u_0, $\psi : \mathbb{R}^n \to \mathbb{R}$ be two functions such that $u_0 \in W^{1,p}_{\mathbb{H}^n}(\Omega_*) \cap L^\infty(\Omega_*)$, with Ω_* smooth domain such that $\Omega \subset\subset \Omega_*$, and $\psi \in C^2(\Omega)$ with $u_0 \leqslant \psi$ a.e. in $\mathbb{R}^n$ and $\Delta_{\mathbb{H}^n} \psi \in L^\infty(\Omega)$.*

If $u : \Omega \to \mathbb{R}$ is a solution of the variational inequality

$$\begin{cases} \displaystyle\int_\Omega \nabla_{\mathbb{H}^n} u(x)(\nabla_{\mathbb{H}^n} v(x) - \nabla_{\mathbb{H}^n} u(x))\, dx \geqslant \int_\Omega f(x)(v(x) - u(x))\, dx \\ \qquad\qquad\qquad \forall\, v \in W^{1,2}_{\mathbb{H}^n}(\Omega),\ v - u_0 \in W^{1,2}_{\mathbb{H}^n,0}(\Omega),\ v \leqslant \psi \text{ a.e. in } \Omega \\ u \in W^{1,2}_{\mathbb{H}^n}(\Omega),\ u - u_0 \in W^{1,2}_{\mathbb{H}^n,0}(\Omega),\ u \leqslant \psi \text{ a.e. in } \Omega, \end{cases}$$

then

$$\begin{aligned} 0 &\leqslant - \int_\Omega \nabla_{\mathbb{H}^n} u(x)\, \nabla_{\mathbb{H}^n} \varphi(x)\, dx + \int_\Omega f(x)\, \varphi(x)\, dx \\ &\leqslant \int_\Omega \big(\Delta_{\mathbb{H}^n} \psi + f\big)^+(x)\, \varphi(x)\, dx \end{aligned} \tag{1.18}$$

for any $\varphi \in W^{1,2}_{\mathbb{H}^n,0}(\Omega)$ with $\varphi \geqslant 0$ a.e. in Ω.

We observe that (1.15), (1.17) and (1.18) may be interpreted in the sense of distributions, by concisely writing

$$\begin{aligned} &0 \leqslant \Delta u + f \leqslant \big(\Delta \psi + f\big)^+, \\ &0 \leqslant \Delta_p u + f \leqslant \big(\Delta_p \psi + f\big)^+ \end{aligned}$$

and

$$0 \leqslant \Delta_{\mathbb{H}^n} u + f \leqslant \big(\Delta_{\mathbb{H}^n} + \psi + f\big)^+,$$

respectively.

Theorem 1.3 was proved in [12] and the literature is rich of many important extensions: see, among the others, [1], [7], [8], [18], [19], [20] and [21], and the references therein. A proof of Theorem 1.4 was recently performed in [4], and we found their approach very inspiring for our setting (see also [23] for related results). The case discussed in Theorem 1.5 was recently considered in [22] (see, e.g., [28] for the basics of the Heisenberg group and of the related Sobolev spaces).

The paper is organized as follows. In Sections 2 and 3 we state and prove a Lewy–Stampacchia estimate in an abstract setting, while Section 4 is devoted to the classical cases of the Laplacian and the p-Laplacian operators, and to the Laplacian in the Heisenberg group. In Section 5 we consider an integrodifferential operator with slow decay and, in this setting, we prove a dual estimate for the solutions of a variational inequality driven by this operator. For us, the main application of this is a Lewy–Stampacchia type estimate for the fractional Laplacian operator.

2. The abstract setting

In what follows let $Q \subset \mathbb{R}^m$, Ω be an open subset of $\mathbb{R}^n$, $m, n \geqslant 1$, and let us denote by $\mathcal{C}\Omega = \mathbb{R}^n \setminus \Omega$. Let μ be a measure on Q (in particular, Q is μ-measurable), and let $L^p(Q, d\mu)$, with $1 \leqslant p \leqslant \infty$, be the standard Lebesgue space with respect to the measure μ. As usual, we also denote $L^p(Q) = L^p(Q, dx)$ the standard Lebesgue space with respect to the Lebesgue measure dx, while $\mathcal{M}(\mathbb{R}^n)$ will be the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$.

We take X to be a linear subspace of $\mathcal{M}(\mathbb{R}^n)$, with the property that (the restriction to Ω of) any function in X belongs to $L^1(\Omega)$ (i.e., in symbols, if $u \in X$, then $u_{|\Omega} \in L^1(\Omega)$).

Moreover, let Y, Z and W be three sets such that it can be defined a product

$$\begin{aligned} \mathcal{P} : Y \times Z &\to W \\ (y, z) &\mapsto \mathcal{P}(y, z) =: yz \end{aligned}$$

with the following property:

$$\forall\, y \in Y \text{ and } \forall\, z \in Z \text{ it holds } yz \in L^1(Q, d\mu)\,. \tag{2.1}$$

We also consider two functions $u_0, \psi \in X$ with $u_0 \leqslant \psi$ a.e. in $\mathbb{R}^n$, and define $\mathcal{W}$ as the set containing all the nonnegative constants and the function $u_0 - \psi$. We also consider a linear subspace X_0 and a set $\widetilde{X}_0$ such that $\widetilde{X}_0 \subseteq X_0 \subseteq X$. We require that X_0 satisfies the following property:

$$\text{if } v \in X_0 \text{ and } w \in \mathcal{W}, \quad \text{then} (v + w)^+ \in X_0\,, \tag{2.2}$$

where g^+ denotes the positive part of a function g, that is $g^+(x) = \max\{g(x), 0\}$.

We also introduce the two functionals

$$a : X \to Y \quad \text{and} \quad b : X \to Z,$$

and we define $\mathcal{A} : X \times X \to \mathbb{R}$ by

$$\mathcal{A}[u, \varphi] := -\int_Q a(u) b(\varphi) d\mu$$

for any $u, \varphi \in X$. We remark that $\mathcal{A}$ is well defined, thanks to (2.1).

If $u \in X$ and if there exists $\Upsilon \in L^\infty(\Omega)$ such that

$$\mathcal{A}[u, \varphi] = \int_\Omega \Upsilon(x)\, \varphi(x)\, dx$$

for any $\varphi \in X_0$, we denote $\mathcal{A}(u) := \Upsilon$ and we say that $\mathcal{A}(u) \in L^\infty(\Omega)$.

Throughout the paper we need the following assumptions on the functionals a, b and $\mathcal{A}$:

$$a(v + \eta) = a(v) \quad \text{for any } v \in X \text{ and } \eta \in \mathbb{R}\,; \tag{2.3}$$

$$b(-v) = -\, b(v) \quad \text{for any } v \in X\,; \tag{2.4}$$

$$\begin{aligned} &\text{if } u, v \in X, \text{ with } (u - v)^+ \in X_0 \text{ and } \mathcal{A}[u, (u - v)^+] \geqslant \mathcal{A}[v, (u - v)^+], \\ &\text{then } u \leqslant v \text{ a.e. in } \Omega\,. \end{aligned} \tag{2.5}$$

Next we prove our Lewy–Stampacchia type estimate in this general framework.

3. Lewy–Stampacchia type estimates

Let us fix $f \in L^\infty(\Omega)$ and suppose that $\mathcal{A}(\psi) \in L^\infty(\Omega)$.

We consider the following variational inequality:

$$\begin{cases} \displaystyle\int_Q a(u)\, b(v-u)\, d\mu \geqslant \int_\Omega f(v-u)dx \quad \forall\, v \in X,\ v-u_0 \in X_0,\ v \leqslant \psi \text{ a.e. in } \Omega \\ u \in X,\ u-u_0 \in X_0,\ u \leqslant \psi \text{ a.e. in } \Omega\,. \end{cases} \tag{3.1}$$

In what follows, let $\eta \in (0,1)$ and let us denote by $h \in L^\infty(\Omega)$ and H_η the functions

$$h = (\mathcal{A}(\psi) + f)^+ \tag{3.2}$$

and

$$H_\eta(t) = \begin{cases} 0 & \text{if} \quad t \leqslant 0, \\ t/\eta & \text{if} \quad 0 < t < \eta, \\ 1 & \text{if} \quad t \geqslant \eta\,. \end{cases} \tag{3.3}$$

We consider the following approximated problem:

$$\begin{cases} \mathcal{A}(u_\eta) = h(1 - H_\eta(\psi - u_\eta)) - f \quad \text{in } \Omega, \\ u_\eta \in X,\ u_\eta - u_0 \in X_0\,. \end{cases} \tag{3.4}$$

More precisely, we say that u_η is a solution of problem (3.4) if

$$\begin{cases} \displaystyle \mathcal{A}[u_\eta, \varphi] = \int_\Omega \big(h(1 - H_\eta(\psi - u_\eta)) - f\big)\varphi\, dx \quad \forall\, \varphi \in X_0 \\ u_\eta \in X,\ u_\eta - u_0 \in X_0\,. \end{cases} \tag{3.5}$$

We assume that the following conditions hold true:

$$\text{for any } \eta > 0 \text{ there exists a solution } u_\eta \text{ of problem (3.5)}; \tag{3.6}$$

$$\begin{gathered} \text{if } u \text{ is a solution of (3.1) and } u_\eta \text{ is a sequence of solutions of (3.5)} \\ \text{such that } u_\eta \to u \text{ uniformly in } \mathbb{R}^n \text{ as } \eta \to 0 \text{ then, up to a subsequence,} \\ \mathcal{A}[u_\eta, \varphi] \to \mathcal{A}[u, \varphi] \text{ for any } \varphi \in \widetilde{X}_0\,. \end{gathered} \tag{3.7}$$

The main result of this section is the following:

Theorem 3.1. *Let $u_0, \psi \in X$, with $u_0 \leqslant \psi$ a.e. in $\mathbb{R}^n$ and $\mathcal{A}(\psi) \in L^\infty(\Omega)$, and let $f \in L^\infty(\Omega)$. Assume conditions (2.1)–(2.5), (3.6) and (3.7) hold true. If $u \in X$ is a solution of the variational inequality (3.1), then*

$$0 \leqslant \mathcal{A}[u, \varphi] + \int_\Omega f\varphi\, dx \leqslant \int_\Omega (\mathcal{A}(\psi) + f)^+ \varphi\, dx \tag{3.8}$$

for all $\varphi \in \widetilde{X}_0$, $\varphi \geqslant 0$ a.e. in Ω.

Proof. Let $\eta \in (0,1)$ and let u_η be a solution of (3.5). First, we prove that $u_\eta \leqslant \psi$ a.e. in Ω. For this, we notice that

$$u_\eta - \psi = u_\eta - u_0 + w\,,$$

with $w = u_0 - \psi \in \mathcal{W}$ by definition of $\mathcal{W}$. Hence

$$(u_\eta - \psi)^+ \in X_0\,,$$

thanks to (2.2) (applied here with $v = u_\eta - u_0 \in X_0$).

Therefore, we can take $\varphi = (u_\eta - \psi)^+$ in (3.5). We get

$$\mathcal{A}[u_\eta, (u_\eta - \psi)^+] = \int_\Omega \big(h(1 - H_\eta(\psi - u_\eta)) - f\big)(u_\eta - \psi)^+ dx\,. \tag{3.9}$$

Similarly,

$$\mathcal{A}[\psi, (u_\eta - \psi)^+] = \int_\Omega \mathcal{A}(\psi)(u_\eta - \psi)^+ dx\,, \tag{3.10}$$

with $\mathcal{A}(\psi) \in L^\infty(\Omega)$ by assumption.

Moreover, taking into account the definition of H_η and of the positive part, we have

$$h(1 - H_\eta(\psi - u_\eta))(u_\eta - \psi)^+ = h(u_\eta - \psi)^+ \quad \text{a.e. in } \Omega\,.$$

So, (3.9) becomes

$$\mathcal{A}[u_\eta, (u_\eta - \psi)^+] = \int_\Omega (h - f)(u_\eta - \psi)^+ dx\,.$$

From this and (3.10), we obtain that

$$\mathcal{A}[u_\eta, (u_\eta - \psi)^+] - \mathcal{A}[\psi, (u_\eta - \psi)^+] = \int_\Omega \big(h - (\mathcal{A}(\psi) + f)\big)(u_\eta - \psi)^+ dx \geqslant 0\,.$$

Notice that the last inequality is a consequence of (3.2). Then, the monotonicity condition (2.5) implies that

$$u_\eta \leqslant \psi \quad \text{a.e. in } \Omega\,. \tag{3.11}$$

Now we claim that $u \geqslant u_\eta$ a.e. in Ω. For this scope, let $\tau := u + (u_\eta - u)^+$. It is easily seen that

$$\tau = \begin{cases} u_\eta & \text{if } u_\eta > u, \\ u & \text{if } u_\eta \leqslant u, \end{cases}$$

so that $\tau \leqslant \psi$ a.e. in Ω. Moreover,

$$u_\eta - u = (u_\eta - u_0) + (u_0 - u) \in X_0\,, \tag{3.12}$$

being X_0 a linear space. As a consequence, applying (2.2) with $w = 0 \in \mathcal{W}$, we obtain that

$$(u_\eta - u)^+ \in X_0\,. \tag{3.13}$$

Therefore, by the definition of τ,

$$\tau - u_0 = (u - u_0) + (u_\eta - u)^+ \in X_0 \,.$$

Thus, we can use (3.1) with $v = \tau$, getting

$$\int_Q a(u)\, b(\tau - u)\, d\mu \geqslant \int_\Omega f(\tau - u)\, dx \,,$$

i.e., using again the definition of τ,

$$\int_Q a(u)\, b((u_\eta - u)^+)\, d\mu \geqslant \int_\Omega f(u_\eta - u)^+\, dx \,. \tag{3.14}$$

Moreover, by (3.13) we can take $\varphi = (u_\eta - u)^+$ in (3.5), so that we get

$$\begin{aligned} &\int_Q a(u_\eta)\, b((u_\eta - u)^+)\, d\mu \\ &\qquad = -\int_\Omega h(1 - H_\eta(\psi - u_\eta))(u_\eta - u)^+ dx + \int_\Omega f(u_\eta - u)^+ dx \\ &\qquad \leqslant \int_\Omega f(u_\eta - u)^+ dx \,, \end{aligned} \tag{3.15}$$

thanks to the definition of H_η. Combining (3.14) and (3.15) we get

$$\int_Q (a(u_\eta) - a(u))\, b((u_\eta - u)^+)\, d\mu \leqslant 0 \,.$$

From this relation and the monotonicity condition (2.5) we conclude that

$$u_\eta \leqslant u \quad \text{a.e. in } \Omega \,, \tag{3.16}$$

as desired.

Now let us show that

$$u \leqslant u_\eta + \eta \quad \text{a.e. in } \Omega \,. \tag{3.17}$$

To this goal, let $\theta := u - (u - u_\eta - \eta)^+$. It is easy to see that

$$\theta \leqslant u \leqslant \psi \quad \text{a.e. in } \Omega \,.$$

Moreover, recalling (3.12) and (2.2) (used here with $w = -\eta \in \mathcal{W}$), we have that

$$(u - u_\eta - \eta)^+ \in X_0 \,. \tag{3.18}$$

Hence

$$\theta - u_0 = (u - u_0) - (u - u_\eta - \eta)^+ \in X_0 \,.$$

Therefore, by taking $v = \theta$ in (3.1), we have

$$\int_Q a(u)\, b(\theta - u)\, d\mu \geqslant \int_\Omega f(\theta - u)\, dx \,,$$

i.e., using the definition of θ and assumption (2.4),

$$\int_Q a(u)\, b((u-u_\eta-\eta)^+)\, d\mu \leqslant \int_\Omega f(u-u_\eta-\eta)^+\, dx\,. \tag{3.19}$$

Also, by (3.18), we can take $\varphi = (u-u_\eta-\eta)^+$ in (3.5). We have

$$\begin{aligned} -\int_Q a(u_\eta)\, b((u-u_\eta-\eta)^+)\, d\mu \\ &= \int_\Omega h(1-H_\eta(\psi-u_\eta))(u-u_\eta-\eta)^+\, dx - \int_\Omega f(u-u_\eta-\eta)^+\, dx \\ &= -\int_\Omega f(u-u_\eta-\eta)^+ dx\,, \end{aligned} \tag{3.20}$$

being

$$\{u-u_\eta-\eta>0\} \subseteq \{\psi-u_\eta>\eta\} \subseteq \{1-H_\eta(\psi-u_\eta)=0\}\,.$$

Using assumptions (2.3) and (3.20) we deduce

$$\begin{aligned} \int_Q a(u_\eta+\eta)\, b((u-u_\eta-\eta)^+)\, d\mu &= \int_Q a(u_\eta)\, b((u-u_\eta-\eta)^+)\, d\mu \\ &= \int_\Omega f(u-u_\eta-\eta)^+\, dx\,. \end{aligned} \tag{3.21}$$

Combining (3.19) and (3.21) we have

$$\int_Q (a(u_\eta+\eta)-a(u))\, b((u-u_\eta-\eta)^+)\, d\mu \geqslant 0\,.$$

Thus, using again the monotonicity assumption (2.5),

$$u \leqslant u_\eta + \eta \quad \text{a.e. in } \Omega\,,$$

so that (3.17) is proved.

By (3.16) and (3.17) we have that

$$u-\eta \leqslant u_\eta \leqslant u \quad \text{a.e. in } \Omega\,.$$

Then

$$u_\eta \to u \quad \text{uniformly in } \mathbb{R}^n, \quad \text{as } \eta \to 0\,. \tag{3.22}$$

Hence, condition (3.7) implies that, up to subsequences, for any $\varphi \in \widetilde{X}_0$,

$$\mathcal{A}[u_\eta,\varphi] \to \mathcal{A}[u,\varphi], \quad \text{as } \eta \to 0\,. \tag{3.23}$$

Taking into account that u_η solves (3.5) and the definition of h and H_η it is easy to check that

$$0 \leqslant \mathcal{A}[u_\eta,\varphi] + \int_\Omega f\varphi\, dx = \int_\Omega h\big(1-H_\eta(\psi-u_\eta)\big)\varphi\, dx \leqslant \int_\Omega h\varphi\, dx$$

for any $\varphi \in X_0$, with $\varphi \geqslant 0$ a.e. in Ω. So, passing to the limit as $\eta \to 0$ and using (3.23) we get

$$0 \leqslant \mathcal{A}[u, \varphi] + \int_\Omega f\, \varphi\, dx \leqslant \int_\Omega h\, \varphi\, dx$$

for any $\varphi \in \widetilde{X}_0$, with $\varphi \geqslant 0$ a.e. in Ω. Recalling the definition of h (see (3.2)), Theorem 3.1 is proved. □

4. Some applications: the Laplacian, the *p*-Laplacian, and the Laplacian in the Heisenberg group

In this section we give some applications of Theorem 3.1 by recovering some known results in the case of the Laplacian, the p-Laplacian, and the Laplacian in the Heisenberg group. The case of the Laplacian is classical and it dates back to the original paper [12], which originated many important extensions and applications (see, among others, [1], [7], [8], [18], [19], [20] and [21]). See [4] and [23] for the p-Laplacian, and [22] for the Laplacian in the Heisenberg group.

Even if these cases have been already treated in the classical or recent literature, we provide our arguments in full detail, in order to clarify the abstract setting. Of course, the expert reader, or the one interested only in the new non-local application, may skip this part and go directly to Section 5.

4.1. The Laplacian operator

Though the case of the Laplacian is known and the techniques exploited are the standard (but tricky) Sobolev tools, we provide full detail of the argument, both for the readers's convenience and in order to make the abstract setting concrete in a model case.

We start with some preliminary observations on the classical Sobolev spaces which will be useful in the sequel (some of these observations are quite elementary and others are likely to be found in some textbook dedicated to Sobolev spaces, but we state them in display for typographical reasons, to facilitate the cross-references in the rest of this paper). First, since the map $\mathbb{R} \ni \tau \mapsto \tau^+ := \max\{\tau, 0\}$ is Lipschitz continuous with constant 1 and (sub)linear at infinity, we obtain:

Lemma 4.1. *Let $\alpha, \beta : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 1$. Then for any $x, y \in \mathbb{R}^n$ there holds*

$$|\alpha^+(x) - \beta^+(y)| \leqslant |\alpha(x) - \beta(y)|\,.$$

Lemma 4.2. *Let $1 \leqslant p < \infty$ and let v_j be a sequence of functions such that $\|v_j - v_\infty\|_{L^p(\Omega)} \to 0$ as $j \to +\infty$. Then,*

$$\|v_j^+ - v_\infty^+\|_{L^p(\Omega)} \to 0 \quad \text{as } j \to +\infty.$$

Lemma 4.3. *Let $1 \leqslant p < \infty$ and let v_j be a sequence of functions such that $\|v_j - v_\infty\|_{W^{1,p}(\Omega)} \to 0$ as $j \to +\infty$. Then (up to a subsequence), as $j \to +\infty$,*

a) $\nabla v_j^+ \to \nabla v_\infty^+$ *a.e. in* Ω;

b) $\|v_j^+ - v_\infty^+\|_{W^{1,p}(\Omega)} \to 0$.

Proof. By the assumption and by Theorem IV.9 in [2] we deduce that, up to a subsequence, as $j \to +\infty$,

$$v_j \to v_\infty \text{ in } L^p(\Omega) \tag{4.1}$$

$$\nabla v_j \to \nabla v_\infty \text{ in } L^p(\Omega) \tag{4.2}$$

$$v_j \to v_\infty \quad\text{and}\quad \nabla v_j \to \nabla v_\infty \text{ in } \Omega \setminus \mathcal{N}_0, \tag{4.3}$$

where $\mathcal{N}_0 \subset \Omega$ is such that $|\mathcal{N}_0| = 0$.

By Lemma 7.7 in [9] we also know that

$$\nabla v_\infty = 0 \text{ for any } x \in \{v_\infty = 0\} \setminus \mathcal{N}_1, \text{ with } |\mathcal{N}_1| = 0 \tag{4.4}$$

and, by Lemma 7.6 in [9], for any $j \in \mathbb{N} \cup \{\infty\}$,

$$\nabla v_j^+(x) = \begin{cases} \nabla v_j(x) & \text{if } x \in \{v_j > 0\} \setminus \mathcal{N}_1^j \\ 0 & \text{if } x \in \{v_j \leqslant 0\} \setminus \mathcal{N}_2^j, \end{cases} \tag{4.5}$$

with $|\mathcal{N}_i^j| = 0$, $i = 1, 2$.

First, let $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1 \cup \left(\cup_{j=1}^\infty \mathcal{N}_1^j\right) \cup \left(\cup_{j=1}^\infty \mathcal{N}_2^j\right)$. Of course $|\mathcal{N}| = 0$. Now, we will show that a) holds for any $x \in \Omega \setminus \mathcal{N}$.

Let $x \in \{v_\infty > 0\} \setminus \mathcal{N}$. Then, for j sufficiently large, $v_j(x) > 0$ by (4.3), so that $\nabla v_j^+(x) = \nabla v_j(x)$ by (4.5). Hence, again by (4.3) and (4.5) we have

$$\lim_{j \to +\infty} \nabla v_j^+(x) = \lim_{j \to +\infty} \nabla v_j(x) = \nabla v_\infty(x) = \nabla v_\infty^+(x).$$

Now, let $x \in \{v_\infty < 0\} \setminus \mathcal{N}$. Then, for j sufficiently large, $v_j(x) < 0$ by (4.3), so that $\nabla v_j^+(x) = 0 = \nabla v_\infty^+(x)$ by (4.5). Hence

$$\lim_{j \to +\infty} \nabla v_j^+(x) = \nabla v_\infty^+(x).$$

Finally, let $x \in \{v_\infty = 0\} \setminus \mathcal{N}$. Then, $\nabla v_\infty(x) = 0$ by (4.4). We also have that

$$|\nabla v_j^+(x)| \leqslant |\nabla v_j(x)| = |\nabla v_j(x) - \nabla v_\infty(x)| \to 0 \quad \text{as } j \to +\infty,$$

thanks to (4.5) and (4.3). Hence,

$$\lim_{j \to +\infty} \nabla v_j^+(x) = 0 = \nabla v_\infty^+(x),$$

thanks to (4.5). Assertion a) is proved.

Now, let us prove b). For this note that by Lemma 4.2 and (4.1) we get

$$\|v_j^+ - v_\infty^+\|_{L^p(\Omega)} \to 0 \tag{4.6}$$

as $j \to +\infty$. Moreover, by (4.2) and Theorem IV.9 in [2], there exists $\ell \in L^p(\Omega)$ such that $|\nabla v_j(x)| \leqslant \ell(x)$ a.e. in Ω for any $j \in \mathbb{N}$. Then, a.e. in Ω,

$$\begin{aligned}|\nabla v_j^+(x) - \nabla v_\infty^+(x)|^p &\leqslant \big(|\nabla v_j^+(x)| + |\nabla v_\infty^+(x)|\big)^p \\ &\leqslant \big(|\nabla v_j(x)| + |\nabla v_\infty(x)|\big)^p \leqslant \big(\ell(x) + |\nabla v_\infty(x)|\big)^p \in L^1(\Omega).\end{aligned}$$

Hence, the dominated convergence Theorem and a) give

$$\|\nabla v_j^+ - \nabla v_\infty^+\|_{L^p(\Omega)} \to 0 \tag{4.7}$$

as $j \to +\infty$. Assertion b) follows by (4.6) and (4.7). □

Corollary 4.4. *Let $1 \leqslant p < \infty$ and let $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, be an open set of class C^1. Let $v \in W_0^{1,p}(\Omega)$ and $w \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, with $w \leqslant 0$ in $\overline{\Omega}$. Then $(v+w)^+ \in W_0^{1,p}(\Omega)$.*

Proof. Of course $v + w \in W^{1,p}(\Omega)$, so that, by Lemma 7.6 in [9] (see also Corollary 6.18 in [13]), we have that $(v + w)^+ \in W^{1,p}(\Omega)$.

Since $v \in W_0^{1,p}(\Omega)$, there exists a sequence $v_j \in C_0^\infty(\Omega)$ such that $\|v_j - v\|_{W^{1,p}(\Omega)} \to 0$ as $j \to +\infty$. Hence, $v_j + w \to v + w$ in $W^{1,p}(\Omega)$ and so, by Lemma 4.3,

$$(v_j + w)^+ \to (v + w)^+ \text{ in } W^{1,p}(\Omega) \quad \text{as } j \to +\infty. \tag{4.8}$$

Moreover, $(v_j + w)^+ \in C(\overline{\Omega})$ and $v_j + w \leqslant v_j = 0$ on $\partial\Omega$, being $w \leqslant 0$ in $\overline{\Omega}$ and $v_j \in C_0^\infty(\Omega)$, that is $(v_j + w)^+ = 0$ on $\partial\Omega$ for any $j \in \mathbb{N}$. Then, by Theorem IX.17 in [2],

$$(v_j + w)^+ \in W_0^{1,p}(\Omega) \quad \text{for any } j \in \mathbb{N}. \tag{4.9}$$

Since $W_0^{1,p}(\Omega)$ is closed, (4.8) and (4.9) imply that $(v + w)^+ \in W_0^{1,p}(\Omega)$. □

With respect to the abstract setting in this subsection we take $Q = \Omega \subset \mathbb{R}^n$, $n \geqslant 1$, Ω open, bounded set of class C^1, $d\mu = dx$ (i.e., the usual Lebesgue measure),

$$\begin{aligned} &X = \{g \in \mathcal{M}(\mathbb{R}^n) : g_{|\Omega} \in H^1(\Omega)\}, \\ &X_0 = \{g \in \mathcal{M}(\mathbb{R}^n) : g_{|\Omega} \in H_0^1(\Omega)\}, \\ &\widetilde{X}_0 = C_0^\infty(\Omega) = \{g : \mathbb{R}^n \to \mathbb{R} : g \in C^\infty(\mathbb{R}^n) \text{ and } g = 0 \text{ in } \mathcal{C}\Omega\}, \\ \text{and}\quad &Y = Z = \{g : \mathbb{R}^n \to \mathbb{R}^n : g_{|\Omega} \in (L^2(\Omega))^n\}, \end{aligned}$$

while $\mathcal{P}$ is the scalar product in $\mathbb{R}^n$. Here $H^1(\Omega)$ and $H_0^1(\Omega)$ are the usual Sobolev spaces endowed with the norm $\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}$.

We also take $u_0 \in \{g \in X : g_{|\Omega} \in C(\overline{\Omega})\}$ and $\psi \in \{g \in \mathcal{M}(\mathbb{R}^n) : g_{|\Omega} \in C^2(\overline{\Omega})\}$ with $u_0 \leqslant \psi$ a.e. in $\mathbb{R}^n$.

Note that the restriction to Ω of any function in X belongs to $L^1(\Omega)$, since Ω is bounded. Moreover, u_0, $\psi \in X$ and $\widetilde{X}_0 \subseteq X_0 \subseteq X$. By Corollary 4.4 (applied here with $p=2$) we get that (2.2) holds true. Here we use that u_0, $\psi \in C(\overline{\Omega})$.

Now, for any $u, \varphi \in X$ we define

$$a(u) = \nabla u, \quad b(\varphi) = \nabla\varphi, \quad \mathcal{A}[u,\varphi] = -\int_\Omega \nabla u(x)\nabla\varphi(x)dx\,.$$

Note that $\mathcal{A}$ is well defined, being $\nabla u, \nabla\varphi \in Y$, that is (2.1) is satisfied. Moreover, it is easy to check that conditions (2.3) and (2.4) are satisfied. In order to show (2.5) it is enough to note that for any $g \in X$

$$\nabla g^+(x) = \begin{cases} \nabla g(x) & \text{a.e. if } g(x) > 0, \\ 0 & \text{a.e. if } g(x) \leqslant 0 \end{cases} \tag{4.10}$$

(see, for instance, Lemma 7.6 in [9]). Indeed, let $u, v \in X$ with $(u-v)^+ \in X_0$ and $\mathcal{A}[u,(u-v)^+] \geqslant \mathcal{A}[v,(u-v)^+]$. Then we have

$$\int_\Omega \nabla(u-v)(x)\nabla(u-v)^+(x)\,dx \leqslant 0\,. \tag{4.11}$$

Using (4.10) and (4.11) we have

$$0 \geqslant \int_\Omega \nabla(u-v)(x)\nabla(u-v)^+(x)\,dx = \int_\Omega |\nabla(u-v)^+(x)|^2\,dx \geqslant 0\,,$$

that is $\nabla(u-v)^+(x) = 0$ a.e. in Ω. Hence, $(u-v)^+$ is constant, say $u - v = c \in \mathbb{R}$ a.e. in Ω. Since $(u-v)^+ \in X_0$, we conclude that $|c| \in H^1_0(\Omega)$, and so $c = 0$. Accordingly, $(u-v)^+ = 0$, and so $u \leqslant v$ a.e. in Ω, to wit (2.5) holds.

Now, let us fix $\eta \in (0,1)$ and let us consider the problem $\Delta u_\eta = g_\eta(x, u_\eta)$ in Ω with $u_\eta = u_0$ in $\mathcal{C}\Omega$, that is

$$\begin{cases} \displaystyle\int_\Omega \nabla u_\eta(x)\nabla\varphi(x)\,dx + \int_\Omega g_\eta(x, u_\eta(x))\varphi(x)\,dx = 0 \quad \forall\, \varphi \in X_0 \\ u_\eta \in X,\ u_\eta - u_0 \in X_0\,, \end{cases} \tag{4.12}$$

where for a.e. $x \in \Omega$ and $t \in \mathbb{R}$

$$g_\eta(x,t) = h(x)\big(1 - H_\eta(\psi(x) - t)\big) - f(x)\,, \tag{4.13}$$

with $h = (\Delta\psi + f)^+$, $f \in L^\infty(\Omega)$ and H_η given in (3.3). Note that $h \in L^\infty(\Omega)$, thanks to the regularity required on ψ, i.e. $\psi \in C^2(\overline{\Omega})$.

We show that assumption (3.6) holds true. For this we consider the space $X_{u_0} = \{u \in X : u - u_0 \in X_0\}$ and the functional $\mathcal{J}_\eta : X_{u_0} \to \mathbb{R}$ defined as follows:

$$\mathcal{J}_\eta(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx + \int_\Omega G_\eta(x, u(x))dx\,,$$

where $G_\eta(x,t) = \int_0^t g_\eta(x,s)ds$. First of all, we note that

$$\inf_{u \in X_{u_0}} \mathcal{J}_\eta(u) > -\infty. \tag{4.14}$$

Indeed, using the definitions of g_η and H_η and the regularity of h and f we get that $g_\eta(\cdot, u(\cdot)) \in L^\infty(\Omega)$ for any $u \in X$ and

$$\|g_\eta(\cdot, u(\cdot))\|_{L^\infty(\Omega)} \leqslant \|h\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)} =: \kappa \tag{4.15}$$

so that

$$|G_\eta(x, u(x))| \leqslant \kappa |u(x)| \quad \text{a.e. } x \in \Omega\,. \tag{4.16}$$

Hence, for any $\delta > 0$ and $u \in X_{u_0}$

$$\begin{aligned} \mathcal{J}_\eta(u) &\geqslant \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \kappa \int_\Omega |u(x)| dx \\ &\geqslant \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \frac{\kappa}{2}\Big(\frac{|\Omega|}{\delta} + \int_\Omega \delta |u(x)|^2 dx\Big)\,, \end{aligned} \tag{4.17}$$

thanks to the Cauchy–Schwarz inequality. Since $u - u_0 \in X_0$, the Poincaré inequality (see, for instance, Corollary IX.19 in [2]) gives

$$\|u - u_0\|_{L^2(\Omega)} \leqslant C\|\nabla u - \nabla u_0\|_{L^2(\Omega)} \leqslant C\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^2(\Omega)}\right)\,, \tag{4.18}$$

where C is a positive constant depending only on Ω. Using the fact that $\|u\|_{L^2(\Omega)} - \|u_0\|_{L^2(\Omega)} \leqslant \|u - u_0\|_{L^2(\Omega)}$, by (4.18) we obtain

$$\|u\|_{L^2(\Omega)} \leqslant \tilde{C}\left(\|\nabla u\|_{L^2(\Omega)} + 1\right)\,, \tag{4.19}$$

where $\tilde{C} = \max\{C, C\|\nabla u_0\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}\}$. Then, by (4.17) and (4.19) we get

$$\mathcal{J}_\eta(u) \geqslant \frac{1}{2}\|\nabla u\|_{L^2(\Omega)} - \frac{\kappa|\Omega|}{2\delta} - \frac{\kappa\delta\tilde{C}}{2}\left(\|\nabla u\|_{L^2(\Omega)} + 1\right)\,. \tag{4.20}$$

Choosing $\delta > 0$ such that $\kappa\delta\tilde{C} < 1$ it easily follows that

$$\mathcal{J}_\eta(u) \geqslant -\frac{\kappa}{2}\Big(\frac{|\Omega|}{\delta} + \delta\tilde{C}\Big) > -\infty\,,$$

for any $u \in X_{u_0}$, so that (4.14) is proved.

Now, let us take a minimizing sequence u_j for $\mathcal{J}_\eta$, i.e. a sequence u_j in X_{u_0} such that

$$\mathcal{J}_\eta(u_j) \to \inf_{u \in X_{u_0}} \mathcal{J}_\eta(u) > -\infty \quad \text{as } j \to +\infty\,.$$

Then the sequence $\mathcal{J}_\eta(u_j)$ is bounded in $\mathbb{R}$, hence, by (4.20) with $u = u_j$,

$$\|\nabla u_j\|_{L^2(\Omega)} \leqslant \tilde{\kappa} \quad \text{for any } j \in \mathbb{N}$$

for some $\tilde{\kappa} > 0$. This and (4.19) give that u_j is also bounded in $L^2(\Omega)$.

Hence, the sequence $u_j - u_0$ is bounded in $H^1_0(\Omega)$. Thus, up to a subsequence, we have that there exists $v_\infty \in H^1_0(\Omega)$ such that

$$\begin{aligned} &\nabla(u_j - u_0) \to \nabla v_\infty \quad \text{weakly in } L^2(\Omega), \\ &u_j - u_0 \to v_\infty \quad \text{in } L^2(\Omega), \\ &u_j - u_0 \to v_\infty \quad \text{a.e. in } \Omega, \end{aligned} \tag{4.21}$$

as $j \to +\infty$ and there exists $\ell \in L^2(\Omega)$ such that

$$|u_j(x)| \leqslant \ell(x) \quad \text{a.e. in } \Omega \quad \text{for any } j \in \mathbb{N} \tag{4.22}$$

(see, for instance Theorem IV.9 in [2]). Now we define

$$u_\infty(x) = \begin{cases} (v_\infty + u_0)(x) & \text{if } x \in \Omega\,, \\ u_0(x) & \text{if } x \in \mathcal{C}\Omega\,. \end{cases}$$

Note that

$$u_\infty \in X_{u_0}\,, \tag{4.23}$$

since $(u_\infty - u_0)_{|\Omega} = v_\infty \in H_0^1(\Omega)$.

Using the weak lower semicontinuity of the norm in $L^2(\Omega)$, the fact that the map $t \mapsto G_\eta(x,t)$ is continuous in $\mathbb{R}$ a.e. $x \in \Omega$, (4.16), (4.21)–(4.23) and the dominated convergence theorem, we deduce that

$$\lim_{j\to+\infty} \mathcal{J}_\eta(u_j) \geqslant \frac{1}{2}\int_\Omega |\nabla u_\infty(x)|^2 dx + \int_\Omega G_\eta(x, u_\infty(x))dx = \mathcal{J}_\eta(u_\infty) \geqslant \inf_{u\in X_{u_0}} \mathcal{J}_\eta(u)\,,$$

so that

$$\mathcal{J}_\eta(u_\infty) = \inf_{u\in X_{u_0}} \mathcal{J}_\eta(u)\,.$$

Then problem (4.12) admits a solution in X_{u_0} and so (3.6) follows.

Now we prove (3.7). Let u_η be a sequence in X converging uniformly to some u in $\mathbb{R}^n$ as $\eta \to 0$. For any $\varphi \in C_0^\infty(\Omega)$, integrating by parts we have

$$\mathcal{A}[u_\eta, \varphi] = -\int_\Omega \nabla u_\eta(x)\,\nabla\varphi(x)\,dx = \int_\Omega u_\eta(x)\,\Delta\varphi(x)\,dx\,.$$

Hence,

$$\begin{aligned} \big|\mathcal{A}[u_\eta, \varphi] - \mathcal{A}[u, \varphi]\big| &= \Big|\int_\Omega (u_\eta(x) - u(x))\Delta\varphi(x)\,dx\Big| \\ &\leqslant \|\Delta\varphi\|_{L^\infty(\mathbb{R}^n)} \int_\Omega |u_\eta(x) - u(x)|\,dx \leqslant \|\Delta\varphi\|_{L^\infty(\mathbb{R}^n)} \|u_\eta - u\|_{L^\infty(\mathbb{R}^n)} |\Omega| \to 0 \end{aligned}$$

as $\eta \to 0$. Thus, condition (3.7) is proved.

Note that, in this proof, we did not need to use that u_η is a solution of (4.12) and that u solves the variational inequality (1.14).

Proof of Theorem 1.3. Let u be a solution of the variational inequality (1.14). In the setting of Theorem 1.3 we can apply Theorem 3.1 so that we get

$$0 \leqslant -\int_\Omega \nabla u(x)\nabla\varphi(x)\,dx \leqslant \int_\Omega (\Delta\psi + f)^+(x)\varphi(x)\,dx \quad \forall\,\varphi \in C_0^\infty(\Omega)\,,\ \varphi \geqslant 0 \text{ in } \Omega\,.$$

Note that, by density, the estimate holds true for any $\varphi \in H_0^1(\Omega)$ with $\varphi \geqslant 0$ a.e. in Ω. □

4.2. The p-Laplacian operator

With respect to the abstract setting here we take $1 < p < \infty$, $Q = \Omega \subset \mathbb{R}^n$, $n \geqslant 1$, Ω open, bounded set with smooth boundary (say, $\partial\Omega \in C^{1,\alpha}$ with $\alpha \in (0,1]$),

$$\begin{aligned}
& d\mu = dx\,,\\
& X = \{g \in \mathcal{M}(\mathbb{R}^n) : g_{|\Omega} \in W^{1,p}(\Omega)\}\,,\\
& X_0 = \{g \in \mathcal{M}(\mathbb{R}^n) : g_{|\Omega} \in W^{1,p}_0(\Omega)\}\,,\\
& \widetilde{X}_0 = C^\infty_0(\Omega)\,,\\
& Y = \{g : \mathbb{R}^n \to \mathbb{R}^n : g_{|\Omega} \in (L^{p'}(\Omega))^n\}\,, \text{ with } 1/p + 1/p' = 1,\\
\text{and}\quad & Z = \{g : \mathbb{R}^n \to \mathbb{R}^n : g_{|\Omega} \in (L^{p}(\Omega))^n\}\,,
\end{aligned}$$

while $\mathcal{P}$ is the scalar product in $\mathbb{R}^n$. Here $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ denote the usual Sobolev spaces endowed with the norm $\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$.

Moreover we fix two functions u_0 and ψ such that

$$u_0 \in \{g \in \mathcal{M}(\mathbb{R}^n) : g_{|\overline{\Omega}} \in C^{1,\alpha}(\overline{\Omega})\} \tag{4.24}$$

and

$$\psi \in \{g \in \mathcal{M}(\mathbb{R}^n) : g_{|\overline{\Omega}} \in C^2(\Omega) \cap C(\overline{\Omega})\} \tag{4.25}$$

with $\Delta_p \psi \in L^\infty(\Omega)$ and $u_0 \leqslant \psi$ a.e. in $\mathbb{R}^n$. Here α is the one appearing in the regularity of $\partial\Omega$. The choice of u_0 is admissible since $C^{1,\alpha}(\overline{\Omega}) \subset W^{1,p}(\Omega)$.

Note that the restriction to Ω of any function in X belongs to $L^1(\Omega)$, since Ω is bounded. Moreover, $\widetilde{X}_0 \subseteq X_0 \subseteq X$. In order to check assumption (2.2) it is enough to use Corollary 4.4 and the fact that $u_0, \psi \in C(\overline{\Omega})$.

For any $u, \varphi \in X$ we take

$$a(u) = |\nabla u|^{p-2}\nabla u, \quad b(\varphi) = \nabla\varphi, \quad \mathcal{A}[u,\varphi] = -\int_\Omega |\nabla u|^{p-2}(x)\nabla u(x)\nabla\varphi(x)dx\,.$$

Note that $\mathcal{A}$ is well defined, thanks to the fact that $\nabla u \in Y$ and $\nabla\varphi \in Z$, so that (2.1) is satisfied. Furthermore, conditions (2.3) and (2.4) are trivially satisfied. In order to show (2.5) we use the following inequality (see, for instance, formula (2.2) in page 210 of [25]):

$$\langle |t|^{p-2}t - |t'|^{p-2}t', t - t'\rangle \geqslant \begin{cases} C\,\dfrac{|t-t'|^2}{(|t|+|t'|)^{2-p}} & \text{if } 1 < p < 2,\\ C\,|t-t'|^p & \text{if } p \geqslant 2\,, \end{cases} \tag{4.26}$$

for any $t, t' \in \mathbb{R}^n$, where C is a positive constant.

Indeed, let $u, v \in X$ with $(u-v)^+ \in X_0$ and $\mathcal{A}[u,(u-v)^+] \geqslant \mathcal{A}[v,(u-v)^+]$. Then we have

$$\int_\Omega \big(|\nabla u(x)|^{p-2}\,\nabla u(x) - |\nabla v(x)|^{p-2}\,\nabla v(x)\big)\,\nabla(u-v)^+(x)\,dx \leqslant 0\,. \tag{4.27}$$

By using (4.10) (which still holds for any $g \in X$, with X defined as in Subsection 4.2, see, for instance, Lemma 7.6 in [9]), (4.26) and (4.27) we get

$$
\begin{aligned}
0 &\geqslant \int_\Omega \big(|\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x)\big)\,\nabla(u-v)^+(x)\,dx \\
&= \int_{\{u>v\}} \big(|\nabla u(x)|^{p-2}\nabla u(x) - |\nabla v(x)|^{p-2}\nabla v(x)\big)\,\nabla(u-v)(x)\,dx \\
&\geqslant \begin{cases} C\displaystyle\int_{\{u>v\}} \frac{|\nabla(u-v)(x)|^2}{(|\nabla u(x)|+|\nabla v(x)|)^{2-p}}\,dx & \text{if } 1<p<2 \\ C\displaystyle\int_{\{u>v\}} |\nabla(u-v)(x)|^p\,dx & \text{if } p\geqslant 2 \end{cases} \\
&= \begin{cases} C\displaystyle\int_\Omega \frac{|\nabla(u-v)^+(x)|^2}{(|\nabla u(x)|+|\nabla v(x)|)^{2-p}}\,dx \geqslant 0 & \text{if } 1<p<2 \\ C\displaystyle\int_\Omega |\nabla(u-v)(x)^+|^p\,dx \geqslant 0 & \text{if } p\geqslant 2\,, \end{cases}
\end{aligned}
$$

which implies that $\nabla(u-v)^+(x) = 0$ a.e. in Ω. Hence, as in the case of the Laplacian, we get that $u \leqslant v$ a.e. in Ω, and so (2.5) holds true.

Now, let us fix $\eta \in (0,1)$ and let us consider the problem $\Delta_p u_\eta = g_\eta(x,u_\eta)$ in Ω with $u_\eta = u_0$ in $\mathcal{C}\Omega$, that is,

$$
\begin{cases} \displaystyle\int_\Omega |\nabla u_\eta|^{p-2}(x)\nabla u_\eta(x)\nabla\varphi(x)\,dx + \int_\Omega g_\eta(x,u_\eta(x))\varphi(x)\,dx = 0 \quad \forall\,\varphi\in X_0 \\ u_\eta \in X,\ u_\eta - u_0 \in X_0\,, \end{cases} \tag{4.28}
$$

where $g_\eta(x,t)$ is given in (4.13), with $h = (\Delta_p\psi + f)^+ \in L^\infty(\Omega)$, being $\Delta_p\,\psi, f \in L^\infty(\Omega)$.

As for condition (3.6) it is enough to argue as in the case of the Laplacian, just substituting the Cauchy–Schwarz inequality with the Young inequality and use the Poincaré inequality in $L^p(\Omega)$.

Now we have to prove the validity of (3.7). The argument for this differs from the one of the Laplacian, since the operator Δ_p is nonlinear: in this case we will make use of the regularity theory for the p-Laplacian.

Let u be a solution of (1.16) and let u_η be a sequence of solutions of (4.28) converging uniformly to u in $\mathbb{R}^n$ as $\eta \to 0$, that is,

$$
\|u_\eta - u\|_{L^\infty(\Omega)} \to 0 \quad \text{as } \eta \to 0\,. \tag{4.29}
$$

First of all, we show that $u_\eta \in L^\infty(\Omega)$ and $\|u_\eta\|_{L^\infty(\Omega)}$ may be bounded independently of η. Let $R>0$ be such that $\Omega \subset B_R$, where $B_R \subset \mathbb{R}^n$ is the ball of radius R centered in 0. For $M>0$ we define

$$
z(x) = e^{M(x_1+R)} - e^{2MR} - \|u_0\|_{L^\infty(\Omega)}\,, \quad x \in B_R\,.
$$

We have that $z \in C^\infty(B_R)$ and

$$\Delta_p z(x) = M^p(p-1)\, e^{M(p-1)(x_1+R)} > M^p(p-1)$$

and so, recalling (4.15), we see that, if M is sufficiently large,

$$\Delta_p z(x) > \|g_\eta(\cdot, u_\eta(\cdot))\|_{L^\infty(\Omega)} \geqslant -g_\eta(x, u_\eta(x)) \quad \text{a.e. in } \Omega\,,$$

that is,

$$\int_\Omega |\nabla z(x)|^{p-2}\, \nabla z(x)\, \nabla \varphi(x) \leqslant \int_\Omega g_\eta(x, u_\eta(x))\, \varphi(x) \tag{4.30}$$

for any $\varphi \in X_0$, $\varphi \geqslant 0$ a.e. in Ω.

Now, we extend z to $\tilde z$ in all $\mathbb{R}^n$ in such a way that $\tilde z(x) = z_{|\Omega}(x)$ if $x \in \Omega$ and $\tilde z(x) \leqslant u_0(x)$ if $x \in \mathcal{C}\Omega$. Of course $\tilde z \in X$, since $z_{|\Omega} \in C^\infty(\overline{\Omega}) \subset W^{1,p}(\Omega)$. Moreover, $\tilde z - u_0 \in X$ and, using the definition of $\tilde z$,

$$\tilde z(x) - u_0(x) \leqslant -\|u_0\|_{L^\infty(\Omega)} - u_0(x) \leqslant u_0(x) - u_0(x) = 0 \quad \text{if } x \in \Omega\,,$$

while

$$\tilde z(x) - u_0(x) \leqslant 0 \quad \text{if } x \in \mathcal{C}\Omega\,,$$

so that $\tilde z - u_0 \leqslant 0$ in $\mathbb{R}^n$. We notice that

$$\tilde z - u_\eta = (\tilde z - u_0) + (u_0 - u_\eta)\,.$$

Therefore, $(\tilde z - u_\eta)^+ \in X_0$, thanks to Corollary 4.4 applied here with $v = u_0 - u_\eta \in X_0$ and $w = \tilde z - u_0 \in X \cap C(\overline{\Omega})$.

Hence, taking $\varphi = (\tilde z - u_\eta)^+$ in (4.28), we have

$$\int_\Omega |\nabla u_\eta(x)|^{p-2}\, \nabla u_\eta(x)\, \nabla(\tilde z - u_\eta)^+(x)\, dx = \int_\Omega g_\eta(x, u_\eta(x))(\tilde z - u_\eta)^+\, dx\,. \tag{4.31}$$

Similarly, using (4.30),

$$\int_\Omega |\nabla z(x)|^{p-2}\, \nabla z(x)\, \nabla(\tilde z - u_\eta)^+(x)\, dx \leqslant \int_\Omega g_\eta(x, u_\eta(x))(\tilde z - u_\eta)^+\, dx\,. \tag{4.32}$$

By (4.31) and (4.32) and using the definition of $\tilde z$ we deduce

$$\int_\Omega \big(|\nabla z(x)|^{p-2}\, \nabla z(x) - |\nabla u_\eta(x)|^{p-2}\, \nabla u_\eta(x)\big)\, \nabla(z - u_\eta)^+(x)\, dx \leqslant 0\,.$$

By the monotonicity condition (2.5) we deduce that

$$z \leqslant u_\eta \quad \text{a.e. in } \Omega\,.$$

Since z is bounded in Ω, we easily get

$$u_\eta \geqslant z \geqslant -\|z\|_{L^\infty(\Omega)} \quad \text{a.e. in } \Omega, \tag{4.33}$$

that is, u_η is bounded from below uniformly in η.

Moreover, since u_η converges uniformly to u, which solves the variational inequality (1.16), we have that, for η sufficiently small,

$$u_\eta \leqslant u + 1 \leqslant \psi + 1 \quad \text{a.e. in } \Omega\,.$$

From this and (4.33), since $\psi \in C(\overline{\Omega})$, we get that $u_\eta \in L^\infty(\Omega)$ and

$$\|u_\eta\|_{L^\infty(\Omega)} \leqslant \hat{\kappa}\,,$$

where $\hat{\kappa}$ is a suitable positive constant, independent of η.

With this result and using the choice of Ω and the regularity of u_0 given in (4.24), we can apply Theorem 1 in [14] with $m = p-2$, $\kappa = 0$, $\lambda = \Lambda = 1$ and $M_0 = \hat{\kappa}$. Then, we have that $u_\eta \in C^{1,\beta}(\overline{\Omega})$ and

$$\|u_\eta\|_{C^{1,\beta}(\overline{\Omega})} \leqslant C \quad \text{for any } \eta > 0\,, \tag{4.34}$$

where $\beta \in (0,1)$ depends only on α, p and n, while C is a positive constants depending only on α, p, n, $\hat{\kappa}$, $\|u_0\|_{C^{1,\alpha}(\overline{\Omega})}$ and Ω.

Hence, by the Ascoli–Arzelà theorem, up to a subsequence, we have that

$$\nabla u_\eta \to g \quad \text{uniformly in } \Omega \text{ as } \eta \to 0\,, \tag{4.35}$$

for some $g \in \big(C^{1,\beta}(\Omega)\big)^n$. By (4.29) and (4.35) it is easy to see that $g = \nabla u$ in Ω and so

$$\nabla u_\eta \to \nabla u \quad \text{uniformly in } \Omega \text{ as } \eta \to 0\,.$$

Hence, in particular, we have that

$$|\nabla u_\eta|^{p-2}\,\nabla u_\eta \to |\nabla u|^{p-2}\,\nabla u \quad \text{a.e. in } \Omega \text{ as } \eta \to 0\,. \tag{4.36}$$

Moreover, by (4.34) and the fact that Ω is bounded we get

$$|\nabla u_\eta(x)| \leqslant C \in L^p(\Omega) \quad \text{a.e. in } \Omega,$$

and so

$$\big|\,|\nabla u_\eta(x)|^{p-2}\,\nabla u_\eta(x)\big| \leqslant C^{p-1} \in L^{p'}(\Omega) \quad \text{a.e. in } \Omega \tag{4.37}$$

for any $\eta > 0$. By (4.36), (4.37) and the dominated convergence theorem we get

$$|\nabla u_\eta|^{p-2}\,\nabla u_\eta \to |\nabla u|^{p-2}\,\nabla u \quad \text{in } L^{p'}(\Omega) \text{ as } \eta \to 0\,. \tag{4.38}$$

Hence, for any $\varphi \in C_0^\infty(\Omega)$ we deduce that

$$\begin{aligned}\mathcal{A}[u_\eta, \varphi] &= \int_\Omega |\nabla u_\eta(x)|^{p-2}\,\nabla u_\eta(x)\,\nabla\varphi(x)\,dx \\ &\to \int_\Omega |\nabla u(x)|^{p-2}\,\nabla u(x)\,\nabla\varphi(x)\,dx = \mathcal{A}[u,\varphi]\end{aligned}$$

as $\eta \to 0$, that is assumption (3.7) is proved.

It is interesting to remark that, differently from the case of the Laplacian, here, in order to check assumption (3.7), we took advantage of the fact that u solves (1.16) and of the fact that u_η is a solution of an equation, so to use the associated regularity theory.

Proof of Theorem 1.4. Let u be a solution of the variational inequality (1.16). In the setting of Theorem 1.4 we can apply Theorem 3.1 to get

$$0 \leqslant -\int_\Omega |\nabla u|^{p-2}(x)\,\nabla u(x)\nabla\,\varphi(x)\,dx \leqslant \int_\Omega (\Delta_p\psi + f)^+(x)\,\varphi(x)\,dx$$

for any $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$ in Ω. By density, it is easily seen that such estimate holds for any $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$ a.e. in Ω. □

4.3. The Laplacian operator in the Heisenberg group

Thanks to our abstract framework, the proof in the Heisenberg group follows exactly that of the standard Laplacian: it is enough to replace ∇ with $\nabla_{\mathbb{H}^n}$, Δ with $\Delta_{\mathbb{H}^n}$ and the Sobolev space H^1 with $W^{1,2}_{\mathbb{H}^n}$. We refer to [28] and the introduction of [22] for further details on the Heisenberg group and on the above mentioned spaces.

In order to check condition (3.6), here we need some regularity assumptions on u_0 and ψ. For instance, it is enough to require that u_0, $\psi : \mathbb{R}^n \to \mathbb{R}$ are such that $u_0 \in W^{1,p}_{\mathbb{H}^n}(\Omega_*) \cap L^\infty(\Omega_*)$, with Ω_* smooth domain such that $\Omega \subset\subset \Omega_*$, and $\psi \in C^2(\Omega)$. For more details we refer to [22].

5. New applications: integral operators with even kernel

In this section we consider a kernel $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ with the properties (1.9)–(1.11). A typical example is given by the fractional Laplace kernel $K(x) = |x|^{-(n+2s)}$, $s \in (0,1)$.

With respect to the abstract setting here we set $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$, where

$$\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^{2n} \tag{5.1}$$

and $\Omega \subset \mathbb{R}^n$, $n > 2s$, is an open bounded set, $d\mu = dx\,dy$ (the standard Lebesgue measure in $\mathbb{R}^{2n}$), X is the linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

$$\text{the map} \quad (x, y) \mapsto (g(x) - g(y))\sqrt{K(x-y)} \quad \text{is in } L^2(Q, dxdy),$$

endowed with the norm defined as

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x) - g(y)|^2\,K(x-y)\,dx\,dy\Big)^{1/2}.$$

It is easy to check that $\|\cdot\|_X$ is a norm on X. We only show that if $\|g\|_X = 0$, then $g = 0$ a.e. in $\mathbb{R}^n$. Indeed, by $\|g\|_X = 0$ we get $\|g\|_{L^2(\Omega)} = 0$, which implies that

$$g = 0 \quad \text{a.e. in } \Omega, \tag{5.2}$$

and

$$\int_Q |g(x)-g(y)|^2\, K(x-y)\, dx\, dy = 0\,. \tag{5.3}$$

By (5.3) we deduce that $g(x) = g(y)$ a.e. $(x,y) \in Q$, that is g is constant a.e. in $\mathbb{R}^n$, say $g = c \in \mathbb{R}$ a.e. in $\mathbb{R}^n$. By (5.2) it easily follows that $c = 0$, so that $g = 0$ a.e. in $\mathbb{R}^n$.

The following lemma is valid:

Lemma 5.1. *Let $\varphi \in C^2_0(\Omega)$. Then the map*

$$\mathbb{R}^{2n} \ni (x,y) \mapsto |\varphi(x)-\varphi(y)|^2\, K(x-y)$$

belongs to $L^1(\mathbb{R}^{2n})$.

Proof. Since φ vanishes outside Ω,

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} |\varphi(x)-\varphi(y)|^2 K(x-y)\, dx\, dy \\
&= \int_{\Omega\times\Omega} |\varphi(x)-\varphi(y)|^2 K(x-y)\, dx\, dy + 2\int_{\Omega\times\mathcal{C}\Omega} |\varphi(x)-\varphi(y)|^2 K(x-y)\, dx\, dy \\
&\leqslant 2\int_{\Omega\times\mathbb{R}^n} |\varphi(x)-\varphi(y)|^2 K(x-y)\, dx\, dy. \qquad (5.4)
\end{aligned}$$

Now, we notice that

$$|\varphi(x)-\varphi(y)| \leqslant \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n)}|x-y| \quad \text{and} \quad |\varphi(x)-\varphi(y)| \leqslant 2\|\varphi\|_{L^\infty(\mathbb{R}^n)}.$$

Accordingly

$$|\varphi(x)-\varphi(y)| \leqslant 2\|\varphi\|_{C^1(\mathbb{R}^n)} \min\{|x-y|, 1\} = 2\|\varphi\|_{C^1(\mathbb{R}^n)}\sqrt{m(x-y)},$$

where m is defined in (1.9). Therefore, from (5.4) we deduce that

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} |\varphi(x)-\varphi(y)|^2 K(x-y)\, dx\, dy &\leqslant 8\,\|\varphi\|^2_{C^1(\mathbb{R}^n)} \int_{\Omega\times\mathbb{R}^n} m(x-y)K(x-y)\, dx\, dy \\
&= 8\,|\Omega|\|\varphi\|^2_{C^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} m(\xi)K(\xi)\, d\xi\,.
\end{aligned}$$

Thus, Lemma 5.1 follows by (1.9) and by the fact that Ω is bounded. □

As a trivial consequence of Lemma 5.1 we get that $C^2_0(\Omega) \subseteq X$. We also note that, since Ω is bounded, the restriction to Ω of any function in X belongs to $L^1(\Omega)$, so that all the assumptions on X are satisfied.

We also take

$$\begin{aligned}
X_0 &= \{g \in X : g = 0 \text{ a.e. in } \mathcal{C}\Omega\}\,, \\
\widetilde{X}_0 &= C^\infty_0(\Omega) \\
Y = Z &= \{g : \mathbb{R}^{2n} \to \mathbb{R} : g_{|Q} \in L^2(Q, dxdy)\}\,,
\end{aligned}$$

while $\mathcal{P}$ is the usual product between functions. Finally, we fix two functions $u_0 \in X \cap L^\infty(\mathcal{C}\Omega)$ and $\psi \in X$, with $u_0 \leqslant \psi$ a.e. in $\mathbb{R}^n$ and $\mathcal{L}_K\psi \in L^\infty(\Omega)$ (see (1.8) for the definition of the operator $\mathcal{L}_K$).

Of course $\widetilde{X}_0 \subseteq X_0 \subseteq X$. In order to verify condition (2.2) we need the following lemma:

Lemma 5.2. *Let v be a function in X. Then $v^+ \in X$.*

Proof. By Lemma 4.1 with $\beta \equiv 0$ we have

$$\int_\Omega |v^+(x)|^2\, dx \leqslant \int_\Omega |v(x)|^2\, dx < +\infty\,,$$

while, taking $\beta = \alpha$ we get

$$\int_Q |v^+(x) - v^+(y)|^2 K(x-y)\, dx\, dy \leqslant \int_Q |v(x) - v(y)|^2 K(x-y)\, dx\, dy < +\infty\,,$$

i.e., $v^+ \in X$. □

With this result, we can prove (2.2). For this let $v \in X_0$ and $w \in X$ with $w \leqslant 0$ a.e. in $\mathbb{R}^n$. Since $v+w \in X$, by Lemma 5.2 we have that $(v+w)^+ \in X$. Moreover, by assumption

$$(v+w)(x) \leqslant v(x) = 0 \quad \text{a.e. } x \in \mathcal{C}\Omega\,,$$

so that $(v+w)^+ = 0$ in $\mathcal{C}\Omega$. Hence $(v+w)^+ \in X_0$, that is (2.2) is verified.

Finally, for any $u, \varphi \in X$ we define $a(u), b(\varphi) : \mathbb{R}^{2n} \to \mathbb{R}$ as follows:

$$\begin{aligned} a(u)(x,y) &= (u(x) - u(y))\sqrt{K(x-y)} \\ b(\varphi)(x,y) &= (\varphi(x) - \varphi(y))\sqrt{K(x-y)} \end{aligned}$$

and

$$\mathcal{A}[u,\varphi] = -\int_Q (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y) dx\, dy\,.$$

Notice that $a(u)$ and $b(\varphi)$ belong to Y, since $u, \varphi \in X$. In the following lemma we prove that $\mathcal{A}$ is well defined.

Lemma 5.3. *Let $v, \varphi \in X$. Then the map $Q \ni (x,y) \mapsto (v(x)-v(y))(\varphi(x)-\varphi(y))$ $K(x-y)$ belongs to $L^1(Q, dxdy)$*

Proof. By the Cauchy–Schwarz inequality we have

$$\begin{aligned} &2\,|v(x) - v(y)|\,|\varphi(x) - \varphi(y)|\,\sqrt{K(x-y)}\,\sqrt{K(x-y)} \\ &\qquad \leqslant |v(x) - v(y)|^2\, K(x-y) + |\varphi(x) - \varphi(y)|^2\, K(x-y). \end{aligned}$$

This and the fact that $v, \varphi \in X$ give the assertion. □

Now, we have to check assumptions (2.3)–(2.5). Using the definitions of a and b and (1.11) it is easily seen that (2.3) and (2.4) holds true. Now, let us show (2.5). Let $u, v \in X$ with $(u-v)^+ \in X_0$ and $\mathcal{A}[u, (u-v)^+] \geqslant \mathcal{A}[v, (u-v)^+]$.

We set $w := u - v$ and we consider the negative part $w^-(x) = \min\{w(x), 0\}$. Notice that

$$w^- \leqslant 0, \quad w = w^+ + w^-, \quad \text{and} \quad w(x)\, w^+(x) = (w^+(x))^2.$$

Therefore,

$$\begin{aligned}
&(w(x) - w(y))\,(w^+(x) - w^+(y)) \\
&\qquad = w(x)\, w^+(x) + w(y)\, w^+(y) - w(x)\, w^+(y) - w^+(x)\, w(y) \\
&\qquad = (w^+(x))^2 + (w^+(y))^2 - w(x)\, w^+(y) - w^+(x)\, w(y).
\end{aligned}$$

Also, by (1.11), we have that

$$\int_Q w(x)\, w^+(y)\, K(x-y)\, dx\, dy = \int_Q w^+(x)\, w(y)\, K(x-y)\, dx\, dy.$$

As a consequence,

$$\begin{aligned}
0 &\geqslant \mathcal{A}[v, (u-v)^+] - \mathcal{A}[u, (u-v)^+] = \int_Q a(w) b(w^+)\, d\mu \\
&= \int_Q (w(x) - w(y))(w^+(x) - w^+(y))\, K(x-y)\, dx\, dy \\
&= \int_Q \big((w^+(x))^2 + (w^+(y))^2 - w(x)w^+(y) - w^+(x)w(y)\big)\, K(x-y)\, dx\, dy \\
&= \int_Q \big((w^+(x))^2 + (w^+(y))^2 - 2w^+(x)w(y)\big)\, K(x-y)\, dx\, dy \\
&= \int_Q \big((w^+(x))^2 + (w^+(y))^2 - 2w^+(x)w^+(y) - 2w^+(x)w^-(y)\big)\, K(x-y)\, dx\, dy \\
&= \int_Q \big((w^+(x) - w^+(y))^2 - 2w^+(x)w^-(y)\big)\, K(x-y)\, dx\, dy \geqslant 0\,,
\end{aligned}$$

being $w^- \leqslant 0$. Then, $(w^+(x) - w^+(y))^2 - 2w^+(x)w^-(y) = 0$ a.e. in Q, that is,

$$0 \leqslant (w^+(x) - w^+(y))^2 = 2w^+(x)w^-(y) \leqslant 0\,.$$

Hence, it follows that $w^+(x) = w^+(y)$ a.e. $(x, y) \in Q$, which implies that w^+ is constant a.e. in $\mathbb{R}^n$, say $w^+(x) = c \geqslant 0$. Since $w^+ \in X_0$ by assumption, then $c = 0$, i.e. $u \leqslant v$ a.e. in $\mathbb{R}^n$. Thus, condition (2.5) is verified.

Now, before going on, we need some preliminary results on X and X_0. In the following we denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the so-called Gagliardo norm:

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x) - g(y)|^2}{|x-y|^{n+2s}}\, dx\, dy\Big)^{1/2}, \tag{5.5}$$

while $H_0^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$. Of course, $H_0^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$.

Lemma 5.4. *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (1.9)–(1.11). Then the following assertions hold true:*

a) *if $v \in X$, then $v \in H^s(\Omega)$. Moreover,*

$$\|v\|_{H^s(\Omega)} \leqslant c(\lambda)\, \|v\|_X\,;$$

b) *if $v \in X_0$, then $v \in H^s(\mathbb{R}^n)$. Moreover,*

$$\|v\|_{H^s(\mathbb{R}^n)} \leqslant c(\lambda)\, \|v\|_X\,.$$

In both cases $c(\lambda) = \max\{1, 1/\sqrt{\lambda}\}$.

Proof. Let us prove part a). By (1.10) we get

$$\begin{aligned}\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\leqslant \frac{1}{\lambda}\int_{\Omega\times\Omega} |v(x)-v(y)|^2 K(x-y)\,dx\,dy\\ &\leqslant \frac{1}{\lambda}\int_{Q} |v(x)-v(y)|^2 K(x-y)\,dx\,dy < +\infty\,,\end{aligned}$$

since $v \in X$. The first assertion is proved.

For part b) note that $v \in X$ and $v = 0$ a.e. in $\mathcal{C}\Omega$. As a consequence,

$$\|v\|_{L^2(\mathbb{R}^n)} = \|v\|_{L^2(\Omega)} < +\infty\,,$$

and

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &= \int_{Q} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &\leqslant \frac{1}{\lambda}\int_{Q} |v(x)-v(y)|^2\, K(x-y)\,dx\,dy < +\infty\,.\end{aligned}$$

Hence $v \in H^s(\mathbb{R}^n)$. The estimate on the norm easily follows. □

Lemma 5.5. *Let v_j be a sequence in X such that*

$$\sup_{j\in\mathbb{N}} \|v_j\|_X < +\infty \tag{5.6}$$

and $v_j \to v_\infty$ a.e. in $\mathbb{R}^n$ as $j \to +\infty$. Then $v_\infty \in X$.

If, in addition, $v_j \in X_0$ for any $j \in \mathbb{N}$, then $v_\infty \in X_0$.

Proof. By (5.6) and the Fatou lemma, we have

$$+\infty > \liminf_{j\to+\infty} \int_\Omega |v_j(x)|^2\,dx \geqslant \int_\Omega |v_\infty(x)|^2\,dx\,,$$

i.e., $v_\infty \in L^2(\Omega)$, and

$$+\infty > \liminf_{j\to+\infty} \int_Q |v_j(x)-v_j(y)|^2 K(x-y)\,dx\,dy \geqslant \int_Q |v_\infty(x)-v_\infty(y)|^2 K(x-y)\,dx\,dy.$$

Thus, $v_\infty \in X$.

Now, suppose that $v_j = 0$ a.e. in $\mathcal{C}\Omega$ for any $j \in \mathbb{N}$. Then, it is easy to see that $v_\infty = 0$ a.e. in $\mathcal{C}\Omega$. Hence, $v_\infty \in X_0$ and the assertion is proved. □

Now, let us fix $\eta \in (0,1)$. We consider the following problem $\mathcal{L}_K u_\eta = g_\eta(x, u_\eta)$ in Ω with $u_\eta = u_0$ in $\mathcal{C}\Omega$, that is,

$$\tag{5.7} \begin{cases} \displaystyle\int_Q (u_\eta(x) - u_\eta(y))\,(\varphi(x) - \varphi(y))\,K(x-y)\,dx\,dy \\ \qquad\qquad\displaystyle + \int_\Omega g_\eta(x, u_\eta(x))\varphi(x)\,dx = 0 \quad \forall\,\varphi \in X_0 \\ u_\eta \in X,\ u_\eta - u_0 \in X_0\,, \end{cases}$$

where g_η is given in (4.13), with $h = (\mathcal{L}_K\,\psi + f)^+ \in L^\infty(\Omega)$, since $f, \mathcal{L}_K\psi \in L^\infty(\Omega)$. For the definition of $\mathcal{L}_K$ see (1.8).

In order to prove (3.6), we consider the space $X_{u_0} = \{u \in X : u - u_0 \in X_0\}$ and the functional $\mathcal{J}_\eta : X_{u_0} \to \mathbb{R}$ defined as follows:

$$\mathcal{J}_\eta(u) = \frac{1}{2}\int_Q |u(x) - u(y)|^2\,K(x-y)\,dx\,dy + \int_\Omega G_\eta(x, u(x))\,dx\,.$$

We recall that G_η is the primitive of g_η with respect to its second variable.

First of all, we note that

$$\tag{5.8} \inf_{u \in X_{u_0}} \mathcal{J}_\eta(u) > -\infty.$$

Indeed, as in the case of the Laplacian (cf. (4.16)–(4.17)) we get that, for any $\delta > 0$ and $u \in X_{u_0}$,

$$\tag{5.9} \begin{aligned} \mathcal{J}_\eta(u) &\geqslant \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)dx\,dy - \kappa\int_\Omega |u(x)|dx \\ &\geqslant \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)dx\,dy - \frac{\kappa\delta}{2}\int_\Omega |u(x)|^2 dx - \frac{\kappa}{2\delta}|\Omega| \\ &\geqslant \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)dx\,dy - \kappa\,\delta\,\|u-u_0\|^2_{L^2(\Omega)} \\ &\qquad\qquad - \kappa\,\delta\,\|u_0\|^2_{L^2(\Omega)} - \frac{\kappa}{2\delta}\,|\Omega| \\ &\geqslant \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \kappa\delta|\Omega|^{(2^*-2)/2^*}\,\|u-u_0\|^2_{L^{2^*}(\Omega)} \\ &\qquad\qquad - \kappa\,\delta\,\|u_0\|^2_{L^2(\Omega)} - \frac{\kappa}{2\delta}\,|\Omega|\,, \end{aligned}$$

thanks to the Cauchy–Schwarz inequality, the Minkowski inequality and to the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ continuously (being Ω bounded and $2 < 2^* = 2n/(n-2s)$). By Lemma 5.4, we know that $u - u_0 \in H^s(\mathbb{R}^n)$ and so, using Theorem 6.5 in [6] (here with $p = 2$), we get

$$\tag{5.10} \|u-u_0\|^2_{L^{2^*}(\mathbb{R}^n)} \leqslant C\int_{\mathbb{R}^{2n}} \frac{|u(x)-u_0(x)-u(y)+u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\,,$$

where C is a positive constant depending only on n and s.

Using (1.10) and again the Minkowski inequality, by (5.9) and (5.10) we have that, for any $\delta > 0$ and $u \in X_{u_0}$,

$$
\begin{aligned}
\mathcal{J}_\eta(u) \geqslant \frac{\lambda}{2} \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \kappa\delta \|u_0\|^2_{L^2(\Omega)} - \frac{\kappa}{2\delta}|\Omega| \\
- \kappa\delta C |\Omega|^{(2^*-2)/2^*} \int_{\mathbb{R}^{2n}} \frac{|u(x)-u_0(x)-u(y)+u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
\geqslant \lambda \int_Q \frac{|u(x)-u_0(x)-u(y)+u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
- \lambda \int_Q \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
- \kappa\,\delta\,C\,|\Omega|^{(2^*-2)/2^*} \int_{\mathbb{R}^{2n}} \frac{|u(x)-u_0(x)-u(y)+u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
- \kappa\,\delta\,\|u_0\|^2_{L^2(\Omega)} - \frac{\kappa}{2\delta}|\Omega| \\
= \left(\lambda - \kappa\delta C|\Omega|^{(2^*-2)/2^*}\right) \int_Q \frac{|u(x)-u_0(x)-u(y)+u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
- \lambda \int_Q \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \kappa\delta\|u_0\|^2_{L^2(\Omega)} - \frac{\kappa}{2\delta}|\Omega|\,.
\end{aligned}
\tag{5.11}
$$

Note that, since $u_0 \in X$ and (1.10) holds true,

$$
\int_Q \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy < +\infty \quad \text{and} \quad \|u_0\|_{L^2(\Omega)} < +\infty\,. \tag{5.12}
$$

Choosing $\delta > 0$ such that $\kappa\,\delta\,C\,|\Omega|^{(2^*-2)/2^*} < \lambda$, by (5.11) and (5.12) it easily follows that

$$
\mathcal{J}_\eta(u) \geqslant -\lambda \int_Q \frac{|u_0(x)-u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \kappa\,\delta\,\|u_0\|^2_{L^2(\Omega)} - \frac{\kappa}{2\delta}\,|\Omega| > -\infty\,,
$$

for any $u \in X_{u_0}$, so that (5.8) is proved.

Now, let us take a minimizing sequence u_j for $\mathcal{J}_\eta$, i.e. a sequence u_j in X_{u_0} such that

$$
\mathcal{J}_\eta(u_j) \to \inf_{u \in X_{u_0}} \mathcal{J}_\eta(u) > -\infty \quad \text{as } j \to +\infty\,.
$$

Then the sequence $\mathcal{J}_\eta(u_j)$ is bounded in $\mathbb{R}$. Hence, using (5.11) with $u = u_j$ and the fact that $u_j = u_0$ a.e. in $\mathcal{C}\Omega$, we have that, for any $j \in \mathbb{N}$,

$$
\begin{aligned}
\int_{\mathbb{R}^{2n}} \frac{|u_j(x)-u_0(x)-u_j(y)+u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
= \int_Q \frac{|u_j(x)-u_0(x)-u_j(y)+u_0(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leqslant \tilde{\kappa}\,,
\end{aligned}
\tag{5.13}
$$

for some $\tilde{\kappa} > 0$. Moreover, since

$$\|u_j - u_0\|_{L^2(\mathbb{R}^n)} = \|u_j - u_0\|_{L^2(\Omega)},$$

the continuity of the embedding $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ and (5.10) give that $u_j - u_0$ is bounded also in $L^2(\mathbb{R}^n)$. Then, $u_j - u_0$ is bounded in $H^s(\mathbb{R}^n)$.

Thus, by Corollary 7.2 in [6], up to a subsequence, there exists $v_\infty \in L^q(\mathbb{R}^n)$ with $q \in (2, 2^*)$ such that

$$u_j - u_0 \to v_\infty \quad \text{in } L^q(\mathbb{R}^n),$$

and so, by Theorem IV.9 in [2],

$$u_j - u_0 \to v_\infty \quad \text{a.e. in } \mathbb{R}^n \tag{5.14}$$

as $j \to +\infty$ and there exists $\ell \in L^q(\mathbb{R}^n)$ such that

$$|u_j(x)| \leqslant \ell(x) \quad \text{a.e. in } \mathbb{R}^n \quad \text{for any } j \in \mathbb{N}. \tag{5.15}$$

Also, (5.6) holds true here for $v_j = u_j - u_0$, thanks to (5.13) and Theorem 6.5 in [6]. As a consequence, by Lemma 5.5 we have that $v_\infty \in X_0$.

Now, let $u_\infty = v_\infty + u_0$. Of course, $u_\infty \in X_{u_0}$ and, by (5.14), (5.15), (4.16), the fact that the map $t \mapsto G_\eta(x,t)$ is continuous in $\mathbb{R}$ a.e. $x \in \Omega$, the Fatou Lemma and the dominated convergence theorem, we also get

$$\begin{aligned}\lim_{j\to+\infty} \mathcal{J}_\eta(u_j) &\geqslant \frac{1}{2}\int_Q |u_\infty(x) - u_\infty(y)|^2 K(x-y)\,dx\,dy + \int_\Omega G_\eta(x, u_\infty(x))\,dx \\ &= \mathcal{J}_\eta(u_\infty),\end{aligned}$$

so that

$$\mathcal{J}_\eta(u_\infty) = \inf_{u \in X_{u_0}} \mathcal{J}_\eta(u).$$

Hence, problem (5.7) has a solution.

In the following lemma we prove a sort of formula of integration by parts in X, which will be useful in the sequel.

Lemma 5.6. *Let $\varphi \in C_0^2(\Omega)$ and $v \in X \cap L^\infty(\mathcal{C}\Omega)$. Then*

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \big(v(x) - v(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy \\ &\qquad = \int_Q \big(v(x) - v(y)\big)\big(\varphi(x) - \varphi(y)\big) K(x-y)\,dx\,dy \\ &\qquad = \int_{\mathbb{R}^{2n}} v(x)\big(2\varphi(x) - \varphi(x+\xi) - \varphi(x-\xi)\big) K(\xi)\,dx\,d\xi.\end{aligned}$$

Proof. In what follows, for notation consistency, it will be convenient to denote

$$\mathcal{D}_0 := Q = \mathbb{R}^{2n} \setminus \mathcal{O}, \tag{5.16}$$

with $\mathcal{O}$ as in (5.1). Also, given $\epsilon \in [0,1)$, we define

$$\mathcal{D}_\epsilon = \big\{(x,y) \in \mathcal{D}_0 \text{ s.t. } |x-y| \geqslant \epsilon\big\}$$

and

$$\mathcal{D}_\epsilon^\pm = \big\{(x,\xi) \in \mathbb{R}^{2n} \text{ s.t. } (x, x\pm\xi) \in \mathcal{D}_0 \text{ and } |\xi| \geqslant \epsilon\big\}.$$

We remark that the above notation for $\mathcal{D}_\epsilon$ is consistent with the one in (5.16) for $\epsilon = 0$. Recalling that φ vanishes outside Ω and (1.11), we have

$$\begin{aligned}
&\int_{\mathcal{D}_\epsilon} |v(x)||\varphi(x)-\varphi(y)|\,K(x-y)\,dx\,dy \\
&\quad = \int_{(\Omega\times\Omega)\cap\{|x-y|\geqslant\epsilon\}} |v(x)||\varphi(x)-\varphi(y)|\,K(x-y)\,dx\,dy \\
&\qquad\quad + \int_{(\Omega\times\mathcal{C}\Omega)\cap\{|x-y|\geqslant\epsilon\}} |v(x)||\varphi(x)-\varphi(y)|\,K(x-y)\,dx\,dy \\
&\qquad\quad + \int_{(\mathcal{C}\Omega\times\Omega)\cap\{|x-y|\geqslant\epsilon\}} |v(x)||\varphi(x)-\varphi(y)|\,K(x-y)\,dx\,dy \\
&\quad = \int_{(\Omega\times\mathbb{R}^n)\cap\{|x-y|\geqslant\epsilon\}} |v(x)||\varphi(x)-\varphi(y)|\,K(x-y)\,dx\,dy \\
&\qquad\quad + \int_{(\Omega\times\mathcal{C}\Omega)\cap\{|x-y|\geqslant\epsilon\}} |v(y)||\varphi(x)-\varphi(y)|\,K(x-y)\,dx\,dy \\
&\quad \leqslant 2\,\|\varphi\|_{L^\infty(\mathbb{R}^n)}\Big[\int_{(\Omega\times\mathbb{R}^n)\cap\{|x-y|\geqslant\epsilon\}} |v(x)|K(x-y)\,dx\,dy \\
&\qquad\qquad + \int_{(\Omega\times\mathcal{C}\Omega)\cap\{|x-y|\geqslant\epsilon\}} |v(y)|K(x-y)\,dx\,dy\Big] \\
&\quad = \frac{2\,\|\varphi\|_{L^\infty(\mathbb{R}^n)}}{\epsilon^2}\Big[\int_{(\Omega\times\mathbb{R}^n)\cap\{|x-y|\geqslant\epsilon\}} |v(x)|m(x-y)K(x-y)\,dx\,dy \\
&\qquad\qquad + \int_{(\Omega\times\mathcal{C}\Omega)\cap\{|x-y|\geqslant\epsilon\}} |v(y)|m(x-y)K(x-y)\,dx\,dy\Big] \\
&\quad \leqslant \frac{2\,\|\varphi\|_{L^\infty(\mathbb{R}^n)}}{\epsilon^2}\Big[\int_{\Omega\times\mathbb{R}^n} |v(x)|m(\xi)K(\xi)\,dx\,d\xi \\
&\qquad\qquad + \|v\|_{L^\infty(\mathcal{C}\Omega)}\int_{\Omega\times\mathbb{R}^n} m(\xi)K(\xi)\,dx\,d\xi\Big] \\
&\quad = \frac{2\,\|\varphi\|_{L^\infty(\mathbb{R}^n)}}{\epsilon^2}\big(\|v\|_{L^1(\Omega)} + |\Omega|\,\|v\|_{L^\infty(\mathcal{C}\Omega)}\big)\int_{\mathbb{R}^n} m(\xi)K(\xi)\,d\xi\,,
\end{aligned}$$

which is finite, thanks to (1.9). As a consequence,

$$\begin{aligned}
&\text{the maps } (x,y)\mapsto v(x)\big(\varphi(x)-\varphi(y)\big)\,K(x-y) \\
&\qquad \text{and } (x,y)\mapsto v(y)\big(\varphi(x)-\varphi(y)\big)\,K(x-y) \text{ belong to } L^1(\mathcal{D}_\epsilon, dxdy).
\end{aligned} \tag{5.17}$$

Thanks to Lemma 5.3 and (5.17) we can split the integrals

$$\begin{aligned}&\int_{\mathcal{D}_\epsilon} \big(v(x)-v(y)\big)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\\&=\int_{\mathcal{D}_\epsilon} v(x)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy+\int_{\mathcal{D}_\epsilon} v(y)\big(\varphi(y)-\varphi(x)\big)\,K(x-y)\,dx\,dy\\&=\int_{\mathcal{D}_\epsilon} v(x)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy+\int_{\mathcal{D}_\epsilon} v(x)\big(\varphi(x)-\varphi(y)\big)\,K(y-x)\,dx\,dy\,.\end{aligned}$$

Now, with the change of variable $\xi=y-x$ in the first integral and $\xi=x-y$ in the second one, and taking into account the fact that K is even (see (1.11)) and the definition of $\mathcal{D}_\epsilon^\pm$, we get

$$\begin{aligned}&\int_{\mathcal{D}_\epsilon} \big(v(x)-v(y)\big)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\\&=\int_{\mathcal{D}_\epsilon} v(x)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy+\int_{\mathcal{D}_\epsilon} v(x)\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\\&=\int_{\mathcal{D}_\epsilon^+} v(x)\big(\varphi(x)-\varphi(x+\xi)\big)\,K(\xi)\,dx\,d\xi+\int_{\mathcal{D}_\epsilon^-} v(x)\big(\varphi(x)-\varphi(x-\xi)\big)\,K(\xi)\,dx\,d\xi.\end{aligned}\tag{5.18}$$

Now, we claim that

$$\int_{\mathcal{D}_\epsilon^-\setminus\mathcal{D}_\epsilon^+} v(x)\big(\varphi(x)-\varphi(x+\xi)\big)\,K(\xi)\,dx\,d\xi=0.\tag{5.19}$$

To check this, let $(x,\xi)\in\mathcal{D}_\epsilon^-\setminus\mathcal{D}_\epsilon^+$. By definitions of $\mathcal{D}_\epsilon^\pm$ it follows that $(x,x-\xi)\in\mathcal{D}_0$, $(x,x+\xi)\notin\mathcal{D}_0$ and $|\xi|\geqslant\epsilon$. Hence $(x,x+\xi)\in\mathcal{O}$, that is

$$x\in\mathcal{C}\Omega,\tag{5.20}$$

and

$$x+\xi\in\mathcal{C}\Omega\,.\tag{5.21}$$

Then, by (5.20) and (5.21), we have that $\varphi(x)=0=\varphi(x+\xi)$, for any $(x,\xi)\in\mathcal{D}_\epsilon^-\setminus\mathcal{D}_\epsilon^+$ which implies (5.19). Analogously, we get that

$$\int_{\mathcal{D}_\epsilon^+\setminus\mathcal{D}_\epsilon^-} v(x)\big(\varphi(x)-\varphi(x-\xi)\big)\,K(\xi)\,dx\,d\xi=0.\tag{5.22}$$

In the light of (5.18), (5.19) and (5.22), we obtain that

$$\begin{aligned}&\int_{\mathcal{D}_\epsilon} \big(v(x)-v(y)\big)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy\\&\qquad=\int_{\mathcal{D}_\epsilon^+\cup\mathcal{D}_\epsilon^-} v(x)\big(\varphi(x)-\varphi(x+\xi)\big)\,K(\xi)\,dx\,d\xi\\&\qquad\qquad+\int_{\mathcal{D}_\epsilon^-\cup\mathcal{D}_\epsilon^+} v(x)\big(\varphi(x)-\varphi(x-\xi)\big)\,K(\xi)\,dx\,d\xi\\&\qquad=\int_{\mathcal{D}_\epsilon^+\cup\mathcal{D}_\epsilon^-} v(x)\big(2\varphi(x)-\varphi(x+\xi)-\varphi(x-\xi)\big)\,K(\xi)\,dx\,d\xi\,.\end{aligned}\tag{5.23}$$

Now, let us define

$$\Phi(x,\xi) = \big(2\varphi(x) - \varphi(x+\xi) - \varphi(x-\xi)\big)\, K(\xi). \tag{5.24}$$

We claim that

$$\Phi \in L^1(\mathbb{R}^{2n}, dxdy). \tag{5.25}$$

To establish this, we observe that

$$|\Phi(x,\xi)| \leqslant 4\, \|\varphi\|_{L^\infty(\mathbb{R}^n)}\, K(\xi), \tag{5.26}$$

and, by a Taylor expansion, that

$$|\Phi(x,\xi)| \leqslant \|D^2\varphi\|_{L^\infty(\mathbb{R}^n)}\, |\xi|^2\, K(\xi).$$

Therefore,

$$|\Phi(x,\xi)| \leqslant 4\, \|\varphi\|_{C^2(\mathbb{R}^n)}\, m(\xi)\, K(\xi). \tag{5.27}$$

Let

$$R > 1 \tag{5.28}$$

be such that $\Omega \subset B_R$. If $x \in \mathcal{C}B_{2R}$ and $x \pm \xi \in \Omega \subset B_R$, then

$$|\xi| \geqslant |x| - |x \pm \xi| \geqslant 2R - R = R > 1 \tag{5.29}$$

by (5.28). Now we define

$$\mathcal{S} = \big\{(x,\xi) \in \mathbb{R}^{2n} : x \in \mathcal{C}B_{2R} \;\text{ and }\; \xi \in B_R(x) \cup B_R(-x)\big\}$$

and

$$\mathcal{S}_* = \big\{(x,\xi) \in \mathbb{R}^{2n} : x \in B_R(\xi) \cup B_R(-\xi) \;\text{ and }\; \xi \in \mathcal{C}B_1\big\}\,.$$

Since (5.29) holds true, then

$$\mathcal{S} \subset \mathcal{S}_*\,. \tag{5.30}$$

Moreover, by the definition of m (see (1.9)),

$$K(\xi) = m(\xi)K(\xi) \quad \text{if } (x,\xi) \in \mathcal{S}_*\,. \tag{5.31}$$

Now, let $(x,\xi) \in \big(\mathcal{C}B_{2R} \times \mathbb{R}^n\big) \setminus \mathcal{S}$. Then $x \in \mathcal{C}B_{2R}$ and $\xi \in \mathcal{C}\big(B_R(x) \cup B_R(-x)\big)$, that is $|x \pm \xi| > R$. As a consequence, since φ vanishes outside Ω

$$\varphi(x) = 0 = \varphi(x \pm \xi) \quad \text{if} \quad (x,\xi) \in \big(\mathcal{C}B_{2R} \times \mathbb{R}^n\big) \setminus \mathcal{S}\,. \tag{5.32}$$

Thus, using (5.26), (5.30)–(5.32) and (1.9) we have

$$\begin{aligned}
\int_{(\mathcal{C}B_{2R})\times\mathbb{R}^n} |\Phi(x,\xi)|\, dx\, d\xi &= \int_{\mathcal{S}} \big|2\varphi(x) - \varphi(x+\xi) - \varphi(x-\xi)\big|\, K(\xi)\, dx\, d\xi \\
&\leqslant 4\, \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathcal{S}_*} K(\xi)\, dx\, d\xi = 4\, \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathcal{S}_*} m(\xi)K(\xi)\, dx\, d\xi \\
&\leqslant 8\, \omega_n\, R^n\, \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} m(\xi)\, K(\xi)\, d\xi = C(n,R,K) < +\infty
\end{aligned} \tag{5.33}$$

for a suitable positive constant $C(n,R,K)$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$, i.e. $\omega_n = 2\pi^{n/2}/(n\Gamma(n/2))$.

By (5.27) and (5.33) we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2n}} |\Phi(x,\xi)|\,dx\,d\xi &\leqslant C(n,R,K) + \int_{B_{2R}\times\mathbb{R}^n} |\Phi(x,\xi)|\,dx\,d\xi \\ &\leqslant C(n,R,K) + 4\,\|\varphi\|_{C^2(\mathbb{R}^n)} \int_{B_{2R}\times\mathbb{R}^n} m(\xi)\,K(\xi)\,dx\,d\xi \\ &\leqslant C(n,R,K) + 4\,\omega_n\,(2R)^n\,\|\varphi\|_{C^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} m(\xi)K(\xi)\,d\xi\,,\end{aligned}$$

which, once more, is finite due to (1.9). This establishes (5.25).

Owing to Lemma 5.3 and (5.25), we can send $\epsilon \to 0^+$ in (5.23): we obtain

$$\begin{aligned}&\int_{\mathcal{D}_0} \big(v(x)-v(y)\big)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy \\ &\qquad = \int_{\mathcal{D}_0^+\cup\mathcal{D}_0^-} v(x)\big(2\varphi(x)-\varphi(x+\xi)-\varphi(x-\xi)\big)\,K(\xi)\,dx\,d\xi.\end{aligned}$$

By using once again that φ vanishes outside Ω, we get the assertion of Lemma 5.6. □

As a consequence of Lemma 5.6, we can prove that $\mathcal{A}$ satisfies condition (3.7). Indeed, the following corollary holds true.

Corollary 5.7. *Let $\varphi \in C_0^2(\Omega)$. Let $v_j \in X$ be a sequence of functions converging uniformly in $\mathbb{R}^n$ to $v_\infty \in X$ as $j \to +\infty$ and such that $v_j - u_0 \in X_0$ for any $j \in \mathbb{N}\cup\{\infty\}$. Then*

$$\begin{aligned}&\lim_{j\to+\infty} \int_Q \big(v_j(x)-v_j(y)\big)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy \\ &\qquad = \int_Q \big(v_\infty(x)-v_\infty(y)\big)\,\big(\varphi(x)-\varphi(y)\big)\,K(x-y)\,dx\,dy.\end{aligned}$$

Proof. We define $w_j := v_j - v_\infty$. Notice that $w_j \in X$, being X a linear space, and $w_j = 0$ a.e. in $\mathcal{C}\Omega$, since $v_j = u_0 = v_\infty$ a.e. in $\mathcal{C}\Omega$. Hence, $w_j \in X \cap L^\infty(\mathcal{C}\Omega)$ for any $j \in \mathbb{N}$. Then, by Lemma 5.6 (applied to w_j), we obtain that

$$\begin{aligned}&\int_Q \big(v_j(x)-v_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ &\qquad - \int_Q \big(v_\infty(x)-v_\infty(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ &= \int_Q \big(w_j(x)-w_j(y)\big)\big(\varphi(x)-\varphi(y)\big)K(x-y)\,dx\,dy \\ &= \int_Q w_j(x)\big(2\varphi(x)-\varphi(x+\xi)-\varphi(x-\xi)\big)K(\xi)\,dx\,d\xi \\ &= \int_Q \big(v_j(x)-v_\infty(x)\big)\big(2\varphi(x)-\varphi(x+\xi)-\varphi(x-\xi)\big)K(\xi)\,dx\,d\xi\,, \qquad (5.34)\end{aligned}$$

for any $\varphi \in C_0^2(\Omega)$ and any $j \in \mathbb{N}$.

Moreover, exploiting the notation in (5.24) and (5.25), we have

$$\begin{aligned}\lim_{j\to+\infty}\Big|\int_{\mathbb{R}^{2n}}\big(v_j(x)-v_\infty(x)\big)\big(2\varphi(x)-\varphi(x+\xi)-\varphi(x-\xi)\big)\,K(\xi)\,dx\,d\xi\Big|\\ =\lim_{j\to+\infty}\Big|\int_{\mathbb{R}^{2n}}\big(v_j(x)-v_\infty(x)\big)\Phi(x,\xi)\,dx\,d\xi\Big|\\ \leqslant\lim_{j\to+\infty}\|v_j-v_\infty\|_{L^\infty(\mathbb{R}^n)}\|\Phi\|_{L^1(\mathbb{R}^{2n})}=0\,,\end{aligned}$$

since $v_j\to v_\infty$ uniformly in $\mathbb{R}^n$ as $j\to+\infty$. This and (5.34) imply the assertion. □

Notice that here, in order to check assumption (3.7) (which follows from Corollary 5.7), we do not take advantage from the fact that u is a solution of the variational inequality (1.12) and that u_η solves the approximated equation (5.7).

Proof of Theorem 1.2. Let u be a solution of the variational inequality (1.12). In the setting of Theorem 1.2 we can apply Theorem 3.1, so that we get

$$\begin{aligned}0&\leqslant-\int_Q(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy+\int_\Omega f(x)\varphi(x)dx\\ &\leqslant\int_\Omega\big(\mathcal{L}_K\,\psi+f\big)^+(x)\varphi(x)dx\end{aligned}\tag{5.35}$$

for any $\varphi\in C_0^\infty(\Omega)$ with $\varphi\geqslant0$ in Ω and $\varphi=0$ in $\mathcal{C}\Omega$. □

5.1. The fractional Laplacian operator

As an application of Theorem 1.2, now we consider the case of the fractional Laplace kernel, i.e., the case when

$$K(x)=|x|^{-(n+2s)},\quad x\in\mathbb{R}^n\setminus\{0\}\,,\tag{5.36}$$

with $s\in(0,1)$, $n\in\mathbb{N}$, $n>2s$.

Proof of Theorem 1.1. By (5.35) and (5.36),

$$\begin{aligned}0&\leqslant-\int_Q\frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy+\int_\Omega f(x)\varphi(x)dx\\ &\leqslant\int_\Omega\big(-(-\Delta)^s\,\psi+f\big)^+(x)\varphi(x)dx\end{aligned}$$

for any $\varphi\in C_0^\infty(\Omega)$ with $\varphi\geqslant0$ in Ω and $\varphi=0$ in $\mathcal{C}\Omega$. Note that, using the definition of $H_0^s(\Omega)$ and a density argument, we get that the estimate holds true for any $\varphi\in H_0^s(\Omega)$ with $\varphi\geqslant0$ a.e. in Ω and $\varphi=0$ a.e. in $\mathcal{C}\Omega$. □

References

[1] BOCCARDO, L. AND GALLOUET, T: Problèmes unilatéraux avec données dans L^1 (Unilateral problems with L^1 data). *C. R. Acad. Sci. Paris, Sér. I* **311** (1990), no. 10, 617–619.

[2] BRÉZIS, H.: *Analyse fonctionelle. Théorie et applications.* Masson, Paris, 1983.

[3] CAFFARELLI, L. A., SALSA, S. AND SILVESTRE, L.: Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. *Invent. Math.* **171** (2008), no. 2, 425–461.

[4] CHALLAL, S., LYAGHFOURI, A. AND RODRIGUES, J. F.: On the A-obstacle problem and the Hausdorff measure of its free boundary. *Ann. Mat. Pura Appl. (4)* **191** (2012), no. 1, 113–165.

[5] DIBENEDETTO, E.: $C^{1+\alpha}$-local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.* **7** (1983), no. 8, 827–850.

[6] DI NEZZA, E., PALATUCCI, G. AND VALDINOCI, E.: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), no. 5, 521–573.

[7] DONATI, F.: A penalty method approach to strong solutions of some nonlinear parabolic unilateral problems. *Nonlinear Anal.* **6** (1982), no. 6, 585–597.

[8] FREHSE, J. AND MOSCO, U.: Irregular obstacles and quasivariational inequalities of stochastic impulse control. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **9** (1982), no. 1, 109–157.

[9] GILBARG, D. AND TRUDINGER, N. S.: *Elliptic partial differential equations of second order,* 2nd ed. Springer Verlag, Berlin, 1983.

[10] HANOUZET, B. AND JOLY, J. L.: Méthodes d'ordre dans l'interprétation de certaines inéquations variationelles et applications. *J. Funct. Anal.* **34** (1979), no. 2, 217–249.

[11] LANDKOF, N. S.: *Foundations of modern potential theory.* Springer Verlag, New York-Heidelberg, 1972.

[12] LEWY, H. AND STAMPACCHIA, G.: On the smoothness of superharmonics which solve a minimum problem. *J. Analyse Math.* **23** (1970), 227–236.

[13] LIEB, E. H. AND LOSS, M.: *Analysis.* Amer. Math. Soc., Providence, RI, 1997.

[14] LIEBERMAN, G. M.: Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.* **12** (1988), no. 11, 1203–1219.

[15] MASTROENI, L. AND MATZEU, M.: An integro-differential parabolic variational inequality connected with the problem of the American option pricing. *Z. Anal. Anwendungen* **14** (1995), no. 4, 869–880.

[16] MASTROENI, L. AND MATZEU, M.: Nonlinear variational inequalities for jump-diffusion processes and irregular obstacles with a financial application. *Nonlinear Anal.* **34** (1998), no. 6, 889–905.

[17] MILAKIS, E. AND SILVESTRE, L.: Regularity for the nonlinear Signorini problem. *Adv. Math.* **217** (2008), no. 3, 1301–1312.

[18] MOKRANE, A. AND MURAT, F.: A proof of the Lewy–Stampacchia's inequality by a penalization method. *Potential Anal.* **9** (1998), no. 2, 105–142.

[19] MOKRANE, A. AND MURAT, F.: The Lewy–Stampacchia inequality for bilateral problems. *Ricerche Mat.* **53** (2004), no. 1, 139–182.

[20] Mokrane, A. and Murat, F.: The Lewy–Stampacchia inequality for the obstacle problem with quadratic growth in the gradient. *Ann. Mat. Pura Appl. (4)* **184** (2005), no. 3, 347–360.

[21] Mosco, U. and Troianiello, G. M.: On the smoothness of solutions of unilateral Dirichlet problems. *Boll. Un. Mat. Ital. (4)* **8** (1973), 57–67.

[22] Pinamonti, A. and Valdinoci, E.: A Lewy–Stampacchia estimate for variational inequalities in the Heisenberg group. *Rend. Istit. Mat. Univ. Trieste* **45** (2013), 1–22.

[23] Rodrigues, J. F.: Stability remarks to the obstacle problem for p-Laplacian type equations. *Calc. Var. Partial Differential Equations* **23** (2005), no. 1, 51–65.

[24] Silvestre, L.: Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* **60** (2007), no. 1, 67–112.

[25] Simon, J.: Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^n$. In *Journées d'Analyse Non Linéaire (Proc. Conf. Besançon, 1977)*, 205–227. Lecture Notes in Math. 665, Springer, Berlin, 1978.

[26] Stein, E. M.: *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press, Princeton, 1993.

[27] Valdinoci, E.: From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl.* **49** (2009), 33–44.

[28] Varopoulos, N. Th., Saloff-Coste, L. and Coulhon, T.: *Analysis and geometry on groups.* Cambridge University Press, UK, 1992.

Received February 2, 2012.

Raffaella Servadei: Dipartimento di Matematica e Informatica, Universitá della Calabria, Ponte Pietro Bucci 31 B, 87036 Arcavacata di Rende (Cosenza), Italy.
E-mail: servadei@mat.unical.it

Enrico Valdinoci: Dipartimento di Matematica, Università di Milano, Via Cesare Saldini 50, 20133 Milano, Italy;

and

Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstraße 39, 10117 Berlin, Germany.
E-mail: enrico.valdinoci@unimi.it

The first author was supported by the MIUR National Research Project *Variational and Topological Methods in the Study of Nonlinear Phenomena*, while the second one by the MIUR National Research Project *Nonlinear Elliptic Problems in the Study of Vortices and Related Topics* and the FIRB project A&B (*Analysis and Beyond*). Both authors were supported by the ERC grant ϵ (*Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities*).