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On varieties with higher osculating defect

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Abstract. In this paper, using the method of moving frames, we generalise some of Terracini's results on varieties with tangent defect. In particular, we characterise varieties with higher order osculating defect in terms of Jacobians of higher fundamental forms and moreover we characterise varieties with "small" higher fundamental forms as contained in scrolls.

Introduction

The starting point of this paper is given by the classical papers [20], [21], [22] and [23] of Terracini on the description of the k-dimensional varieties V of $\mathbb{P}^N(\mathbb{C})$, $(N > 2k)$, such that the embedded tangent variety $\mathrm{Tan}(V)$ is defective, i.e. it has dimension less than $2k$ ($2k-\ell$ with $\ell > 0$). In [21], Terracini links this problem to the determination of the linear systems of quadrics for which the Jacobian matrix has rank $k-\ell$. After Terracini, there have been many papers on this subject: here we cite as examples only [2], [16], [17] and [18].

Terracini proved results bounding the tangent defect of V and on the structure of the varieties satisfying a certain number of Laplace equations. Given a local parametrisation $\mathbf{x}(t_1,\dots,t_k) = (x_1(t_1,\dots,t_k),\dots,x_N(t_1,\dots,t_k))$ and denoting by $\mathbf{x}^I = \frac{\partial^{|I|}\mathbf{x}(t_1,\dots,t_k)}{\partial t_1^{i_1}\dots\partial t_k^{i_k}}$ the partial derivatives of $\mathbf{x}$, we will say that V *satisfies* δ_s *Laplace equations of order* s if there hold the following partial differential equations:

$$\sum_{0\le|I|\le s} E_I^{(h)}\,\mathbf{x}^I = 0, \quad h = 1,\dots,\delta_s\,,$$

where at least a $E_I^{(h)} \neq 0$ with $|I| = s$ and these equations are linearly independent.

In this paper, we apply the method of moving frames, developed by Darboux, Cartan and others, to understand the relationship between the algebraic geometry of subvarieties of projective space and their local projective differential geometry.

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This was a project of classical geometers, revived by Akivis and Goldberg (see [1] and references therein) and Griffiths and Harris (see [6]) and more recently by Landsberg (see for example [12], [13], and with other authors, [11] and [14]).

We generalise Terracini's Theorem to varieties with defect of higher order by studying linear systems of hypersurfaces (the fundamental forms) instead of the Laplace equations of every order satisfied by the variety. We prove the following.

Theorem. *Let $V \subseteq \mathbb{P}^N$ be a k-dimensional irreducible variety whose t-th fundamental form has dimension $k-\ell-1$, with $\ell > 0$; then V has $(t-1)$-osculating defect $\geq \ell$ and moreover there hold:*

1) *V is contained in a d-dimensional scroll $S(\Sigma_r^h)$ in $\mathbb{P}^r$, with $0 \leq h \leq k-\ell$ and $k-h \leq r$.*
2) *The tangent $\mathbb{P}^d$'s to $S(\Sigma_r^h)$ at the smooth points of a generic $\mathbb{P}^r$ of $S(\Sigma_r^h)$ are contained in a linear space of dimension $d_t - h = d_{t-1} + k - \ell - h$, where d_t is the dimension of the t-th osculating space to V at its general point. In particular, $r \leq d \leq d_{t-1} + k - \ell - h$.*

See Theorem 2.4.

Moreover, we have obtained classifications for the extremal cases of the preceding theorem; for example, we show that, if $\ell = k-1$ and $t = 2$, then V is either a hypersurface or a developable $\mathbb{P}^{k-1}$-bundle.

Later, in [21], Terracini studied again varieties with tangent defect, but satisfying a number of Laplace equations less than $\binom{k}{2} + \ell$.

We also generalise this result as follows, in terms of fundamental forms:

Theorem. *Let $V \subseteq \mathbb{P}^N$ be a k-dimensional irreducible variety. V has t-th osculating defect $o_t = \ell > 0$ and the $(t+1)$-st fundamental form has dimension at least $k-\ell$ if and only if the Jacobian matrix of the $(t+1)$-st fundamental form of V has rank $k-\ell$.*

See Theorem 2.8.

Rational varieties satisfying one Laplace equation are studied also in [5], [9] and [10] or, more recently, in [3] and [15].

The article is structured as follows. In Section 1 we give the basic notations and preliminaries, and we show some results that we need. Many of them either are natural generalisations of known results (mainly from [6]) or are not very surprising; nevertheless, we think that including them is useful because of the lack of adequate references. More precisely, after fixing some notation and recalling basic definitions, such as Laplace equations, Darboux frames, the second fundamental form and apolarity, we prove the relation between the dimension of the second fundamental form and the number of Laplace equations of order two for a k-dimensional projective variety $V \subset \mathbb{P}^N$. More precisely, if V satisfies δ_2 Laplace equations, then the second fundamental form has dimension $\binom{k+1}{2} - 1 - \delta_2$.

Then, after recalling the definition of osculating spaces of higher orders, we link them to the higher fundamental forms, proving in particular that the Jacobian system of the t-th fundamental form is contained in the $(t-1)$-st fundamental form.

We also prove the equivalence between the dimension of the t-th fundamental form and the number of Laplace equations of order t, extending the above result for the second fundamental form.

We recall the definition of the t-th Gauss map and we show that its differential can be interpreted as the t-th fundamental form. Finally, we introduce the definition of the t-th dual variety of V and we prove some lemmas about it.

In Section 2 we state and prove the main theorems of the article, i.e., Theorems 2.4 and 2.8. In order to do so, we also prove a lemma on the tangent space of the higher osculating variety of V.

1. Notation and preliminaries

We use notation as in [6] and [8]. Let $V \subset \mathbb{P}^N$ be a projective variety of dimension k over $\mathbb{C}$, that will be always irreducible. For any point $P \in V$ we use the following notation: $\widetilde{T}_P(V) \subset \mathbb{P}^N$ is the embedded tangent projective space to V in P and $T_P(V)$ is the Zariski tangent space.

As in [6], we abuse notation by identifying the embedded tangent space in $\mathbb{P}^N$ with the affine cone over it in $\mathbb{C}^{N+1}$. With this convention, $T_P(V) \cong \widetilde{T}_P(V)/\mathbb{C}$. We denote by $\mathbb{G}(N,t)$ the Grassmannian of t-planes of $\mathbb{P}^N$.

We define $\mathrm{Tan}(V) := \overline{\bigcup_{P\in V_0} \widetilde{T}_P(V)}$ where $V_0 \subset V$ is the smooth locus of V. $\mathrm{Tan}(V)$ has expected dimension $2k$, and the case in which $\mathrm{Tan}(V)$ is less than expected has been studied by many algebraic geometers: classically Terracini [21] linked the dimension $2k-\ell$ of $\mathrm{Tan}(V)$ with the number of Laplace equations that the variety V satisfies, and more recently Griffiths and Harris [6] analysed the same dimension in terms of second fundamental form II.

Actually, for studying Laplace equations, it is standard to consider a parametric representation of V; in [6] and [11] the authors instead use the language of Darboux frames. So, our first step is to understand in this language what it means that V satisfies a Laplace equation.

We begin by defining the Laplace equations. Let $V \subseteq \mathbb{P}^N$ and let $\mathbf{x} = \mathbf{x}(t_1,\dots,t_k) = \mathbf{x}(\mathbf{t})$ be a local affine parametrisation of V centred at the smooth point $P = [p_0 : p_1 : \cdots : p_N]$, with, for example, $p_0 \neq 0$ and $\mathbf{x}(\mathbf{0}) = P$. Let $I = (i_1,\dots,i_k)$ be a multi-index, that is a k-tuple of nonnegative integers. We shall denote by $|I|$ the sum of the components of I, i.e., $|I| = i_1 + \cdots + i_k$. If $\mathbf{x}(t_1,\dots,t_k) = (x_1(t_1,\dots,t_k),\dots,x_N(t_1,\dots,t_k))$ is the above vector-valued function, we shall denote by $\mathbf{x}^I$ the partial derivatives of $\mathbf{x}$:

$$\mathbf{x}^I = \frac{\partial^{|I|}\mathbf{x}(t_1,\dots,t_k)}{\partial t_1^{i_1}\dots\partial t_k^{i_k}}.$$

Definition 1.1. By saying that V *satisfies* δ_s *Laplace equations of order* s we mean that, with the above local parametrisation $\mathbf{x}$ of V, $\mathbf{x}$ satisfies the following system of partial differential equations:

$$\sum_{0\le|I|\le s} E_I^{(h)}\,\mathbf{x}^I = 0, \quad E_I^{(h)} \in \mathbb{C}, \quad h = 1,\dots,\delta_s \tag{1.1}$$

where there is at least one $E_I^{(h)}$ with $|I| = s$ that does not vanish, and that these equations are linearly independent. Equivalently, we say that V represents the system of differential equations (1.1) or that V is an integral variety for it.

It is not restrictive to suppose that $P = [1 : 0 : \cdots : 0]$, and therefore we have $\mathbf{x}(\mathbf{0}) = P = \mathbf{0} \in \mathbb{A}^N$, and

$$x_i(t_1, \ldots, t_k) = t_i \quad \forall i \leq k, \tag{1.2}$$

i.e., $x_{k+1} = \cdots = x_N$ defines $\widetilde{T}_P(V) \subset \mathbb{P}^N$. With these hypotheses, the equations (1.1) become

$$\sum_{2 \leq |I| \leq s} E_I^{(h)} \mathbf{x}^I = 0, \quad h = 1, \ldots, \delta_s. \tag{1.3}$$

In what follows, we will make these assumptions.

At the same time, to study the behaviour of V in P, following [6] (and references [2], [6], [7] and [10] therein) and [12], we consider the manifold $\mathcal{F}(V)$ of frames in V. An element of $\mathcal{F}(V)$ is a Darboux frame centred in P. This means an $(N+1)$-tuple

$$\{A_0; A_1, \ldots, A_k; \ldots, A_N\}$$

which is a basis of $\mathbb{C}^{N+1}$ such that, if $\pi \colon \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ is the canonical projection,

$$\pi(A_0) = P \quad \text{and} \quad \pi(A_0), \pi(A_1), \ldots, \pi(A_k) \ \text{ span } \widetilde{T}_P(V).$$

Let this frame move in $\mathcal{F}(V)$; then we have the following structure equations (in terms of the restrictions to V of the Maurer–Cartan 1-forms ω_i, $\omega_{i,j}$ on $\mathcal{F}(\mathbb{P}^N)$) for the exterior derivatives of this moving frame:

$$\begin{cases} \omega_\mu = 0 & \forall \mu > k \\ dA_0 = \displaystyle\sum_{i=0}^{k} \omega_i A_i, & \\ dA_i = \displaystyle\sum_{j=0}^{N} \omega_{i,j} A_j & i = 1, \ldots, N, \\ d\omega_j = \displaystyle\sum_{h=0}^{k} \omega_h \wedge \omega_{h,j} & j = 0, \ldots, k, \\ d\omega_{i,j} = \displaystyle\sum_{h=0}^{N} \omega_{i,h} \wedge \omega_{h,j} & i = 1, \ldots, N, \ j = 0, \ldots, N. \end{cases} \tag{1.4}$$

Remark 1.2. Geometrically, the frame $\{A_i\}$ defines a coordinate simplex in $\mathbb{P}^N$. The 1-forms $\omega_i, \omega_{i,j}$ give the rotation matrix when the coordinate simplex is infinitesimally displaced; in particular, modulo A_0, as $dA_0 \in T_P^*(\mathbb{P}^N)$ (the cotangent space), the 1-forms $\omega_1, \ldots, \omega_k$ give a basis for the cotangent space $T_P^*(V)$, the corresponding $\pi(A_i) = v_i \in T_P(V)$ give a basis for $T_P(V)$ such that v_i is tangent to the line $\overline{A_0 A_i}$, and $\omega_{k+1} = \cdots = \omega_N = 0$ on $T_P(V)$.

Using this notation, we can define the second fundamental form locally:

Definition 1.3. The *second fundamental form of V in P* is the linear system $|II|$ in the projective space $\mathbb{P}(T_P(V)) \cong \mathbb{P}^{k-1}$ of the quadrics defined by the equations:

$$\sum_{i,j=1}^{k} q_{i,j,\mu}\,\omega_i\,\omega_j = 0, \quad \mu = k+1, \dots, N,$$

where the coefficients $q_{i,j,\mu}$ are defined by the relations

$$\omega_{i,\mu} = \sum_{j=1}^{k} q_{i,j,\mu}\,\omega_j, \quad q_{i,j,\mu} = q_{j,i,\mu} \tag{1.5}$$

obtained from $d\omega_\mu = 0$, $\forall \mu > k$, via the Cartan lemma (see (1.17) in [6]).

We may write the second fundamental form symbolically (as in (1.20) of [6]) as

$$d^2 A_0 \equiv \sum_{\substack{0\le i,j\le k\\ k+1\le \mu\le N}} q_{i,j,\mu}\,\omega_i\,\omega_j\,A_\mu \mod \tilde{T}(V), \tag{1.6}$$

or more intrinsically, as the (global) map

$$II\colon \operatorname{Sym}^{(2)} T(V) \to N(V)\,, \tag{1.7}$$

where $N(V)$ is the normal bundle ($N_P(V) := \mathbb{C}^{N+1}/\tilde{T}_P(V)$ as in [6]) which in coordinates is

$$II\Big(\sum_{i,j} a_{i,j}\,v_i\,v_j\Big) = \sum_{\substack{0\le i,j\le k\\ k+1\le \mu\le N}} q_{i,j,\mu}\,a_{i,j}\,A_\mu.$$

To relate the second fundamental form to the Laplace equations (1.1), for ease of exposition, we consider the case $s = 2$. If there are δ_2 independent relations of the form

$$\sum_{i,j=1}^{k} a_{i,j}^{(\alpha)}\,\mathbf{x}^{(ij)} + \sum_{i=1}^{k} b_i^{(\alpha)}\,\mathbf{x}^{(i)} + c^{(\alpha)}\,\mathbf{x} = 0, \quad \alpha = 1, \dots, \delta_2,$$

that, with our assumptions on the coordinates can be rewritten as

$$\sum_{i,j=1}^{k} a_{i,j}^{(\alpha)}\,\mathbf{x}^{(ij)} = 0, \quad \alpha = 1, \dots, \delta_2, \tag{1.8}$$

we can consider the linear system of quadrics of $\mathbb{P}(T_P(V)^*)$ of dimension $\delta_2 - 1$, generated by the quadrics of equations

$$\sum_{i,j=1}^{k} a_{i,j}^{(\alpha)}\,v_i\,v_j = 0, \quad \alpha = 1, \dots, \delta_2. \tag{1.9}$$

This defines the linear system of quadrics *associated* to the system of Laplace equations.

We recall now some notions of apolarity. Since our definitions are base dependent, for ease of exposition, we say that two forms $f = \sum_I a_I \mathbf{x}^I \in \mathbb{C}[x_0, \dots, x_N]$ and $g = \sum_I b_I \mathbf{y}^I \in \mathbb{C}[y_0, \dots, y_N] = C[x_0, \dots, x_N]^*$ of the same degree n, are *apolar* if

$$\sum_{I=(i_0,\dots,i_N)} a_I \, b_I = 0.$$

Since f and g define hypersurfaces $F := V(f) \subset \mathbb{P}^N = \mathrm{Proj}(\mathbb{C}[x_0, \dots, x_N])$ and $G := V(g) \subset \mathbb{P}^{N*} = \mathrm{Proj}(\mathbb{C}[y_0, \dots, y_N])$, we will say also that F and G are *apolar* if f and g are apolar.

Given a system h of hypersurfaces in $\mathbb{P}^N$, we say that the linear system K in $\mathbb{P}^{N*}$ given by the hypersurfaces which are apolar to all hypersurfaces in H is the *apolar system* of H.

The following result is classical:

Proposition 1.4. *$|II|$ is the apolar system to the system of quadrics (1.9); so, if V satisfies δ_2 independent Laplace equations, then* $\dim |II| = \binom{k+1}{2} - 1 - \delta_2$.

Proof. Since we can identify the parametrisation $\mathbf{x}$ around P with $\pi(A_0)$, by (1.6),

$$d^2 A_0 \Big(\sum_{i,j=1}^{k} a_{i,j}^{(\alpha)} \, v_i \, v_j \Big) = \sum_{1 \le i,j \le k} q_{i,j,\mu} \, a_{i,j}^{(\alpha)}, \quad \alpha = 1, \dots, \delta_2, \quad \mu = k+1, \dots, N;$$

for our choice of the coordinates. On the other hand,

$$d^2 A_0 \Big(\sum_{i,j=1}^{k} a_{i,j}^{(\alpha)} \, v_i \, v_j \Big) = \sum_{i,j=1}^{k} a_{i,j}^{(\alpha)} \frac{d^2 A_0}{dv_i \, dv_j} = \sum_{i,j=1}^{k} a_{i,j}^{(\alpha)} \, \mathbf{x}^{(ij)}, \quad \alpha = 1, \dots, \delta_2.$$

□

The second fundamental form can be related also with the second osculating space that we define as follows:

Definition 1.5. Let $P \in V$. The *second osculating space* to V at P is the subspace $\tilde{T}_P^{(2)}(V) \subset \mathbb{P}^N$ spanned by A_0 and by all the derivatives $dA_0/dv_\alpha = A_\alpha$ and $dA_\alpha/dv_\beta = dA_\beta/dv_\alpha$ for $1 \le \alpha, \beta \le k$.

From now on we can consider the Darboux frame

$$\{A_0; A_1, \dots, A_k; A_{k+1}, \dots, A_{k+r}; A_{k+r+1}, \dots, A_N\}$$

so that $A_0; A_1, \dots, A_k; A_{k+1}, \dots, A_{k+r}$ in P span $\tilde{T}_P^{(2)}(V)$. It is straightforward to see, for example from the proof of Proposition 1.4, that

$$\dim |II| = r - 1 \iff \dim \tilde{T}_P^{(2)}(V) = k + r.$$

Generalising Definition 1.3, we can define the t-th fundamental form and the t-th osculating space at $P \in V$, for $t \ge 3$, and relate them with (1.1).

Definition 1.6. Let $P \in V$, let $t \geq 3$ be an integer and let $I = (i_1, \ldots, i_k)$ be such that $|I| \leq t$. The *$t-th$ osculating space* to V at P is the subspace $\tilde{T}_P^{(t)}(V) \subset \mathbb{P}^N$ spanned by A_0 and by all the derivatives $d^{|I|}A_0/dv_1^{i1} \cdots dv_k^{i_k}$, where $v_1, \ldots, v_k$ span $T_P(V)$. We will put

$$d_t := \dim\big(\tilde{T}_P^{(t)}(V)\big),$$
$$e_t := \operatorname{expdim}\big(\tilde{T}_P^{(t)}(V)\big) = \min\Big(N, d_{t-1} + \binom{k-1+t}{t}\Big).$$

Remark 1.7. If V satisfies δ_t Laplace equations of order t, we have $d_t = e_t - \delta_t$. Moreover, since a Laplace equation of order t contains at least one of the $\binom{k-1+t}{t}$ partial derivatives of order t, we have $\delta_t \leq \binom{k-1+t}{t}$.

Put $k_t := \binom{k+t}{t} - 1$. Obviously, $d_t \leq \min(k_t, N)$. If $N < k_t$, then $V \subseteq \mathbb{P}^N$ represents at least $k_t - N$ Laplace equations of order t. These Laplace equations are called *trivial*.

Definition 1.8. Let $t \geq 2$ and let $V_0 \subseteq V$ be the quasi projective variety of points where $\tilde{T}_P^{(t)}(V)$ has maximal dimension. The variety

$$\operatorname{Tan}^t(V) := \overline{\bigcup_{P \in V_0} \tilde{T}_P^{(t)}(V)}$$

is called the *variety of osculating t-spaces to V*. Its expected dimension is

$$\operatorname{expdim} \operatorname{Tan}^t(V) := \min(k + d_t, N)$$

The *t-th osculating defect of V* is the integer

$$o_t := \operatorname{expdim} \operatorname{Tan}^t(V) - \dim \operatorname{Tan}^t(V).$$

If $t = 1$, we call o_1 the *tangent defect*.

Remark 1.9. Obviously we have

$$d_t \leq d_{t-1} + \binom{k-1+t}{t} \leq \cdots \leq \sum_{i=1}^{t} \binom{k+i-1}{i} = k_t.$$

We will study the osculating defects related to the fundamental forms. Following [6] and recalling (1.6) we give:

Definition 1.10. The *t-th fundamental form* of V in P is the linear system $|I^t|$ in the projective space $\mathbb{P}(T_P(V)) \cong \mathbb{P}^{k-1}$ of hypersurfaces of degree t defined symbolically by the equations:

$$d^t A_0 = 0.$$

More intrinsically, we write I^t as the map

$$I^t \colon \operatorname{Sym}^{(t)} T(V) \to N^t(V)$$

where $N^t(V)$ is the bundle defined locally by $N^t_P(V) := \mathbb{C}^{N+1}/\tilde{T}^{(t-1)}_P(V)$ and the map I^t is defined locally at each $v \in T_P(V)$ by

$$v^t \mapsto \frac{d^t A_0}{dv^t} \mod \tilde{T}^{(t-1)}_P(V).$$

Choose a Darboux frame

$$\{A_0; A_1, \dots, A_k; A_{k+1}, \dots, A_{d_2}; A_{d_2+1}, \dots, A_{d_s}; \dots, A_{d_t}; \dots, A_N\} \tag{1.10}$$

such that $A_0, A_1, \dots, A_{d_s}$ span $\tilde{T}^{(s)}_P(V)$ for all $s = 1, \dots, t$, with $d_1 := k$. By the definition of $\tilde{T}^{(s)}_P(V)$, we have that

$$dA_{\alpha_{s-1}} \equiv 0 \mod \tilde{T}^{(s)}_P(V), \quad \alpha_{s-1} = d_{s-2}+1, \dots, d_{s-1}, \quad s = 2, \dots, t-1, \tag{1.11}$$

where we put $d_0 = 0$, and from (1.4) we have

$$\omega_{\alpha_{s-1},\mu_s} = 0, \quad \alpha_{s-1} = d_{s-2}+1, \dots, d_{s-1}, \quad \mu_s > d_s, \quad s = 2, \dots t-1, \tag{1.12}$$

from which we infer, after some computation,

$$d^t A_0 \equiv \sum_{\substack{d_{s-1}+1 \le \alpha_s \le d_s \\ s=1,\dots,t-1 \\ d_{t-1}+1 \le \alpha_t \le N}} \omega_{\alpha_1}\omega_{\alpha_1,\alpha_2}\cdots\omega_{\alpha_s,\alpha_{s+1}}\cdots\omega_{\alpha_{t-1},\alpha_t}A_{\alpha_t} \mod \tilde{T}^{(t-1)}_P(V). \tag{1.13}$$

Alternatively, using the Cartan lemma,

$$\begin{aligned} d^t A_0 &\equiv \sum_{\substack{1 \le i_1,\dots,i_t \le k \\ d_{t-1}+1 \le \alpha_t \le N}} q_{i_1,\dots,i_t,\alpha_t}\,\omega_{i_1}\cdots\omega_{i_t}\,A_{\alpha_t} \\ &= \sum_{\substack{|I|=t \\ d_{t-1}+1 \le \alpha_t \le N}} q_{I,\alpha_t}\,\omega_I\,A_{\alpha_t} \mod \tilde{T}^{(t-1)}_P(V), \end{aligned} \tag{1.14}$$

with the natural symmetries for the indices $i_1, \dots, i_t$ of $q_{i_1,\dots,i_t,\alpha_t}$ which can be expressed as

$$\frac{d^{t-1}A_i}{dv_j} \equiv \frac{d^{t-1}A_j}{dv_i} \mod \tilde{T}^{(t-1)}_P(V), \quad i,j = 1, \dots, k. \tag{1.15}$$

From (1.12),

$$0 = d\omega_{\alpha_{s-1},\mu_s} = \sum_{h_s = d_{s-1}+1}^{d_s} \omega_{\alpha_{s-1},h_s} \wedge \omega_{h_s,\mu_s}$$
$$\alpha_{s-1} = d_{s-2}+1, \dots, d_{s-1}, \quad \mu_s > d_s, \quad s = 2, \dots t-1.$$

Now, $\omega_{\alpha_{s-1},h_s}$ and ω_{h_s,μ_s} are horizontal for the fibration $\tilde{T}_P^{(t-1)}(V) \to V$, and therefore, by induction on s, since the case $s=2$ is proved on page 374 of [6], they are a linear combination of $\omega_1,\dots,\omega_k$. Then, we have

$$0 = d\omega_{\alpha_{s-1},\mu_s}\Big(\frac{\partial}{\partial\omega_\gamma}\Big) = \sum_{h_s=d_{s-1}+1}^{d_s}\Big(\frac{\partial\omega_{\alpha_{s-1},h_s}}{\partial\omega_\gamma}\omega_{h_s,\mu_s} - \omega_{\alpha_{s-1},h_s}\frac{\partial\omega_{h_s,\mu_s}}{\partial\omega_\gamma}\Big)$$
$$\alpha_{s-1} = d_{s-2}+1,\dots,d_{s-1},\ \mu_s > d_s,$$
$$\gamma = 1,\dots,k,\ s = 2,\dots t-1.$$

which means

$$\sum_{h_s=d_{s-1}+1}^{d_s}\Big(\frac{\partial\omega_{\alpha_{s-1},h_s}}{\partial\omega_\gamma}\omega_{h_s,\mu_s}\Big) = \sum_{h_s=d_{s-1}+1}^{d_s}\Big(\frac{\partial\omega_{h_s,\mu_s}}{\partial\omega_\gamma}\omega_{\alpha_{s-1},h_s}\Big) \tag{1.16}$$
$$\alpha_{s-1} = d_{s-2}+1,\dots,d_{s-1},\ \mu_s > d_s,$$
$$\gamma = 1,\dots,k,\ s = 2,\dots t-1.$$

Since by relation (1.13) the linear system $|I^t|$ is generated by the degree t polynomials

$$V_{\alpha_t} := \sum_{\substack{d_{s-1}+1\le\alpha_s\le d_s\\ s=1,\dots,t-1}} \omega_{\alpha_1}\omega_{\alpha_1,\alpha_2}\cdots\omega_{\alpha_s,\alpha_{s+1}}\cdots\omega_{\alpha_{t-1},\alpha_t},\quad d_{t-1}+1\le\alpha_t\le N,$$

we can prove Theorem 1.12. In order to do so, we recall:

Definition 1.11. Let Σ be the linear system of dimension d of hypersurfaces of degree $n>1$ in $\mathbb{P}^N$ $(N>1)$, generated by the $d+1$ hypersurfaces $f_0=0,\dots,f_d=0$. The Jacobian matrix of the forms $f_0,\dots,f_d$,

$$J(\Sigma) := (\partial f_i/\partial x_j)_{i=0,\dots,d;j=0,\dots,r}$$

is called the *Jacobian matrix* of the system Σ.

The *Jacobian system* of Σ is the linear system of the minors of maximum order of $J(\Sigma)$. Obviously, the Jacobian system depends not on the choice of $f_0,\dots,f_d$, but only on Σ.

Theorem 1.12. *Given a k-dimensional projective variety $V\subset\mathbb{P}^N$, its t-th fundamental form $|I^t|$ is a linear system of polynomials of degree t whose Jacobian system is contained in the $(t-1)$-st fundamental form $|I^{t-1}|$.*

Proof. With the notation as above, we start considering, with $d_{t-1}+1\le\alpha_t\le N$,

$$\frac{\partial V_{\alpha_t}}{\partial\omega_\gamma} = \sum_{\substack{d_{s-1}+1\le\alpha_s\le d_s\\ s=2,\dots,t-1}} \omega_{\gamma,\alpha_2}\cdots\omega_{\alpha_s,\alpha_{s+1}}\cdots\omega_{\alpha_{t-1},\alpha_t} + \cdots$$
$$+ \sum_{\substack{d_{s-1}+1\le\alpha_s\le d_s\\ s=1,\dots,t-1}}\Big(\omega_{\alpha_1}\cdots\frac{\partial\omega_{\alpha_s,\alpha_{s+1}}}{\partial\omega_\gamma}\cdots\omega_{\alpha_{t-1},\alpha_t} + \cdots + \omega_{\alpha_1}\cdots\omega_{\alpha_s,\alpha_{s+1}}\cdots\frac{\partial\omega_{\alpha_{t-1},\alpha_t}}{\partial\omega_\gamma}\Big).$$

Then, from (1.16), we deduce

$$\frac{\partial V_{\alpha_t}}{\partial \omega_\gamma} = \sum_{\substack{d_{s-1}+1\le\alpha_s\le d_s \\ s=2,\dots,t-1}} \omega_{\gamma,\alpha_2}\cdots\omega_{\alpha_s,\alpha_{s+1}}\cdots\omega_{\alpha_{t-1},\alpha_t} + (t-1)\sum_{\substack{d_{s-1}+1\le\alpha_s\le d_s \\ s=1,\dots,t-1}} \omega_{\alpha_1}\cdots\omega_{\alpha_s,\alpha_{s+1}}\cdots\frac{\partial\omega_{\alpha_{t-1},\alpha_t}}{\partial\omega_\gamma}.$$

Then, for example by (1.5),

$$\omega_{\gamma,\alpha_2} = \sum_{\alpha_1=1}^{k} q_{\gamma,\alpha_1,\alpha_2}\omega_{\alpha_1} = \sum_{\alpha_1=1}^{k} q_{\alpha_1,\gamma,\alpha_2}\omega_{\alpha_1} = \sum_{\alpha_1=1}^{k}\frac{\partial\omega_{\alpha_1,\alpha_2}}{\partial\omega_\gamma}\omega_{\alpha_1}$$

and again by (1.16),

$$\frac{\partial V_{\alpha_t}}{\partial \omega_\gamma} = t\sum_{\substack{d_{s-1}+1\le\alpha_s\le d_s \\ s=1,\dots,t-1}} \omega_{\alpha_1}\cdots\omega_{\alpha_s,\alpha_{s+1}}\cdots\frac{\partial\omega_{\alpha_{t-1},\alpha_t}}{\partial\omega_\gamma}. \qquad \square$$

Actually, as for the second fundamental form, Proposition 1.13 holds with a proof adapted from the one given for Proposition 1.4. In order to do so, we fix a Darboux frame as in (1.10). Then, if we have a system of δ_t Laplace equations of order t as in (1.1), they can be expressed as

$$\sum_{|I|=t} E_I^{(h)}\,\mathbf{x}^I = 0, \quad h = 1,\dots,\delta_t. \tag{1.17}$$

As in (1.9), we can define the linear systems of homogeneous polynomials of degree t *associated* to (1.17) by

$$\sum_{|I|=t} E_I^{(h)}\,\mathbf{v}_I = 0, \quad h = 1,\dots,\delta_t, \quad \text{where} \quad \mathbf{v}_I = \prod_{\substack{i=1,\dots,k \\ i_1+\cdots+i_k=t}} v_i^{i_j}. \tag{1.18}$$

Proposition 1.13. *If V satisfies δ_t Laplace equations of order t as in (1.1), the t-th fundamental form is the apolar system to the system of the hypersurfaces of degree t associated to the system of Laplace equations (i.e., the hypersurfaces in (1.18)), and vice versa.*

Proof. It is enough to repeat the proof of Proposition 1.4 with an adapted local coordinate system. More precisely, we can choose a Darboux frame as in (1.10). Since we can identify the parametrisation $\mathbf{x}$ around P with $\pi(A_0)$, then, by our hypothesis, the Laplace equations of order t become

$$\sum_{|I|=t} E_I^{(h)}\,\mathbf{x}^I = 0, \quad h = 1,\dots,\delta_t. \tag{1.19}$$

By (1.14), we have

$$d^t A_0\Big(\sum_{|I|=t} E_I^{(h)}\,\mathbf{v}_I\Big) = \sum_{|I|=t} q_{I,\beta}\,E_I^{(h)}, \quad h=1,\dots,\delta_t, \quad \beta = d_{t-1}+1,\dots,N,$$

and, on the other hand,

$$d^t A_0\Big(\sum_{|I|=t} E_I^{(h)}\,\mathbf{v}_I\Big) = \sum_{|I|=t} E_I^{(h)}\frac{d^t A_0}{(d\mathbf{v})^I} = \sum_{|I|=t} E_I^{(h)}\,\mathbf{x}^I, \quad h=1,\dots,\delta_t. \qquad \square$$

From Proposition 1.13 we recover immediately the following:

Corollary 1.14. *If V satisfies δ_t Laplace equations of order t as in (1.1), the t-th fundamental form has dimension $\binom{k-1+t}{t} - 1 - \delta_t$, and vice versa.*

We will denote the dimension of the t-th fundamental form by Δ_t:

$$\Delta_t := \dim(|I^t|).$$

Corollary 1.15. *If $N \geq k_t$, we have that*

$$d_t = d_{t-1} + \Delta_t + 1,$$

and vice versa: if $d_t = d_{t-1} + \Delta + 1$, then the t-th fundamental form has dimension Δ.

From now on, we will suppose that our Darboux frame is as in (1.10).

In order to prove the results of the following section, we recall also the following notation and definitions.

Let $\Sigma_t^h \subset \mathbb{G}(N,t)$ be a subvariety of pure dimension h. Let $I_{\Sigma_t^h} \subset \Sigma_t^h \times \mathbb{P}^N$ be the incidence variety of the pairs (σ, q) such that $q \in \sigma$ and let $p_1 \colon I_{\Sigma_t^h} \to \Sigma_t^h$ and $p_2 \colon I_{\Sigma_t^h} \to \mathbb{P}^N$ be the maps induced by restricting to $I_{\Sigma_t^h}$ the canonical projections of $\Sigma_t^h \times \mathbb{P}^N$ to its factors.

The morphism $p_1 \colon I_{\Sigma_t^h} \to \Sigma_t^h$ is said to be a *family of t-dimensional linear subvarieties of* $\mathbb{P}^N$. While Σ_t^h is the parameter space of the family, for brevity we will often refer to it as to the family itself. Obviously,

$$\dim(I_{\Sigma_t^h}) = t + \dim(\Sigma_t^h).$$

Let us suppose that Σ_t^h is irreducible. We will denote by $S(\Sigma_t^h)$ the image of $I_{\Sigma_t^h}$ under p_2. By definition, $S(\Sigma_t^h)$ is a *scroll in* $\mathbb{P}^r$ of $\mathbb{P}^N$. The previous notation will be useful to study the osculating variety.

Definition 1.16. Let $t \geq 1$. The *t-th projective Gauss map* is the rational map

$$\begin{aligned} \gamma^t \colon V &\dashrightarrow \mathbb{G}(\mathbb{P}^N, d_t) \\ P &\mapsto \tilde{T}_P^{(t)}(V). \end{aligned}$$

For the classification of surfaces with second Gauss map not birational on the image see [4].

Remark 1.17. The *t-th osculating variety* is

$$\widetilde{\mathrm{Tan}^t}(V) = \overline{\bigcup_{P\in V_0} \gamma^t(P)} \subset \mathbb{G}(\mathbb{P}^N, k_t),$$

where, as before, V_0 denotes the open subset of the variety V comprising the points for which $\dim \tilde{T}_P^{(t)}(V) = d_t$ and then $\mathrm{Tan}^t(V)$ is the scroll $S(\widetilde{\mathrm{Tan}^t}(V))$ of dimension

$$\dim \mathrm{Tan}^t(V) \leq \dim \mathrm{Im}\, \gamma^t + d_t = k + d_t - \dim((\gamma^t)^{-1}(\Pi)),$$

where Π is a general element of $\widetilde{\mathrm{Tan}^t}(V)$.

We prove now:

Theorem 1.18. *The first differential of γ^t at P is the $(t+1)$-st fundamental form at P.*

Proof. We have, by definition of γ^t, that

$$d\gamma_P^t \colon T_P V \dashrightarrow T_{\tilde{T}_P^{(t)}V}\mathbb{G}(\mathbb{P}^N, d_t),$$

and we recall that $T_{\tilde{T}_P^{(t)}V}\mathbb{G}(\mathbb{P}^N, d_t) \cong \mathrm{Hom}(\tilde{T}_P^{(t)}V, N_P^{t+1}(V))$. Moreover, if we choose a Darboux frame as in (1.10), we have that $dA_0 \in \tilde{T}_P V \subset \tilde{T}_P^{(t)}V$ and

$$\frac{\tilde{T}_P^{(t)}V}{\mathbb{C}A_0} = T_P^{(t)}V$$

and therefore $d\gamma_P^t \in \mathrm{Hom}(T_P V \otimes T_P^{(t)}V, N_P^{t+1}(V))$.

Now, we remark that, in our Darboux frame, we can interpret γ^t as

$$\gamma^t(P) = A_0 \wedge \cdots \wedge A_{d_t},$$

and therefore by (1.4),

$$d\gamma_P^t \equiv \sum_{\substack{1\leq i\leq d_t \\ d_t+1\leq j\leq N}} (-1)^{d_t-i+1}\omega_{i,j} A_0 \wedge \cdots \wedge \hat{A}_i \wedge \cdots \wedge A_{d_t} \wedge A_j \mod \tilde{T}_P^{(t)}V.$$

Now, a basis for $T_P V \otimes T_P^{(t)}V$ can be written as $(A_\alpha \otimes A_\mu)_{\substack{\alpha=1,\ldots,k \\ \mu=1,\ldots,d_t}}$, and

$$d\gamma_P^t(A_\alpha \otimes A_\mu) = \sum_{d_t+1\leq j\leq N} \omega_{\mu,j}(A_\alpha) A_j \in N_P^{t+1}(V).$$

On the other hand, for the $(t+1)$-st fundamental form we have

$$\frac{dA_\mu}{dv_\alpha} \equiv \sum_{d_t+1\leq j\leq N} \omega_{\mu,j}(A_\alpha) A_j \mod \tilde{T}_P^{(t)}(V). \qquad \square$$

We recall now the definition of higher order dual varieties (see [19]), which is the natural extension of the definition of the dual variety.

Definition 1.19. Let $V \subset \mathbb{P}^N$ be a projective variety. By the *t-th dual variety* $\check{V}^{(t)}$ of V we mean

$$\check{V}^{(t)} = \overline{\bigcup_{P \in V_0} C_P^{(t)}(V)}, \tag{1.20}$$

where, as before, $V_0 \subset V$ is the set of the points for which $\dim T_P^{(t)}(V) = d_t$, and $C_p^{(t)}(V)$ is

$$C_P^{(t)}(V) := \bigcap_{K \in T_P^{(t)}(V)} K = \left\{ H \in \mathbb{P}^{N^*} \;\middle|\; H \supset \tilde{T}_P^{(t)}(V) \right\} \subset \mathbb{P}^{N^*}.$$

This $C_p^{(t)}(V)$ is classically called the *t-th characteristic space* of V in P.

We now make an observation similar to the one in §3(a) of [6]: elements of $C_P^{(t)}(V)$ can naturally be identified with hyperplanes in $\mathbb{P}(N_P^{t+1}(V))$ and therefore $\check{V}^{(t)}$ is just the image of the map

$$\delta^t \colon \mathbb{P}(N^{t+1}(V)^*) \to \mathbb{P}^{N^*},$$

analogous to the one in (3.1) of [6]. In terms of frames, a hyperplane ξ of $\mathbb{P}(N_P^{t+1}(V))$ can be given by choosing $A_{d_t+1}, \ldots, A_{N-1}$ such that their projections in $N_P^{t+1}(V) = \mathbb{C}^{N+1}/\tilde{T}_P^{(t)}(V)$ span ξ. Therefore, in terms of coordinates, δ^t can be expressed as

$$\delta^t(P, \xi) = A_0 \wedge A_1 \wedge \cdots \wedge A_{N-1},$$

(see (3.2) in [6]) or, if we choose dual coordinates

$$A_i^* := (-1)^{N-i} A_0 \wedge \cdots \wedge A_{i-1} \wedge A_{i+1} \wedge \cdots \wedge A_N,$$

$\delta^t(P, \xi) = A_N^*$. From the relations (1.4) we deduce

$$dA_j^* = \sum_{i \neq j} \big(-\omega_{i,j} A_i^* + \omega_{i,i} A_j^* \big),$$

and in particular

$$dA_N^* = \sum_{i=0}^{N-1} \big(-\omega_{i,N} A_i^* + \omega_{i,i} A_N^* \big) = \sum_{i=1}^{N-1} (-\omega_{i,N} A_i^*) + \Big(\omega_0 + \sum_{i=1}^{N-1} \omega_{i,i} \Big) A_N^*.$$

By definition of $\check{V}^{(t)}$, we have for its dimension

$$N - d_t - 1 \leq \dim \check{V}^{(t)} =: d_{t,1} \leq N - d_t - 1 + k.$$

Choosing a Darboux frame as in (1.10), these formulas become, thanks to (1.12),

$$dA_j^* = \sum_{\substack{i\neq j\\ i>u-2}} \big(-\omega_{i,j}A_i^* + \omega_{i,i}A_j^*\big), \quad \begin{cases} j = d_{u-1}+1,\dots,d_u & \text{if } u=0,\dots,t-1,\\ j = d_{t-1}+1,\dots,N & \text{if } u=t,\end{cases}$$

where we put $d_{-1} := -1$ when we vary j. In particular,

$$dA_N^* = \sum_{i=t-1}^{N-1}\big(-\omega_{i,N}A_i^* + \omega_{i,i}A_N^*\big) = \sum_{i=t-1}^{N-1}(-\omega_{i,N}A_i^*) + \Big(\omega_0 + \sum_{i=t-1}^{N-1}\omega_{i,i}\Big)A_N^*,$$

and therefore

$$dA_N^* \equiv \sum_{i=t-1}^{N-1}(-\omega_{i,N}A_i^*) \mod A_N^*. \tag{1.21}$$

Definition 1.20. We say that $\check V^{(t)}$ is *degenerate* if it has dimension less than expected: $d_{t,1} < N-1-d_t+k$.

In relation (1.21) the last $N-d_t-1$ forms $\omega_{i,N}$, $i=d_t+1,\dots,N-1$, restrict to a basis for the forms of the fibres $\mathbb{P}(N_P^{t+1})^* = \mathbb{P}^{N-1-d_t}$; in fact, they describe the variation of ξ when P is held fixed. The first $\omega_{i,N}$, with $i\le d_t$ are horizontal for the fibreing $\mathbb{P}(N^{t+1}(V)^*)\to V$, and therefore $\check V^{(t)}$ is degenerate if and only if

$$\omega_{i_1,N}\wedge\cdots\wedge\omega_{i_k,N}=0 \quad \forall i_1,\dots i_k \text{ with } t-1\le i_1<\cdots<i_k\le d_t.$$

If we put $d_{t,s} := \dim \tilde T_\xi^{(s)}(\check V^{(t)})$, if $N-d_{t,s}\ge d_{t-1}+1$, i.e., $d_{t,s}\le N-d_{t-1}-1$ (otherwise $\tilde T_\xi^{(s)}(\check V^{(t)}) = \mathbb{P}^{N*}$), we can choose a Darboux frame such that $T_\xi^{(s)}(\check V^{(t)})$ is generated by A_N^* and $A_{N-1}^*,\dots,A_{N-d_{t,s}}^*$.

Let us now define the characteristic varieties of a projective variety $V\subset\mathbb{P}^N$.

Definition 1.21. The *variety of the s-th characteristic spaces of the t-th osculating spaces* of V is the s-th dual of the t-th dual variety of V, that is,

$$\mathrm{Car}_t^s(V) := \overline{\bigcup_{\xi\in\check V_0^{(t)}} C_\xi^{(s)}(\check V^{(t)})} \subset \mathbb{P}^N,$$

where $\check V_0^{(t)}$ is the open subset of $\check V^{(t)}$ comprising the points ξ such that $\xi\supset\tilde T_P^{(t)}(V)$ and $\dim\tilde T_P^{(t)}(V)=d_t$. In the following we will denote this s-th characteristic space of $\check V^{(t)}$ in a general $\xi\supset\tilde T_P^{(t)}(V)$ by $C_{t,P}^{(s)}(V) := C_\xi^{(s)}(\check V^{(t)})$. Then, using the above notation, $\dim(C_{t,P}^{(s)}(V)) = N-1-d_{t,s}$.

Lemma 1.22. *With notation as above, if $P\in V$, we have:*

a) $\tilde T_P^{(t-1)}(V)\subset C_{t,P}^{(1)}(V)$;

b) $P\in C_{t,P}^{(s)}(V)$;

c) *if* $\xi\in C_P^{(t)}(V)$, $\tilde T_\xi(\check V^{(t)})\subset C_P^{(t-1)}(V)$.

Proof. a) Using the above notations, we have that, since $\tilde{T}_\xi(\check{V}^{(t)})$ is generated by A^*_N and $A^*_{N-1},\dots,A^*_{t-1}$, we can choose a frame such that we have that the first $d_{t,1}$, $A^*_{N-1},\dots,A^*_{N-d_{t,1}}$ are basis of $\tilde{T}_\xi(\check{V}^{(t)})$. Then, $C^{(1)}_\xi(\check{V}^{(t)})$ contains A_0 and $A_1,\dots,A_{N-d_{t,1}-1}$, and since $d_{t-1}\le d_t-k\le N-d_{t,1}-1$, we have the assertion.

b) Since $\tilde{T}^{(s)}_\xi(\check{V}^{(t)})$ is generated, as usual, in an appropriate frame, by A^*_N and $A^*_{N-1},\dots,A^*_{N-d_{t,s}}$, we have that $C^{(s)}_\xi(\check{V}^{(t)})$ contains A_0.

c) This is just a) in the dual space. □

Corollary 1.23. *With the notation above, if $P\in V$, $\xi\in\check{V}^{(t)}$ and $Q\in C^{(s)}_{t,P}(V)$ are general points, then $\tilde{T}_\xi(\mathrm{Car}^s_t(V))\subset C^{(s-1)}_{t,P}(V)$.*

Proof. This is simply the dual of Lemma 1.22 c). □

2. Terracini's theorems and generalisations

In this section we generalise the classical results of Terracini in terms of the osculating defect and higher fundamental forms instead of the Laplace equations, so that we forget the parametrisation of V. First of all, by Corollary 1.14 we rewrite the results of [20] and Section 3 of [21] as follows:

Theorem 2.1. *Let $V\subseteq\mathbb{P}^N$ be a k-dimensional irreducible variety whose second fundamental form has dimension $k-\ell-1$, with $\ell>0$. Then V has tangent defect at least ℓ and it is contained in a scroll $S(\Sigma^h_t)$ in $\mathbb{P}^t$ such that $T_{\mathbb{P}^t_v}(S(\Sigma^h_t))\subset\mathbb{P}^{2k-h-\ell}$ with $0\le h\le k-\ell$, where $v\in\Sigma^h_t$ is a general point, and $\mathbb{P}^t_v$ is the corresponding fibre of the scroll.*

Theorem 2.2. *Let $V\subseteq\mathbb{P}^N$ be a k-dimensional irreducible variety. Then V has tangent defect $o_1=\ell>0$ and the second fundamental form has dimension at least $k-\ell$ if and only if the Jacobian matrix of the second fundamental form of V has rank $k-\ell$.*

We will prove Theorems 2.4 and 2.8. Theorems 2.1 and 2.2 are just corollaries of them.

Lemma 2.3. *Let $V\subseteq\mathbb{P}^N$ be a k-dimensional irreducible variety, and let $P\in V$. Then, the tangent cone to $\mathrm{Tan}^{t-1}(V)$ in P is contained in $\tilde{T}^{(t)}_P(V)$, and therefore $\tilde{T}_P(\mathrm{Tan}^{t-1}(V))\subset\tilde{T}^{(t)}_P(V)$.*

Proof. Let us take a frame in V as above, i.e., $\{A_0;A_1,\dots,A_{k_t};\dots,A_N\}$, where the first k-elements $A_1,\dots,A_k$ generate $T_P(V)$, and so on, and therefore $A_1,\dots,A_{k_t}$ generate $T^{(t)}_P(V)$.

Let us take also a frame $\{B_0;B_1,\dots,B_\ell;\dots,B_N\}$ on $\mathrm{Tan}^{t-1}(V)$, centred at P, such that B_0 represents $P\in\mathrm{Tan}^{t-1}(V)$ and $B_1,\dots,B_\ell$ generate $T_P(\mathrm{Tan}^{t-1}(V))$. By definition, we have

$$B_0=C_0A_0+\sum_{i=1}^{k_{t-1}}C_iA_i.$$

Taking the exterior derivative,

$$dB_0 = dC_0A_0 + C_0dA_0 + \sum_{i=1}^{k_{t-1}} \big(dC_iA_i + C_idA_i\big).$$

From this we infer that the tangent cone to $\mathrm{Tan}^{t-1}(V)$ in P is contained in $\tilde{T}_P^{(t)}(V)$. Since the tangent cone spans the tangent space, we have $\tilde{T}_P(\mathrm{Tan}^{t-1}(V)) \subset \tilde{T}_P^{(t)}(V)$. □

Theorem 2.4. *Let $V \subseteq \mathbb{P}^N$ be a k-dimensional irreducible variety whose t-th fundamental form has dimension $k-\ell-1$, with $\ell > 0$. Then:*

a) *V has $(t-1)$-osculating defect $o_{t-1} \geq \ell$.*

b) *V is contained in a d-dimensional scroll $S(\Sigma_r^h)$, $(d \leq h+r)$, in linear spaces of dimension r, with $0 \leq h \leq k-\ell$ and $k-h \leq r$.*

c) *Let $\mathbb{P}^r \subset S(\Sigma_r^h)$ be a general r-dimensional space of the scroll $S(\Sigma_r^h)$. Then $\big\langle \cup_{A\in\mathbb{P}^r} \tilde{T}_A(S(\Sigma_r^h))\big\rangle$ is contained in a linear space of dimension*

$$d_t - h = d_{t-1} + k - \ell - h \leq \binom{k+t-1}{t-1} - 1 + k - \ell - h.$$

In particular, $r \leq d \leq d_{t-1} + k - \ell - h$.

Proof. a) By hypothesis, Lemma 2.3 and Corollary 1.15 (and with the above notations)

$$\begin{aligned}\dim \mathrm{Tan}^{t-1}(V) &\leq \dim T_P(\mathrm{Tan}^{t-1}(V)) \\ &\leq \dim(\tilde{T}_P^{(t)}(V)) = d_{t-1} + \Delta_t + 1 \leq \mathrm{expdim}\,\mathrm{Tan}^{t-1}(V) - \ell.\end{aligned}$$

b) As above, let $\gamma^t \colon V \dashrightarrow \mathbb{G}(N, d_t)$ be the t-th Gauss map. Let $h := \dim \mathrm{Im}(\gamma^t)$, so that $k-h$ is the dimension of the general fibre of γ^t. Let $\Phi_{k-h}(\Pi) := (\gamma^t)^{-1}(\Pi)$ be a general fibre; this is just the set of points $Q \in V$ for which $\Pi = \tilde{T}_Q^{(t)}(V)$. Then $\Phi_{k-h}(\Pi)$ generates a linear space $\mathbb{P}^r$, $k-h \leq r \leq d_t$. Let us consider the scroll $S(\Sigma_r^h)$ over $\mathrm{Im}(\gamma^t) =: \Sigma_r^h$ of these spaces. By definition, $V \subset S(\Sigma_r^h)$.

Let $\check{V}^{(t)} \subset \mathbb{P}^{N^*}$ be the t-th dual variety. We have

$$\dim(\check{V}^{(t)}) = h + N - 1 - \dim(\tilde{T}_P^{(t)}(V)).$$

Moreover, by Lemma 1.22 a), $\tilde{T}_P^{(t-1)}(V)) \subset C_{t,P}^{(1)}(V)$, so that, by Corollary 1.15,

$$d_{t-1} \leq N - 1 - d_{t,1} = d_t - h = d_{t-1} + \delta_t + 1 - h = d_{t-1} + k - \ell - h, \tag{2.1}$$

and therefore $h \leq k - \ell$.

c) We have, by Lemma 1.22 b), $Q \in C_{t,P}^{(t)}(V)$ if $Q \in \Phi_{k-h}(\Pi)$, $\Pi = \tilde{T}_P^{(t)}(V)$. Since $C_{t,P}^{(t)}(V)$ is a linear space, we have that $\langle \Phi_{k-h}(\Pi)\rangle = \mathbb{P}^r \subset C_{t,P}^{(t)}(V)$, and

therefore $S(\Sigma_r^h) \subset \mathrm{Car}_t^t(V)$. Finally, apply Corollary 1.23 to get that if $R \in \mathbb{P}^r$ is a general point, $\tilde{T}_R S(\Sigma_r^h) \subset C_{t,P}^{(t-1)}(V)$ and, moreover, since $\dim(\check{V}^{(t)}) = N - 1 - d_t + h$, we have that

$$\dim C_{t,P}^{(t-1)}(V) \leq d_t - h = d_{t-1} + k - \ell - h \leq \binom{k+t-1}{t-1} - 1 + k - \ell - h. \quad \square$$

We give some applications of this theorem.

Example 2.5. Clearly, when $h = 0$, V is contained in a $\mathbb{P}^{d_t}$. For example, this is the only possibility when $k = 1$, i.e, the case of curves. However, in this case we can say even more: we have $\ell = 1$ and $k - \ell = 0 = h$, and from (2.1) we deduce that the curve is contained in a $\mathbb{P}^{d_{t-1}}$. So, if the theorem holds for $k = 1$ and $t = 2$, $V = \mathbb{P}^1$ and for $k = 1$ and $t = 3$, V is a plane curve, etc.

Example 2.6. More generally, if $\ell = k$ and $h = 0 = k - \ell$, thanks to (2.1), we deduce that V is contained in a $\mathbb{P}^{d_{t-1}}$. In particular, if the theorem holds for $t = 2$, we deduce $V = \mathbb{P}^k$.

Example 2.7. Let us pass to the next case $\ell = k - 1$; in this case $h = 0, 1$. If $h = 0 < 1 = k - \ell$, thanks to (2.1), we infer that $d_t = d_{t-1} + 1$. Hence $V \subset \mathbb{P}^{d_{t-1}+1}$ by Example 2.5. For $t = 2$, we deduce that V is a hypersurface in a $\mathbb{P}^{k+1}$.

If $h = 1 = k - \ell$, again by (2.1), we infer that $d_t = d_{t-1} + 1$. Since, $k - 1 \leq r \leq d_t - 1$ for $t = 2$, we have that $k - 1 \leq r \leq d \leq k$, but we cannot have $r = k$, since otherwise we would have that $V = \mathbb{P}^r = S(\Sigma_r^h)$ and then we would have $h = 0$. Therefore, $r = k - 1$, $\Phi_{k-h}(\Pi) = \mathbb{P}^{k-1}$ and V is a developable $\mathbb{P}^{k-1}$-bundle.

Our result generalising Theorem 2.2 is the following.

Theorem 2.8. *Let $V \subseteq \mathbb{P}^N$ be a k-dimensional irreducible variety. Then V has t-th osculating defect $o_t = \ell > 0$ and the $(t+1)$-st fundamental form has dimension at least $k-\ell$ if and only if the Jacobian matrix of the $(t+1)$-st fundamental form of V has rank $k - \ell$.*

Proof. Let us fix as usual a Darboux frame for V as in (1.10). If $P \in V$ is a general point, then, by Definition 1.6,

$$\tilde{T}_P^t(V) = \left\langle \left(\frac{d^{|I|} A_0}{dv_1^{i_1} \dots dv_k^{i_k}} \right)_{|I| \leq t} \right\rangle,$$

with the convention that $d^0 A_0 = A_0$. Therefore, we can fix a Darboux frame $\{B_0; B_1, \dots, B_d; B_{d+1}, \dots, B_N\}$ ($d := \dim \mathrm{Tan}^{(t)}(V) = \mathrm{expdim}\, \mathrm{Tan}^{(t)}(V) - \ell$) for $\mathrm{Tan}^{(t)}(V)$ centred at $Q \in \tilde{T}_P^t(V)$, where $B_1, \dots, B_d$ span $T_Q(\mathrm{Tan}^{(t)}(V))$, and so

$$B_0 = A_0 \sum_{|I|=t} \lambda^{(I)} \frac{d^{|I|} A_0}{dv_1^{i_1} \dots dv_k^{i_k}}. \tag{2.2}$$

Saying $\dim \mathrm{Tan}^{(t)}(V) = \mathrm{expdim}\,\mathrm{Tan}^{(t)}(V) - \ell$ means that there are ℓ linearly independent linear homogeneous relations between the (first) partial derivatives of B_0 with respect to the v_j's and the $\lambda^{(I)}$'s:

$$\sum_{j=1}^{k} a_{\alpha,j}\,\frac{\partial B_0}{\partial v_j} + \sum_{|I|=t} a_{\alpha,I}\,\frac{\partial B_0}{\partial \lambda^{(I)}} = 0, \quad \alpha = 1,\dots,\ell.$$

Then, by (2.2),

$$\sum_{j=1}^{k} a_{\alpha,j}\Big(\sum_{|I|=t} \lambda^{(I)}\,\frac{d^{|I|+1}A_0}{dv_j\,d\mathbf{v}^I}\Big) \equiv 0, \quad \alpha = 1,\dots,\ell, \quad \mod T_P^{(t)}(V).$$

In other words, these relations are indeed relations between the partial derivatives up to order $t+1$ of A_0, and we can think of them as a system of Laplace equations of order $t+1$:

$$\sum_{j=1}^{k} a_{\alpha,j}\Big(\sum_{|I|=t} \lambda^{(I)}\,\mathbf{x}^{I+j}\Big) = 0, \quad \alpha = 1,\dots,\ell,$$

and their associated polynomials

$$\Big(\sum_{j=1}^{k} a_{\alpha,j}\,v_j\Big)\Big(\sum_{|I|=t} \lambda^{(I)}\,\mathbf{v}_I\Big) = 0, \quad \alpha = 1,\dots,\ell, \tag{2.3}$$

are all reducible with the same factor of degree t,

$$\sum_{|I|=t} \lambda^{(I)}\,\mathbf{v}_I.$$

Since these homogeneous polynomials are independent, the ℓ linear forms

$$\sum_{j=1}^{k} a_{\alpha,j}\,v_j, \quad \alpha = 1,\dots,\ell,$$

are independent. In particular, up to a change of coordinates, it is not restrictive to suppose that these forms are $v_1,\dots,v_\ell$.

By Proposition 1.13, we have that the $(t+1)$-fundamental form is the apolar system associated to (2.3); in particular, we have that all the partial derivatives of the $(t+1)$-fundamental form with respect to $v_1,\dots,v_\ell$, are zero, from which we get that the rank of the Jacobian is $k-\ell$.

Since all the above can be reversed, the converse follows easily. □

This theorem suggests the project of classifying varieties with tangent, or more generally, higher osculating defect. We will study this in a future paper. This relies on the study of linear systems of quadrics, or more generally, of higher degree hypersurfaces, with Jacobian matrices having ranks lower than expected.

References

[1] Akivis, M. A. and Goldberg, V. V.: *Differential geometry of varieties with degenerate Gauss maps*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 18, Springer-Verlag, New York, 2004.

[2] Degoli, L.: Sulle varietà V_6 i cui spazi tangenti ricoprono una varietà W di dimensione inferiore all'ordinaria. *Atti Sem. Mat. Fis. Univ. Modena* **16** (1967), 89–99.

[3] Di Gennaro, R., Ilardi, G. and Vallès, J.: Singular hypersurfaces characterizing the Lefschetz properties. *J. London Math. Soc. (2)*, first published online October 15, 2013, doi:10.1112/jlms/jdt053.

[4] Franco, D. and Ilardi, G.: On multiosculating spaces. *Comm. Algebra* **29** (2001), no. 7, 2961–2976.

[5] Franco, D. and Ilardi, G.: On a theorem of Togliatti. *Int. Math. J.* **2** (2002), no. 4, 379–397.

[6] Griffiths, P. and Harris, J.: Algebraic geometry and local differential geometry. *Ann. Sci. École Norm. Sup. (4)* **12** (1979), no. 3, 355–452.

[7] Harris, J.: *Algebraic geometry. A first course*. Graduate Texts in Mathematics 133, Springer-Verlag, New York, 1992.

[8] Hartshorne, R.: *Algebraic Geometry*. Graduate Text in Mathematics 52, Springer-Verlag, New York, 1977.

[9] Ilardi, G.: Rational varieties satisfying one or more Laplace equations. *Ricerche Mat.* **48** (1999), no. 1, 123–137.

[10] Ilardi, G.: Togliatti systems. *Osaka J. Math.* **43** (2006), no. 1, 1–12.

[11] Ivey, T. A. and Landsberg, J. M.: *Cartan for beginners: differential geometry via moving frames and exterior differential systems*. Graduate Studies in Mathematics 61, American Mathematical Society, Providence, RI, 2003.

[12] Landsberg, J. M.: On second fundamental forms of projective varieties. *Invent. Math.* **117** (1994), no. 2, 303–315.

[13] Landsberg, J. M.: Differential-geometric characterizations of complete intersections. *J. Differential Geom.* **44** (1996), no. 1, 32–73.

[14] Landsberg, J. M. and Robles, C.: Lines and osculating lines of hypersurfaces. *J. London Math. Soc. (2)* **82** (2010), no. 3, 733–746.

[15] Mezzetti, E., Miró-Roig, R. and Ottaviani, G.: Laplace equations and the weak Lefschetz property. *Canad. J. Math.* **65** (2013), no. 3, 634–654.

[16] Muracchini, L.: Le varietà V_5 i cui spazi tangenti ricoprono una varietà W di dimensione inferiore alla ordinaria. II. *Rivista Mat. Univ. Parma* **3** (1952), 75–89.

[17] Muracchini, L.: Le varietà V_5 i cui spazi tangenti ricoprono una varietà W di dimensione inferiore alla ordinaria. I. *Rivista Mat. Univ. Parma* **2** (1951), 435–462.

[18] Muracchini, L.: Le varietà V_5 i cui spazi tangenti ricoprono una varietà W di dimensione inferiore alla ordinaria. *Boll. Un. Mat. Ital. (3)* **6** (1951), 97–103.

[19] Piene, R.: A note on higher order dual varieties, with an application to scrolls. In *Singularities, Part 2 (Arcata, Calif., 1981)*, 335–342. Proc. Sympos. Pure Math. 40. Amer. Math. Soc., Providence, RI, 1983.

[20] Terracini, A.: Sulle V_k che rappresentano più di $\frac{1}{2}k(k-1)$ equazioni di Laplace linearmente indipendenti. *Rend. Circ. Mat. Palermo* **33** (1912), 176–186.

[21] Terracini, A.: Alcune questioni sugli spazi tangenti ed osculatori ad una varietà. I. *Atti R. Accad. Sc. Torino* **49** (1914), 214–247.

[22] Terracini, A.: Alcune questioni sugli spazi tangenti ed osculatori ad una varietà. II. *Atti R. Accad. Sc. Torino* **51** (1915–6), 695–714.

[23] Terracini, A.: Alcune questioni sugli spazi tangenti ed osculatori ad una varietà. III. *Atti R. Accad. Sc. Torino* **55** (1919–20), 480–500.

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