



Exponential growth of rank jumps for A -hypergeometric systems

María-Cruz Fernández-Fernández

Abstract. The dimension of the space of holomorphic solutions at non-singular points (also called the holonomic rank) of an A -hypergeometric system $M_A(\beta)$ is known to be bounded above by $2^{2d} \text{vol}(A)$, where d is the rank of the matrix A and $\text{vol}(A)$ is its normalized volume. This bound was thought to be much too large because it is exponential in d . Indeed, all the examples we have found in the literature satisfy $\text{rank}(M_A(\beta)) < 2\text{vol}(A)$. We construct here, in a very elementary way, some families of matrices $A_{(d)} \in \mathbb{Z}^{d \times n}$ and parameter vectors $\beta_{(d)} \in \mathbb{C}^d$, $d \geq 2$, such that $\text{rank}(M_{A_{(d)}}(\beta_{(d)})) \geq a^d \text{vol}(A_{(d)})$ for some $a > 1$.

1. Introduction

Let $A = (a_{ij}) = (a_1 \ a_2 \ \dots \ a_n)$ be a full rank matrix with columns $a_j \in \mathbb{Z}^d$ and $d \leq n$. Following Gel'fand, Graev, Kapranov and Zelevinsky (see [5] and [6]) we define the A -hypergeometric system with parameter $\beta \in \mathbb{C}^d$ as the left ideal $H_A(\beta)$ of the Weyl algebra $D = \mathbb{C}[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle$ generated by the following set of linear partial differential operators:

$$(1.1) \quad \square_u := \left(\prod_{i: u_i > 0} \partial_i^{u_i} \right) - \left(\prod_{i: u_i < 0} \partial_i^{-u_i} \right) \quad \text{for all } u \in \mathbb{Z}^n \text{ such that } Au = 0$$

and

$$(1.2) \quad E_i - \beta_i := \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i \quad \text{for } i = 1, \dots, d$$

The operators given in (1.1) generate the so-called toric ideal $I_A \subseteq \mathbb{C}[\partial_1, \dots, \partial_n]$ associated with A and the d operators given in (1.2) are called the Euler operators associated with the pair (A, β) . The hypergeometric D -module associated

with (A, β) , namely the quotient $M_A(\beta) = D/DH_A(\beta)$, is a holonomic D -module (see [6], [1]). In particular, the space of holomorphic solutions of $M_A(\beta)$ at a non-singular point has finite dimension. This dimension or, equivalently, the holonomic rank of $M_A(\beta)$ equals the normalized volume $\text{vol}_{\mathbb{Z}A}(A)$ of the matrix A (see (2.1)) when either I_A is Cohen-Macaulay or β is generic (see [6], [1], [14]).

The first example of a pair (A, β) for which $\text{rank}(M_A(\beta)) > \text{vol}_{\mathbb{Z}A}(A)$ was described in [15] (see Example 2.5). A complete description of the case $d = 2$ appears in [4], revealing that in this case the rank of $M_A(\beta)$ can be only $\text{vol}_{\mathbb{Z}A}(A)$ (the generic value) or $\text{vol}_{\mathbb{Z}A}(A) + 1$ (the exceptional value).

In general it is known that $\text{rank}(M_A(\beta)) \geq \text{vol}_{\mathbb{Z}A}(A)$ for all β [14], [10]. In fact, it is proved in [10] that the map $\beta \in \mathbb{C}^d \mapsto \text{rank}(M_A(\beta))$ is upper semicontinuous in the Zarisky topology and they also provide an explicit description of the exceptional set

$$\varepsilon(A) = \{\beta \in \mathbb{C}^d : \text{rank}(M_A(\beta)) > \text{vol}_{\mathbb{Z}A}(A)\},$$

that turns out to be an affine subspace arrangement with codimension at least 2. Previous descriptions of the exceptional set in particular cases appear in [4], [7], [13], [8], and [9].

If for a fixed matrix A we have that $j_A(\beta) = \text{rank}(M_A(\beta)) - \text{vol}_{\mathbb{Z}A}(A) > 0$ then it is said that the A -hypergeometric system has a *rank jump* of $j_A(\beta)$ at β or that β is a rank jumping parameter for A .

The paper [11] provides the first family of hypergeometric systems with rank jump greater than 2. Indeed, the authors of [11] construct a family of pairs $(A_{(d)}, \beta_{(d)})$ with $A_{(d)} \in \mathbb{Z}^{d \times 2d}$ and $\beta_{(d)} \in \mathbb{C}^d$ such that $j_{A_{(d)}}(\beta_{(d)}) = d - 1$. However, for this family $\text{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)}) = d + 2$ and thus

$$\frac{\text{rank}(M_{A_{(d)}}(\beta_{(d)}))}{\text{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)})} = 2 - \frac{3}{d + 2} < 2.$$

More recently, in [2] a general combinatorial formula is given for the rank jump $j_A(\beta)$ of the A -hypergeometric system at a given β . However, the formula is very complicated and, in fact, all the examples included in [2] satisfy $\text{rank}(M_A(\beta)) < 2\text{vol}_{\mathbb{Z}A}(A)$ as well. Previous computations of $j_A(\beta)$ in particular cases appear for example in [4], [13], and [12].

In the case when the toric ideal is standard homogeneous, the following upper bound for the holonomic rank of a hypergeometric system is proved in [14]:

$$\text{rank}(M_A(\beta)) \leq 2^{2d} \text{vol}_{\mathbb{Z}A}(A).$$

However, it is mentioned in page 159 of [14] that this upper bound is most likely far from optimal and that it would be desirable to know whether the ratio $\text{rank}(M_A(\beta))/\text{vol}_{\mathbb{Z}A}(A)$ can be bounded above by some polynomial function in d . Here we provide a very elementary construction of some families of hypergeometric systems for which the ratio $\text{rank}(M_A(\beta))/\text{vol}_{\mathbb{Z}A}(A)$ is exponential in d , giving a negative answer to this last question.

Moreover, for one of the families constructed the dimension of the space of Laurent polynomial solutions is lower than the rank jump (see Remark 2.8). This is in contrast with what occurs for the examples found in the literature (see for example [4] and [11]).

I am grateful to Christine Berkesch for many helpful conversations about her paper [2].

2. Construction of the examples

Recall that the normalized volume of a full rank matrix $A \in \mathbb{Z}^{d \times n}$ is given by

$$(2.1) \quad \text{vol}_{\mathbb{Z}A}(A) = d! \frac{\text{vol}_{\mathbb{R}^d}(\Delta_A)}{[\mathbb{Z}^d : \mathbb{Z}A]},$$

where $[\mathbb{Z}^d : \mathbb{Z}A]$ is the index of the subgroup $\mathbb{Z}A := \sum_{i=1}^n \mathbb{Z}a_i \subseteq \mathbb{Z}^d$, Δ_A is the convex hull of the columns of A and the origin in \mathbb{R}^d , and $\text{vol}_{\mathbb{R}^d}(\Delta_A)$ denotes the Euclidean volume of the polytope Δ_A .

Let us also recall that the direct sum of two matrices $A_1 \in \mathbb{Z}^{d_1 \times n_1}$, $A_2 \in \mathbb{Z}^{d_2 \times n_2}$ is the $(d_1 + d_2) \times (n_1 + n_2)$ matrix

$$A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0_{d_1 \times n_2} \\ 0_{d_2 \times n_1} & A_2 \end{pmatrix},$$

where $0_{d \times n}$ denotes the $d \times n$ zero matrix.

The following two lemmas are easy to prove.

Lemma 2.1. *If A is the direct sum of two matrices $A_1 \in \mathbb{Z}^{d_1 \times n_1}$, $A_2 \in \mathbb{Z}^{d_2 \times n_2}$ then $\text{vol}_{\mathbb{Z}A}(A) = \text{vol}_{\mathbb{Z}A_1}(A_1) \cdot \text{vol}_{\mathbb{Z}A_2}(A_2)$.*

Lemma 2.2. *Let $A_i \in \mathbb{Z}^{d_i \times n_i}$ be full rank matrices, $d_i \leq n_i$, and $\beta_{(i)} \in \mathbb{C}^{d_i}$ for $i = 1, 2$. If $A = A_1 \oplus A_2$ and $\beta = (\beta_{(1)}, \beta_{(2)})$ then we have that $H_A(\beta) = DH_{A_1}(\beta_{(1)}) + DH_{A_2}(\beta_{(2)})$ where $H_{A_1}(\beta_{(1)})$ is a left ideal of the Weyl Algebra $D_{A_1} = \mathbb{C}[x_1, \dots, x_{n_1}][\partial_1, \dots, \partial_{n_1}]$ and $H_{A_2}(\beta_{(2)})$ is a left ideal of the Weyl Algebra $D_{A_2} = \mathbb{C}[x_{n_1+1}, \dots, x_{n_1+n_2}][\partial_{n_1+1}, \dots, \partial_{n_1+n_2}]$ (equivalently, $M_A(\beta)$ is the exterior tensor product of $M_{A_1}(\beta_{(1)})$ and $M_{A_2}(\beta_{(2)})$).*

The following corollary follows from Lemma 2.2 by general properties of the exterior tensor product of holonomic D -modules.

Corollary 2.3. *Under the assumptions of Lemma 2.2 we have:*

- i) $\text{rank}(M_A(\beta)) = \text{rank}(M_{A_1}(\beta_{(1)})) \cdot \text{rank}(M_{A_2}(\beta_{(2)}))$.
- ii) *If Ω_i is a basis for the space of holomorphic (respectively Laurent polynomial) solutions of the hypergeometric system $M_{A_i}(\beta_{(i)})$ at a point $p_i \in \mathbb{C}^{n_i}$, then the set*

$$\Omega = \{f_1(x_1, \dots, x_{n_1}) \cdot f_2(x_{n_1+1}, \dots, x_{n_1+n_2}) : f_i \in \Omega_i, i = 1, 2\}$$

is a basis for the space of holomorphic (respectively Laurent polynomial) solutions of $M_A(\beta)$ at $p = (p_1, p_2) \in \mathbb{C}^{n_1+n_2}$.

In view of Corollary 2.3, we can already give a first type of families of hypergeometric systems for which the rank jump grows exponentially with d .

Theorem 2.4. *Let $A \in \mathbb{Z}^{d \times n}$ and $\beta \in \mathbb{C}^d$ be such that $M_A(\beta)$ has a rank jump, i.e., $\text{rank}(M_A(\beta))/\text{vol}_{\mathbb{Z}A}(A) = q > 1$. Consider, for $d_r = rd$ with $r \geq 1$, the matrix $A_r \in \mathbb{Z}^{d_r \times n_r}$ ($n_r = rn$), defined as the direct sum of r copies of A , and the parameter vector $\beta_r = (\beta, \dots, \beta) \in \mathbb{C}^{d_r}$, defined by r copies of β . We have that the family (A_r, β_r) satisfies $\text{rank}(M_{A_r}(\beta_r))/\text{vol}_{\mathbb{Z}A_r}(A_r) = a^{d_r}$, where $a = \sqrt[r]{q} > 1$.*

Example 2.5. For $d = 2$ we will consider the first example of a hypergeometric system with rank jump described in [15]. Consider the pair $(A_{(2)}, \beta_{(2)})$, where

$$(2.2) \quad A_{(2)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix} \quad \text{and} \quad \beta_{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The toric ideal associated with $A_{(2)}$ is

$$I_{A_{(2)}} = (\partial_1 \partial_4 - \partial_2 \partial_3, \partial_1^2 \partial_3 - \partial_2^3, \partial_2 \partial_4^2 - \partial_3^3, \partial_1 \partial_3^2 - \partial_2^2 \partial_4),$$

and the Euler operators are $E_1 - \beta_{(2),1} = x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 - 1$ and $E_2 - \beta_{(2),2} = x_2 \partial_2 + 3x_3 \partial_3 + 4x_4 \partial_4 - 2$.

For this example $\text{rank}(M_{A_{(2)}}(\beta)) = \text{vol}_{\mathbb{Z}A_{(2)}}(A_{(2)}) = 4$ for all $\beta \in \mathbb{C}^2 \setminus \{\beta_{(2)}\}$, but $\text{rank}(M_{A_{(2)}}(\beta_{(2)})) = 5$. A basis of the space of solutions of $M_{A_{(2)}}(\beta_{(2)})$ can also be found in [15]. Let us point out that this basis consists of the two Laurent polynomials $p_1 = x_2^2/x_1$, $p_2 = x_3^2/x_4$, and three other functions that are not Laurent polynomials.

Notice that if we apply Theorem 2.4 to this example we get a base $a = \sqrt{5}/2$ of the exponential in $d_r = 2r$, $r \geq 1$.

Example 2.6. For $d = 3$, we will consider the hypergeometric system of the family $\{M_{A_{(d)}}(\beta_{(d)})\}_{d \geq 2}$ described in [11]. It is the one associated with the pair

$$(2.3) \quad A_{(3)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 4 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta_{(3)} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

The volume of $A_{(3)}$ is $d+2 = 5$, while the rank of $M_{A_{(3)}}(\beta(3))$ is $2d+1 = 7$. Thus, if we apply Theorem 2.4 to this example we get a base $a = \sqrt[3]{7/5}$ of the exponential in $d_r = 3r$, $r \geq 1$. We remark that this base $a = \sqrt[3]{7/5}$ is the maximum we have found after computing the quantity $\sqrt[d]{\text{rank}(M_A(\beta))/\text{vol}_{\mathbb{Z}A}(A)}$ for many pairs (A, β) with A a full rank integer $d \times n$ matrix, $\beta \in \mathbb{C}^d$ and $d = 2, 3, 4$.

In the sequel, we will first construct a family of examples similar to the ones given by Theorem 2.4, but using direct sums of the two matrices $A_{(2)}$ and $A_{(3)}$. After that, we will modify these examples in order to exhibit another family of A -hypergeometric systems which are not exterior tensor products of smaller hypergeometric systems, and whose holonomic ranks also grow exponentially in d .

Example 2.7. For any $d \geq 4$, let $r, s \in \mathbb{N}$ be such that $2r + 3s = d$. We will choose s as high as possible in order to fix uniquely $r, s \in \mathbb{N}$ for each d (in particular $0 \leq r \leq 4$).

We define $A_{(d)} \in \mathbb{Z}^{d \times 2d}$ to be the direct sum of r copies of the matrix $A_{(2)}$ and s copies of the matrix $A_{(3)}$. By Lemma 2.1 and examples 2.5 and 2.6 we have that $\text{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)}) = 4^r 5^s$.

On the other hand, let $\beta_{(d)} \in \mathbb{C}^d$ be the complex vector with coordinates:

$$\begin{aligned} \beta_{(d),2i-1} &= 1 \quad \text{and} \quad \beta_{(d),2i} = 2 \quad \text{for } 1 \leq i \leq r, \\ \beta_{(d),2r+3j-2} &= 1, \quad \beta_{(d),2r+3j-1} = 0, \quad \beta_{(d),2r+3j} = 2 \quad \text{for } 1 \leq j \leq s \end{aligned}$$

(i.e., $\beta_{(d)}$ has a copy of $\beta_{(2)}$ for each copy of $A_{(2)}$ and a copy of $\beta_{(3)}$ for each copy of $A_{(3)}$). With this definition of $(A_{(d)}, \beta_{(d)})$ and using Corollary 2.3 and Examples 2.5 and 2.6 we have that $\text{rank}(M_{A_{(d)}}(\beta_{(d)})) = 5^r 7^s$. Thus,

$$\text{rank}(M_{A_{(d)}}(\beta_{(d)})) / \text{vol}_{\mathbb{Z}A_{(d)}}(A_{(d)}) = (5/4)^r (7/5)^s \geq (\sqrt{5}/2)^d.$$

Remark 2.8. Example 2.7 also shows that the rank jump $j_A(\beta)$ can be greater than the dimension of the space of Laurent polynomial solutions of $M_A(\beta)$. Indeed, since the space of Laurent polynomial solutions of $M_{A_{(2)}}(\beta_{(2)})$ has dimension 2 ([15]) and the space of Laurent polynomial solutions of $M_{A_{(3)}}(\beta_{(3)})$ has dimension 4 (see [11]) then, by Corollary 2.3, the space of Laurent polynomial solutions of $M_{A_{(d)}}(\beta_{(d)})$ has dimension $2^r 4^s < j_{A_{(d)}}(\beta_{(d)}) = 5^r 7^s - 4^r 5^s$ for all $d = 2r + 3s \geq 4$.

Let us see how to modify Example 2.7 in order to get hypergeometric systems that are not exterior tensor products of smaller hypergeometric systems, and such that the corresponding ratio $\text{rank}(M_A(\beta)) / \text{vol}(A)$ is still exponential in d .

Consider the following matrices and parameters:

$$(2.4) \quad \hat{A}_{(2)} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix} \quad \text{and} \quad \hat{\beta}_{(2)} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

$$(2.5) \quad \hat{A}_{(3)} = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 3 & 4 & 0 & 1 \end{pmatrix} \quad \text{and parameter} \quad \hat{\beta}_{(3)} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$$

Notice that $\hat{A}_{(2)}$ and $\hat{A}_{(3)}$ are obtained from $A_{(2)}$ and $A_{(3)}$ respectively by multiplying the first row by 2 (this doesn't change the hypergeometric system) and then by adding a first column with its first coordinate equal to 1 and the other coordinates equal to zero. After these modifications we get that $\text{vol}_{\mathbb{Z}\hat{A}_{(2)}}(\hat{A}_{(2)}) = 2 \cdot 4 = 8$ and that $\text{vol}_{\mathbb{Z}\hat{A}_{(3)}}(\hat{A}_{(3)}) = 2 \cdot 5 = 10$. However, since $\hat{\beta}_{(i)}$ is a hole in $\mathbb{N}\hat{A}_{(i)}$ (meaning that $\hat{\beta}_{(i)} \notin \mathbb{N}\hat{A}_{(i)}$ but $\hat{\beta}_{(i)} + (\mathbb{N}\hat{A}_{(i)} \setminus \{0\}) \subseteq \mathbb{N}\hat{A}_{(i)}$) we have by Remark 4.14 in [12] that $\text{rank}(M_{\hat{A}_{(i)}}(\hat{\beta}_{(i)})) = \text{vol}_{\mathbb{Z}\hat{A}_{(i)}}(\hat{A}_{(i)}) + (i - 1)$, $i = 2, 3$.

The following lemma follows from the results in [2].

Lemma 2.9. Let $A \in \mathbb{Z}^{d \times n}$ and $B \in \mathbb{Z}^{d \times m}$ be two matrices satisfying $\mathbb{N}A = \mathbb{N}B$ and $\Delta_A = \Delta_B$. Then $\text{rank}(M_A(\beta)) = \text{rank}(M_B(\beta))$ for all $\beta \in \mathbb{C}^d$.

For $d = 2r + 3s \geq 2$, $r, s \in \mathbb{N}$ (with s as high as possible), let $\hat{\beta}_{(d)} \in \mathbb{C}^d$ be the complex vector that is given by r copies of $\hat{\beta}_{(2)}$ and s copies of $\hat{\beta}_{(3)}$. The new matrix $\hat{A}_{(d)} \in \mathbb{Z}^{d \times (6r+8s-1)}$ is constructed as follows.

Let $a_1, a_2, \dots, a_{5r+7s} \in \mathbb{Z}^d$ be the columns of the matrix

$$A_{r,s} = \hat{A}_{(2)} \oplus \overbrace{\dots}^r \oplus \hat{A}_{(2)} \oplus \hat{A}_{(3)} \oplus \overbrace{\dots}^s \oplus \hat{A}_{(3)} \in \mathbb{Z}^{d \times (5r+7s)}.$$

We will construct a matrix $\hat{A}_{(d)}$ by adding $r + s - 1$ column vectors to the matrix $A_{r,s}$. These vectors will belong to both $\Delta_{A_{r,s}}$ and $\mathbb{N}A_{r,s}$. These conditions guarantee that $\text{vol}_{\mathbb{Z}\hat{A}_{(d)}}(\hat{A}_{(d)}) = \text{vol}_{\mathbb{Z}A_{r,s}}(A_{r,s}) = 8^r 10^s$ and by Lemma 2.9, we will also have that $\text{rank}(M_{\hat{A}_{(d)}}(\beta)) = \text{rank}(M_{A_{r,s}}(\beta))$ for all $\beta \in \mathbb{C}^d$. In particular, for $\beta = \hat{\beta}_{(d)}$, we have $\text{rank}(M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)})) = 9^r 12^s$.

If $r \geq 2$ then for $1 \leq i \leq r - 1$ we define

$$a_{5r+7s+i} = a_1 + a_{5i+1} = \frac{1}{2} a_2 + \frac{1}{2} a_{5i+2} \in \mathbb{N}A_{r,s} \cap \Delta_{A_{r,s}}.$$

Notice that $(a_{5r+7s+i})_j$ equals 1 for $j = 1$ or $j = 2i + 1$, and 0 otherwise.

If $r, s \geq 1$ then for $1 \leq i \leq s$ we define

$$a_{5r+7s+r-1+i} = a_1 + a_{5r+7i+1} = \frac{1}{2} a_2 + \frac{1}{2} a_{5r+7i+2} \in \mathbb{N}A_{r,s} \cap \Delta_{A_{r,s}}.$$

If $r = 0$ and $s \geq 2$ then for $1 \leq i \leq s - 1$ we define

$$a_{7s+i} = a_1 + a_{7i+1} = \frac{1}{2} a_2 + \frac{1}{2} a_{7i+2} \in \mathbb{N}A_{r,s} \cap \Delta_{A_{r,s}}.$$

Let us define $\hat{A}_d = (a_1 \ a_2 \ \dots \ a_{6r+8s-1})$ and recall that $\hat{\beta}_{(d)} \in \mathbb{C}^d$ is given by r copies of $\hat{\beta}_{(2)}$ and s copies of $\hat{\beta}_{(3)}$. The hypergeometric system $M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)})$ is not an exterior tensor product of smaller hypergeometric systems and we have proved the following.

Theorem 2.10. *With the notations above we have*

$$\frac{\text{rank}(M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)}))}{\text{vol}_{\mathbb{Z}\hat{A}_{(d)}}(\hat{A}_{(d)})} = (9/8)^r (12/10)^s \geq (\sqrt{9/8})^d.$$

Remark 2.11. Notice that the toric ideal associated with $\hat{A}_{(d)}$ is not homogeneous. However, by Theorem 7.3 in [3], if we consider the associated homogeneous matrix $\hat{A}_{(d)}^h$ (that is obtained by adding to the matrix $\hat{A}_{(d)}$ an initial column of zeroes and, after that, an initial row of ones) and the parameter $\hat{\beta}_{(d)}^h = (\beta_0, \hat{\beta}_{(d)})$ with $\beta_0 \in \mathbb{C}$ then the rank of $M_{\hat{A}_{(d)}^h}(\hat{\beta}_{(d)}^h)$ equals the rank of $M_{\hat{A}_{(d)}}(\hat{\beta}_{(d)})$ if $\beta_0 \in \mathbb{C}$ is generic. This implies that for some particular β_0 (for example $\beta_0 = 0$) the rank

of $M_{\hat{A}(d)}^{\hat{\beta}(d)}$ will be greater than or equal to the rank of $M_{\hat{A}(d)}(\hat{\beta}(d))$ by the upper semi-continuity of the rank [10]. Moreover,

$$\text{vol}_{\hat{A}(d)}(\hat{A}(d)) = \text{vol}_{\hat{A}(d)}(\hat{A}(d)).$$

Thus, we also have that the ratio

$$\text{rank}(M_{\hat{A}(d)}^{\hat{\beta}(d)})/\text{vol}_{\mathbb{Z}\hat{A}(d)}(\hat{A}(d))$$

is exponential in d .

References

- [1] ADOLPHSON, A.: Hypergeometric functions and rings generated by monomials. *Duke Math. J.* **73** (1994), no. 2, 269–290.
- [2] BERKESCH, CH.: The rank of a hypergeometric system. *Compositio Math.* **147** (2011), no. 1, 284–318.
- [3] BERKESCH, CH.: *Euler Koszul methods in algebra and geometry*. PhD thesis, Purdue University, 2010.
- [4] CATTANI, E., D’ANDREA, C. AND DICKENSTEIN, A.: The \mathcal{A} -hypergeometric system associated with a monomial curve. *Duke Math. J.* **99** (1999), no. 2, 179–207.
- [5] GEL’FAND, I. M., GRAEV, M. I. AND ZELEVINSKY, A. V.: Holonomic systems of equations and series of hypergeometric type. *Dokl. Akad. Nauk SSSR* **295** (1987), no. 1, 14–19.
- [6] GEL’FAND, I. M., ZELEVINSKY, A. V. AND KAPRANOV, M. M.: Hypergeometric functions and toric varieties. *Funktsional Anal. i Prilozhen* **23** (1989), no. 2, 12–26.
- [7] MATUSEVICH, L. F.: Rank jumps in codimension 2 A -hypergeometric systems. Effective methods in rings of differential operators. *J. Symbolic Comput.* **32** (2001), no. 6, 619–641.
- [8] MATUSEVICH, L. F.: Exceptional parameters for generic A -hypergeometric systems. *Int. Math. Res. Not.* (2003), no. 22, 1225–1248.
- [9] MATUSEVICH, L. F. AND MILLER, E.: Combinatorics of rank jumps in simplicial hypergeometric systems. *Proc. Amer. Math. Soc.* **134** (2006), no. 5, 1375–1381.
- [10] MATUSEVICH, L. F., MILLER, E. AND WALTHER, U.: Homological methods for hypergeometric families. *J. Amer. Math. Soc.* **18** (2005), no. 4, 919–941.
- [11] MATUSEVICH, L. F. AND WALTHER, U.: Arbitrary rank jumps for A -hypergeometric systems through Laurent polynomials. *J. London Math. Soc. (2)* **75** (2007), no. 1, 213–224.
- [12] OKUYAMA, G.: A -hypergeometric ranks for toric threefolds. *Int. Math. Res. Not.* (2006), Art. ID 70814, 38 pp.
- [13] SAITO, M.: Logarithm-free A -hypergeometric functions. *Duke Math. J.* **115** (2002), no. 1, 53–73.
- [14] SAITO, M., STURMFELS, B. AND TAKAYAMA, N.: *Gröbner deformations of hypergeometric differential equations*. Algorithms and Computation in Mathematics 6, Springer-Verlag, Berlin, 2000.

- [15] STURMFELS, B. AND TAKAYAMA, N.: Gröbner bases and hypergeometric functions. In *Gröbner bases and applications (Linz, 1998)*, 246–258. London Math. Soc. Lecture Note Ser. 251, Cambridge Univ. Press, Cambridge, 1998.

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MARÍA-CRUZ FERNÁNDEZ-FERNÁNDEZ: Departamento de Álgebra, Universidad de Sevilla, C/ Tarfia s/n, 41012 Sevilla, Spain.

E-mail: mcferfer@us.es