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Defining functions for unbounded C^m domains

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Abstract. For a domain $\Omega \subset \mathbb{R}^n$, we introduce the concept of a uniformly C^m defining function. We characterize uniformly C^m defining functions in terms of the signed distance function for the boundary and provide a large class of examples of unbounded domains with uniformly C^m defining functions. Some of our results extend results from the bounded case.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. A C^m *defining function*, $m \geq 1$, for Ω is a real-valued C^m function ρ defined on a neighborhood U of $\partial\Omega$ such that $\{x \in U : \rho(x) < 0\} = \Omega \cap U$ and $\nabla\rho \neq 0$ on $\partial\Omega$. If Ω has a C^m defining function, we say that Ω is a C^m *domain*.

For many applications on unbounded domains, the preceding definition is inadequate. For example, to work in local coordinates that are adapted to the boundary, it is necessary to work in a neighborhood whose size depends on the C^2 norm of the defining function. If the C^2 norm is not uniformly bounded, then such neighborhoods may be arbitrarily small, which means that a partition of unity subordinate to these neighborhoods might not have uniform bounds on the derivatives. Problems may also arise in constructions which involve choosing a constant large enough to bound quantities depending on derivatives of the defining function. Typical results on C^m domains will require the following:

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$, and let ρ be a C^m defining function for Ω defined on a neighborhood U of $\partial\Omega$ such that

1) $\operatorname{dist}(\partial\Omega, \partial U) > 0$,
2) $\|\rho\|_{C^m(U)} < \infty$,
3) $\inf_U |\nabla\rho| > 0$.

We say that such a defining function is *uniformly* C^m. If ρ on U is uniformly C^m for all $m \in \mathbb{N}$, we say ρ is *uniformly* C^∞.

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On bounded domains, compactness of the boundary implies that every bounded C^m domain has a uniformly C^m defining function. On unbounded C^m domains with noncompact boundaries, these properties may not hold. For example, consider $\Omega \subset \mathbb{R}^3$ defined by $\Omega = \{z < xy^2\}$. This is a C^∞ domain, and any C^2 defining function ρ for Ω will take the form $\rho(x, y, z) = h(x, y, z)(z - xy^2)$ for a C^1 function h satisfying $h > 0$ on $\partial\Omega$. If we restrict to the line $\ell = \{y = z = 0\} \subset \partial\Omega$, we see that $|\nabla\rho||_\ell = h$ and $\frac{\partial^2\rho}{\partial y^2}|_\ell = -2xh$. If $|\nabla\rho| > C_1 > 0$ on U then $h > C_1$ on ℓ, but if $\|\rho\|_{C^2(U)} < C_2$ then $2|x|h < C_2$. This is impossible if $|x| \geq C_2/(2C_1)$, so no defining function for Ω is uniformly C^2, even though the domain itself is C^∞.

A natural choice for a defining function is the signed distance function. For $\Omega \subset \mathbb{R}^n$ with C^m boundary, define the signed distance function for Ω by

$$\tilde{\delta}(x) = \begin{cases} d(x, \partial\Omega) & x \notin \Omega, \\ -d(x, \partial\Omega) & x \in \overline{\Omega}. \end{cases}$$

Note that the distance function $\delta(x) := d(x, \partial\Omega)$ equals $|\tilde{\delta}(x)|$ for any $x \in \mathbb{R}^n$. Let

$$\mathrm{Unp}(\partial\Omega) = \{x \in \mathbb{R}^n : \text{ there is a unique point } y \in \partial\Omega \text{ such that } \delta(x) = |y - x|\}.$$

The following concepts were introduced in [2].

Definition 1.2. If $y \in \partial\Omega$, then define the *reach* of $\partial\Omega$ at y by

$$\mathrm{Reach}(\partial\Omega, y) = \sup\{r \geq 0 : B(y, r) \subset \mathrm{Unp}(\partial\Omega)\}$$

and the *reach* of $\partial\Omega$ to be

$$\mathrm{Reach}(\partial\Omega) = \inf\{\mathrm{Reach}(\partial\Omega, y) : y \in \partial\Omega\}.$$

Our main result is the following:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a C^m domain, $m \geq 2$. Then the following are equivalent:*

1. *Ω has a uniformly C^m defining function.*

2. *$\partial\Omega$ has positive reach, and for any $0 < \epsilon < \mathrm{Reach}(\partial\Omega)$, the signed distance function satisfies $\|\tilde{\delta}\|_{C^m(U_\epsilon)} < \infty$ on $U_\epsilon = \{x \in \mathbb{R}^n : \delta(x) < \epsilon\}$.*

3. *There exists a C^m defining function ρ for Ω and a constant $C > 0$ such that for every point $p \in \partial\Omega$ with local coordinates $\{y_1, \ldots, y_n\}$ satisfying $\partial\rho/\partial y_j(p) = 0$ for $1 \leq j \leq n-1$, we have*

$$|\nabla\rho(p)|^{-1}\left|\frac{\partial^k\rho(p)}{\partial y^I}\right| < C$$

where I is a multi-index of length k, $2 \leq k \leq m$, and $0 \leq I_n \leq \min\{m-k, k\}$.

Remark 1.4. An important consequence of this theorem is that our definition of uniformly C^∞ is not too strong. If for every $m \in \mathbb{N}$ there exists a defining function ρ_m on U_m such that ρ_m is uniformly C^m on U_m, then there exists a uniformly C^∞ defining function ρ, and we can take ρ to be the signed distance function.

Remark 1.5. In [9], Krantz and Parks show that if Ω is a C^m domain, $m \geq 2$, then there exists a neighborhood $U \supset \partial\Omega$ on which $\tilde{\delta}$ is C^m. Part (2) of Theorem 1.3 extends their result by showing that $\tilde{\delta}$ is C^m up to $\mathrm{Reach}(\partial\Omega)$.

Proof. That (2) implies (1) and (1) implies (3) are immediate from the definitions. That (3) implies (2) will follow from Lemmas 2.1 and 2.4, proved in Section 2. □

When studying the asymptotic behavior of a domain, it is natural to consider the domain after embedding $\mathbb{R}^n \subset \mathbb{RP}^n$, and we will do so in Section 3. Our theorem will make it easy to check that any C^m domain in $\mathbb{R}^n$ which can be extended to a C^m domain in $\mathbb{RP}^n$ under this embedding will have a uniformly C^m defining function. However, we will also show that there are examples which are not even C^1 in $\mathbb{RP}^n$ but still have uniformly C^m defining functions.

We conclude the paper in Section 4 with two specific applications of uniformly C^m defining functions. The first is the construction of weighted Sobolev spaces on unbounded domains, and the second is a brief example from several complex variables to illustrate the advantages of uniformly C^m defining functions in generalizing some well-known constructions.

Over the course of several papers, we will study domains Ω that admit a uniformly C^m defining function, build weighted Sobolev spaces on them, and develop the elliptic theory associated to the Sobolev spaces [8]. We will then be in a position to investigate the $\bar{\partial}$-Neumann and $\bar{\partial}_b$-problems in weighted L^2 on $\Omega \subset \mathbb{C}^n$. Gansberger has obtained compactness results for the $\bar{\partial}$-Neumann operator in weighted L^2 [4], but (at the time) there was available neither the elliptic theory nor a Sobolev space theory suitable for studying the $\bar{\partial}$-Neumann problem in H^s or to facilitate the passage from the $\bar{\partial}$-Neumann operator at the Sobolev scale $s = 1/2$ to the complex Green operator on $\partial\Omega$ in weighted L^2. There are other results about solution operators for $\bar{\partial}$ in the unbounded setting but for the case $\Omega = \mathbb{C}^n$, rendering any boundary discussion moot [6], [3].

2. Basic results

A *multi-index* is an ordered n-tuple of nonnegative integers. For $I = (I_1, \ldots, I_n)$, we define $|I| = \sum_{j=1}^n I_j$. We say that a multi-index I has *length* k if $|I| = k$ and let $\mathcal{I}_k$ be the space of all multi-indices of length k. Denote the jth component of a multi-index I by I_j so that, for example, $\partial^{|I|}/\partial x^I = \partial^{I_1}/\partial x_1^{I_1} \cdots \partial^{I_n}/\partial x_n^{I_n}$. We equip the space of all multi-indices with the partial ordering $J \leq I$ if $J_j \leq I_j$ for every $1 \leq j \leq n$. For $J \leq I$, the difference $I - J$ is also a multi-index. In order to apply the product rule, we define $\binom{I}{J} = \prod_{j=1}^n \binom{I_j}{J_j}$.

Below, we will take the C^k norm of a function on $\partial\Omega$. We take an extrinsic view, and for a C^k function f defined on a neighborhood of $\partial\Omega$, we set

$$\|f\|^2_{C^k(\partial\Omega)} = \sup_{p\in\partial\Omega} \sum_{j=0}^{k} \sum_{I\in\mathcal{I}_j} \left|\frac{\partial^j f(p)}{\partial x^I}\right|^2 = \inf_{U\supset\partial\Omega} \|f\|^2_{C^k(U)}\,.$$

The intrinsic C^k norm of a defining function (defined in terms of tangential derivatives on the boundary) is always zero, hence our use of the extrinsic norm.

For $p \in \partial\Omega$, let $e_n = \nabla\tilde{\delta}(p)$, and let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis for the orthogonal complement of e_n (ordered to preserve orientation). In our given coordinate system $(x_1, \dots, x_n)$ we can write $e_j = (e_j^1, \dots, e_j^n)$. The matrix $E = \left(e_j^k\right)$ is orthogonal, so we can define orthonormal coordinates $(y_1, \dots, y_n)$ by $y_j = \sum_{k=1}^n e_j^k(x_k - p_k)$. Then in these new coordinates we can write $p = 0$ and $\nabla_y\tilde{\delta}(p) = (0, \dots, 0, 1)$. Throughout this paper, we will use $(x_1, \dots, x_n)$ for our given coordinate system and $(y_1, \dots, y_n)$ for the special coordinate system satisfying $\nabla_y\tilde{\delta}(p) = (0, \dots, 0, 1)$ and $p = 0$ for a point $p \in \partial\Omega$.

For functions f defined in a neighborhood of p, we define a family of special C^k norms that is adapted to the boundary. For any integer $k \geq 0$, define

$$|f|^2_{C_b^k(p)} = \sum_{k'=0}^{k} \sum_{\substack{I\in\mathcal{I}_{k'}\\ I_n\leq\min\{k-k',k'\}}} \left|\frac{\partial^{k'} f(p)}{\partial y^I}\right|^2.$$

The C_b^k norms provide a balance between computability (derivatives are only with respect to $\{y_j\}$) and theoretical elegance (intrinsic tangential derivatives and the normal). In particular, terms in the C_b^k norm agree with terms in the expansion of a k-fold composition of tangential differential operators with respect to local coordinates. For the purposes of induction, we also define, for any integers $k \geq 1$ and $k \geq 2j \geq 0$,

$$|f|^2_{C_b^{k,j}(p)} = |f|^2_{C_b^{k-1}(p)} + \sum_{j'=j}^{\lfloor k/2\rfloor} \sum_{\substack{I\in\mathcal{I}_{k-j'}\\ I_n=j}} \left|\frac{\partial^{k-j'} f(p)}{\partial y^I}\right|^2.$$

The $C_b^{k,j}$ are intermediate norms between C_b^k and C_b^{k-1}. In particular, $|f|^2_{C_b^{k,0}(p)} = |f|^2_{C_b^k(p)}$. Also, when k is even,

$$|f|^2_{C_b^{k,k/2}(p)} = |f|^2_{C_b^{k-1}(p)} + \left|\frac{\partial^{k/2} f(p)}{\partial y_n^{k/2}}\right|^2,$$

and when k is odd,

$$|f|^2_{C_b^{k,(k-1)/2}(p)} = |f|^2_{C_b^{k-1}(p)} + \sum_{\ell=1}^{n-1} \left|\frac{\partial^{(k+1)/2} f(p)}{\partial y_\ell \partial y_n^{(k-1)/2}}\right|^2.$$

In general, if I is a multi-index, then

$$\left|\frac{\partial^{|I|} f(p)}{\partial y^I}\right| \leq |f|_{C_b^{|I|+I_n, I_n}(p)} \leq |f|_{C_b^{|I|+I_n}(p)}\,. \tag{2.1}$$

The utility of this norm can be seen from the following lemma.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ have C^m boundary, $m \geq 2$. Let ρ be a C^m defining function for Ω and let h be the positive C^{m-1} function defined in a neighborhood of $\partial\Omega$ by $\tilde{\delta} = h\rho$. Then*

$$\sup_{p\in\partial\Omega} \frac{|\rho|_{C_b^m(p)}}{|\nabla\rho(p)|} < \infty$$

if and only if

$$\|\tilde{\delta}\|_{C^m(\partial\Omega)} < \infty \quad \text{and} \quad \sup_{p\in\partial\Omega} \frac{|h|_{C_b^{m-2}(p)}}{h(p)} < \infty.$$

Remark 2.2. When $m = 2$ the statement about h is trivial, so the conditions on ρ and $\tilde{\delta}$ are equivalent. We will see in (2.9) that something stronger is true in this case.

Proof. Let $(x_1, \ldots, x_n)$ be arbitrary coordinates on $\mathbb{R}^n$. Since $|\nabla\tilde{\delta}|^2 = 1$ on a neighborhood of $\partial\Omega$ (see [9] and Theorem 4.8 (3) in [2]), for $I \in \mathcal{I}_k$ with $1 \leq k \leq m-1$, we can apply $\partial^k/\partial x^I$ to this equality to obtain

$$\sum_{j=1}^{n} \sum_{J\leq I} \binom{I}{J} \frac{\partial}{\partial x_j}\left(\frac{\partial^{|J|}\tilde{\delta}}{\partial x^J}\right) \frac{\partial}{\partial x_j}\left(\frac{\partial^{k-|J|}\tilde{\delta}}{\partial x^{I-J}}\right) = 0$$

on $\partial\Omega$ (note that this method is also used with $k = 1$ in Corollary 5.3 of [7]). For fixed $p \in \partial\Omega$, choose coordinates $(y_1, \ldots, y_n)$ so that $p = 0$ and $\nabla_y\tilde{\delta}(p) = (0, \ldots, 0, 1)$. In these coordinates,

$$\sum_{j=1}^{n} \sum_{J\neq 0, J<I} \binom{I}{J} \frac{\partial}{\partial y_j}\left(\frac{\partial^{|J|}\tilde{\delta}}{\partial y^J}\right) \frac{\partial}{\partial y_j}\left(\frac{\partial^{k-|J|}\tilde{\delta}}{\partial y^{I-J}}\right)(p) + 2\frac{\partial}{\partial y_n}\left(\frac{\partial^k\tilde{\delta}}{\partial y^I}\right)(p) = 0. \tag{2.2}$$

From this, we conclude that

$$\left|\frac{\partial}{\partial y_n}\left(\frac{\partial^k\tilde{\delta}}{\partial y^I}\right)(p)\right| \leq C_1\|\tilde{\delta}\|^2_{C^k(\partial\Omega)} \tag{2.3}$$

for some constant $C_1 > 0$ and for any $I \in \mathcal{I}_k$ with $1 \leq k \leq m-1$.

Since h is C^{m-1}, we may apply $\partial^k/\partial x^I$, for $I \in \mathcal{I}_k$, to $\tilde{\delta} = h\rho$ in a neighborhood of $\partial\Omega$ to obtain

$$\frac{\partial^k\tilde{\delta}}{\partial x^I} = \sum_{J\leq I} \binom{I}{J} \frac{\partial^{|J|}h}{\partial x^J} \frac{\partial^{k-|J|}\rho}{\partial x^{I-J}}.$$

This can not be differentiated directly again if $k = m-1$ because h is only C^{m-1}, but we may form a difference quotient at $p \in \partial\Omega$ and take the limit to obtain

$$\frac{\partial^{k+1}\tilde{\delta}(p)}{\partial x^I} = \sum_{J<I} \binom{I}{J} \frac{\partial^{|J|}h(p)}{\partial x^J} \frac{\partial^{k+1-|J|}\rho(p)}{\partial x^{I-J}} \tag{2.4}$$

for any $I \in \mathcal{I}_{k+1}$, since $\rho(p) = 0$. Switching from generic coordinates $(x_1, \ldots, x_n)$ to our special coordinates $(y_1, \ldots, y_n)$, we have $\partial\rho/\partial y_j(p) = 0$ if $j \neq n$, so if $I_n = 0$ in these coordinates, all of the terms with first derivatives of ρ will also vanish, leaving us with

$$\left|\frac{\partial^{k+1}\tilde{\delta}(p)}{\partial y^I}\right| \leq C_2\,|h|_{C_b^{k-1}(p)}\,|\rho|_{C_b^{k+1}(p)} \tag{2.5}$$

for some constant $C_2 > 0$.

For $0 \leq k' \leq m$ and $I' \in \mathcal{I}_{k'}$, we obtain from (2.4) the equation

$$\frac{\partial^{|I'|}\tilde{\delta}(p)}{\partial y^{I'}} = \sum_{\substack{J<I'\\ J_n<I'_n}} \binom{I'}{J}\frac{\partial^{|J|}h(p)}{\partial y^J}\,\frac{\partial^{|I'|-|J|}\rho(p)}{\partial y^{I'-J}} + \sum_{\substack{J<I'\\ J_n=I'_n}} \binom{I'}{J}\frac{\partial^{|J|}h(p)}{\partial y^J}\,\frac{\partial^{|I'|-|J|}\rho(p)}{\partial y^{I'-J}}.$$

Subtracting the terms of highest order in h (with respect to the C_b^k norm) and setting e_j to be the jth standard basis vector in $\mathbb{R}^n$ with respect to $\{y_1, \ldots, y_n\}$, we can use (2.1) to estimate the remainder by

$$\left|\frac{\partial^{|I'|}\tilde{\delta}(p)}{\partial y^{I'}} - I'_n\frac{\partial^{|I'|-1}h(p)}{\partial y^{I'-e_n}}\,\frac{\partial\rho(p)}{\partial y_n} - \sum_{\substack{J<I',J_n=I'_n\\ |J|=|I'|-2}} \binom{I'}{J}\frac{\partial^{|I'|-2}h(p)}{\partial y^J}\,\frac{\partial^2\rho(p)}{\partial y^{I'-J}}\right| \\ \leq C_3\,|h|_{C_b^{|I'|+I'_n-3}(p)}\,|\rho|_{C_b^{|I'|+I'_n}(p)}.$$

for some constant $C_3 > 0$. If $I'_n = |I'|$ or $I'_n = |I'| - 1$, we have simply

$$\left|\frac{\partial^{|I'|}\tilde{\delta}(p)}{\partial y^{I'}} - I'_n\frac{\partial^{|I'|-1}h(p)}{\partial y^{I'-e_n}}\,\frac{\partial\rho(p)}{\partial y_n}\right| \leq C_3\,|h|_{C_b^{|I'|+I'_n-3}(p)}\,|\rho|_{C_b^{|I'|+I'_n}(p)}.$$

Suppose that $0 \leq j \leq (k-1)/2$ and that $I \in \mathcal{I}_{k-j-1}$ satisfies $I_n = j$. Note that $\partial\rho/\partial y_n(p) = |\nabla\rho(p)|$ and $\partial\tilde{\delta}/\partial y_n(p) = 1$, so by (2.4) with $k = 0$ we have

$$h(p)|\nabla\rho(p)| = 1. \tag{2.6}$$

If we let $I' = I + e_n$, it follows that

$$(j+1)\left|\frac{\partial^{k-j-1}h(p)}{\partial y^I}\right|h(p)^{-1} \leq \left|\frac{\partial^{k-j}\tilde{\delta}(p)}{\partial y^I\partial y_n}\right| \\ + \sum_{\substack{J<I+e_n,J_n=j+1\\ |J|=k-j-2}} \binom{I+e_n}{J}\left|\frac{\partial^{k-j-2}h(p)}{\partial y^J}\,\frac{\partial^2\rho(p)}{\partial y^{I-J}\partial y_n}\right| \\ + C_3\,|h|_{C_b^{k-2}(p)}\,|\rho|_{C_b^{k+1}(p)} \tag{2.7}$$

if $k \geq 2j+3$, and

$$(j+1)\left|\frac{\partial^{k-j-1}h(p)}{\partial y^I}\right|h(p)^{-1} \leq \left|\frac{\partial^{k-j}\tilde{\delta}(p)}{\partial y^I\partial y_n}\right| + C_3\,|h|_{C_b^{k-2}(p)}\,|\rho|_{C_b^{k+1}(p)}. \tag{2.8}$$

if $k < 2j+3$.

We now proceed by induction on k. Assume $\sup_{p\in\partial\Omega}|\rho|_{C_b^m(p)}/|\nabla\rho(p)|<\infty$. Suppose that for some $m-1\geq k\geq 1$, $\|\tilde{\delta}\|_{C^k(\partial\Omega)}<\infty$ and $\sup_{p\in\partial\Omega}|h|_{C_b^{k-2}(p)}/h(p)$ $<\infty$. When $k=1$, this is clear since $\|\tilde{\delta}\|_{C^1(\partial\Omega)}=1$ and the condition on h is vacuous. Using $j=\lfloor(k-1)/2\rfloor$ with (2.8) and the induction hypothesis we can show that $\sup_{p\in\partial\Omega}|h|_{C_b^{k-1,\lfloor(k-1)/2\rfloor}(p)}/h(p)<\infty$. Suppose that for some $0\leq j\leq(k-3)/2$ we know that $\sup_{p\in\partial\Omega}|h|_{C_b^{k-1,j+1}(p)}/h(p)<\infty$. Using (2.7), we know now that $\sup_{p\in\partial\Omega}|h|_{C_b^{k-1,j}(p)}/h(p)<\infty$ since

$$|h|_{C_b^{k-1,j}(p)}=|h|_{C_b^{k-1,j+1}(p)}+\sum_{\substack{I\in\mathcal{I}_{k-j-1}\\ I_n=j}}\left|\frac{\partial^{k-j-1}h(p)}{\partial y^I}\right|.$$

Proceeding by downward induction on j we have $\sup_{p\in\partial\Omega}|h|_{C_b^{k-1}(p)}/h(p)<\infty$.

Using (2.3) and (2.5), we conclude $\|\tilde{\delta}\|_{C^{k+1}(\partial\Omega)}<\infty$. The result follows by induction on k.

For the converse, we simply subtract the term of highest degree in ρ from (2.4) with $I\in\mathcal{I}_{k'+1}$ for $0\leq k'\leq m-1$ to obtain

$$\left|\frac{\partial^{k'+1}\tilde{\delta}(p)}{\partial y^I}-h(p)\frac{\partial^{k'+1}\rho(p)}{\partial y^I}\right|\leq C_4\,|h|_{C_b^{k'+I_n-1}(p)}\,|\rho|_{C_b^{k'+I_n}(p)}\,,$$

for some constant $C_4>0$. For any $0\leq j\leq(k+1)/2$ and $I\in\mathcal{I}_{k-j+1}$ with $I_n=j$, if we set $k'=k-j$ then we have

$$|\nabla\rho(p)|^{-1}\left|\frac{\partial^{k-j+1}\rho(p)}{\partial y^I}\right|\leq\left|\frac{\partial^{k-j+1}\tilde{\delta}(p)}{\partial y^I}\right|+C_4\,|h|_{C_b^{k-1}(p)}\,|\rho|_{C_b^{k}(p)}\,.$$

The result follows by induction on k. □

Although Lemma 2.1 may not apply to all C^m defining functions, it will suffice to prove the main theorem. However, the inductive procedure used to prove this lemma may also be used to construct a system of boundary invariants for any defining function. We illustrate this by considering the $m=2$ and $m=3$ cases. By (2.3), it will suffice to consider derivatives in tangential directions. Fix $1\leq j,k,\ell\leq n-1$. In the special coordinates of Lemma 2.1 at p we apply (2.4) repeatedly to obtain

$$1=h(p)|\nabla\rho(p)|;\ \frac{\partial^2\tilde{\delta}(p)}{\partial y_j\partial y_k}=h(p)\frac{\partial^2\rho(p)}{\partial y_j\partial y_k};\ \frac{\partial^2\tilde{\delta}(p)}{\partial y_j\partial y_n}=h(p)\frac{\partial^2\rho(p)}{\partial y_j\partial y_n}+\frac{\partial h(p)}{\partial y_j}|\nabla\rho(p)|;$$
$$\frac{\partial^3\tilde{\delta}(p)}{\partial y_j\partial y_k\partial y_\ell}=h(p)\frac{\partial^3\rho(p)}{\partial y_j\partial y_k\partial y_\ell}+\frac{\partial h(p)}{\partial y_j}\frac{\partial^2\rho(p)}{\partial y_k\partial y_\ell}+\frac{\partial h(p)}{\partial y_k}\frac{\partial^2\rho(p)}{\partial y_j\partial y_\ell}+\frac{\partial h(p)}{\partial y_\ell}\frac{\partial^2\rho(p)}{\partial y_j\partial y_k}.$$

By (2.2), $\partial^2\tilde{\delta}(p)/\partial y_j\partial y_n=0$, so we may use (2.6) and the previous equalities to conclude

$$\frac{\partial^2\tilde{\delta}(p)}{\partial y_j\partial y_k}=|\nabla\rho|^{-1}\frac{\partial^2\rho(p)}{\partial y_j\partial y_k}, \tag{2.9}$$

and

$$\frac{\partial^3\tilde{\delta}(p)}{\partial y_j\partial y_k\partial y_\ell} = |\nabla\rho|^{-1}\frac{\partial^3\rho(p)}{\partial y_j\partial y_k\partial y_\ell} - |\nabla\rho|^{-2}\Big(\frac{\partial^2\rho(p)}{\partial y_j\partial y_n}\frac{\partial^2\rho(p)}{\partial y_k\partial y_\ell} + \frac{\partial^2\rho(p)}{\partial y_k\partial y_n}\frac{\partial^2\rho(p)}{\partial y_j\partial y_\ell} + \frac{\partial^2\rho(p)}{\partial y_\ell\partial y_n}\frac{\partial^2\rho(p)}{\partial y_j\partial y_k}\Big). \tag{2.10}$$

Once we have completed the proof of the main theorem, we can derive necessary and sufficient conditions for the existence of uniformly C^2 (resp. C^3) defining functions by checking the boundedness of (2.9) (resp. (2.9) and (2.10)). Higher order conditions can be derived as well, but these will be progressively more complicated. We note that (2.9) also follows from Remark 4.2 (i) of [7], as does the observation that successive differentiations can be used to obtain higher order equalities.

To facilitate formulas without special coordinates, we define

$$T_p(\partial\Omega) = \Big\{t \in \mathbb{R}^n : \sum_{j=1}^n t_j \frac{\partial\tilde{\delta}}{\partial x_j}(p) = 0\Big\}.$$

We also use the notation $y = (y', y_n)$ for $y' \in \mathbb{R}^{n-1}$ and $y_n \in \mathbb{R}$.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ have a C^2 boundary. Then for any C^2 defining function ρ we have*

$$\sup_{p\in\partial\Omega}\ \sup_{\substack{t\in T_p(\partial\Omega)\\ |t|=1}} |\nabla\rho|^{-1}\Big|\sum_{j,k=1}^n t_j\frac{\partial^2\rho}{\partial x_j\partial x_k}(p)t_k\Big| < \infty \tag{2.11}$$

if and only if $\partial\Omega$ has positive reach, and

$$\mathrm{Reach}(\partial\Omega) = \Big(\sup_{p\in\partial\Omega}\ \sup_{\substack{t\in T_p(\partial\Omega)\\ |t|=1}} |\nabla\rho|^{-1}\Big|\sum_{j,k=1}^n t_j\frac{\partial^2\rho}{\partial x_j\partial x_k}(p)t_k\Big|\Big)^{-1}. \tag{2.12}$$

Proof. For $p \in \partial\Omega$, choose local coordinates $(y_1, \ldots, y_n)$ so that $p = 0$ and $\nabla\tilde{\delta}(p) = (0', 1)$. Suppose that for some $r > 0$, $B((0', r), r) \subset \Omega^c$ and $B((0', -r), r) \subset \Omega$. Then for $q \in \partial\Omega$, $|q - (0', \pm r)|^2 \geq r^2$, so $|q|^2 \mp 2q_n r \geq 0$. Hence $|q|^2/(2r) \geq |q_n|$. By Theorem 4.18 in [2], this can be accomplished at every $p \in \partial\Omega$ if and only if $\mathrm{Reach}(\partial\Omega) \geq r$.

In our special coordinates, note that for $s \in \mathbb{R}^{n-1}$ we have

$$\sum_{j,k=1}^{n-1} s_j\frac{\partial^2\tilde{\delta}(0)}{\partial y_j\partial y_k}s_k = \lim_{h\to 0}\frac{2\tilde{\delta}(hs, 0)}{h^2}.$$

If $|q|^2/(2r) \geq |q_n|$ for all $q \in \partial\Omega$, then by comparing $(hs, 0)$ to the boundary point which shares its first $n-1$ coordinates with hs we see that $\delta(hs, 0) \leq |hs|^2/(2r)$, so

$$\sum_{j,k=1}^{n-1}\Big|s_j\frac{\partial^2\tilde{\delta}(0)}{\partial y_j\partial y_k}s_k\Big| \leq \frac{|s|^2}{r}. \tag{2.13}$$

On the other hand, Taylor's theorem implies that if (2.13) holds in a neighborhood of 0, then $|\tilde{\delta}(q) - q_n| \leq |q|^2/(2r)$ for q sufficiently close to 0. Thus, if $q \in \partial\Omega$, then $|q|^2/(2r) \geq |q_n|$.

Converting (2.13) to generic coordinates $(x_1, \dots, x_n)$, we see that Reach$(\partial\Omega) \geq r$ if and only if

$$\sum_{j,k=1}^{n} \left| t_j \frac{\partial^2 \tilde{\delta}(p)}{\partial x_j \partial x_k} t_k \right| \leq \frac{|t|^2}{r}$$

on $\partial\Omega$ for any vector $t \in T_p(\partial\Omega)$. By (2.9), this is equivalent to

$$|\nabla\rho|^{-1} \left| \sum_{j,k=1}^{n} t_j \frac{\partial^2 \rho(p)}{\partial x_j \partial x_k} t_k \right| \leq \frac{|t|^2}{r}$$

for all $p \in \partial\Omega$ and $t \in T_p(\partial\Omega)$. If we take the supremum over all possible $r > 0$ satisfying these inequalities, the result follows. □

For $m = 2$, the following result follows from computations ((2.14) and (2.15)) that can be found in several sources (see in particular [10], Lemma 14.17 in [5], and Section 5 in [7]). In (2.16) and (2.17) we generalize these to $m \geq 2$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ have a C^m boundary for some $m \geq 2$ and suppose that the signed distance function for Ω satisfies $\|\tilde{\delta}\|_{C^m(\partial\Omega)} < \infty$. Then for any $0 < \epsilon <$ Reach$(\partial\Omega)$ the signed distance function satisfies $\|\tilde{\delta}\|_{C^m(U)} < \infty$ on $U_\epsilon = \{x \in \mathbb{R}^n : \delta(x) < \epsilon\}$.*

Proof. By the previous lemma, $\partial\Omega$ has positive reach, so $\tilde{\delta}$ is a C^m function on a neighborhood $U' \supset \partial\Omega$ [9]. Note that the result of Krantz and Parks is essentially local, so it is possible that dist$(\partial U', \partial\Omega) = 0$ if $\partial\Omega$ is not compact. Set

$$U = \left\{x \in \mathbb{R}^n : \delta(x) < \text{Reach}(\partial\Omega)\right\}.$$

By Theorem 4.8 (3) and (5) in [2], for any $x \in U$ we have $\nabla\tilde{\delta}(x) = \nabla\tilde{\delta}(\pi(x))$, where $\pi(x) = x - \tilde{\delta}(x)\nabla\tilde{\delta}(x) \in \partial\Omega$ is the unique boundary point nearest to x. This is differentiable, and solving the derivative for $\nabla^2\tilde{\delta}$ gives us

$$\frac{\partial^2 \tilde{\delta}(x)}{\partial x_j \partial x_\ell} = \sum_{\ell'=1}^{n} \frac{\partial^2 \tilde{\delta}(\pi(x))}{\partial x_j \partial x_{\ell'}} \left(Id + \tilde{\delta}(x) \nabla^2 \tilde{\delta}(\pi(x)) \right)^{-1}_{\ell'\ell} \tag{2.14}$$

for $x \in U$, where Id is the identity matrix (see [10] and [7]; see also (2.15) below). Note that $Id + \tilde{\delta}(x)\nabla^2\tilde{\delta}(x)$ is invertible on U by (2.9) and (2.12). This formula shows that $\tilde{\delta}$ is C^2 on U (we already know that $\tilde{\delta}$ is C^2 near $\partial\Omega$ and $\pi(x) \in \partial\Omega$). Since this formula relates derivatives away from $\partial\Omega$ to derivatives on $\partial\Omega$ (which exist since $\partial\Omega \subset U'$), we may continue to differentiate and use induction to show that $\tilde{\delta}$ is C^m on U.

Fix $p \in \partial\Omega$ and choose new coordinates $(y_1, \dots, y_n)$ so that $p = 0$, $\nabla\tilde{\delta}(p) = (0', 1)$, and $\nabla^2\tilde{\delta}(p)$ is diagonalized with eigenvalues $\kappa_1(0), \dots, \kappa_n(0)$. By Theorem 4.8 (3) in [2], when $y' = 0'$ and $|y_n| <$ Reach$(\partial\Omega)$, we have $\tilde{\delta}(y) = y_n$ and

$\nabla\tilde{\delta}(y) = (0',1)$. Differentiating $|\nabla\tilde{\delta}|^2 = 1$ once demonstrates that $\kappa_n = 0$. For $m \geq 3$, differentiating $|\nabla\tilde{\delta}|^2 = 1$ twice yields

$$2\sum_{\ell=1}^{n}\Big(\frac{\partial\tilde{\delta}}{\partial x_\ell}\frac{\partial^3\tilde{\delta}}{\partial x_\ell\partial x_j\partial x_k} + \frac{\partial^2\tilde{\delta}}{\partial x_\ell\partial x_j}\frac{\partial^2\tilde{\delta}}{\partial x_\ell\partial x_k}\Big) = 0$$

on U. From (2.14), we can see that eigenvectors of $\nabla^2\tilde{\delta}$ are preserved along the normal direction. Rewriting the above equation in our y-coordinates, when $j = k$ and $y' = 0'$, we have $2(\partial\kappa_j/\partial y_n + \kappa_j^2)(y) = 0$ on U. The unique solution to this equation is given by

$$\kappa_j(y) = \frac{\kappa_j(0)}{1 + y_n\kappa_j(0)} \tag{2.15}$$

(see also Lemma 14.17 in [5], but with the opposite sign convention). Since $\operatorname{Reach}(\partial\Omega) \leq |\kappa_j(0)|^{-1}$ (see (2.9) and (2.12)) for all $1 \leq j \leq n-1$ with $\kappa_j \neq 0$, κ_j is uniformly bounded on U_ϵ.

For $3 \leq k \leq m-1$, let $I \in \mathcal{I}_k$. Then differentiating $|\nabla\tilde{\delta}|^2 = 1$ gives us

$$\sum_{j=1}^{n}\sum_{J\leq I}\binom{I}{J}\frac{\partial}{\partial x_j}\Big(\frac{\partial^{|J|}\tilde{\delta}}{\partial x^J}\Big)\frac{\partial}{\partial x_j}\Big(\frac{\partial^{k-|J|}\tilde{\delta}}{\partial x^{I-J}}\Big) = 0.$$

on U. In our diagonalized coordinates, we can evaluate terms involving only first or second derivatives separately to obtain

$$2\frac{\partial^{k+1}\tilde{\delta}}{\partial y_n\partial y^I} + 2\Big(\sum_{j=1}^{n} I_j\kappa_j\Big)\frac{\partial^k\tilde{\delta}}{\partial y^I} + \sum_{j=1}^{n}\sum_{\substack{J<I\\2\leq|J|\leq k-2}}\binom{I}{J}\frac{\partial}{\partial y_j}\Big(\frac{\partial^{|J|}\tilde{\delta}}{\partial y^J}\Big)\frac{\partial}{\partial y_j}\Big(\frac{\partial^{k-|J|}\tilde{\delta}}{\partial y^{I-J}}\Big) = 0$$

on U when $y' = 0'$ (when $k = 3$ the final sum can be omitted). If we set

$$\mu_I(y_n) = \prod_{j=1}^{n}\big(1 + y_n\kappa_j(0)\big)^{I_j},$$

then by (2.15), $\mu_I(y_n)$ solves the initial value problem

$$\frac{\partial\mu_I}{\partial y_n}(y_n) = \mu_I(y_n)\sum_{j=1}^{n} I_j\kappa_j(0',y_n) \quad \text{and} \quad \mu_I(0) = 1,$$

so

$$2\frac{\partial}{\partial y_n}\Big(\mu_I\frac{\partial^k\tilde{\delta}}{\partial y^I}\Big) + \mu_I\sum_{j=1}^{n}\sum_{\substack{J<I\\2\leq|J|\leq k-2}}\binom{I}{J}\frac{\partial}{\partial y_j}\Big(\frac{\partial^{|J|}\tilde{\delta}}{\partial y^J}\Big)\frac{\partial}{\partial y_j}\Big(\frac{\partial^{k-|J|}\tilde{\delta}}{\partial y^{I-J}}\Big) = 0$$

on U when $y' = 0'$. Hence, we may integrate to obtain

$$\frac{\partial^k \tilde{\delta}}{\partial y^I}(0', y_n) = \frac{1}{\mu_I(y_n)} \frac{\partial^k \tilde{\delta}}{\partial y^I}(0) - \frac{1}{2\mu_I(y_n)} \int_0^{y_n} \mu_I(t) \sum_{j=1}^{n} \sum_{\substack{J<I \\ 2\le |J| \le k-2}} \binom{I}{J} \frac{\partial}{\partial y_j}\Big(\frac{\partial^{|J|}\tilde{\delta}}{\partial y^J}\Big) \frac{\partial}{\partial y_j}\Big(\frac{\partial^{k-|J|}\tilde{\delta}}{\partial y^{I-J}}\Big)(0', t)dt. \tag{2.16}$$

When $k = 3$ the integrated sum is omitted, so we have the formula

$$\frac{\partial^3 \tilde{\delta}}{\partial y^I}(0', y_n) = \frac{1}{\mu_I(y_n)} \frac{\partial^3 \tilde{\delta}}{\partial y^I}(0). \tag{2.17}$$

Since $\mu_I(y_n)$ is uniformly bounded below on U_ϵ and the terms in the integral are differentiated at most $k-1$ times, we may use induction on k to obtain uniform bounds on $\partial^k \tilde{\delta}/\partial y^I$ on U_ϵ for all I with $3 \le k \le m-1$.

Now, we wish to differentiate our formulas for $k = m-1$ to show that they also hold for $k = m$. By differentiating (2.14), we can obtain formulas for the first m derivatives of $\tilde{\delta}$ on U in terms of derivatives restricted to $\partial\Omega$. By formal manipulations, these must be equivalent to those obtained in (2.16) and (2.17), and hence the first m derivatives remain uniformly bounded on U_ϵ. □

3. Examples in projective space

A large class of examples of domains with uniformly C^m defining functions can be found by considering $\mathbb{R}^n \subset \mathbb{RP}^n$. Recall that $\mathbb{RP}^n = (\mathbb{R}^n \setminus \{0\})/\sim$ under the equivalence relation $x \sim y$ if $x = \lambda y$ for $\lambda \in \mathbb{R}\setminus\{0\}$. If we denote coordinates on $\mathbb{RP}^n$ by $[x_1 : \ldots : x_{n+1}]$, the canonical embedding of $\mathbb{R}^n$ in $\mathbb{RP}^n$ is given by $(x_1, \ldots, x_n) \mapsto [x_1 : \ldots : x_n : 1]$. Every unbounded domain Ω in $\mathbb{R}^n$ can be extended to a bounded domain $\tilde{\Omega}$ in $\mathbb{RP}^n$ with respect to this embedding, although smooth unbounded domains will generally not embed as smooth domains in $\mathbb{RP}^n$. Conversely, from a domain $\tilde{\Omega} \subset \mathbb{RP}^n$, we can canonically produce a (possibly unbounded) domain $\Omega \subset \mathbb{R}^n$ under the mapping $[x_1 : \ldots : x_n : 1] \mapsto (x_1, \ldots, x_n)$.

Corollary 3.1. *Let $\tilde{\Omega} \subset \mathbb{RP}^n$ be a C^m domain. Then the domain $\Omega \subset \mathbb{R}^n$ obtained by pulling back along the canonical embedding has a uniformly C^m defining function.*

Proof. For $S^n \subset \mathbb{R}^{n+1}$, we can define $\tilde{\Omega}$ by a C^m defining function $\tilde{\rho} : S^n \to \mathbb{R}$ such that $\tilde{\rho}(-\tilde{x}) = \tilde{\rho}(\tilde{x})$. Extend $\tilde{\rho}$ to all of $\mathbb{R}^{n+1}\setminus\{0\}$ by $\tilde{\rho}(\tilde{x}) = \tilde{\rho}\big(\tilde{x}/|\tilde{x}|\big)$. Since we have $\tilde{\rho}(\tilde{x}) = \tilde{\rho}(\lambda\tilde{x})$ for any $\lambda \in \mathbb{R}\setminus\{0\}$, we also obtain $\nabla^k\tilde{\rho}(\tilde{x}) = \lambda^k(\nabla^k\tilde{\rho})(\lambda\tilde{x})$. If we assume that $\left|\nabla^k\tilde{\rho}\right| < C_k$ and $|\nabla\tilde{\rho}| > C_0$ on $\partial\tilde{\Omega} \cap S^n$ for any integer $1 \le k \le m$ and some constants $C_0, C_k > 0$, then substituting $\lambda = 1/|\tilde{x}|$ we have in general $\left|\nabla^k\tilde{\rho}(\tilde{x})\right| < C_k\,|\tilde{x}|^{-k}$ and $|\nabla\tilde{\rho}(\tilde{x})| > C_0\,|\tilde{x}|^{-1}$ whenever $\tilde{\rho}(\tilde{x}) = 0$.

A defining function ρ for the domain $\Omega \subset \mathbb{R}^n$ can now be obtained by considering $\rho(x_1, \ldots, x_n) = \tilde{\rho}(x_1, \ldots, x_n, 1)$. Since we are using $\tilde{x} = (x_1, \ldots, x_n, 1)$, we have $|\tilde{x}| = (1+|x|^2)^{1/2}$. Thus $\left|\nabla^k\rho(x)\right| < C_k(1+|x|^2)^{-k/2}$ and $|\nabla\rho(x)| > C_0(1+|x|^2)^{-1/2}$ on $\partial\Omega$, so

$$\frac{\left|\nabla^k\rho\right|}{|\nabla\rho|}(x) < \frac{C_k}{C_0(1+|x|^2)^{(k-1)/2}}$$

on $\partial\Omega$ for all $1 \leq k \leq m$. By our main theorem, this implies that Ω has a uniformly C^m defining function. □

Note that this proof can still be used if $\tilde{\rho}$ is C^m when $x_{n+1} \neq 0$ and $\left|\nabla^k\tilde{\rho}(x)\right| < C_k x_{n+1}^{1-k}$ for $x \in S^n$ with $x_{n+1} \neq 0$, so a uniformly C^m defining function in $\mathbb{R}^n$ covers a much larger class of examples than those given by C^m domains in $\mathbb{RP}^n$.

For example, consider the domain $\Omega_1 \subset \mathbb{R}^2$ defined by

$$\Omega_1 = \left\{y < x^{-1}\sin x, x \neq 0\right\} \cup \left\{y < 1, x = 0\right\}.$$

Then Ω_1 is a C^∞ domain. Let $\rho_1(x,y) = y - x^{-1}\sin x$ when $x \neq 0$ and $\rho_1(0,y) = y-1$. By considering the Maclaurin series of $\sin x$ we can see that ρ_1 is real-analytic (hence smooth) in a neighborhood of the set where $x = 0$. When $x \neq 0$, we have $\nabla\rho_1 = (x^{-2}\sin x - x^{-1}\cos x, 1)$, so $|\nabla\rho_1|$ is uniformly bounded from above and away from zero. Differentiating m times, we have $|\partial^m\rho_1/\partial x^m| \leq O(x^{-1})$, so this is also uniformly bounded. Hence ρ_1 is a uniformly C^m defining function for any integer m. In $\mathbb{RP}^2$, this defining function can be written $\rho_1([x:y:z]) = y/z - z/x\sin(x/z)$. On the coordinate patch where $x \neq 0$, this can be written $\rho_1(y,z) = y/z - z\sin(1/z)$. To normalize this near $z = 0$, we use $\tilde{\rho}_1(y,z) = y - z^2\sin(1/z)$. Note that for fixed y this is a classic example of a function which is differentiable at $z = 0$ but not C^1 in a neighborhood of $z = 0$. We conclude that $\tilde{\Omega}_1 \subset \mathbb{RP}^2$ is not a C^1 domain.

On the other hand, consider $\Omega_2 \subset \mathbb{R}^2$ defined by

$$\Omega_2 = \left\{y < x^{-2}\sin x^2, x \neq 0\right\} \cup \left\{y < 1, x = 0\right\}.$$

Let $\rho_2(x,y) = y - x^{-2}\sin x^2$ when $x \neq 0$ and $\rho_2(0,y) = y-1$. Again, the Maclaurin series will show that all derivatives are uniformly bounded near $x = 0$, so we focus on $x \neq 0$. Since $\nabla\rho_2 = (2x^{-3}\sin x^2 - 2x^{-1}\cos x^2, 1)$, we define

$$D_1 = \frac{1}{|\nabla\rho_2|}\frac{\partial}{\partial x} - \frac{2x^{-3}\sin x^2 - 2x^{-1}\cos x^2}{|\nabla\rho_2|}\frac{\partial}{\partial y}$$
$$\text{and}\quad D_2 = \frac{2x^{-3}\sin x^2 - 2x^{-1}\cos x^2}{|\nabla\rho_2|}\frac{\partial}{\partial x} + \frac{1}{|\nabla\rho_2|}\frac{\partial}{\partial y}$$

to represent the directions tangent and normal to the boundary. Since $\partial^2\rho_2/\partial x^2 = 4\sin x^2 + O(x^{-2})$, ρ_2 is a uniformly C^2 defining function for Ω, and hence $\partial\Omega$ has positive reach. However, $\partial^3\rho_2/\partial x^3 = 8x\cos x^2 + O(x^{-1})$. If we fix $p = (p_x, p_y) \in \partial\Omega$ with $p_x \neq 0$, then (2.10) tells us that

$$\begin{aligned}(D_1|_p)^3\tilde{\delta}(p) &= |\nabla\rho_2|^{-1}(D_1|_p)^3\rho_2(p) - 3\,|\nabla\rho_2|^{-2}\big((D_1|_p)(D_2|_p)\rho_2(p)\big)\big((D_1|_p)^2\rho_2(p)\big)\\ &= 8x\cos x^2 + O(x^{-1}).\end{aligned}$$

This is not uniformly bounded, so there does not exist a uniformly C^3 defining function for $\partial\Omega$. Hence, positive reach does not suffice to extend C^m defining functions as uniformly C^m defining functions.

Finally, let $h(x,y) = e^{x^2}$. Then $\sup |h|_{C^1_b}/h = \infty$ with respect to either of the previous two examples (since $\partial/\partial x$ is asymptotically the tangential direction in these examples). By Lemma 2.1, the defining function $\rho_1^h(x,y) = (y - x^{-1}\sin x)h(x,y)$ fails to satisfy $\sup \left|\rho_1^h\right|_{C^3_b}/\left|\nabla\rho_1^h\right| < \infty$ even though this defines a domain with a uniformly C^3 defining function. Hence, not all defining functions need satisfy the conditions of Lemma 2.1. Turning to our other example, $\rho_2^h(x,y) = (y - x^{-2}\sin x^2)h(x,y)$ still satisfies $\sup \left|\rho_2^h\right|_{C^2_b}/\left|\nabla\rho_2^h\right| < \infty$ even though $\|\rho_2^h\|_{C^2(\partial\Omega)} = \infty$. Thus, it is helpful to consider the special C^k_b norm in place of the standard extrinsic C^k norm.

4. Applications of uniformly C^m defining functions

In [8], we define weighted Sobolev spaces on the boundaries of unbounded domains. From the standpoint of the present paper, the weight function is irrelevant. However, it seems difficult to obtain elliptic regularity results without a weight, so for the sake of defining a meaningful space of functions we will include the weight. The weight functions that we use satisfy a number of technical hypotheses (similar to those in [6], [4], and [3]) all satisfied by $\varphi_t(x) = t|x|^2$, $t \in \mathbb{R}\setminus\{0\}$. Let $\Omega \subset \mathbb{R}^n$ have a C^m boundary, $m \geq 2$, that admits a uniformly C^m defining function. We define weighted Sobolev spaces both on the boundary and near the boundary.

Suppose $\Omega \subset \mathbb{R}^n$ admits a uniformly C^2 defining function. By Theorem 1.3, $\partial\Omega$ has positive reach, so for $\frac{1}{2}\operatorname{Reach}(\partial\Omega) > \epsilon > 0$ we set

$$\Omega_\epsilon = \left\{x \in \Omega : \delta(x) < \epsilon\right\}.$$

Since $\|\tilde\delta\|_{C^2(\Omega_{2\epsilon})} < \infty$, there exists a radius $\frac{1}{4}\operatorname{Reach}(\partial\Omega) > r > 0$ such that whenever $B(p,r)\cap\Omega_\epsilon \neq \emptyset$, there exist coordinates on $B(p,r)$ such that the level curves of $\tilde\delta$ can be written as graphs. Hence, there exists an orthonormal basis $L_1,\ldots,L_{n-1}$ of the tangent space to the level curves of $\tilde\delta$ on $B(p,r)$. We also let $L_n = \nu$ be the unit outward normal to the level curves of $\tilde\delta$. For $1 \leq j \leq n$, set

$$T_j = L_j - L_j(\varphi_t).$$

We call a first order differential operator T *tangential* if the first order component of T is tangential. Note that we use T_j instead of L_j for technical reasons involving integration by parts in weighted norms, but these are not relevant for the present paper.

Let $\{p_j\}$ be an enumeration of all points in $\mathbb{R}^n$ whose coordinates are integral multiples of $r/\sqrt{n}$. Then $\{\overline{B(p_j,r/2)}\}$ is a locally finite cover of $\mathbb{R}^n$, with a uniform upper bound on the number of sets covering each point. If $\chi \in C_0^\infty(B(0,r))$ satisfies $\chi = 1$ on $B(0,r/2)$ and $1 \geq \chi \geq 0$, we can construct a partition of unity subordinate to $\{B(p_j,r)\}$ by using $\chi_j(x)=\chi(x-p_j)/\left(\sum_k \chi(x-p_k)\right)$. Because there is a uniform upper bound on the number of nonzero terms in the denominator, we have a uniform bound on $\|\chi_j\|_{C^m}$ for any $m\geq 0$.

Let $\{U_j\}$ be a restriction of this cover to include only those sets covering Ω_ϵ, with the corresponding modification to χ_j. For any distribution v on Ω_ϵ, we set $v_j=v\chi_j$, so $v=\sum_{j=1}^\infty v_j$. If Ω admits a uniformly C^m defining function, $m\geq 2$, we define the weighted Sobolev space $W^{k,p}(\Omega_\epsilon,\varphi_t,\nabla\varphi_t)$, $0\leq k\leq m$, as the space of distributions v on Ω_ϵ whose partial derivatives up to order k agree with functions and for which the norm

$$|||v|||^p_{W^{k,p}(\Omega_\epsilon,\varphi_t,\nabla\varphi_t)}=\sum_{j=1}^\infty\sum_{|\alpha|\leq k}\|T^\alpha v_j\|^p_{L^p(\Omega_\epsilon,\varphi_t)}$$

is finite, where $T_j=L_j-L_j(\varphi_t)$ is well defined on U_j and the composition T^α is defined by $T^\alpha=T_{\alpha_1}\cdots T_{\alpha_{|\alpha|}}$.

For the boundary Sobolev space, set

$$W^{k,p}(\partial\Omega,\varphi_t,\nabla\varphi_t)=\big\{f\in L^p(\partial\Omega,\varphi_t):T^\alpha f\in L^p(\partial\Omega,\varphi_t),\ |\alpha|\leq k \\ \text{and } T_{\alpha_j} \text{ is tangential for } 1\leq j\leq|\alpha|\big\}.$$

Choosing uniform neighborhoods with good local coordinates only makes sense on domains with positive reach, and compositions of derivatives would be extremely difficult to control without a uniformly C^m defining function. When $p=2$, we define fractional Sobolev spaces via interpolation and can prove many of the standard Sobolev space results.

We also provide an example from several complex variables. The following theorem is well known in the bounded case (see for example Theorem 3.4.4 in [1]).

Theorem 4.1. *Let $\Omega\subset\mathbb{C}^n$ be a domain with a C^2 defining function r and a constant $C>0$ satisfying*

$$\sum_{j,k=1}^n\frac{\partial^2 r}{\partial z_j\partial\bar z_k}t_j\bar t_k\geq C\,|\nabla r|\sum_{j=1}^n|t_j|^2 \tag{4.1}$$

on $\partial\Omega$ for $t\in T_p(\partial\Omega)$.

1. *If Ω admits a uniformly C^2 defining function, then there exists a defining function which is strictly plurisubharmonic on $\partial\Omega$.*
2. *If Ω admits a uniformly C^3 defining function, then there exists a defining function which is plurisubharmonic on Ω and strictly plurisubharmonic on $\{z\in\Omega:\delta(z)<\varepsilon\}$ for some $\varepsilon>0$.*

Remark 4.2. The assumption (4.1) implies that Ω is strictly pseudoconvex, but the uniform lower bound on the Levi form is not true for all strictly pseudoconvex domains in the unbounded case. For an example that satisfies our condition, consider the tube in $\mathbb{C}^n$ defined by the defining function $r(z)=|z'|^2+(\operatorname{Im} z_n)^2-1$.

Remark 4.3. The second statement is not sharp in the bounded case, where (4.1) alone (without the C^3 assumption) guarantees the existence of a strictly plurisubharmonic defining function. We require C^3 to govern the decay rate of (4.1) off of $\partial\Omega$, and our resulting function is merely plurisubharmonic because we can not use $|z|^2$ to obtain strict plurisubharmonicity in the interior (it is no longer a bounded function).

Proof. Since (4.1) is independent of the choice of defining function, it will be satisfied by the signed distance function. For $\lambda > 0$ to be determined later, define

$$\rho(z) = e^{\lambda\tilde{\delta}(z)} - 1$$

for z in a small neighborhood of $\partial\Omega$. For $v : \Omega_\epsilon \to \mathbb{C}^n$, we may decompose $v = \tau + \nu$, where $\sum_{j=1}^n \frac{\partial\tilde{\delta}}{\partial z_j}\tau_j = 0$ and ν is a scalar multiple of $\left(\frac{\partial\tilde{\delta}}{\partial\bar{z}_1}, \dots, \frac{\partial\tilde{\delta}}{\partial\bar{z}_n}\right)$. Then since $\left|\sum_{j=1}^n \frac{\partial\tilde{\delta}}{\partial z_j}v_j\right| = \frac{1}{2}\left(\sum_{j=1}^n |\nu_j|^2\right)^{1/2}$ we have

$$\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}v_j\bar{v}_k = \lambda e^{\lambda\tilde{\delta}} \sum_{j,k=1}^n \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_k}v_j\bar{v}_k + \lambda^2 e^{\lambda\tilde{\delta}}\frac{1}{4}\sum_{j=1}^n |\nu_j|^2 .$$

Since $\tilde{\delta}$ satisfies (4.1) on $\partial\Omega$, we have

$$\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}v_j\bar{v}_k \geq \lambda C\sum_{j=1}^n |\tau_j|^2 + \lambda \sum_{j,k=1}^n 2\,\mathrm{Re}\left(\frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_k}\tau_j\bar{\nu}_k\right) + \lambda\sum_{j,k=1}^n \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_k}\nu_j\bar{\nu}_k + \lambda^2\frac{1}{4}\sum_{j=1}^n |\nu_j|^2$$

on $\partial\Omega$. Since $\tilde{\delta}$ is uniformly C^2, there exists a constant $C_2 > 0$ such that $\left|\frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_k}\right| \leq C_2$ on $\partial\Omega$. Hence, the Cauchy–Schwarz inequality gives us

$$\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}v_j\bar{v}_k \geq \lambda C|\tau|^2 - 2\lambda C_2|\tau|\,|\nu| - \lambda C_2|\nu|^2 + \lambda^2\frac{1}{4}|\nu|^2 .$$

This is strictly positive provided that $C\left(\frac{1}{4}\lambda - C_2\right) > 4C_2^2$. Hence, we may choose λ sufficiently large so that ρ is strictly plurisubharmonic on $\partial\Omega$.

If we assume that $\tilde{\delta}$ is uniformly C^3, then from (4.1) there exists some uniform neighborhood U of $\partial\Omega$ on which

$$\sum_{j,k=1}^n \frac{\partial^2\tilde{\delta}}{\partial z_j\partial\bar{z}_k}\tau_j\bar{\tau}_k \geq \frac{1}{2}C\sum_{j=1}^n |\tau_j|^2 .$$

We may assume $|\partial^2\tilde{\delta}/\partial z_j\partial\bar{z}_k| \leq C_2$ on U, so that

$$\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}v_j\bar{v}_k \geq \lambda\, e^{\lambda\tilde{\delta}}\,\frac{1}{2}\,C\,|\tau|^2 + 2\,\lambda\, e^{\lambda\tilde{\delta}}\,C_2\,|\tau|\,|\nu| + \lambda\, e^{\lambda\tilde{\delta}}\,C_2\,|\nu|^2 + \lambda^2\, e^{\lambda\tilde{\delta}}\,\frac{1}{4}\,|\nu|^2 .$$

This is positive provided that $\frac{1}{2}C\left(\frac{1}{4}\lambda - C_2\right) \geq 4C_2^2$, so we may again choose λ sufficiently large so that ρ is plurisubharmonic on $\partial\Omega$.

To extend ρ to all of Ω, let $A = \sup_{\Omega\setminus U}\tilde{\delta}$. Since we were able to choose a uniform neighborhood U, $A < 0$. $\hat{\rho} = \max\{\rho, A\}$ will be a Lipschitz plurisubharmonic defining function for Ω, and a smooth convex approximation to max can be used to obtain a smooth plurisubharmonic defining function for Ω. □

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