Rev. Mat. Iberoam. **29** (2013), no. 4, 1421–1436
DOI 10.4171/RMI/763

Ground states for pseudo-relativistic Hartree equations of critical type

Vittorio Coti Zelati and Margherita Nolasco

Abstract. We study the existence of ground state solutions for a class of nonlinear pseudo-relativistic Schrödinger equations with critical two-body interactions. Such equations are characterized by a nonlocal pseudo-differential operator closely related to the square root of the Laplacian. We investigate this problem using variational methods after transforming the problem to an elliptic equation with a nonlinear Neumann boundary conditions.

1. Introduction

The relativistic Hamiltonian for N identical particles of mass m, position x_i and momentum p_i interacting through the two-body potential $\alpha W(|x_i - x_j|)$ is given by

$$\mathcal{H} = \sum_{i=1}^{N} \Big(\sqrt{p_i^2 c^2 + m^2 c^4} - mc^2 \Big) - \alpha \sum_{i \neq j} W\big(|x_i - x_j| \big).$$

where c is the speed of light and $\alpha > 0$ is a coupling constant.

According to the usual quantization rules the dynamics of the corresponding system of N-identical quantum spinless particles (a *Bose gas*) is described by the complex wave function $\Psi_N = \Psi_N(t, x_1, \ldots, x_N)$ governed by the Schrödinger equation

$$i\hbar \partial_t \Psi_N = \mathcal{H}_N \Psi_N$$

where $\hbar$ is the Planck's constant. Here $\mathcal{H}_N \colon \mathcal{D} \subset L^2(\mathbb{R}^3)^{\otimes_s N} \to L^2(\mathbb{R}^3)^{\otimes_s N}$ is the *quantum mechanics* Hamiltonian operator, obtained from the classical Hamiltonian via the usual quantization rule $p \mapsto -i\hbar\nabla$, and defined in a suitable dense

Mathematics Subject Classification (2010): Primary 35Q55; Secondary 35J61.
Keywords: Nonlinear Schrödinger equation, pseudo-relativistic Hartree approximation, solitary waves, ground states.

domain $\mathcal{D}$. In the case of interest here, $\mathcal{H}_N$ is

$$\mathcal{H}_N = \Big(\sum_{j=1}^{N} \sqrt{-\hbar^2 c^2 \Delta_j + m^2 c^4} - mc^2 \Big) - \alpha \sum_{i \neq j}^{N} W\big(|x_i - x_j|\big),$$

where W is the multiplication operator corresponding to the two-body interaction potential, (e.g., $W(|x|) = |x|^{-1}$ for gravitational interactions).

The operator (from now on we will take $\hbar = 1$ and $c = 1$)

$$\sqrt{-\Delta + m^2} \tag{1.1}$$

can be defined for all $f \in H^1(\mathbb{R}^N)$ as the inverse Fourier transform of the L^2 function $\sqrt{|k|^2 + m^2}\, \mathcal{F}[f](k)$ (here $\mathcal{F}[f]$ denotes the Fourier transform of f) and it is also associated to the quadratic form

$$\mathcal{Q}(f, g) = \int_{\mathbb{R}^N} \sqrt{|k|^2 + m^2}\ \mathcal{F}[f]\, \mathcal{F}[g]\, dk$$

which can be extended to the space

$$H^{1/2}(\mathbb{R}^N) = \Big\{ f \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |k|\, |\mathcal{F}[f](k)|^2\, dk < +\infty \Big\}$$

(see, e.g., [10] for more details).

In the mean field limit approximation (i.e., $\alpha N \simeq O(1)$ as $N \to +\infty$) of a quantum relativistic Bose gas, one is lead to study the nonlinear mean field equation −called *the pseudo-relativistic Hartree equation*− given by

$$i\partial_t \psi = \big(\sqrt{-\Delta + m^2} - m\big)\psi - \big(W * |\psi|^2\big)\psi. \tag{1.2}$$

where $*$ denotes convolution. We will consider attractive two-body interaction, and hence W will always be a nonnegative function.

See [11] for the study of this equation when W is the gravitational interaction, and [4] for a rigorous derivation of the mean field equation (1.2) as an $N \to +\infty$ limit of the Schrödinger equation for N quantum particles, and [3] for more recent developments for models involving the pseudo-relativistic operator $\sqrt{-\Delta + m^2}$.

It has recently been proved that for Newton or Yukawa type two-body interactions (i.e., $W(|x|) = |x|^{-1}$ or $|x|^{-1} e^{-|x|}$ in $\mathbb{R}^3$) such an equation is locally well posed in H^s, $s \geq 1/2$, and that the solution is global in time for small initial data in L^2 (see [8]). Blowup has been proved in [6] and [7].

Due to the *focusing* nature of the nonlinearity (attractive two-body interaction) there exist *solitary waves* solutions given by

$$\psi(t, x) = \mathrm{e}^{i\mu t}\, \varphi(x)\,,$$

where φ satisfies the nonlinear eigenvalue equation

$$\sqrt{-\Delta + m^2}\, \varphi - m\varphi - (W * |\varphi|^2)\varphi = -\mu\varphi. \tag{1.3}$$

In [11] the existence of such solutions (in the case $W(x) = |x|^{-1}$) was proved provided that $M < M_c$, M_c being the *Chandrasekhar limit mass.*

More precisely, the authors have shown the existence in $H^{1/2}(\mathbb{R}^3)$ of a radial, real-valued nonnegative minimizer (*ground state*) of

$$\mathcal{E}[\psi] = \frac{1}{2}\int_{\mathbb{R}^3} \bar{\psi}\big(\sqrt{-\Delta + m^2} - m\big)\psi\, dx - \frac{1}{4}\int_{\mathbb{R}^3} \big(|x|^{-1} * |\psi|^2\big)|\psi|^2\, dx. \tag{1.4}$$

with given fixed "mass-charge" $M = \int_{\mathbb{R}^3} |\psi|^2\, dx < M_c$. We call mass-critical the potentials W whose associated functional $\mathcal{E}$ exhibits this kind of phenomenon.

More recently, in [5] it has been proved that the ground state solution is regular ($H^s(\mathbb{R}^3)$, for all $s \geq 1/2$), strictly positive, and exponentially decaying. Moreover the solution is unique, at least for small L^2 norm ([9]).

Let us remark that these last results are heavily based on the specific form (Newton or Yukawa type) of the two-body interactions in the Hartree nonlinearity. Indeed in these cases the estimates of the nonlinearity rely on the following facts:

- for this class of potentials one has that

$$\frac{e^{-\mu|x|}}{4\pi\,|x|} * f = (\mu^2 - \Delta)^{-1} f \qquad \text{for } f \in \mathcal{S}(\mathbb{R}^3),\ \mu \geq 0;$$

- the use of a generalized Leibnitz rule for Riesz and Bessel potentials;
- there holds the estimate

$$\left\| \frac{1}{|x|} * |u|^2 \right\|_{L^\infty} \leq \frac{\pi}{2} \left\| (-\Delta)^{1/4} u \right\|_{L^2}^2.$$

In [2] there has been proved an existence and regularity result for the solutions of (1.3) for a wider class of nonlinearities by exploiting the relation of equation (1.3) with an elliptic equation on $\mathbb{R}^{N+1}_+$ with a nonlinear Neumann boundary condition. Such a relation has been recently used to study several problems involving fractional powers of the Laplacian (see e.g. [1] and references therein) and it is based on an alternative definition of the operator (1.1) that can be described as follows. Given any function $u \in \mathcal{S}(\mathbb{R}^N)$ there is a unique function $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ (here $\mathbb{R}^{N+1}_+ = \{\, (x, y) \in \mathbb{R} \times \mathbb{R}^N \;\big|\; x > 0 \,\}$) such that

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ v(0, y) = u(y) & \text{for } y \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+. \end{cases}$$

Setting

$$Tu(y) = -\frac{\partial v}{\partial x}(0, y),$$

we have that the equation

$$\begin{cases} -\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w(0, y) = Tu(y) = -\frac{\partial v}{\partial x}(0, y) & \text{for } y \in \mathbb{R}^N, \end{cases}$$

has the solution $w(x, y) = -\frac{\partial v}{\partial x}(x, y)$. From this we have that

$$T(Tu)(y) = -\frac{\partial w}{\partial x}(0, y) = \frac{\partial^2 v}{\partial x^2}(0, y) = \big(-\Delta_y v + m^2 v\big)(0, y)$$

and hence $T^2 = (-\Delta_y + m^2)$.

In [2] we studied the equation

$$\sqrt{-\Delta + m^2}\, v = \mu v + \nu\, |v|^{p-2}\, v + \sigma(W * |v|^2)v \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $p \in (2, 2N/(N-1))$, $\mu < m$ is fixed , ν, $\sigma \geq 0$ (but not both equal to 0), $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, $r > N/2$, $W \geq 0$, and $W(x) = W(|x|) \to 0$ as $|x| \to +\infty$.

The results are obtained, following the approach outlined above, by studying the equivalent elliptic problem with nonlinear boundary condition

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\frac{\partial v}{\partial x} = \mu v + \nu\, |v|^{p-2}\, v + \sigma\big(W * |v|^2\big)v & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+, \end{cases} \tag{1.6}$$

and the associated functional on $H^1(\mathbb{R}^{N+1}_+)$.

Let us point out that in [2] the L^2 norm of the solution is not prescribed. In such a case existence of a (positive, radially symmetric) solution can be proved for a class of potentials W and exponents p which is larger than the one we deal with here.

When the L^2 norm is prescribed to be M (the most relevant problem from a physical point of view), as in [11], then the Newtonian potential ($|x|^{-1}$ in $\mathbb{R}^3$) is critical, in the sense that minimization of $\mathcal{E}$ given by (1.4) is possible only when $M < M_c$ (see Theorem 1.1).

The main purpose of this paper is to exploit this approach also for the problem of finding minimizer of the static energy

$$\mathcal{E}[u] = \frac{1}{2}\int_{\mathbb{R}^N} u\big(\sqrt{-\Delta + m^2} - m\big)u\,dx + \frac{\eta}{p}\int_{\mathbb{R}^N} |u|^p\,dx - \frac{\sigma}{4}\int_{\mathbb{R}^N} \big(W * |u|^2\big)|u|^2\,dx \tag{1.7}$$

with prescribed L^2 norm, for a wider class of attractive two-body potential including the critical case.

To be more precise, we consider a class of two-body potentials $W \in L^q_w(\mathbb{R}^N)$, with $q \geq N$. We recall that $L^q_w(\mathbb{R}^N)$, the weak L^q space, is the space of all measurable functions f such that

$$\sup_{\alpha > 0} \alpha \big|\, \{x \,|\, |f(x)| > \alpha\} \big|^{1/q} < +\infty,$$

where $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{R}^N$. Note that $W(x) = |x|^{-1}$ does not belong to any L^q-space but it belongs to $L^N_w(\mathbb{R}^N)$. We say that a potential W is *critical* if $W \in L^N(\mathbb{R}^N)$.

Our main result is the following.

Theorem 1.1. *Let $W \in L^q_w(\mathbb{R}^N)$, where $q \geq N \geq 2$, and $W(y) \geq 0$ for all $y \in \mathbb{R}^N$, and suppose that*

$$W(\lambda^{-1}y) \geq \lambda^\alpha W(y), \quad \textit{for all } \lambda \in (0,1) \textit{ and for some } \alpha > 0. \tag{1.8}$$

We also assume that $W(x) = W(|x|)$ is rotationally symmetric and that $W(r) \to 0$ as $r \to +\infty$.

Take $\eta \geq 0$, $\sigma > 0$ and $p \in \big(2 + 2/q, 2 + 2/(N-1) = 2N/(N-1)\big]$. Then:

- *if $\eta > 0$ or $\eta = 0$ and $q > N$, then for all $M > 0$ there is a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{E}[u]$ such that $\int_{\mathbb{R}^N} u^2 = M$;*
- (mass-critical case) *if $\eta = 0$ and $q = N$, there is a critical value $M_c > 0$ such that for all $0 < M < M_c$ there is a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{E}[u]$ such that $\int_{\mathbb{R}^N} u^2 = M$.*

Moreover there exists $\mu > 0$ such that u is a smooth, exponentially decaying at infinity, solution of

$$\big(\sqrt{-\Delta + m^2} - m\big)u = -\mu u - \eta\, |u|^{p-2}\, u + \sigma\big(W * |u|^2\big)u \quad \textit{in } \mathbb{R}^N,$$

and u is radial if $W = W(r)$ is a decreasing function of $r > 0$.

Remark 1.2. The nonlinear term $|u|^{p-2}\,u$ is a defocusing nonlinearity, the convolution term is a focusing nonlinearity. An open problem is to understand if solitons exist also for other ranges of p, in particular for $2 < p \leq 2 + 2/q$ and $W \in L^q_w$.

Remark 1.3. If $W \in L^q_w$ and (1.8) holds for some $\alpha > 0$, then necessarily $\alpha \in (0, N/q]$. If $W(x) = |x|^{-\alpha}$, then $W \in L^q_w$ if and only if $\alpha = N/q$.

Remark 1.4. μ is a Lagrange multiplier.

2. Preliminaries

Let $(x,y) \in \mathbb{R}\times\mathbb{R}^N$. We have already introduced $\mathbb{R}^{N+1}_+ = \{\,(x,y) \in \mathbb{R}^{N+1} \bigm| x>0\,\}$. We will always denote the norm of $u \in L^p(\mathbb{R}^{N+1}_+)$ by $\|u\|_p$, the norm of $u \in H^1(\mathbb{R}^{N+1}_+)$ by $\|u\|$, and the norm of $v \in L^p(\mathbb{R}^N)$ by $|v|_p$.

We recall that, for all $v \in H^1(\mathbb{R}^{N+1}) \cap C^\infty_0(\mathbb{R}^{N+1})$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |v(0,y)|^p\, dy &= \int_{\mathbb{R}^N} dy \int_{+\infty}^0 \frac{\partial}{\partial x}\, |v(x,y)|^p\, dx \\
&\leq p \iint_{\mathbb{R}^{N+1}_+} |v(x,y)|^{p-1} \left| \frac{\partial v}{\partial x}(x,y) \right| dx\, dy \\
&\leq p \Big(\iint_{\mathbb{R}^{N+1}_+} |v(x,y)|^{2(p-1)}\, dx\, dy \Big)^{1/2} \Big(\iint_{\mathbb{R}^{N+1}_+} \left| \frac{\partial v}{\partial x}(x,y) \right|^2 dx\, dy \Big)^{1/2}.
\end{aligned}$$

That is,

$$|v(0,\cdot)|_p^p \leq p\, \|v\|_{2(p-1)}^{p-1} \left\| \frac{\partial v}{\partial x} \right\|_2, \tag{2.1}$$

which, by Sobolev embedding, is finite for all $2 \leq 2(p-1) \leq 2(N+1)/((N+1)-2)$, that is $2 \leq p \leq 2^\sharp$, where we have set $2^\sharp = 2N/(N-1)$. By density of $H^1(\mathbb{R}^{N+1}) \cap C_0^\infty(\mathbb{R}^{N+1})$ in $H^1(\mathbb{R}_+^{N+1})$ such an estimate allows us to define the trace $\gamma(v)$ of v for all $v \in H^1(\mathbb{R}_+^{N+1})$. The inequality

$$|\gamma(v)|_p^p \leq p\, \|v\|_{2(p-1)}^{p-1} \left\| \frac{\partial v}{\partial x} \right\|_2, \tag{2.2}$$

holds then for all $v \in H^1(\mathbb{R}_+^{N+1})$.

It is known that the traces of functions in $H^1(\mathbb{R}_+^{N+1})$ belong to $H^{1/2}(\mathbb{R}^N)$ and that every function in $H^{1/2}(\mathbb{R}^N)$ is the trace of a function in $H^1(\mathbb{R}_+^{N+1})$. Then (2.2) is in fact equivalent to the well-known fact that $\gamma(v) \in H^{1/2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ provided $q \in [2, 2^\sharp]$. Here we also recall that

$$\|w\|_{H^{1/2}}^2 = \inf \left\{ \|u\|^2 \,\middle|\, u \in H^1(\mathbb{R}_+^{N+1}),\ \gamma(u) = w \right\} = \int_{\mathbb{R}^N} (1 + |\xi|)\, |\mathcal{F}w(\xi)|^2 \, d\xi.$$

Let us also introduce the norm of the weak L^q-space as follows:

$$\|f\|_{q,w} = \sup_A |A|^{-1/r} \int_A |f(x)|\, dx$$

where $1/q + 1/r = 1$ and A denotes any measurable set of finite measure $|A|$ (see, e.g., [10] for more details). Using this norm we can state the *weak Young inequality.* If $g \in L_w^q(\mathbb{R}^N)$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ where $1 < q, p, r < +\infty$ and $1/q + 1/p + 1/r = 2$, then

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(y)\, g(y-z)\, h(y)\, dy\, dz \leq C_{p,q,r}\, \|g\|_{q,w}\, |f|_p\, |h|_r. \tag{2.3}$$

We consider the class of two-body interactions $W \in L_w^q(\mathbb{R}^N)$ for $q \geq N$. By the weak Young inequality and the Hölder inequality we have for $r = 4q/(2q-1)$ ($\in (2, 2^\sharp)$ since $q \geq N$) and for all $p \in (4q/(2q-1), 2^\sharp]$,

$$\int_{\mathbb{R}^N} (W * |u|^2)\, |w|^2 \, dy \leq C \|W\|_{q,w}\, |w|_r^4 \leq C \|W\|_{q,w}\, |w|_2^{4 - \frac{2p}{q(p-2)}}\, |w|_p^{\frac{2p}{q(p-2)}}. \tag{2.4}$$

For $p = 2^\sharp$ we get

$$\int_{\mathbb{R}^N} \left(W * |w|^2\right) |w|^2 \, dy \leq C\, \|W\|_{q,w}\, |w|_2^{4-2N/q}\, |w|_{2^\sharp}^{2N/q}. \tag{2.5}$$

In the (critical) case $q = N$ this gives

$$\int_{\mathbb{R}^N} \left(W * |w|^2\right) |w|^2 \, dy \leq C\, \|W\|_{N,w}\, |w|_2^2\, |w|_{2^\sharp}^2. \tag{2.6}$$

We point out that one cannot deduce (2.6) from the weak Young's inequality (2.3) directly, and that it is not true, in general, that the L^∞ norm of $W * |u|^2$ can be bounded by the $L^{2^\sharp}$ norm of u if $W \in L_w^N$.

For all $v \in H^1(\mathbb{R}^{N+1}_+)$, we consider the functional given by

$$\mathcal{I}(v) = \frac{1}{2}\Big(\iint_{\mathbb{R}^{N+1}_+} \big(\,|\nabla v|^2 + m^2\,|v|^2\,\big)\,dx\,dy - \int_{\mathbb{R}^N} m\,|\gamma(v)|^2\;dy\Big)$$
$$+\frac{\eta}{p}\int_{\mathbb{R}^N} |\gamma(v)|^p\;dy - \frac{\sigma}{4}\int_{\mathbb{R}^N} \big(W * |\gamma(v)|^2\,\big)\,|\gamma(v)|^2\;dy.$$

In view of (2.2) and (2.4), all the terms in the functional $\mathcal{I}$ are well defined if $p \in (2, 2^\sharp]$ and $W \in L^q_w(\mathbb{R}^N)$ with $q \geq N$.

Note that from (2.1), with $p = 2$, it follows that

$$m\int_{\mathbb{R}^N} |\gamma(v)|^2\,dy \leq 2(m\|v\|_2)\|\nabla v\|_2 \leq \iint_{\mathbb{R}^{N+1}_+} \big(\,|\nabla v|^2 + m^2\,|v|^2\,\big)\,dx\,dy, \tag{2.7}$$

showing that the quadratic part of the functional $\mathcal{I}$ is nonnegative.

Moreover the following property can be checked easily:

Lemma 2.1. *For $u \in H^1(\mathbb{R}^{N+1}_+)$, let $w = \gamma(u) \in H^{1/2}(\mathbb{R}^N)$, $\hat{w} = \mathcal{F}(w)$ and*

$$v(x,y) = \mathcal{F}^{-1}(e^{-x\sqrt{m^2+|\cdot|^2}}\hat{w}) = \int_{\mathbb{R}^N} e^{-x\sqrt{m^2+|\xi|^2}}\hat{w}(\xi)e^{i\xi y}\,d\xi.$$

Then $v \in H^1(\mathbb{R}^{N+1}_+)$, $\|v\| = \|w\|_{H^{1/2}}$, $\mathcal{I}(v) \leq \mathcal{I}(u)$ and $\mathcal{I}(v) = \mathcal{E}[w]$.

3. Minimization problem

We consider the minimization problem

$$I(M) = \inf\big\{\mathcal{I}(v)\;:\;v \in \mathcal{M}_M\big\}, \tag{3.1}$$

where the manifold $\mathcal{M}_M$ is given by

$$\mathcal{M}_M = \Big\{v \in H^1(\mathbb{R}^{N+1}_+)\;:\;\int_{\mathbb{R}^N} |\gamma(v)|^2 = M\Big\}$$

Remark 3.1. The term $m\int_{\mathbb{R}^N} |\gamma(v)|^2$ in the functional $\mathcal{I}(v)$ is constant for all $v \in \mathcal{M}_M$. The presence of such a term will allow us to show that the infimum of the functional $\mathcal{I}$ on $\mathcal{M}_M$ is negative.

Concerning the existence of a minimizer for problem (3.1) we start by proving, in the following lemmas, boundedness from below on $\mathcal{M}_M$ of the functional $\mathcal{I}$, and some properties of the infimum $I(M)$.

Lemma 3.2. *The functional $\mathcal{I}$ is bounded from below and coercive on $\mathcal{M}_M \subset H^1(\mathbb{R}^{N+1}_+)$ for all $M > 0$ if $\eta > 0$ or $q > N$ and for all M small enough if $\eta = 0$ and $q = N$.*

Proof. First we examine first the convolution term. If $\eta > 0$, from (2.4) and $|\gamma(u)|_2^2 = M$ we have

$$0 \le \int_{\mathbb{R}^N} \left(W * |\gamma(u)|^2\right) |\gamma(u)|^2 \le C \, \|W\|_{q,w} \, |\gamma(u)|_2^{4-\frac{2p}{q(p-2)}} \, |\gamma(u)|_p^{\frac{2p}{q(p-2)}} = C \, \|W\|_{q,w} \, M^{2-\frac{p}{q(p-2)}} \, |\gamma(u)|_p^{\frac{2p}{q(p-2)}} . \tag{3.2}$$

Since by assumption $\frac{2p}{q(p-2)} < p$, this is enough to prove coercivity if $\eta > 0$. Indeed in such a case we have that

$$\mathcal{I}(u) \ge \frac{1}{2}\|u\|^2 - \frac{1}{2}mM + C_1 \, |\gamma(u)|_p^p - C_2 \, |\gamma(u)|_p^{\frac{2p}{q(p-2)}} \ge \frac{1}{2}\|u\|^2 - C_3.$$

In the case $\eta = 0$ we deduce from (2.6) and $|\gamma(u)|_{2^\sharp} \le C \, \|u\|$ that

$$\mathcal{I}(u) \ge \|u\|^2 - mM - C \, \|W\|_{q,w} \, M^{2-N/q} \, \|u\|^{2N/q}.$$

It is then clear that the functional is bounded from below and coercive whenever $q > N$ and, when $q = N$, if $\|W\|_{N,w} M$ is small enough. □

Lemma 3.3. *$I(M) < 0$ for all $M > 0$.*

Proof. Take any function $u \in C_0^\infty(\mathbb{R}^N)$ such that $|u|_2^2 = M$, and let $w(x, y) = e^{-mx} u(y)$. Then,

$$\begin{aligned}
I(M) &= \inf_{v \in \mathcal{M}_M} \mathcal{I}(v) \le \mathcal{I}(w) \\
&= \frac{1}{2} \iint_{\mathbb{R}^{N+1}_+} \left(|\partial_x w|^2 + |\nabla_y w|^2 + m^2 \, |w|^2\right) dx \, dy - \frac{m}{2} \int_{\mathbb{R}^N} |u|^2 \, dy + G(u) \\
&= \frac{m}{4} \int_{\mathbb{R}^N} |u|^2 dy + \frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 dy + \frac{m}{4} \int_{\mathbb{R}^N} |u|^2 dy - \frac{m}{2} \int_{\mathbb{R}^N} |u|^2 dy + G(u) \\
&= \frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 \, dy + G(u),
\end{aligned}$$

where

$$G(u) = \frac{\eta}{p} \int_{\mathbb{R}^N} |u|^p \, dy - \frac{\sigma}{4} \int_{\mathbb{R}^N} \left(W * |u|^2\right) |u|^2 \, dy$$

For $\lambda > 0$ take $u_\lambda(y) = \lambda^{N/2} u(\lambda y)$ and $w_\lambda(x, y) = e^{-mx} u_\lambda(y) \in \mathcal{M}_M$. We find that

$$\begin{aligned}
I(M) &\le \inf_{\lambda > 0} \mathcal{I}(w_\lambda) \\
&\le \inf_{\lambda \in (0,1)} \Big[\frac{1}{4m} \int_{\mathbb{R}^N} |\nabla_y u_\lambda|^2 + \frac{\eta}{p} \int_{\mathbb{R}^N} |u_\lambda|^p - \frac{\sigma}{4} \int_{\mathbb{R}^N} \left(W * |u_\lambda|^2\right) |u_\lambda|^2 \Big] \\
&\le \inf_{\lambda \in (0,1)} \Big[\frac{\lambda^2}{4m} \int_{\mathbb{R}^N} |\nabla_y u|^2 + \frac{\eta \lambda^{N(\frac{p}{2}-1)}}{p} \int_{\mathbb{R}^N} |u|^p - \frac{\sigma \lambda^\alpha}{4} \int_{\mathbb{R}^N} \left(W * |u|^2\right) |u|^2 \Big],
\end{aligned}$$

and since $\alpha < N(p/2 - 1) < 2$, the infimum is negative. □

Lemma 3.4. *For all $M > 0$ and $\beta \in (0, M)$ we have that $I(M) < I(M-\beta)+I(\beta)$. Moreover, $I(M)/M$ is a concave function of M and hence $I(M)$ is a continuous function of M.*

Proof. The subadditivity is a consequence of the fact that, for all $\theta > 1$,

$$I(\theta M) < \theta I(M), \quad \text{which implies} \quad \frac{1}{\theta} I(M) < I(M/\theta). \tag{3.3}$$

Indeed, taking $\theta_1 = M/\beta$ and $\theta_2 = M/(M-\beta)$, we have that

$$I(M) = \frac{\beta}{M} I(M) + \frac{M-\beta}{M} I(M) < I(\beta) + I(M-\beta).$$

To prove that (3.3) holds, we remark that for all $v \in \mathcal{M}_M$ and $\lambda = \theta^{1/2} > 1$ we have, thanks to (2.7),

$$\mathcal{I}(\lambda v) = \frac{\lambda^2}{2}\Big[\iint_{\mathbb{R}^{N+1}_+} \big(|\nabla v|^2 + m^2 |v|^2\big)\, dx\, dy - m \int_{\mathbb{R}^N} |\gamma(v)|^2\, dy\Big] + \frac{\eta\lambda^p}{p}\int_{\mathbb{R}^N} |\gamma(v)|^p\, dy - \frac{\sigma\lambda^4}{4}\int_{\mathbb{R}^N}\big(W * |\gamma(v)|^2\big)|\gamma(v)|^2\, dy \le \lambda^4 \mathcal{I}(v).$$

Hence, since $I(M) < 0$,

$$I(\theta M) = \inf_{|\gamma(v)|_2^2 = \theta M} \mathcal{I}(v) = \inf_{|\gamma(v)|_2 = M} \mathcal{I}(\theta^{1/2} v) \le \theta^2 \inf_{|\gamma(v)|_2 = M} \mathcal{I}(v) = \theta^2 I(M) < \theta I(M) < I(M).$$

To prove the concavity of $I(M)/M$, we remark that

$$\frac{I(M)}{M} = \frac{1}{M}\inf_{u\in\mathcal{M}_M} \mathcal{I}(u) = \inf_{u\in\mathcal{M}_1} \frac{\mathcal{I}(\sqrt{M}u)}{M}.$$

We now show that, for all $u \in \mathcal{M}_1$, $M \mapsto \mathcal{I}(\sqrt{M}u)/M$ is a concave function of M. This will immediately prove that $I(M)/M$ is a concave function. Since

$$\frac{\mathcal{I}(\sqrt{M}v)}{M} = \frac{1}{2}\Big(\iint_{\mathbb{R}^{N+1}_+} \big(|\nabla v|^2 + m^2 v^2\big)\, dx\, dy - \int_{\mathbb{R}^N} m\,|\gamma(v)|^2\, dy\Big) + \frac{\eta M^{p/2-1}}{p}\int_{\mathbb{R}^N} |\gamma(v)|^p\, dy - \frac{\sigma M}{4}\int_{\mathbb{R}^N}\big(W * |\gamma(v)|^2\big)|\gamma(v)|^2\, dy,$$

it is immediate to check that the second derivative with respect to the variable M is negative for all $M > 0$ when $p/2 < 2$ and that the function is linear when $p = 4$ (namely the critical exponent for $N = 2$). □

We are now ready to prove the existence of a minimizer for the functional $\mathcal{I}$ on $\mathcal{M}_M$.

Proposition 3.5. *For every $M > 0$ there is a function $u \in H^1(\mathbb{R}^{N+1}_+)$ such that*

$$\begin{cases} \mathcal{I}(u) = I(M), \\ \int_{\mathbb{R}^N} |\gamma(u)|^2 \, dy = M, \end{cases}$$

i.e., a minimizer for $\mathcal{I}$ in $\mathcal{M}_M$.

Proof. Let $\{u_n\} \subset \mathcal{M}_M$ be a minimizing sequence. It follows from Lemma 2.1 that

$$v_n(x,y) = \mathcal{F}^{-1}\big(e^{-x\sqrt{m^2+|\cdot|^2}}\mathcal{F}(\gamma(u_n))\big)$$

is also minimizing. From Lemma 3.2 we deduce that v_n is bounded in $H^1(\mathbb{R}^{N+1}_+)$ and that $w_n \equiv \gamma(v_n) = \gamma(u_n)$ is bounded in $H^{1/2}(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} |w_n|^2 \, dy = M$.

We will now use the concentration-compactness method of P. L. Lions [12]. Namely, one of the following cases must occur:

(vanishing) for all $R > 0$,

$$\lim_{n\to+\infty} \sup_{z\in\mathbb{R}^N} \int_{z+B_R} |w_n|^2 \, dy = 0;$$

(dichotomy) for a subsequence $\{n_k\}$,

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \sup_{z\in\mathbb{R}^N} \int_{z+B_R} |w_{n_k}|^2 \, dy = \alpha \in (0, M);$$

(compactness) for all $\epsilon > 0$ there is $R > 0$, a sequence $\{y_k\}$ and a subsequence $\{w_{n_k}\}$ such that

$$\int_{y_k+B_R} |w_{n_k}|^2 \, dy \geq M - \epsilon.$$

Following the usual strategy we will show that the vanishing and dichotomy cases cannot occur.

Lemma 3.6. *If vanishing occurs, then*

$$\int_{\mathbb{R}^N} \big(W * |w_n|^2\big) |w_n|^2 \, dy \to 0.$$

Proof. Take any $\delta > 0$ and $R > 0$. Define $W_\delta = W\mathbb{I}_{\{W\geq\delta\}}$ and

$$W_\delta^R(|y|) = \big(W_\delta(|y|) - R\big)^+\mathbb{I}_{\{|y|<R\}} + W_\delta(|y|)\mathbb{I}_{\{|y|\geq R\}},$$

where $\mathbb{I}_A$ is the characteristic function of the set A . Then it easy to check that $W \in L^q_w(\mathbb{R}^N)$ implies that $W_\delta \in L^s(\mathbb{R}^N)$ for any $s \in [1, q)$ and moreover that $|W_\delta^R|_s \to 0$ as $R \to +\infty$ for any $\delta > 0$. Also define $\Gamma_\delta^R = W_\delta - W_\delta^R$. It is clear that

$$0 \leq (W - W_\delta)(|y|) \leq \delta, \quad 0 \leq \Gamma_d^R(|y|) \leq R \quad \forall y \in \mathbb{R}^N$$

Then, for any given $\delta > 0$ and $R > 0$ and for some $s \geq N/2$ (which implies that $2 < 4s/(2s-1) \leq 2N/(N-1)$), we get from the Young inequality (also taking into account that by the Sobolev embedding theorem the sequence $\{w_n\}$ is bounded in L^p for $p \in [2, 2N/(N-1)]$),

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(W * |w_n|^2\right) |w_n|^2 \\
&\leq \int_{\mathbb{R}^N} \left((W - W_\delta) * |w_n|^2\right) |w_n|^2 + \int_{\mathbb{R}^N} \left(W_\delta^R * |w_n|^2\right) |w_n|^2 + \int_{\mathbb{R}^N} \left(\Gamma_\delta^R * |w_n|^2\right) |w_n|^2 \\
&\leq \delta\, |w_n|_2^4 + |W_\delta^R|_s |w_n|_{4s/(2s-1)}^4 + R \iint_{\mathbb{R}^N \times \mathbb{R}^N} |w_n(y)|^2\, |w_n(z)|^2\, \mathbb{I}_{|z-y| \leq R}\, dy\, dz \\
&\leq \delta M^2 + C|W_\delta^R|_s + RM \sup_{z \in \mathbb{R}^N} \int_{z + B_R} |w_n|^2\, dy.
\end{aligned}$$

Now, first letting $n \to +\infty$, then letting $R \to +\infty$, and finally letting $\delta \to 0^+$, we conclude the proof of the lemma. □

Lemma 3.7. *If dichotomy occurs, then for any $\alpha \in (0, M)$ we have*

$$I(M) \geq I(\alpha) + I(M - \alpha).$$

Proof. If dichotomy occurs, then there is a sequence $\{n_k\} \subset \mathbb{N}$ such that, for any $\epsilon > 0$, there exists $R > 0$ and a sequence $\{z_k\} \subset \mathbb{R}^N$ such that

$$\lim_{k \to +\infty} \int_{z_k + B_R} |w_{n_k}|^2\, dy \in (\alpha - \epsilon, \alpha + \epsilon).$$

Define $\tilde{w}_k = w_{n_k}(\cdot + z_k)$ and

$$\tilde{u}_k(x, y) = \mathcal{F}^{-1}\left(e^{-x\sqrt{m^2 + |\cdot|^2}} \mathcal{F}(\tilde{w}_k)\right),$$

so that $\{\tilde{u}_k\}$ is a minimizing sequence for $\mathcal{I}$ on $\mathcal{M}_M$ such that

$$\lim_{k \to +\infty} \int_{B_R} |\gamma(\tilde{u}_k)|^2\, dy \in (\alpha - \epsilon, \alpha + \epsilon).$$

Since $\{\tilde{u}_k\}$ is a bounded sequence in $H^1(\mathbb{R}_+^{N+1})$, $\tilde{u}_k \to u$ weakly in $H^1(\mathbb{R}_+^{N+1})$ and $\tilde{w}_k = \gamma(\tilde{u}_k) \to w = \gamma(u)$ weakly in $H^{1/2}$ and strongly in $L^p_{loc}(\mathbb{R}^N)$ for $p \in \left[2, 2N/(N-1)\right)$. Hence, for all $\epsilon > 0$ there is $R > 0$ such that

$$\int_{B_R} |\gamma(u)|^2\, dy = \lim_{k \to +\infty} \int_{B_R} |\gamma(\tilde{u}_k)|^2\, dy \in (\alpha - \epsilon, \alpha + \epsilon)$$

and

$$\int_{\mathbb{R}^N} |\gamma(u)|^2\, dy = \lim_{R \to +\infty} \int_{B_R} |\gamma(u)|^2\, dy = \alpha.$$

We set $v_k = \tilde{u}_k - u$ and $\beta_k = \int_{\mathbb{R}^N} |\gamma(v_k)|^2\, dy$. By weak convergence of the sequence $\{\gamma(\tilde{u}_k)\}$in L^2 we get $\lim_{k \to +\infty} \beta_k = M - \alpha$.

Now we claim that

$$I(M) = \lim_{k \to +\infty} \mathcal{I}(\tilde{u}_k) = \mathcal{I}(u) + \lim_{k \to +\infty} \mathcal{I}(v_k) \geq I(\alpha) + \lim_{k \to +\infty} I(\beta_k).$$

Then, by the continuity of the function I, as stated in Lemma 3.4, the lemma follows.

Now we prove the claim. We will show that

$$\lim_{k\to+\infty}(\mathcal{I}(\tilde{u}_k)-\mathcal{I}(v_k))\to\mathcal{I}(u)$$

Indeed, by weak convergence in $H^1(\mathbb{R}^{N+1}_+)$, we immediately get

$$\lim_{k\to+\infty}\Big(\iint_{\mathbb{R}^{N+1}_+}|\nabla\tilde{u}_k|^2-\iint_{\mathbb{R}^{N+1}_+}|\nabla v_k|^2\Big)=\iint_{\mathbb{R}^{N+1}_+}|\nabla u|^2$$
$$\lim_{k\to+\infty}\Big(\iint_{\mathbb{R}^{N+1}_+}|\tilde{u}_k|^2-\iint_{\mathbb{R}^{N+1}_+}|v_k|^2\Big)=\iint_{\mathbb{R}^{N+1}_+}|u|^2$$

and by the Brezis–Lieb lemma

$$\lim_{k\to+\infty}\Big(\int_{\mathbb{R}^N}|\gamma(\tilde{u}_k)|^p-\int_{\mathbb{R}^N}|\gamma(v_k)|^p\Big)=\int_{\mathbb{R}^N}|\gamma(u)|^p$$

for $2\le p\le 2N/(N-1)$. Hence we have to investigate the last nonlinear term. We will show in Appendix A that

$$\lim_{k\to+\infty}\Big(\int_{\mathbb{R}^N}(W*|\tilde{w}_k|^2)\,|\tilde{w}_k|^2-\int_{\mathbb{R}^N}(W*|\gamma(v_k)|^2)\,|\gamma(v_k)|^2\Big)=\int_{\mathbb{R}^N}(W*|w|^2)\,|w|^2,$$

from which the claim follows. □

Finally, since we have ruled out both vanishing and dichotomy, then we may conclude that indeed *compactness* occurs, namely that for all $\epsilon>0$ there is $R>0$, a sequence $\{y_k\}$ and a subsequence $\{w_{n_k}\}$ such that

$$\int_{y_k+B_R}|w_{n_k}|^2\,dy\ge M-\epsilon.$$

Define as before $\tilde{w}_k=w_{n_k}(\cdot+y_k)$ and $\tilde{u}_k(x,y)=\mathcal{F}^{-1}\big(e^{-x\sqrt{m^2+|\cdot|^2}}\mathcal{F}(\tilde{w}_k)\big)$. Then $\tilde{u}_k$ is a minimizing sequence for $\mathcal{I}$ on $\mathcal{M}_M$ such that

$$\int_{B_R}|\gamma(\tilde{u}_k)|^2\ge M-\epsilon.$$

Since $\{\tilde{u}_k\}$ is a bounded sequence in $H^1(\mathbb{R}^{N+1}_+)$, $\tilde{u}_k\to u$ weakly in $H^1(\mathbb{R}^{N+1}_+)$ and $\tilde{w}_k=\gamma(\tilde{u}_k)\to w=\gamma(u)$ weakly in $H^{1/2}$ and strongly in $L^p_{loc}(\mathbb{R}^N)$ for $p\in\big[2,2N/(N-1)\big)$. As in the proof of Lemma 3.7 we deduce that $\int_{\mathbb{R}^N}|\gamma(u)|^2=M$.

Moreover we claim that, as $k\to+\infty$,

$$\int_{\mathbb{R}^N}\big(W*|\tilde{w}_k|^2\big)\,|\tilde{w}_k|^2\to\int_{\mathbb{R}^N}(W*w^2)w^2.$$

Indeed, by the weak Young inequality and the Hölder inequality we have

$$\Big|\int_{\mathbb{R}^N}(W*\tilde{w}_k^2)\tilde{w}_k^2-\int_{\mathbb{R}^N}(W*w^2)w^2\Big|\le\int_{\mathbb{R}^N}(W*(\tilde{w}_k^2+w^2))|\tilde{w}_k^2-w^2|$$
$$\le C\|W\|_{q,w}|\tilde{w}_k^2+w^2|_s|\tilde{w}_k^2-w^2|_s\le C|\tilde{w}_k-w|_{2s}\to 0$$

since $2<2s=4q/(2q-1)<2N/(N-1)$.

Hence, finally, by the weakly lower semicontinuity of the H^1 and L^p norms (the positive terms of the functional $\mathcal{I}$), we conclude that

$$\mathcal{I}(u) \le \liminf_{k\to+\infty} \mathcal{I}(\tilde{u}_k) = I(M),$$

which implies the u is a minimizer for $\mathcal{I}$ in $\mathcal{M}_M$. □

Now we collect all the results obtained to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. By Proposition 3.5 there exists a function $u \in H^1(\mathbb{R}^{N+1}_+)$ which minimizes $\mathcal{I}$ in $\mathcal{M}_M$. Therefore u can always be assumed nonnegative and, by Lemma 2.1, to have the form

$$u(x,y) = \mathcal{F}^{-1}\big(e^{-x\sqrt{m^2+|\cdot|^2}}\mathcal{F}(w)\big),$$

where $w = \gamma(u) \in H^{1/2}(\mathbb{R}^N)$.

If W is a nonincreasing radial function, then w can be assumed to be a radial nonincreasing function. Indeed let w^* be the spherically symmetric decreasing rearrangement of w and define

$$u^*(x,y) = \mathcal{F}^{-1}\big(e^{-x\sqrt{m^2+|\cdot|^2}}\mathcal{F}(w^*)\big).$$

Then $\mathcal{I}(u^*) = \mathcal{E}[w^*]$ (this also follows from Lemma 2.1). We can then use the properties of the spherically symmetric decreasing rearrangement, namely

(i) w^* is a nonnegative, radial function;

(ii) $w \in L^p(\mathbb{R}^N)$ implies $w^* \in L^p(\mathbb{R}^N)$ and $|w^*|_p = |w|_p$;

(iii) *symmetric decreasing rearrangement decreases kinetic energy* (Lemma 7.17 in [10]), that is,

$$\int_{\mathbb{R}^N} w^*\big(\sqrt{-\Delta+m^2}-m\big)w^*\,dy \le \int_{\mathbb{R}^N} w\big(\sqrt{-\Delta+m^2}-m\big)w\,dy;$$

(iv) *Riesz's rearrangement inequality* (see Theorem 3.7 in [10])),

$$\int_{\mathbb{R}^N} \big(W * |w^*|^2\big)|w^*|^2\,dy \ge \int_{\mathbb{R}^N} \big(W * |w|^2\big)|w|^2\,dy$$

if $W(y) = W^*(|y|)$ (in particular if W is radial and nonincreasing);

to deduce that

$$\mathcal{I}(u^*) = \mathcal{E}[w^*] \le \mathcal{E}[w] = \mathcal{I}(u) = I(M).$$

Moreover, by the theory of Lagrange multipliers, any minimizer $u \in H^1(\mathbb{R}^{N+1}_+)$ of the functional $\mathcal{I}$ on $\mathcal{M}_M$ is such that

$$\iint_{R^{N+1}_+} \big(\nabla u \nabla w + m^2 uw\big)\,dx\,dy - \int_{\mathbb{R}^N} m\gamma(u)\gamma(w)\,dy + \mu\int_{\mathbb{R}^N} \gamma(u)\gamma(w)\,dy$$
$$+\,\eta\int_{\mathbb{R}^N} |\gamma(u)|^{p-2}\gamma(u)\gamma(w)\,dy - \sigma\int_{\mathbb{R}^N} \big(W * |\gamma(u)|^2\big)\gamma(u)\gamma(w)\,dy = 0 \tag{3.4}$$

for all $w \in H^1(\mathbb{R}^{N+1}_+)$, i.e., u is a weak solution of the nonlinear Neumann boundary condition problem

$$\begin{cases} -\Delta u + m^2 u = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\frac{\partial u}{\partial x} + \mu u = mu - \eta\,|u|^{p-2}\,u + \sigma(W * |u|^2)u & \text{on } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+, \end{cases} \tag{3.5}$$

for some Lagrange multiplier $\mu \in \mathbb{R}$. To prove that $\mu > 0$ we take $w = u$ in (3.4) to get

$$\begin{aligned} 0 &= \iint_{\mathbb{R}^{N+1}_+} \big(\,|\nabla u|^2 + m^2\,|u|^2\,\big)\,dx\,dy - \int_{\mathbb{R}^N} m\,|\gamma(u)|^2\;dy + \mu \int_{\mathbb{R}^N} |\gamma(u)|^2\;dy \\ &\quad + \eta \int_{\mathbb{R}^N} |\gamma(u)|^p\;dy - \sigma \int_{\mathbb{R}^N} \big(W * |\gamma(u)|^2\,\big)\,|\gamma(u)|^2\;dy \\ &= 2\mathcal{I}(u) + \mu \int_{\mathbb{R}^N} |\gamma(u)|^2\;dy + \eta\Big(1 - \frac{2}{p}\Big)\int_{\mathbb{R}^N} |\gamma(u)|^p\;dy \\ &\quad - \frac{\sigma}{2}\int_{\mathbb{R}^N} \big(W * |\gamma(u)|^2\,\big)\,|\gamma(u)|^2\;dy. \end{aligned}$$

Since $\mathcal{I}(u) < 0$, we have in particular that

$$\frac{\eta}{p}\int_{\mathbb{R}^N} |\gamma(u)|^p\;dy < \frac{\sigma}{4}\int_{\mathbb{R}^N} \big(W * |\gamma(u)|^2\,\big)\,|\gamma(u)|^2\;dy$$

and hence, since $p \le 2N/(N-1) \le 4$, for $N \ge 2$, we get

$$\begin{aligned} \mu \int_{\mathbb{R}^N} |\gamma(u)|^2\,dy &= -2\mathcal{I}(u) - \eta\Big(1 - \frac{2}{p}\Big)\int_{\mathbb{R}^N} |\gamma(u)|^p + \frac{\sigma}{2}\int_{\mathbb{R}^N} \big(W * |\gamma(u)|^2\,\big)\,|\gamma(u)|^2\,dy \\ &> \eta\Big(\frac{4}{p} - 1\Big)\int_{\mathbb{R}^N} |\gamma(u)|^p\;dy \ge 0. \end{aligned}$$

Finally the regularity, the strictly positivity and the exponential decay at infinity of the weak nonnegative solutions of (3.5) follow straightforwardly from Theorems 3.14 and 5.1 in [2]. □

4. Appendix A

We prove that

$$\begin{aligned} &\int_{\mathbb{R}^N} \big|\big(W * w\gamma(v_k)\big)w\gamma(v_k)\big| + \int_{\mathbb{R}^N} \big|\big(W * \gamma(v_k)^2\big)w^2\big| + \int_{\mathbb{R}^N} \big|\big(W * w\gamma(v_k)\big)w^2\big| \\ &\qquad + \int_{\mathbb{R}^N} \big|\big(W * \gamma(v_k)^2\big)w\gamma(v_k)\big| \to 0 \;\text{ as } k \to +\infty, \end{aligned}$$

as claimed in the proof of Lemma 3.7. Indeed we have the following result.

Lemma 4.1. *For any $w \in H^{1/2}(\mathbb{R}^N)$ and for sequences $\{f_n, g_n, h_n\}$ bounded in $H^{1/2}(\mathbb{R}^N)$ and such that $f_n \to 0$ in L^2_{loc} we have*

$$\int_{\mathbb{R}^N} \big(W * |f_n g_n|\big)|w h_n| \to 0 \quad \textit{as } n \to +\infty.$$

Proof. It is convenient to introduce, for any given $\delta > 0$ and $R > 0$, $W_\delta = W \mathbb{I}_{W \geq \delta}$ and

$$W_\delta^R(y) = (W_\delta - R)^+ \mathbb{I}_{|y|<R} + W_\delta \mathbb{I}_{|y| \geq R}.$$

Then for $W \in L^q_w(\mathbb{R}^N)$ we have $W_\delta \in L^p(\mathbb{R}^N)$ for any $p \in [1, q)$ and moreover that $|W_\delta^R|_p \to 0$ as $R \to +\infty$ for any $\delta > 0$. Define again also $\Gamma_\delta^R = W_\delta - W_\delta^R$. Note that $\operatorname{supp} \Gamma_\delta^R \subset B_R$ and $0 \leq \Gamma_\delta^R \leq R$.

From the Young inequality (with $p = N/2$, $r = 2p/(2p-1) = N/(N-1)$), the Hölder inequality and the Sobolev embedding theorem we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \big(W * |f_n g_n|\,\big)\, |w h_n| \\
&\quad \leq \int_{\mathbb{R}^N} \big((W - W_\delta) * |f_n g_n|\,\big)\, |w h_n| + \int_{\mathbb{R}^N} \big(W_\delta^R * |f_n g_n|\,\big)\, |w h_n| \\
&\qquad + \int_{\mathbb{R}^N} \big(\Gamma_\delta^R * |f_n g_n|\,\big)\, |w h_n| \\
&\quad \leq \delta\, |f_n g_n|_1\, |w h_n|_1 + |W_\delta^R|_{N/2}\, |f_n g_n|_r\, |w h_n|_r + \int_{\mathbb{R}^N} \big(\Gamma_\delta^R * |f_n g_n|\big)\, |w h_n| \\
&\quad \leq C\big(\delta + |W_\delta^R|_{N/2}\big) + \int_{\mathbb{R}^N} \big(\Gamma_\delta^R * |f_n g_n|\big)|w h_n|. \qquad (4.1)
\end{aligned}$$

First we claim that

$$\int_{\mathbb{R}^N} \big(\Gamma_\delta^R * |f_n g_n|\big)|w h_n| \to 0 \quad \text{as } n \to +\infty.$$

Indeed, for any $\epsilon > 0$ we fix $R_1 > 0$ such that $|\mathbb{I}_{\mathbb{R}^N \setminus B_1} w|_2 < \epsilon$, where $B_1 = B_{R_1}$. We define also $R_2 = R_1 + R$ and $B_2 = B_{R_2}$ so that for any $y \in B_1$ and $z \in \mathbb{R}^N \setminus B_2$, we have $|z - y| \geq R$ and hence $\Gamma_\delta^R(z - y) = 0$.

Now we estimate the term as follows:

$$\begin{aligned}
\int_{\mathbb{R}^N} \big(\Gamma_\delta^R * |f_n g_n|\big)|w h_n| &= \int_{B_1} \big(\Gamma_\delta^R * (\mathbb{I}_{B_2} |f_n g_n|)\big)|w h_n| + \int_{\mathbb{R}^N \setminus B_1} \big(\Gamma_\delta^R * |f_n g_n|\big)|w h_n| \\
&\leq R \,\big|\, \mathbb{I}_{B_2} f_n g_n \big|_1 \big|\, \mathbb{I}_{B_1} w h_n \big|_1 + \big|\, \Gamma_\delta^R * (f_n g_n) \big|_\infty \big|\, \mathbb{I}_{\mathbb{R}^N \setminus B_1} h_n \big|_2 \big|\, \mathbb{I}_{\mathbb{R}^N \setminus B_1} w \big|_2 \\
&\leq R\, |g_n|_2\, |h_n|_2 \left(\big|\, \mathbb{I}_{B_2} f_n \big|_2 |w|_2 + R\, |f_n|_2 \big|\, \mathbb{I}_{\mathbb{R}^N \setminus B_1} w \big|_2 \right) \\
&\leq C\, R \left(\big|\, \mathbb{I}_{B_2} f_n \big|_2 + \big|\, \mathbb{I}_{\mathbb{R}^N \setminus B_1} w \big|_2 \right).
\end{aligned}$$

Since $f_n \to 0$ as $n \to +\infty$ in $L^2(B_2)$, the claim is proved.

We conclude the proof of the lemma letting first $n \to +\infty$, then $R \to +\infty$ and finally $\delta \to 0$ in (4.1). □

References

[1] Cabré, X. and Solà-Morales, J.: Layer solutions in a half-space for boundary reactions. *Comm. Pure Appl. Math.* **58** (2005), no. 12, 1678–1732.

[2] Coti-Zelati, V. and Nolasco, M.: Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **22** (2011), no. 1, 51–72.

[3] Dall'Acqua, A., Sørensen, T. Ø. and Stockmeyer, E.: Hartree–Fock theory for pseudo-relativistic atoms. *Ann. Henri Poincaré* **9** (2008), no. 4, 711–742.

[4] Elgart, A. and Schlein, B.: Mean field dynamics of boson stars. *Comm. Pure Appl. Math.* **60** (2007), no. 4, 500–545.

[5] Fröhlich, J., Jonsson, B. L. G. and Lenzmann, E.: Boson stars as solitary waves. *Comm. Math. Phys.* **274** (2007), no. 1, 1–30.

[6] Fröhlich, J., Jonsson, B. L. G. and Lenzmann, E.: Blowup for nonlinear wave equations describing boson stars. *Comm. Pure Appl. Math.* **60** (2007), no. 11, 1691–1705.

[7] Fröhlich, J., Jonsson, B. L. G. and Lenzmann, E.: Dynamical collapse of white dwarfs in Hartree and Hartree–Fock theory. *Comm. Math. Phys.* **274** (2007), no. 3, 737–750.

[8] Lenzmann, E.: Well-posedness for semi-relativistic Hartree equations of critical type. *Math. Phys. Anal. Geom.* **10** (2007), no. 1, 43–64.

[9] Lenzmann, E.: Uniqueness of ground states for pseudo-relativistic Hartree equations. *Anal. PDE* **2** (2009), no. 1, 1–27.

[10] Lieb, E. H. and Loss, M.: *Analysis.* Graduate Studies in Mathematics 14, American Mathematical Society, Providence, RI, 1997.

[11] Lieb E. H. and Yau, H. T.: The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. *Comm. Math. Phys.* **112** (1987), no. 1, 147–174.

[12] Lions, P. L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984), no. 2, 109–145.

Received February 8, 2012.

Vittorio Coti Zelati: Dipartimento di Matematica Pura e Applicata "R. Caccioppoli", Università di Napoli "Federico II", Via Cintia, M. S. Angelo, 80126 Napoli, Italy.
E-mail: zelati@unina.it

Margherita Nolasco: Dipartimento di Matematica Pura e Applicata, Università dell'Aquila, Via Vetoio, Loc. Coppito, 67010 L'Aquila AQ, Italia.
E-mail: nolasco@univaq.it

Work partially supported by the PRIN2009 grant "Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations".