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A general form of the weak maximum principle and some applications

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Abstract. The aim of this paper is to introduce new forms of the weak and Omori–Yau maximum principles for linear operators, notably for trace type operators, and show their usefulness, for instance, in the context of PDEs and in the theory of hypersurfaces. In the final part of the paper we consider a large class of nonlinear operators and we show that our previous results can be appropriately generalized to this case.

1. Introduction

A well known result due to Omori [21] and Yau [27], [9], called from now on the Omori–Yau maximum principle, states that on a complete Riemannian manifold $(M, \langle\ ,\ \rangle)$ with Ricci tensor bounded from below, for any function $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\} \subset M$ with the properties

$$\text{(1.1)} \qquad \text{a) } u(x_k) > u^* - \frac{1}{k}, \quad \text{b) } \Delta u(x_k) < \frac{1}{k}, \quad \text{and c) } |\nabla u|(x_k) < \frac{1}{k}$$

for eack $k \in \mathbb{N}$.

In 2002, Pigola, Rigoli and Setti [23] introduced what has been called the weak maximum principle with the following definition: one says that the weak maximum principle holds on a Riemannian manifold $(M, \langle\ ,\ \rangle)$ if for any function $u \in \mathcal{C}^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence $\{x_k\} \subset M$ with the properties a) and b) in (1.1).

This seemingly simple-minded definition is in fact deep: it turns out to be equivalent to the stochastic completeness of the Riemannian manifold $(M, \langle\ ,\ \rangle)$ as was shown in [23]. This latter concept does not require the manifold to be complete from the Riemannian point of view and a simple useful condition to guarantee stochastic completeness is given by the Khas'minskiĭ test [17], that is,

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by the existence of a function $\gamma \in \mathcal{C}^2(M)$ such that

$$\begin{cases} \text{i)} \quad \gamma(x) \to +\infty & \text{as } x \to \infty, \\ \text{ii)} \quad \Delta\gamma \le \lambda\gamma & \text{outside a compact subset of } M \end{cases} \tag{1.2}$$

for some positive constant $\lambda > 0$.

Thus, we do not necessarily require any curvature conditions to guarantee the applicability of the principle. This observation applies to the Omori–Yau maximum principle too, as shown in Theorem 1.9 of [24]. We remark that, very recently, the sufficient condition for stochastic completeness given by the Khas'minskiĭ test has been shown to be in fact also necessary [20].

This approach, based on the existence of some auxiliary function satisfying appropriate conditions, has proved to be of great versatility in geometric applications; for instance, in the geometry of submanifolds [2], [1], [3], [5], [6] and in the study of soliton structures [12], [19], [22].

The purpose of this paper is to prove a weak maximum principle (Theorem A), an Omori–Yau type maximum principle (Theorem B) and further related results for a large class of linear differential operators of geometrical interest.

From now on $(M, \langle\ ,\ \rangle)$ will denote a connected, Riemannian manifold of dimension $m \ge 2$. To describe our first result, let T be a symmetric positive semi-definite $(2,0)$-tensor field on M and let X be a vector field. We write $L = L_{T,X}$ to denote the differential operator acting on $u \in \mathcal{C}^2(M)$ by

$$Lu = \operatorname{div}(T(\nabla u,\ )^\sharp) - \langle X, \nabla u\rangle = \operatorname{tr}(T \circ \operatorname{Hess}(u)) + \operatorname{div} T(\nabla u) - \langle X, \nabla u\rangle, \tag{1.3}$$

where ${}^\sharp\colon T^*M \to TM$ is the musical isomorphism. For instance, if $T = \langle\ ,\ \rangle$ and X is a vector field on M, for $u \in \mathcal{C}^2(M)$ we have

$$Lu = \Delta u - \langle X, \nabla u\rangle \tag{1.4}$$

and L coincides with the X-Laplacian, denoted by Δ_X, used in the study of general soliton structures, [19]; in particular if $X = \nabla f$ then $L = \Delta_f$ is the f-Laplacian, appearing also as the natural symmetric diffusion operator in the study of the weighted Riemannian manifold $(M, \langle,\rangle, e^{-f} d\mathrm{vol})$, [16]. On the other hand, if T is as above and $X = (\operatorname{div} T)^\sharp$, then for $u \in \mathcal{C}^2(M)$, Lu reduces to

$$Lu = \operatorname{tr}(T \circ \operatorname{Hess}(u)) \tag{1.5}$$

and it is a typical trace operator.

Theorem A. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and $L = L_{T,X}$ as above. Let $q(x) \in \mathcal{C}^0(M)$, $q(x) \ge 0$ and suppose that*

$$q(x) > 0 \text{ outside a compact set.} \tag{1.6}$$

Let $\gamma \in \mathcal{C}^2(M)$ be such that

$$\begin{cases} \text{i)} \quad \gamma(x) \to +\infty & \text{as } x \to \infty, \\ \text{ii)} \quad q(x) L\gamma(x) \le B & \text{outside a compact set} \end{cases} \tag{Γ}$$

for some constant $B > 0$*. If* $u \in \mathcal{C}^2(M)$ *and* $u^* < +\infty$*, then there exists a sequence* $\{x_k\} \subset M$ *with the properties*

$$\text{(1.7)} \qquad \text{a)}\ u(x_k) > u^* - \frac{1}{k}, \quad \textit{and} \quad \text{b)}\ q(x_k)\, Lu(x_k) < \frac{1}{k}$$

for each $k \in \mathbb{N}$*.*

If the conclusion of the theorem holds on $(M, \langle\ ,\ \rangle)$ we shall say that the q-weak maximum principle for the operator L holds on $(M, \langle\ ,\ \rangle)$. If $q \equiv 1$ we shall say that the weak maximum principle for the operator L holds on $(M, \langle\ ,\ \rangle)$. Obviously, if the q-weak maximum principle holds for L and $0 \leq \hat{q}(x) \leq q(x)$, $\hat{q}(x)$ satisfying (1.6), then the $\hat{q}$-weak maximum principle also holds for the operator L.

Note that, if $T = p(x)\langle\ ,\ \rangle$ for some $p \in \mathcal{C}^1(M)$, $p > 0$ on M, and $X \equiv 0$, then $q(x)L$ is (at least on the set where q is positive) a typical (nonsymmetric) diffusion operator.

We stress that the Riemannian manifold M is not assumed to be (geodesically) complete. This matches with the fact that for $L = \Delta$ and $q(x) \equiv 1$, i), ii) of condition (Γ) (see also Remark (1.1)) are exactly the Khas'minskiĭ condition that we have mentioned above.

Remark 1.1. As we shall show below, condition ii) in (Γ) can be replaced, for instance, by

$$(\Gamma) \qquad \text{ii)}' \quad q(x)\, L\gamma(x) \leq G(\gamma(x)) \quad \text{outside a compact subset of } M,$$

where $G \in \mathcal{C}^1(\mathbb{R}^+)$ is nonnegative and satisfies

$$\text{(1.8)} \qquad \text{i)}\ \frac{1}{G} \notin L^1(+\infty); \quad \text{ii)}\ G'(t) \geq -A(\log t + 1),$$

for $t >> 1$ and some constant $A \geq 0$. For instance, the functions

$$G(t) = t, \quad G(t) = t\log t, \quad t >> 1, \quad G(t) = t \log t \log\log t, \quad t >> 1,$$

and so on, satisfy i) and (ii) in (1.8) with $A = 0$.

It seems worth to underline the following fact. In [23] the third author, jointly with Pigola and Setti, proved that the weak maximum principle for Δ is equivalent to the stochastic completeness of the manifold M via the known characterization (see Grigor'yan [15] or [24]) that $(M, \langle , \rangle)$ is stochastically complete if and only if for each $\lambda > 0$ the only non-negative bounded solution of $\Delta u = \lambda u$ is $u \equiv 0$. The work of Mari and Valtorta [20] shows that the weak maximum principle implies the existence of a function γ satisfying the Khas'minskiĭ criterion (1.2). This latter classically implies stochastic completeness (see [24] for a simple proof using the equivalence mentioned above). Theorem A above provides a direct proof of the weak maximum principle starting from the Kash'minski test.

The "Omori–Yau" type version of Theorem A is as follows.

Theorem B. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and let L be as above. Let $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, and suppose*

$$q(x) > 0 \quad \text{outside a compact set.} \tag{1.9}$$

Let $\gamma \in \mathcal{C}^2(M)$ be such that

$$(\Gamma_B) \qquad \begin{cases} \text{i)} \quad \gamma(x) \to +\infty & \text{as } x \to \infty, \\ \text{ii)} \quad q(x)L\gamma \leq B & \text{outside a compact subset of } M, \\ \text{iii)} \quad |\nabla\gamma| \leq A & \text{outside a compact subset of } M \end{cases}$$

for some constants $A, B > 0$. If $u \in \mathcal{C}^2(M)$ and $u^* < +\infty$ then there exists a sequence $\{x_k\} \subset M$ with the properties

$$\text{a) } u(x_k) > u^* - \frac{1}{k}, \qquad \text{b) } q(x_k)Lu(x_k) < \frac{1}{k}, \qquad \text{and} \quad \text{c) } |\nabla u(x_k)| < \frac{1}{k} \tag{1.10}$$

for each $k \in \mathbb{N}$.

Remark 1.2. In this case conditions ii) and iii) in (Γ_B) can be replaced by the apparently weaker conditions

$$(\Gamma_B) \qquad \begin{cases} \text{ii)}' \quad q(x)L\gamma \leq G(\gamma) \\ \text{iii)}' \quad |\nabla\gamma| \leq G(\gamma) \end{cases}$$

outside a compact subset of M, where $G \in \mathcal{C}^1(R_0^+)$ is a positive function satisfying i) and ii) of (1.8) .

We observe that when $(M, \langle\ ,\ \rangle)$ is a complete, noncompact Riemannian manifold a special candidate for γ, in both Theorems A and B, is the distance function $r(x)$ from a fixed origin $o \in M$. Of course $r(x)$ is smooth only outside $\{o\} \cup \mathrm{cut}(o)$, where $\mathrm{cut}(o)$ is the cut locus of o, but, as we shall show at the end of the proof of Theorem B, this problem can be bypassed using an old trick of Calabi [8]. Needless to say, the inequalities involving $r(x)$ and the operator L have to be understood in the weak-Lip sense. We underline that the arguments we shall give below, via a comparison principle, also shows that if $\gamma \in \mathcal{C}^1(M)$ satisfies (Γ_B) i), iii), and is a classical weak solution of (Γ_B) ii), then Theorem B is still valid. The same, of course, applies to Theorem A and to the regularity of u (but in this case with the further assumption $1/q \in L^1_{\mathrm{loc}}(M)$ and the application of Theorem 5.6 of [26] when proving that u^* is not attained on M; see the proof of Theorem A″).

On the other hand, given T and X as above we introduce the operator $H = H_{T,X}$ acting on $\mathcal{C}^2(M)$ by

$$Hu = H_{T,X}u = T(\mathrm{hess}(u)\cdot, \cdot) + (\mathrm{div}T - X^\flat) \otimes du.$$

Observe that $Lu = \mathrm{tr}(Hu)$. Then, the above theorems admit the following generalizations.

Theorem A′. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and let $H = H_{T,X}$ be as above. Let $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, and suppose that*

$$q(x) > 0 \text{ outside a compact set.} \tag{1.11}$$

Let $\gamma \in \mathcal{C}^2(M)$ be such that

$$(\Gamma_C) \qquad \begin{cases} \text{i)} \quad \gamma(x) \to +\infty \quad \text{as } x \to \infty, \\ \text{ii)} \quad q(x) H\gamma(x)(v,v) \leq B|v|^2, \end{cases}$$

for some constant $B > 0$ and for every $x \in M \setminus K$, for some compact $K \subset M$, and for every $v \in T_xM$. If $u \in \mathcal{C}^2(M)$ and $u^* < +\infty$, then there exists a sequence $\{x_k\} \subset M$ with the properties

$$\text{i)}\ u(x_k) > u^* - \frac{1}{k}, \quad \text{and} \quad \text{ii)}\ q(x_k)Hu(x_k)(v,v) < \frac{1}{k}|v|^2 \tag{1.12}$$

for each $k \in \mathbb{N}$ and every $v \in T_{x_k}M, v \neq 0$.

Theorem B′. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and let $H = H_{T,X}$ be as above. Let $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, and suppose that*

$$q(x) > 0 \text{ outside a compact set.} \tag{1.13}$$

Let $\gamma \in \mathcal{C}^2(M)$ be such that

$$(\Gamma_D) \qquad \begin{cases} \text{i)} \quad \gamma(x) \to +\infty \quad \text{as } x \to \infty, \\ \text{ii)} \quad q(x) H\gamma(x)(v,v) \leq B|v|^2, \\ \text{iii)} \quad |\nabla\gamma(x)| \leq A, \end{cases}$$

for some constants $A, B > 0$, for every $x \in M \setminus K$, for some compact $K \subset M$, and for every $v \in T_xM$. If $u \in \mathcal{C}^2(M)$ and $u^* < +\infty$, then there exists a sequence $\{x_k\} \subset M$ with the properties

$$\text{i)}\ u(x_k) > u^* - \frac{1}{k}, \quad \text{ii)}\ q(x_k)Hu(x_k)(v,v) < \frac{1}{k}|v|^2, \quad \text{and} \quad |\nabla u(x_k)| < \frac{1}{k} \tag{1.14}$$

for each $k \in \mathbb{N}$ and every $v \in T_{x_k}M, v \neq 0$.

In Section 6 below we generalize Theorems A and B to a large class of nonlinear operators containing, for instance, the p-Laplacian, with $p > 1$, the mean curvature operator and so on. Of course Theorems A′ and B′ admit similar generalizations to the nonlinear case for $\mathcal{C}^2$-solutions. We leave the interested reader to state the results and provide her/his own proofs following arguments similar to those of Theorems A″ and B″.

2. Proof of Theorem A and related results

In this section we give the proof of Theorems A and of some companion results.

Proof of Theorem A. We fix $\eta > 0$ and let

$$A_\eta = \{x \in M : \ u(x) > u^* - \eta\}. \tag{2.1}$$

We claim that

$$\inf_{A_\eta} \{q(x)Lu(x)\} \leq 0. \tag{2.2}$$

Note that (2.2) is equivalent to conclusion (1.7) of Theorem A.

We reason by contradiction and we suppose that

$$q(x)Lu(x) \geq \sigma_0 > 0 \quad \text{on } A_\eta. \tag{2.3}$$

First we observe that u^* cannot be attained at any point $x_0 \in M$, for otherwise $x_0 \in A_\eta$, $\nabla u(x_0) = 0$, and $Lu(x_0)$ reduces to $Lu(x_0) = \mathrm{tr}(T \circ \mathrm{Hess}(u))(x_0)$, so that, since T is positive semi-definite, $q(x_0)Lu(x_0) \leq 0$ contradicting (2.3).

Next we let

$$\Omega_t = \{x \in M : \ \gamma(x) > t\}, \tag{2.4}$$

and define

$$u_t^* = \sup_{x \in \Omega_t^c} u(x). \tag{2.5}$$

Clearly Ω_t^c is closed; we show that it is also compact. In fact, by (Γ) i) there exists a compact set K_t such that $\gamma(x) > t$ for every $x \notin K_t$. In other words, $\Omega_t^c \subset K_t$ and hence it is also compact. In particular, $u_t^* = \max_{x \in \Omega_t^c} u(x)$.

Since u^* is not attained in M and $\{\Omega_t^c\}$ is a nested family exhausting M, we find a divergent sequence $\{t_j\} \subset \mathbb{R}_0^+$ such that

$$u_{t_j}^* \to u^* \quad \text{as } j \to +\infty, \tag{2.6}$$

and we can choose $T_1 > 0$ sufficiently large in such a way that

$$u_{T_1}^* > u^* - \frac{\eta}{2}. \tag{2.7}$$

Furthermore we can suppose chosen T_1 sufficiently large that $q(x) > 0$ and (Γ) ii) holds on Ω_{T_1}. We choose α such that $u_{T_1}^* < \alpha < u^*$. Because of (2.6) we can find j so large that

$$T_2 = t_j > T_1 \quad \text{and} \quad u_{T_2}^* > \alpha. \tag{2.8}$$

We select $\overline{\eta} > 0$ small enough that

$$\alpha + \overline{\eta} < u_{T_2}^*. \tag{2.9}$$

For $\sigma \in (0, \sigma_0)$ we define

$$\gamma_\sigma(x) = \alpha + \sigma(\gamma - T_1). \tag{2.10}$$

We note that

$$\gamma_\sigma(x) = \alpha \quad \text{for every } x \in \partial\Omega_{T_1}, \tag{2.11}$$

and

$$q(x)L\gamma_\sigma(x) = \sigma q(x)L\gamma(x) \le \sigma B < \sigma_0 \quad \text{on } \Omega_{T_1}, \tag{2.12}$$

up to having chosen σ sufficiently small.

Since on $\Omega_{T_1} \setminus \Omega_{T_2}$ we have

$$\alpha \le \gamma_\sigma(x) \le \alpha + \sigma(T_2 - T_1), \tag{2.13}$$

we can choose $\sigma \in (0, \sigma_0)$ sufficiently small, so that

$$\sigma(T_2 - T_1) < \overline{\eta} \tag{2.14}$$

and then

$$\alpha \le \gamma_\sigma(x) < \alpha + \overline{\eta} \quad \text{on} \quad \Omega_{T_1} \setminus \Omega_{T_2}. \tag{2.15}$$

For any such σ, on $\partial\Omega_{T_1}$ we have

$$\gamma_\sigma(x) = \alpha > u^*_{T_1} \ge u(x), \tag{2.16}$$

so that

$$(u - \gamma_\sigma)(x) < 0 \quad \text{on } \partial\Omega_{T_1}. \tag{2.17}$$

Furthermore, if $\overline{x} \in \Omega_{T_1} \setminus \Omega_{T_2}$ is such that

$$u(\overline{x}) = u^*_{T_2} > \alpha + \overline{\eta}$$

then

$$(u - \gamma_\sigma)(\overline{x}) \ge u^*_{T_2} - \alpha - \sigma(T_2 - T_1) > u^*_{T_2} - \alpha - \overline{\eta} > 0$$

by (2.9) and (2.14). Finally, (Γ) i) and the fact that $u^* < +\infty$ imply

$$(u - \gamma_\sigma)(x) < 0 \quad \text{on } \Omega_{T_3} \tag{2.18}$$

for $T_3 > T_2$ sufficiently large. Therefore,

$$m = \sup_{x \in \overline{\Omega}_{T_1}} (u - \gamma_\sigma)(x) > 0,$$

and in fact a positive maximum is attained at a certain point z_0 in the compact set $\overline{\Omega}_{T_1} \setminus \Omega_{T_3}$. In particular, $\nabla(u - \gamma_\sigma)(z_0) = 0$ and $L(u - \gamma_\sigma)(z_0)$ reduces to $\operatorname{tr}(T \circ \operatorname{Hess}(u - \gamma_\sigma))(z_0)$. Therefore, since T is positive semi-definite we have that $Lu(z_0) \le L\gamma_\sigma(z_0)$.

By (2.17) we know that $\gamma(z_0) > T_1$. Therefore, at z_0 we have

$$u(z_0) = \gamma_\sigma(z_0) + m > \gamma_\sigma(z_0) > \alpha > u^*_{T_1} > u^* - \frac{\eta}{2}, \tag{2.19}$$

and hence $z_0 \in A_\eta \cap \Omega_{T_1}$. In particular $q(z_0) > 0$ and (Γ) ii) holds at z_0. From (2.3) we have

$$0 < \sigma_0 \le q(z_0)Lu(z_0) \le q(z_0)L\gamma_\sigma(z_0) \le \sigma B < \sigma_0, \tag{2.20}$$

which is a contradiction. □

We observe that, in Theorem A, we can relax the assumption on the boundedness of the function u from above to control on u at infinity via the function γ. This is the content of the next result.

Theorem $\hat{\mathbf{A}}$. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and let $L = L_{T,X}$ be as above. Let $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, and suppose that*

$$q(x) > 0 \quad \textit{outside a compact set.} \tag{2.21}$$

Let $\gamma \in \mathcal{C}^2(M)$ be such that

$$\begin{cases} \text{i)} \quad \gamma(x) \to +\infty & \textit{as } x \to \infty, \\ \text{ii)} \quad q(x)L\gamma(x) \leq B & \textit{outside a compact set} \end{cases} \tag{Γ}$$

for some constant $B > 0$. If $u \in \mathcal{C}^2(M)$ and

$$u(x) = o(\gamma(x)) \ \textit{as } x \to \infty, \tag{2.22}$$

then for each μ such that

$$A_\mu = \big\{x \in M : u(x) > \mu\big\} \neq \emptyset$$

we have

$$\inf_{A_\mu} \big\{q(x)Lu(x)\big\} \leq 0.$$

Proof. Of course we consider here the case $u^* = +\infty$. We reason by contradiction as in the proof of Theorem A and we suppose the validity of (2.3) on A_μ. Proceed as in the above proof (obviously in this case u^* is not attained on M) to arrive to a modification of (2.6), that now takes the form

$$u^*_{t_j} \to +\infty \text{ as } j \to \infty, \tag{2.23}$$

and choose $T_1 > 0$ sufficiently large in such a way that (2.7) becomes now

$$u^*_{T_1} > 2\mu. \tag{2.24}$$

Furthermore we can suppose to have chosen T_1 so large that $q(x) > 0$ and (Γ) ii) holds on Ω_{T_1}. We choose α such that $\alpha > u^*_{T_1}$. Because of (2.23) we can find j sufficiently large that

$$T_2 = t_j > T_1 \quad \text{and} \quad u^*_{T_2} > \alpha. \tag{2.25}$$

Proceed now up to (2.18) which is now true on Ω_{T_3} for T_3 so large since

$$(u - \gamma_\sigma)(x) = \gamma_\sigma\Big(\frac{u}{\gamma_\sigma} - 1\Big)(x),$$

expression which becomes negative on Ω_{T_3}, for T_3 sufficiently large, because of condition (2.22). The rest of the proof is as in that of Theorem A. □

We now show the validity of Remark 1.1. Thus we assume (Γ) ii)' with G as in (1.8). We set

$$\varphi(t) = \int_{t_0}^{t} \frac{ds}{G(s) + A\, s \log s} \tag{2.26}$$

on $[t_0, +\infty)$ for some $t_0 > 0$. Note that, by (1.8) i), $\varphi(t) \to +\infty$ as $t \to +\infty$. Thus, defining $\widehat{\gamma} = \varphi(\gamma)$, ($\Gamma$) i) implies that

$$\widehat{\gamma}(x) \to +\infty \quad \text{as } x \to \infty. \tag{2.27}$$

Next, using that

$$L(\varphi(u)) = \varphi'(u)Lu + \varphi''(u)T(\nabla u, \nabla u),$$

a computation gives

$$\begin{aligned} q(x)L\widehat{\gamma}(x) = {} & \frac{q(x)L\gamma(x)}{G(\gamma(x)) + A\gamma(x)\log\gamma(x)} \\ & - \frac{G'(\gamma(x)) + A\big(1 + \log\gamma(x)\big)}{\big(G(\gamma(x)) + A\gamma(x)\log\gamma(x)\big)^2}\, q(x)T\big(\nabla\gamma(x), \nabla\gamma(x)\big) \end{aligned}$$

outside a sufficiently large compact set. Since $T(\nabla\gamma, \nabla\gamma) \geq 0$, $q(x) \geq 0$ and (1.8) ii) holds, we deduce

$$q(x)L\widehat{\gamma}(x) \leq \frac{q(x)L\gamma(x)}{G(\gamma(x)) + A\gamma(x)\log\gamma(x)} \tag{2.28}$$

if $\gamma(x)$ is sufficiently large. Thus, from (Γ) ii)' and $G \geq 0$ we finally obtain

$$q(x)L\widehat{\gamma}(x) \leq B \tag{2.29}$$

outside a compact set. Then (2.27) and (2.29) show the validity of (Γ) i), ii) for the function $\widehat{\gamma}$.

This finishes the proof of Remark 1.1. Regarding Theorem $\hat{\text{A}}$, if we replace (Γ) ii) with (Γ) ii)', G satisfying (1.8), then condition (2.22) has to be replaced by

$$u(x) = o\Big(\int_0^{\gamma(x)} \frac{ds}{G(s) + As\log s}\Big) \quad \text{as } x \to \infty. \tag{2.30}$$

Thus for instance if $G(t) = t$, so that we can choose $A = 0$, (Γ) ii)' is $q(x)L\gamma(x) \leq \gamma(x)$, but (2.30) becomes $u(x) = o(\log\gamma(x))$ as $x \to \infty$, showing a balancing effect between the two conditions.

Proof of Theorem A$'$. For the proof of Theorem A$'$ we proceed as in the proof of Theorem A letting

$$A_\eta = \big\{x \in M : u(x) > u^* - \eta\big\}. \tag{2.31}$$

We claim that for every $\varepsilon > 0$ there exists $x \in A_\eta$ such that

$$q(x)Hu(x)(v, v) < \varepsilon$$

for each $v \in T_xM$ with $|v| = 1$. Aiming for a contradiction, suppose that there exists $\sigma_0 > 0$ such that, for every $x \in A_\eta$ there exists $\bar{v} \in T_xM$, $|\bar{v}| = 1$, such that

$$q(x)Hu(x)(\bar{v},\bar{v}) \geq \sigma_0. \tag{2.32}$$

Now we follow the argument of the proof of Theorem A up to equation (2.12), which is now replaced by

$$q(x)H\gamma_\sigma(x)(\bar{v},\bar{v}) = \sigma q(x)H\gamma(x)(\bar{v},\bar{v}) \leq \sigma B < \sigma_0 \quad \text{on } \Omega_{T_1}, \tag{2.33}$$

up to having chosen σ sufficiently small. We then proceed up to the existence of a certain point z_0 in the compact set $\overline{\Omega}_{T_1} \setminus \Omega_{T_3}$ where the function $u - \gamma_\sigma$ attains its positive maximum. In particular, $\nabla(u-\gamma_\sigma)(z_0) = 0$ and $H(u-\gamma_\sigma)(z_0)$ reduces to

$$H(u-\gamma_\sigma)(z_0)(v,v) = T(\operatorname{hess}(u-\gamma_\sigma)(z_0)v,v) \quad \text{for every } v \in T_{z_0}M.$$

Therefore, since T is positive semi-definite we have

$$Hu(z_0)(v,v) \leq H\gamma_\sigma(z_0)(v,v)$$

for every $v \in T_{z_0}M$.

As in the proof of Theorem A, we have that $z_0 \in A_\eta \cap \Omega_{T_1}$. In particular $q(z_0) > 0$ and (Γ) ii)' holds at z_0. On the other hand, from (2.32) we have

$$0 < \sigma_0 \leq q(z_0)Hu(z_0)(\bar{v},\bar{v}) \leq q(z_0)H\gamma_\sigma(z_0)(\bar{v},\bar{v}) \leq \sigma B < \sigma_0, \tag{2.34}$$

which is a contradiction. □

3. Proof of Theorem B and some related results

We use the notation of the previous section and give the proof of Theorem B.

Proof of Theorem B. We first observe that, although it is not required in the statement, the two assumptions (Γ_B) i) and iii) imply that the manifold M is geodesically complete. To see this, let $\varsigma : [0,\ell) \to M$ be any divergent path parametrized by arc-length. Here by *divergent* path we mean a path that eventually lies outside any compact subset of M. From (Γ_B) iii) we have that $|\nabla\gamma| \leq A$ outside a compact subset K of M. We set $h(t) = \gamma(\varsigma(t))$ on $[t_0,\ell)$, where t_0 has been chosen so that $\varsigma(t) \notin K$ for all $t_0 \leq t < \ell$. Then, for every $t \in [t_0,\ell)$ we have

$$|h(t) - h(t_0)| = \left|\int_{t_0}^{t} h'(s)ds\right| \leq \int_{t_0}^{t} |\nabla\gamma(\varsigma(s))|ds \leq A(t-t_0).$$

Since ς is divergent, then $\varsigma(t) \to \infty$ as $t \to \ell^-$, so that $h(t) \to +\infty$ as $t \to \ell^-$ because of assumption (Γ_B) i). Therefore, letting $t \to \ell^-$ in the inequality above, we conclude that $\ell = +\infty$. This shows that divergent paths in M have infinite length. In other words, the metric on M is complete.

As in the proof of Theorem A we fix $\eta > 0$ but, instead of the set A_η of (2.1), we now consider the set

$$B_\eta = \{x \in M : \ u(x) > u^* - \eta \text{ and } |\nabla u(x)| < \eta\}. \tag{3.1}$$

Since the manifold is complete, by applying the Ekeland quasi-minimum principle (see for instance [10]) we deduce that $B_\eta \neq \emptyset$. We claim that

$$\inf_{B_\eta} \{q(x)Lu(x)\} \leq 0. \tag{3.2}$$

Note that (3.2) is equivalent to conclusion (1.10) of Theorem B. We reason by contradiction and suppose that

$$q(x)Lu(x) \geq \sigma_0 > 0 \quad \text{on } B_\eta. \tag{3.3}$$

Now the proof follow the pattern of that of Theorem A with the choice of T_1, such that also (Γ) iii) holds on Ω_{T_1}. We observe that in this case

$$\gamma_\sigma(x) = \alpha \quad \text{for every } x \in \partial\Omega_{T_1}, \tag{3.4}$$

$$q(x)L\gamma_\sigma(x) = \sigma q(x)L\gamma(x) \leq \sigma B < \sigma_0 \quad \text{on } \Omega_{T_1}, \tag{3.5}$$

and

$$|\nabla\gamma_\sigma(x)| = \sigma|\nabla\gamma(x)| \leq \sigma A < \eta \quad \text{on } \Omega_{T_1}, \tag{3.6}$$

up to having chosen σ sufficiently small.

Therefore, we find a point $z_0 \in \overline{\Omega}_{T_1} \setminus \Omega_{T_3}$ where $u - \gamma_\sigma$ attains a positive absolute maximum m. As in the proof of Theorem A, $z_0 \in \Omega_{T_1}$ and at z_0 we have

$$u(z_0) > \gamma_\sigma(z_0) > \alpha > u^*_{T_1} > u^* - \frac{\eta}{2} > u^* - \eta; \tag{3.7}$$

furthermore

$$|\nabla u(z_0)| = |\nabla\gamma_\sigma(z_0)| = \sigma|\nabla\gamma(z_0)| \leq \sigma A < \eta, \tag{3.8}$$

by our choice of σ. Thus $z_0 \in B_\eta \cap \Omega_{T_1}$ and a contradiction is achieved as at the end of the proof of Theorem A. □

We note that the validity of Remark 1.2 is immediate. Indeed defining $\widehat{\gamma} = \varphi(\gamma)$ as in the previous subsection, conditions (Γ_B) i), ii) are satisfied for $\widehat{\gamma}$; as for condition (Γ_B) iii), using (Γ_B) iii)' and $G \geq 0$, we have

$$|\nabla\widehat{\gamma}| = \frac{|\nabla\gamma|}{G(\gamma) + A\gamma\log\gamma} \leq \frac{G(\gamma)}{G(\gamma) + A\gamma\log\gamma} \leq 1 \tag{3.9}$$

outside a compact set. Thus, we also have the validity of (Γ_B) iii) for $\widehat{\gamma}$.

Remark 3.1. As mentioned in the Introduction, if $(M, \langle\ ,\ \rangle)$ is a complete, non-compact Riemannian manifold then a natural candidate for $\gamma(x)$ is $r(x)$ (or a composition of $r(x)$ with an appropriate function). However, $r(x)$ is not $\mathcal{C}^2$ in $C = \{o\} \cup \mathrm{cut}(o)$ and assumptions (Γ) ii) (in Theorem A) and (Γ_B) ii) and iii) (in Theorem B) have to be understood and assumed in the weak sense. Nevertheless the proof of Theorem A (and that of Theorem B) still works in this case, adding in both the assumption $1/q \in L^1_{\mathrm{loc}}(M)$ (see Section 6 for more details). Indeed, the only problem is at the end of the proof if the point z_0 where $u-\gamma_\sigma$ attains its positive absolute maximum $m>0$ is in C. However, $u-\gamma_\sigma$ is now given by $f=u-\alpha-\sigma(r-T_1)$ and to avoid the problem we use a trick of Calabi [8] as follows. Take any point z where the function f attains its positive absolute maximum. If $z\notin C$ then

$$|\nabla u(z)| = \sigma|\nabla r(z)| = \sigma < \eta.$$

Otherwise, if $z \in C$, let ς be a minimizing geodesic, parametrized by arc-length, and joining o to z. For $\varepsilon>0$ suitably small let $o_\varepsilon = \varsigma(\varepsilon)$ and $r_\varepsilon(x) = \mathrm{dist}_M(x, o_\varepsilon)$. Thus $z \notin \mathrm{cut}(o_\varepsilon)$ and $r_\varepsilon(x)$ is smooth around z. Consider the function

$$f_\varepsilon = u - \alpha - \sigma(r_\varepsilon + \varepsilon - T_1). \tag{3.10}$$

Using the triangle inequality we have

$$f_\varepsilon(x) - f(x) = \sigma(r(x) - r_\varepsilon(x) - \varepsilon) \le 0 \tag{3.11}$$

in a neighborhood of z. But on $\varsigma|_{[\varepsilon, r(z)]}$, $f_\varepsilon = f$ since

$$r(\varsigma(t)) = \mathrm{dist}_M(o, o_\varepsilon) + \mathrm{dist}_M(o_\varepsilon, \varsigma(t)) = r_\varepsilon(x) + \varepsilon.$$

Therefore z is also a local maximum for f_ε which is $\mathcal{C}^2$ in a neighborhood of z. Thus, at z

$$|\nabla u(z)| = \sigma|\nabla r_\varepsilon(z)| = \sigma < \eta \tag{3.12}$$

up to having chosen σ sufficiently small.

To complete the proof of Theorem A in this case we proceed as follows. We let

$$K = \big\{x \in \Omega_{T_1} :\ (u-\gamma_\sigma)(x) = f(x) = m\big\}, \tag{3.13}$$

where now $\Omega_t = \{x \in M : r(x) > t\}$. For every $x \in K$ we have

$$u(x) = \alpha + \sigma(r(x) - T_1) + m > \alpha > u^* - \frac{\eta}{2},$$

so that $K \subset A_\eta$. Fix $z_0 \in K$ and choose $0<\mu<m$ sufficiently close to m that the connected component Λ_{z_0} of the set

$$\big\{x \in \Omega_{T_1} :\ (u-\gamma_\sigma)(x) > \mu\big\} \tag{3.14}$$

containing z_0 is contained in A_η. Note that Λ_{z_0} is bounded by (2.18). From (2.3) and (2.12), we have

$$Lu(x) \ge \frac{\sigma_0}{q(x)} > L\gamma_\sigma(x) \tag{3.15}$$

on $A_\eta \cap \Omega_{T_1}$ in the weak sense. Moreover, $u = \gamma_\sigma + \mu$ on the boundary of Λ_{z_0}. Applying Theorem 5.3 of [26] (the requirement $v<\delta$ is vacuous in our case) we deduce that $u \le \gamma_\sigma + \mu$ on Λ_{z_0}. However, $z_0 \in \Lambda_{z_0}$, and from the above we have $m \le \mu$, contradiction.

As for completing the proof of Theorem B, we follow the same reasoning replacing A_η by B_η. To do it, simply observe that $K \subset B_\eta$ by (3.12).

We omit the details of the proof of Theorem B′, which follows similarly from the proof of Theorem B.

A typical application of Theorem B is the following "a priori" estimate. Note that condition (3.19) below coincides (for $f = F$) with the Keller–Osserman condition for the Laplace–Beltrami operator (see [13]) showing that in this type of results what really matters is the structure, in this case linear, of the differential operator.

Theorem 3.2. *Assume the validity of the q-maximum principle for the operator $L = L_{T,X}$ on $(M, \langle\ ,\ \rangle)$ and suppose that*

$$q(x)T(\cdot,\cdot) \le C\langle\cdot,\cdot\rangle \tag{3.16}$$

for some $C > 0$. Let $u \in \mathcal{C}^2(M)$ be a solution of the differential inequality

$$q(x)Lu \ge \varphi(u, |\nabla u|) \tag{3.17}$$

with $\varphi(t, y)$ continuous in t, $\mathcal{C}^2$ in y and such that

$$\frac{\partial^2 \varphi}{\partial y^2}(t, y) \ge 0. \tag{3.18}$$

Set $f(t) = \varphi(t, 0)$. Then a condition sufficient to guarantee that

$$u^* = \sup_M u < +\infty$$

is the existence of a continuous function F positive on $[a, +\infty)$ for some $a \in \mathbb{R}$, and satisfying

$$\left(\int_a^t F(s)ds\right)^{-1/2} \in L^1(+\infty), \tag{3.19}$$

$$\limsup_{t\to+\infty} \frac{\int_a^t F(s)ds}{tF(t)} < +\infty, \tag{3.20}$$

$$\liminf_{t\to+\infty} \frac{f(t)}{F(t)} > 0, \tag{3.21}$$

$$\liminf_{t\to+\infty} \frac{\left(\int_a^t F(s)ds\right)^{-1/2}}{F(t)} \frac{\partial\varphi}{\partial y}(t, 0) > -\infty. \tag{3.22}$$

Furthermore, in this case, we have

$$f(u^*) \le 0. \tag{3.23}$$

Proof. Following the proof of Theorem 1.31 in [24] we choose $g \in \mathcal{C}^2(\mathbb{R})$ to be increasing from 1 to 2 on $(-\infty, a+1)$ and defined by

$$g(t) = \int_{a+1}^t \frac{ds}{\left(\int_a^s F(r)dr\right)^{1/2}} + 2 \quad \text{on } [a+1, +\infty).$$

Observe that

$$g'(t) = \Big(\int_a^t F(s)ds\Big)^{-1/2} \quad \textit{and} \quad g''(t) = -\frac{F(t)}{2}g'(t)^3 < 0 \tag{3.24}$$

on $(a+1, +\infty)$. We reason by contradiction and assume that $u^* = +\infty$. Since g is increasing,

$$\inf_M \frac{1}{g(u)} = \frac{1}{g(u^*)} = \frac{1}{g(+\infty)} > 0.$$

By applying the q-maximum principle for L to $1/g$, there exists a sequence $\{x_k\} \subset M$ such that

$$\lim_{k\to+\infty} \frac{1}{g(u(x_k))} = \frac{1}{g(+\infty)}, \tag{3.25}$$

or equivalently

$$\lim_{k\to+\infty} u(x_k) = +\infty, \tag{3.26}$$

$$\Big|\nabla\frac{1}{g(u)}(x_k)\Big| = \frac{g'(u(x_k))}{g(u(x_k))^2}\big|\nabla u(x_k)\big| < \frac{1}{k} \tag{3.27}$$

and finally

$$\begin{aligned} -\frac{1}{k} < q(x_k)\, L\Big(\frac{1}{g(u)}\Big)(x_k) = q(x_k)\Big\{ -\frac{g'(u(x_k))}{g(u(x_k))^2}\, Lu(x_k) \\ + \Big(\frac{2g'(u(x_k))^2}{g(u(x_k))^3} - \frac{g''(u(x_k))}{g(u(x_k))^2}\Big)T\big(\nabla u(x_k), \nabla u(x_k)\big)\Big\}. \end{aligned} \tag{3.28}$$

Because of (3.26), we can suppose that the sequence $\{x_k\}$ satsifies $u(x_k) > a+1$, so that (3.24) holds along the sequence $u(x_k)$. Multiplying (3.28) by

$$\frac{g'(u(x_k))^2}{-g(u(x_k))^2 g''(u(x_k))} > 0$$

and using (3.17), we obtain

$$\begin{aligned} \frac{g'(u(x_k))^3}{g(u(x_k))^4|g''(u(x_k))|}\,\varphi\big(u(x_k), |\nabla u(x_k)|\big) \le \frac{1}{k}\frac{g'(u(x_k))^2}{g(u(x_k))^2|g''(u(x_k))|} \\ + \Big(\frac{2g'(u(x_k))^4}{g(u(x_k))^5|g''(u(x_k))|} + \frac{g'(u(x_k))^2}{g(u(x_k))^4}\Big)q(x_k)T\big(\nabla u(x_k), \nabla u(x_k)\big). \end{aligned} \tag{3.29}$$

Since $g \ge 1$, then $1/g^2 \le 1/g$ and

$$\frac{g'(u(x_k))^2}{g(u(x_k))^2|g''(u(x_k))|} \le \frac{g'(u(x_k))^2}{g(u(x_k))|g''(u(x_k))|}.$$

On the other hand, by (3.16) we also have

$$q(x_k)T\big(\nabla u(x_k), \nabla u(x_k)\big) \le C|\nabla u(x_k)|^2.$$

Using these two facts in (3.29), jointly with (3.27), yields

$$\frac{g'(u(x_k))^3}{g(u(x_k))^4|g''(u(x_k))|}\,\varphi\big(u(x_k),|\nabla u(x_k)|\big)\le\frac{g'(u(x_k))^2}{g(u(x_k))|g''(u(x_k))|}\Big(\frac{1}{k}+\frac{2C}{k^2}\Big)+\frac{C}{k^2}.$$

Next, we use the Taylor formula with respect to y centered at $(u(x_k),0)$ and (3.18) to obtain

$$\varphi\big(u(x_k),|\nabla u(x_k)|\big)\ge f\big(u(x_k)\big)+\frac{\partial\varphi}{\partial y}\big(u(x_k),0\big)\big|\,\nabla u(x_k)\big|,$$

so that

$$\frac{g'(u(x_k))^3 f(u(x_k))}{g(u(x_k))^4|g''(u(x_k))|}+A_k\le\frac{g'(u(x_k))^2}{g(u(x_k))|g''(u(x_k))|}\Big(\frac{1}{k}+\frac{2C}{k^2}\Big)+\frac{C}{k^2}, \tag{3.30}$$

where

$$A_k:=\min\Big\{0,\frac{1}{k}\frac{\partial\varphi}{\partial y}\big(u(x_k),0\big)\frac{g'(u(x_k))^2}{g(u(x_k))^2|g''(u(x_k))|}\Big\}.$$

In what follows, we always assume that t is taken sufficiently large. Observe that we have

$$\frac{g'(t)^2}{g(t)|g''(t)|}=2\,\frac{\big(\int_a^t F(s)ds\big)^{1/2}}{g(t)F(t)}=2\,\frac{\int_a^t F(s)ds}{g(t)\big(\int_a^t F(s)ds\big)^{1/2}F(t)},$$

and

$$g(t)\ge\frac{t-a-1}{\big(\int_a^t F(s)ds\big)^{1/2}},$$

so that

$$\frac{g'(t)^2}{g(t)|g''(t)|}\le c\,\frac{\int_a^t F(s)ds}{tF(t)},\qquad t\gg 1,$$

for some positive constant c. Therefore, using (3.20) we deduce

$$\limsup_{k\to+\infty}\frac{g'(u(x_k))^2}{g(u(x_k))|g''(u(x_k))|}<+\infty,$$

and then

$$\limsup_{k\to+\infty}\frac{g'(u(x_k))^2}{g(u(x_k))|g''(u(x_k))|}\Big(\frac{1}{k}+\frac{2C}{k^2}\Big)+\frac{C}{k^2}=0. \tag{3.31}$$

On the other hand,

$$\frac{g'(t)^3\,f(t)}{g(t)^4\,|g''(t)|}=\frac{2f(t)}{g(t)^4F(t)}\ge c,\frac{f(t)}{F(t)}$$

for some $c>0$, since $\sup_M g<+\infty$ by (3.19). Therefore, using (3.21) we have

$$\liminf_{k\to+\infty}\frac{g'(u(x_k))^3,f(u(x_k))}{g(u(x_k))^4\,|g''(u(x_k))|}>0. \tag{3.32}$$

Finally, observe that

$$\frac{\partial\varphi}{\partial y}(t,0)\,\frac{g'(t)^2}{g(t)^2\,|g''(t)|}=\frac{1}{g(t)^2}\Big(\frac{\partial\varphi}{\partial y}(t,0)\,\frac{\big(\int_a^t F(s)ds\big)^{1/2}}{F(t)}\Big)$$

whence, using $\sup_M g<+\infty$ and (3.22), we get

$$\liminf_{t\to+\infty}\Big(\frac{\partial\varphi}{\partial y}(t,0)\,\frac{g'(t)^2}{g(t)^2\,|g''(t)|}\Big)>-\infty.$$

Thus,

$$\liminf_{k\to+\infty}A_k=0. \tag{3.33}$$

Therefore, taking $k\to+\infty$ in (3.30) and using (3.31), (3.32) and (3.33) we obtain the desired contradiction.

As for the conclusion $f(u^*)\le 0$, we note that if φ were continuous in both variables, then to reach the desired conclusion it would be enough to apply the q-maximum principle to u to get a sequence $\{y_k\}$ with $\lim u(y_k)=u^*$, $\lim|\nabla u(y_k)|=0$ and

$$\frac{1}{k}>q(y_k)Lu(y_k)\ge\varphi\big(u(y_k),|\nabla u(y_k)|\big).$$

Taking the limit as $k\to+\infty$ we would get $f(u^*)\le 0$. On the other hand, with our more general assumptions, we can argue in the following way. We redefine the function $g(t)$ at the very beginning of the proof in such a way that it changes concavity only once at the point $T=\min\{u^*,a\}-1$. We emphasize that with this choice $g''<0$ on $(T,+\infty)$. We now proceed as in the proof of the first part of the theorem, applying the q-maximum principle to the function $1/g(u)$, and get the existence of a sequence $\{x_k\}$ as before, with $g''(u(x_k))<0$ if k is sufficiently large. That is all we need to arrive at (3.30). Taking the limit as $k\to+\infty$ in this last expression and using $\lim_{k\to+\infty}u(x_k)=u^*<+\infty$, we conclude that $f(u^*)\le 0$. □

4. An application to hypersurfaces into non-degenerate Euclidean cones

We begin with a general observation. Consider a complete, noncompact Riemannian manifold $(M,\langle\ ,\ \rangle)$, let $o\in M$ be a reference point, denote by $r(x)$ the Riemannian distance from o, and let $D_o=M\setminus\mathrm{cut}(o)$ be the domain of the normal geodesic coordinates centered at o. Assume that

$$K_{\mathrm{rad}}\ge -G(r)^2,$$

where K_{rad} denotes the radial sectional curvature of M, and $G\in\mathcal{C}^1(\mathbb{R}_0^+)$ satisfies

$$\text{i) } G(0)>0,\ \ \text{ii) } G'(t)\ge 0,\ \ \text{and}\ \ \text{iii) } \frac{1}{G}\notin L^1(+\infty) \tag{4.1}$$

Using the general Hessian comparison theorem of [25] one has

$$\mathrm{Hess}(r)\le\frac{g'(r)}{g(r)}\big(\langle,\rangle-dr\otimes dr\big) \tag{4.2}$$

on D_o, where $g(t)$ is the (positive on $\mathbb{R}^+$) solution of the Cauchy problem

$$\begin{cases} g''(t) - G(t)^2 g(t) = 0, \\ g(0) = 0, \quad g'(0) = 1. \end{cases} \tag{4.3}$$

Now let

$$\psi(t) = \frac{1}{G(0)}\left(e^{\int_0^t G(s)ds} - 1\right). \tag{4.4}$$

Then $\psi(0) = 0$, $\psi'(0) = 1$ and

$$\psi''(t) - G(t)^2\psi(t) = \frac{1}{G(0)}\left(G(t)^2 + G'(t)\, e^{\int_0^t G(s)ds}\right) \geq 0, \tag{4.5}$$

that is, ψ is a subsolution of (4.3). By the Sturm comparison theorem,

$$\frac{g'(t)}{g(t)} \leq \frac{\psi'(t)}{\psi(t)} \leq CG(t)\,, \tag{4.6}$$

where the last inequality holds for a constant $C > 0$ and t sufficiently large. Hence, from (4.2) and for r sufficiently large

$$\mathrm{Hess}(r) \leq CG(r)\langle\ ,\ \rangle. \tag{4.7}$$

Thus, given the symmetric positive semi-definite $(2,0)$-tensor T we have

$$Lr = \mathrm{tr}(T \circ \mathrm{Hess}(r)) \leq C(\mathrm{tr}\, T)\, G(r) \quad \text{for } r >> 1. \tag{4.8}$$

Assume that $\mathrm{tr}\, T > 0$ (equivalently, $T \neq 0$) outside a compact set of M. Then

$$\frac{1}{\mathrm{tr}\, T} Lr \leq CG(r) \tag{4.9}$$

on D_o for r sufficiently large. Since $|\nabla r| = 1$, if $\mathrm{cut}(o) = \emptyset$ condition (Γ_B) of Theorem B is satisfied; otherwise we have to prove the validity of (4.9) weakly outside a sufficiently large ball B_R. Since

$$Lu = \mathrm{tr}(T \circ \mathrm{Hess}(u)) = \mathrm{div}\left(T(\nabla u,\ )^\sharp\right) - \mathrm{div}\, T(\nabla u),$$

we have to show that, for every $\psi \in \mathcal{C}_0^\infty(M \setminus \overline{B}_R)$, $\psi \geq 0$,

$$-\int_{M\setminus \overline{B}_R} \left(T(\nabla r, \nabla\psi) + \mathrm{div}\, T(\nabla r)\psi\right) \leq C\int_{M\setminus\overline{B}_R} \mathrm{tr}\, T G(r)\psi,$$

and this can be obtained as in the proof of Lemma 4.1 in [26] under the assumption that

$$T(\nabla r, \nu) \geq 0 \quad \text{ in } \Omega, \tag{4.10}$$

for an exhaustion of $M \setminus \mathrm{cut}(o)$ by smooth bounded domains Ω, star-shaped with respect to o, where ν denotes the outward unit normal along $\partial\Omega$. We have thus proved the validity of the following:

Proposition 4.1. *Let $(M, \langle\ ,\ \rangle)$ be a complete, noncompact Riemannian manifold whose radial sectional curvature satisfies*

$$K_{\mathrm{rad}} \geq -G(r)^2 \tag{4.11}$$

with $G \in \mathcal{C}^1(\mathbb{R}^+)$ as in (4.1). Let T be a symmetric, positive semi-definite, $(2,0)$-tensor field such that $T \neq 0$ outside a compact set of M. Assume that either $\operatorname{cut}(o) = \emptyset$ or otherwise that (4.10) holds. Then the q-Omori–Yau maximum principle holds on M for the operator $L = \operatorname{tr}(T \circ \operatorname{Hess})$ with $q = 1/\operatorname{tr} T$.

Now we shall apply Proposition 4.1 when T is the k-th Newton tensor of an isometrically immersed oriented hypersurface into Euclidean space for which, from now and till the end of this section, we assume the validity of $\operatorname{cut}(o) = \emptyset$ or otherwise that of (4.10).[1] Note that for $T = I$ (4.10) is automatically satisfied. Let $\varphi\colon M^m \to \mathbb{R}^{m+1}$ denote such an immersion of a connected, m-dimensional Riemannian manifold and assume that it is oriented by a globally defined normal unit vector N. Let A denote the second fundamental form of the immersion with respect to N. Then, the k-mean curvatures of the hypersurface are given by

$$H_k = \binom{m}{k}^{-1} S_k,$$

where $S_0 = 1$ and, for $k = 1, \ldots, m$, S_k is the k-th elementary symmetric function of the principal curvatures of the hypersurface. In particular, $H_1 = H$ is the mean curvature, H_m is the Gauss–Kronecker curvature, and H_2 is, up to a constant, the scalar curvature of M.

The Newton tensors $P_k : TM \to TM$ associated to the immersion are defined inductively by $P_0 = I$ and

$$P_k = S_k I - A P_{k-1}, \quad 1 \leq k \leq m.$$

Note, for further use, that

$$\operatorname{Tr} P_k = (m-k) S_k = c_k H_k \quad \text{and} \quad \operatorname{Tr} A P_k = (k+1) S_{k+1} = c_k H_{k+1},$$

where

$$c_k = (m-k)\binom{m}{k} = (k+1)\binom{m}{k+1}.$$

Associated to each globally defined Newton tensor $P_k : TM \to TM$, we may consider the second order differential operator $L_k : \mathcal{C}^2(M) \to \mathcal{C}(M)$ given by

$$L_k = \operatorname{Tr}(P_k \circ \operatorname{Hess}) = \operatorname{div}\left(P_k(\nabla u,\ )^\sharp\right) - \left\langle \operatorname{div} P_k, \nabla u \right\rangle,$$

where $\operatorname{div} P_k = \operatorname{Tr} \nabla P_k$. In particular, L_0 is the Laplace–Beltrami operator Δ. Observe that L_k is semi-elliptic (respectively, elliptic) if and only if P_k is positive semi-definite (respectively, positive definite).

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[1] *Added in proof.* After the completion of this paper we have been able to get rid of assumption (4.10) in Proposition 4.1 (see Theorem 3 in [4]). Hence in Proposition 4.1 as well as in the statements of the remaining results in this Section we can remove the assumption: "Assume that either $\operatorname{cut}(o) = \emptyset$ or otherwise that (4.10) holds."

Remark 4.2. In this respect, it is worth pointing out that the ellipticity of the operator L_1 is guaranteed by the assumption $H_2 > 0$. Indeed, if this happens the mean curvature does not vanish on M, because of the basic inequality $H_1^2 \geq H_2$. Therefore, we can choose the unit normal vector N on M so that $H_1 > 0$. Furthermore,

$$m^2 H_1^2 = \sum_{j=1}^{m} \kappa_j^2 + m(m-1)H_2 > \kappa_i^2$$

for every $i = 1, \ldots, m$, and then the eigenvalues of P_1 satisfy $\mu_{i,1} = mH_1 - \kappa_i > 0$ for every i (see, for instance, Lemma 3.10 in [11]). This shows ellipticity of L_1. Regarding the operator L_j when $j \geq 2$, a natural hypothesis to guarantee ellipticity is the existence an elliptic point in M, that is, a point $x \in M$ at which the second fundamental form A is positive definite (with respect to the appropriate orientation). In fact, it follows from the proof of Proposition 3.2 in [7] that if M has an elliptic point and $H_{k+1} \neq 0$ on M, then each L_j, $1 \leq j \leq k$ is elliptic.

Fix an origin $o \in \mathbb{R}^{m+1}$ and a unit vector $a \in \mathbb{S}^m$. For $\theta \in (0, \pi/2)$, we denote by $\mathcal{C} = \mathcal{C}_{o,a,\theta}$ the non-degenerate cone with vertex o, direction a and width θ, that is,

$$\mathcal{C} = \mathcal{C}_{o,a,\theta} = \Big\{ p \in \mathbb{R}^{m+1} \backslash \{o\} : \Big\langle \frac{p-o}{|p-o|}, a \Big\rangle \geq \cos\theta \Big\}.$$

By non-degenerate we mean that it is strictly smaller than a half-space. We consider here isometrically immersed hypersurfaces $\varphi\colon M^m \to \mathbb{R}^{m+1}$ with images inside a non-degenerate cone of $\mathbb{R}^{m+1}$ and, as an application of Proposition 4.1 and motivated by the results in [18], we provide a lower bound for the width of the cone in terms of higher order mean curvatures of the hypersurface. Specifically, we obtain the following result.

Theorem 4.3. *Let $\varphi\colon M^m \to \mathbb{R}^{m+1}$ be an oriented isometric immersion of a complete noncompact Riemannian manifold M^m whose radial sectional curvatures satisfy*

$$K_{\mathrm{rad}} \geq -G(r)^2$$

with $G \in \mathcal{C}^1(\mathbb{R}^+)$ as in (4.1). Assume that P_k is positive semi-definite and H_k does not vanish on M, and assume that either $\mathrm{cut}(o) = \emptyset$ or otherwise that (4.10) holds for $T = P_k$. If $\varphi(M)$ is contained into a non-degenerate cone $\mathcal{C} = \mathcal{C}_{o,a,\theta}$ as above with vertex at $o \in \mathbb{R}^{m+1} \backslash \varphi(M)$, then

$$\sup\Big(\frac{|H_{k+1}|}{H_k}\Big) \geq A_0 \frac{\cos^2\theta}{d\big(\Pi_a, \varphi(M)\big)}, \tag{4.12}$$

where $A_0 = 6\sqrt{3}/(25\sqrt{5}) \approx 0.186$, Π_a denote the hyperplane orthogonal to a passing through o and $d(\Pi_a, \varphi(M))$ is the Euclidean distance between this hyperplane and $\varphi(M)$.

Proof. To prove the theorem we shall follow the ideas of and make use of some computations performed in the proof of Theorem 1.4 in [18]. We may assume without loss of generality that the vertex of the cone is the origin $0 \in \mathbb{R}^{m+1}$, so that there exists $a \in \mathbb{S}^m$ and $0 < \theta < \pi/2$ such that

$$\left\langle \frac{\varphi(x)}{|\varphi(x)|}, a \right\rangle \geq \cos\theta \tag{4.13}$$

for every $x \in M$. Observe that

$$d\big(\Pi_a, \varphi(M)\big) = \inf_{x\in M} \langle \varphi(x), a\rangle.$$

We reason by contradiction and assume that (4.12) does not hold. Therefore, there exists $x_0 \in M$ such that

$$\big\langle \varphi(x_0), a\big\rangle \sup\Big(\frac{|H_{k+1}|}{H_k}\Big) < A\cos^2\theta$$

for a positive constant $A < A_0$. For ease of notation we set $\alpha = \langle\varphi(x_0), a\rangle > 0$, let $\beta \in (0,1)$ and define the function

$$u(x) = \sqrt{\alpha^2 + \beta^2\cos^2\theta|\varphi(x)|^2} - \langle\varphi(x), a\rangle$$

for every $x \in M$. Note that, by construction, $u(x_0) > 0$. We claim that

$$u(x) \leq \alpha$$

for every $x \in M$. Indeed, an algebraic manipulation shows that this is equivalent to

$$\langle\varphi(x), a\rangle^2 + 2\alpha\langle\varphi(x), a\rangle - \beta^2\cos^2\theta|\varphi(x)|^2 \geq 0,$$

which holds true by (4.13) since

$$\langle\varphi(x), a\rangle^2 + 2\alpha\langle\varphi(x), a\rangle - \beta^2\cos^2\theta|\varphi(x)|^2 \geq \langle\varphi(x), a\rangle^2 - \cos^2\theta|\varphi(x)|^2 \geq 0.$$

Next, we consider the closed nonempty set

$$\overline{\Omega}_0 = \big\{x \in M : u(x) \geq u(x_0)\big\}.$$

For every $x \in \overline{\Omega}_0$, using (4.13) one has

$$\sqrt{\alpha^2 + \beta^2\cos^2\theta\,|\varphi(x)|^2} \geq u(x_0) + \langle\varphi(x), a\rangle \geq u(x_0) + \cos\theta\,|\varphi(x)| > 0.$$

Squaring this inequality yields

$$(1-\beta^2)\cos^2\theta\,|\varphi(x)|^2 + 2u(x_0)\cos\theta\,|\varphi(x)| + u(x_0)^2 - \alpha^2 \leq 0$$

for every $x \in \overline{\Omega}_0$. The left half of the above inequality is a quadratic polynomial in $|\varphi(x)|$ with two distinct roots $\alpha_- < 0 < \alpha_+$ given by

$$\alpha_\pm = \frac{\pm\sqrt{\beta^2 u(x_0)^2 + (1-\beta^2)\alpha^2} - u(x_0)}{(1-\beta^2)\cos\theta}.$$

Therefore, for every $x \in \overline{\Omega}_0$, there holds

$$0 < |\varphi(x)| \leq \alpha_+ = \frac{\sqrt{\beta^2 u(x_0)^2 + (1-\beta^2)\alpha^2} - u(x_0)}{(1-\beta^2)\cos\theta}.$$

Using the elementary inequality $\sqrt{1+t^2} \leq 1+t$ for $t \geq 0$, we have

$$\begin{aligned}
\alpha_+ &= \frac{1}{(1-\beta^2)\cos\theta}\left(\sqrt{\beta^2 u(x_0)^2\left(1+\frac{(1-\beta^2)\alpha^2}{\beta^2 u(x_0)^2}\right)} - u(x_0)\right)\\
&= \frac{\beta u(x_0)}{(1-\beta^2)\cos\theta}\sqrt{1+\frac{(1-\beta^2)\alpha^2}{\beta^2 u(x_0)^2}} - \frac{u(x_0)}{(1-\beta^2)\cos\theta}\\
&\leq \frac{\beta u(x_0)}{(1-\beta^2)\cos\theta}\left(1+\frac{\sqrt{1-\beta^2}\alpha}{\beta u(x_0)}\right) - \frac{u(x_0)}{(1-\beta^2)\cos\theta}\\
&= \frac{\alpha}{\sqrt{1-\beta^2}\cos\theta} - \frac{u(x_0)}{(1+\beta)\cos\theta} \leq \frac{\alpha}{\sqrt{1-\beta^2}\cos\theta}.
\end{aligned}$$

Therefore,

$$|\varphi(x)| \leq \frac{\alpha}{\sqrt{1-\beta^2}\cos\theta} \quad \text{on } \overline{\Omega}_0. \tag{4.14}$$

To compute $L_k u = \operatorname{tr}(P_k \circ \operatorname{Hess} u)$ when P_k is the k-th Newton tensor, we first observe that

$$\nabla u = -a^\top + \frac{\beta^2\cos^2\theta}{\sqrt{\alpha^2+\beta^2\cos^2\theta\,|\varphi|^2}}\,\varphi^\top, \tag{4.15}$$

where, as usual, $^\top$ denotes the tangential component along the immersion φ. That is,

$$a = a^\top + \langle a, N\rangle N \quad \text{and} \quad \varphi = \varphi^\top + \langle \varphi, N\rangle N.$$

Using that

$$\nabla_X a^\top = \langle a, N\rangle AX \quad \text{and} \quad \nabla_X \varphi^\top = X + \langle \varphi, N\rangle AX$$

for every $X \in TM$, we get from (4.15) that

$$\begin{aligned}
\nabla^2 u(X,Y) = \langle \nabla_X \nabla u, Y\rangle &= \frac{\beta^2\cos^2\theta}{\sqrt{\alpha^2+\beta^2\cos^2\theta\,|\varphi|^2}}\,\langle X, Y\rangle\\
&\quad + \left\langle \frac{\beta^2\cos^2\theta}{\sqrt{\alpha^2+\beta^2\cos^2\theta\,|\varphi|^2}}\,\varphi - a, N\right\rangle \langle AX, Y\rangle\\
&\quad + \frac{-\beta^4\cos^4\theta}{(\alpha^2+\beta^2\cos^2\theta\,|\varphi|^2)^{3/2}}\,\langle X, \varphi^\top\rangle\langle Y, \varphi^\top\rangle,
\end{aligned} \tag{4.16}$$

for every $X, Y \in TM$. Hence,

$$L_k u = \sum_{i=1}^{m} \nabla^2 u(e_i, Pe_i) = \Big\langle \frac{\xi}{|\varphi|}\,\varphi - a, N\Big\rangle\,\mathrm{tr}(A \circ P_k) + \frac{\xi}{|\varphi|}\,\mathrm{tr}(P_k) \tag{4.17}$$

$$- \frac{\xi^2}{|\varphi|^2}\frac{1}{\sqrt{\alpha^2 + \beta^2\cos^2\theta\,|\varphi|^2}}\langle P_k\varphi^\top, \varphi^\top\rangle, \tag{4.18}$$

where

$$\xi(x) = \frac{\beta^2\cos^2\theta\,|\varphi(x)|}{\sqrt{\alpha^2 + \beta^2\cos^2\theta\,|\varphi(x)|^2}}.$$

That is,

$$\begin{aligned} L_k u = {} & c_k\Big\langle \frac{\xi}{|\varphi|}\,\varphi - a, N\Big\rangle H_{k+1} + c_k\frac{\xi}{|\varphi|}H_k \\ & - \frac{\xi^2}{|\varphi|^2}\frac{1}{\sqrt{\alpha^2 + \beta^2\cos^2\theta\,|\varphi|^2}}\langle P_k\varphi^\top, \varphi^\top\rangle. \end{aligned} \tag{4.19}$$

Observe that, by (4.13),

$$\Big| \frac{\xi}{|\varphi|}\,\varphi - a\Big|^2 = \xi^2 - 2\xi\frac{\langle\varphi, a\rangle}{|\varphi|} + 1 \le \xi^2 - 2\cos\theta\xi + 1 \le 1, \tag{4.20}$$

since $0 < \xi(x) < \beta\cos\theta$ for every $x \in M$. On the other hand, since P_k is positive semi-definite we have

$$0 \le \langle P_k\varphi^\top, \varphi^\top\rangle \le \mathrm{tr}(P_k)|\varphi^\top|^2 \le c_k H_k |\varphi|^2. \tag{4.21}$$

Since, by hypothesis, $H_k > 0$ on M, we obtain from here that

$$\begin{aligned} \frac{1}{c_k H_k} L_k u & \ge -\frac{|H_{k+1}|}{H_k} + \frac{\xi}{|\varphi|} - \frac{\xi^2}{\sqrt{\alpha^2 + \beta^2\cos^2\theta\,|\varphi|^2}} \\ & \ge -\sup\frac{|H_{k+1}|}{H_k} + \frac{\alpha^2\beta^2\cos^2\theta}{(\alpha^2 + \beta^2\cos^2\theta\,|\varphi|^2)^{3/2}} \end{aligned} \tag{4.22}$$

on M. Recall that, by our choice of x_0, we have

$$\sup\frac{|H_{k+1}|}{H_k} < A\frac{\cos^2\theta}{\alpha}$$

for a positive constant $A < A_0 = 6\sqrt{3}/(25\sqrt{5})$. On the other hand, by (4.14) we also have

$$|\varphi|^2 < \frac{\alpha^2}{(1-\beta^2)\cos^2\theta} \tag{4.23}$$

on $\overline{\Omega}_0$. This yields

$$\frac{\alpha^2\beta^2\cos^2\theta}{(\alpha^2 + \beta^2\cos^2\theta\,|\varphi|^2)^{3/2}} \ge \frac{\cos^2\theta}{\alpha}\beta^2(1-\beta^2)^{3/2}$$

on $\overline{\Omega}_0$. Choose $\beta = \sqrt{2/5}$. Then, $\beta^2(1-\beta^2)^{3/2} = A_0$ and

$$\frac{1}{c_k H_k} L_k u \geq \frac{\cos^2\theta}{\alpha}(A_0 - A) > 0 \quad \text{on } \overline{\Omega}_0. \tag{4.24}$$

There are now two possibilities:

i) x_0 is an absolute maximum for u on M. Then, $L_k u(x_0) \leq 0$, contradicting (4.24).

ii) $\Omega_0 = \{x \in M : u(x) > u(x_0)\} \neq \emptyset$. In this case, since $u(x)$ is bounded from above on M, it is enough to evaluate inequality (4.24) along a sequence $\{x_k\}$ realizing the $1/c_k H_k$-weak maximum principle for the operator L_k on M. This maximum principle applies because of Proposition 4.1 and the assumptions of the theorem. We thus have $u(x_k) > u^* - 1/k$ and therefore $x_k \in \Omega_0$ for k sufficiently large and

$$0 < \frac{\cos^2\theta}{\alpha}(A_0 - A) \leq \frac{1}{c_k H_k} L_k u(x_k) < \frac{1}{k}.$$

By taking $\lim_{k\to\infty}$ in this inequality we get a contradiction.

This completes the proof of the theorem. □

Corollary 4.4. *Let $\varphi : M^m \to \mathbb{R}^{m+1}$ be an oriented isometric immersion of a complete noncompact Riemannian manifold M^m whose radial sectional curvatures satisfy*

$$K_{\mathrm{rad}} \geq -G(r)^2$$

with $G \in \mathcal{C}^1(\mathbb{R}^+)$ as in (4.1). Assume that P_k is positive semi-definite, and assume that either $\mathrm{cut}(o) = \emptyset$ or otherwise that (4.10) holds for $T = P_k$. If $\varphi(M)$ is contained in a non-degenerate cone $\mathcal{C} = \mathcal{C}_{o,a,\theta}$ as above with vertex at $o \in \mathbb{R}^{m+1} \backslash \varphi(M)$, then

$$\sup |H_{k+1}| \geq A_0 \frac{\cos^2\theta}{d(\Pi_a, \varphi(M))} \inf H_k, \tag{4.25}$$

where $A_0 = 6\sqrt{3}/(25\sqrt{5}) \approx 0.186$, Π_a denote the hyperplane orthogonal to a passing through o, and $d(\Pi_a, \varphi(M))$ is the Euclidean distance between this hyperplane and $\varphi(M)$.

For the proof of Corollary 4.4 observe that (4.25) holds trivially if $\inf_M H_k = 0$. If $\inf_M H_k > 0$, then $H_k > 0$ everywhere and the result follows directly from Theorem 4.3 since the estimate (4.25) is weaker than (4.12).

On the other hand, in the case of $k = 1$ we can slightly improve our Theorem 4.3 with respecto to both the condition on the ellipticity of P_1 and the value of the constant A_0 in (4.12). Specifically we prove the following.

Corollary 4.5. *Let $\varphi : M^m \to \mathbb{R}^{m+1}$ be an oriented isometric immersion of a complete noncompact Riemannian manifold M^m whose radial sectional curvatures satisfy*

$$K_{\mathrm{rad}} \geq -G(r)^2$$

with $G \in \mathcal{C}^1(\mathbb{R}^+)$ as in (4.1). Assume that either $\mathrm{cut}(o) = \emptyset$ or otherwise that (4.10) *holds for $T = P_1$. If $H_2 > 0$ (equivalently, the scalar curvature of M is positive) and $\varphi(M)$ is contained in a non-degenerate cone $\mathcal{C} = \mathcal{C}_{o,a,\theta}$ as above with vertex at $o \in \mathbb{R}^{m+1} \backslash \varphi(M)$, then*

$$\sup \sqrt{H_2} \geq \sup\left(\frac{H_2}{H_1}\right) \geq B_m \frac{\cos^2\theta}{d\big(\Pi_a, \varphi(M)\big)}, \tag{4.26}$$

where $B_2 = B_3 = A_0 = 6\sqrt{3}/(25\sqrt{5}) \approx 0.186$, and, for $m \geq 4$,

$$B_m = \max_{0<\varrho<1}\left(\varrho^2\sqrt{1-\varrho^2}\,(1 - \frac{3}{m}\varrho^2)\right)$$

We emphasize that $B_m > A_0$ and $B_m \sim 2/(3\sqrt{3}) \approx 0.385$ when m goes to infinity.

Proof. According to Remark 4.2, the assumptions $H_2 > 0$ and $m^2H_1^2 - |A|^2 = m(m-1)H_2 > 0$ guarantee that P_1 is positive definite for an appropriate choice of the unit normal N, so that $H_1 > 0$ and $mH_1 - |A| > 0$ on M.

By the Cauchy–Schwarz inequality,

$$H_1^2 - H_2 = \frac{1}{m(m-1)}\left(\sum_{i=1}^m \kappa_i^2 - \frac{1}{m}\left(\sum_{i=1}^m \kappa_i\right)^2\right) \geq 0.$$

This immediately yields $H_2/H_1 \leq \sqrt{H_2}$ and gives the first inequality in (4.26).

As for the second inequality in (4.26), arguing as in the proof of Theorem 4.3, we reason by contradiction and assume that there exists a point $x_0 \in M$ such that

$$\alpha \sup\left(\frac{H_2}{H_1}\right) < A\cos^2\theta \tag{4.27}$$

for a positive constant $A < B_m$, where $\alpha = \langle \varphi(x_0), a\rangle$. We then follow the proof of Theorem 4.3 until we reach (4.19), which jointly with (4.20) yields

$$L_1 u \geq -c_1 H_2 + c_1 \frac{\xi}{|\varphi|} H_1 - \frac{\xi^2}{|\varphi|^2}\frac{1}{\sqrt{\alpha^2 + \beta^2\cos^2\theta\,|\varphi|^2}}\langle P_1\varphi^\top, \varphi^\top\rangle.$$

The idea for improving the value of the constant A_0 in (4.12) is to improve the estimate (4.21) in the following way. Using that $P_1 = mH_1 I - A$ we have

$$\langle P_1\varphi^\top, \varphi^\top\rangle = mH_1|\varphi^\top|^2 - \langle A\varphi^\top, \varphi^\top\rangle \leq 2mH_1|\varphi|^2, \tag{4.28}$$

because of the fact that

$$\big|\langle A\varphi^\top, \varphi^\top\rangle\big| \leq |A|\,|\varphi^\top|^2 \leq mH_1|\varphi|^2.$$

Note that (4.28) gives a better estimate than (4.21) for $k = 1$ when $m \geq 4$. In this case, making use of (4.28) we obtain

$$\begin{aligned}\frac{1}{c_1H_1}L_1 u &\geq -\frac{H_2}{H_1} + \frac{\xi}{|\varphi|} - \frac{2}{m-1}\frac{\xi^2}{\sqrt{\alpha^2+\beta^2\cos^2\theta\,|\varphi|^2}}\\ &\geq -\sup\frac{H_2}{H_1} + \frac{\alpha^2\beta^2\cos^2\theta + \frac{m-3}{m}\beta^4\cos^4\theta\,|\varphi|^2}{\big(\alpha^2+\beta^2\cos^2\theta\,|\varphi|^2\big)^{3/2}}\end{aligned}$$

on M, instead of (4.22). It follows from (4.23) that

$$\frac{\alpha^2\beta^2\cos^2\theta + \frac{m-3}{m}\beta^4\cos^4\theta\,|\varphi|^2}{\left(\alpha^2+\beta^2\cos^2\theta\,|\varphi|^2\right)^{3/2}} \ge \frac{\cos^2\theta}{\alpha}\beta^2\sqrt{1-\beta^2}\Big(1-\frac{3}{m}\beta^2\Big)$$

on $\overline{\Omega}_0$. Choose $\beta\in(0,1)$ to maximize $\varrho^2\sqrt{1-\varrho^2}\,(1-3\varrho^2/m)$. That is,

$$\beta^2 = \frac{4+m-\sqrt{(4+m)^2-40m/3}}{10}$$

and

$$B_m = \beta^2\sqrt{1-\beta^2}\Big(1-\frac{3}{m}\beta^2\Big).$$

Then,

$$\frac{1}{c_1H_1}L_1u \ge \frac{\cos^2\theta}{\alpha}\,(B_m - A) > 0 \quad \text{on } \overline{\Omega}_0. \tag{4.29}$$

The proof then finishes as for Theorem 4.3. □

For the case $k\ge 2$ there is an inequality corresponding to the first one in (4.26), given by

$$\sup_M \sqrt[k+1]{H_{k+1}} \ge \sup_M\Big(\frac{H_{k+1}}{H_k}\Big).$$

However, to guarantee its validity ones needs to assume the existence of an elliptic point (see [4] for details).

5. An application to PDEs

We give a typical application of Theorem A to PDEs in the following comparison theorem. To this end, we make the following definition: a function $f\colon\mathbb{R}^+\to\mathbb{R}^+$ is said to be ζ-increasing if for every $\zeta>1$ and for every closed interval $I\subset\mathbb{R}^+$ there exists $A=A(\zeta,I)>0$ such that

$$\frac{f(\zeta t)}{f(t)} \ge 1+A \tag{5.1}$$

for every $t\in I$. Note that this implies that $tf(t)$ is strictly increasing on $\mathbb{R}^+$. Typical examples of ζ-increasing functions are $f(t)=t^\sigma\log^a(1+t)$ with $\sigma\ge 1$, $a\ge 0$, $f(t)=t^\sigma e^{at}$ with $\sigma\ge 0, a>0$, and so on.

Theorem 5.1. *Let $a(x),b(x)\in\mathcal{C}^0(M)$ and let $f\in\mathcal{C}^1(\mathbb{R}^+)$ be a ζ-increasing function. Assume that*

$$\text{i)}\ \ b(x)>0 \quad \text{on } M \quad \text{and} \quad \text{ii)}\ \ \sup_M\frac{a_-}{b}<+\infty, \tag{5.2}$$

where, as usual, a_- denotes the negative part of a.

For $L = L_{T,X}$ as in our previous notation, let $u, v \in \mathcal{C}^2(M)$ be non-negative solutions of

$$Lu + a(x)u - b(x)uf(u) \geq 0 \geq Lv + a(x)v - b(x)vf(v) \tag{5.3}$$

on M satisfying

$$\text{i)}\ \ v(x) \geq C_1, \quad \text{ii)}\ \ u(x) \leq C_2 \tag{5.4}$$

outside some compact set $K \subset M$ for some positive constants C_1, C_2. Then

$$u(x) \leq v(x)$$

on M provided that the $1/b$-weak maximum principle holds for L.

As an immediate consequence, we have:

Corollary 5.2. *With the assumptions of Theorem 5.1, the equation*

$$Lu + a(x)u - b(x)uf(u) = 0$$

has at most one nonnegative, nontrivial, and bounded solution u with

$$\liminf_{x \to \infty} u(x) > 0.$$

Proof of Theorem 5.1. We can assume that $u \not\equiv 0$, otherwise, there is nothing to prove. Next, the differential inequality

$$Lv + a(x)v - b(x)vf(v) \leq 0$$

and (5.4) i), together with the strong maximum principle (see the observation after the proof of Theorem 3.5 on page 35 of [14]), imply $v > 0$ on M. This fact and (5.4) tell us that

$$\zeta = \sup_M \frac{u}{v} \tag{5.5}$$

satisfies

$$0 < \zeta < +\infty.$$

If $\zeta \leq 1$ then $u \leq v$ on M. Aiming of a contradiction, let us assume that $\zeta > 1$ and define

$$\varphi = u - \zeta v.$$

Note that $\varphi \leq 0$ on M. It is not hard to see, using (5.4) and (5.5), that

$$\sup_M \varphi = 0. \tag{5.6}$$

We now use (5.3) and the linearity of L to compute

$$L\varphi \geq -a(x)\varphi + b(x)\big[uf(u) - \zeta vf(\zeta v)\big] + b(x)\zeta v\big[f(\zeta v) - f(v)\big]. \tag{5.7}$$

Let

$$h(x) = \begin{cases} \big[f(u) + uf'(u)\big](x) & \text{if} \quad u(x) = \zeta v(x) \\ \dfrac{1}{u(x) - \zeta v(x)} \displaystyle\int_{\zeta v(x)}^{u(x)} \big[f(t) + tf'(t)\big]dt & \text{if} \quad u(x) < \zeta v(x). \end{cases}$$

Observe that h is continuous on M and non-negative, since

$$(tf(t))' = f(t) + tf'(t) \geq 0 \quad \text{on } \mathbb{R}^+.$$

Furthermore, we can rewrite (5.7) in the form

$$L\varphi \geq \big[-a(x) + b(x)h(x)\big]\varphi + b(x)\zeta v\big[f(\zeta v) - f(v)\big],$$

and using $-a(x)\varphi \geq a_-(x)\varphi$ we get

$$L\varphi \geq \big[a_-(x) + b(x)h(x)\big]\varphi + b(x)\zeta v\big[f(\zeta v) - f(v)\big]. \tag{5.8}$$

Let

$$\Omega_{-1} = \big\{x \in M : \varphi(x) > -1\big\}.$$

On Ω_{-1} we have

$$v(x) = \frac{1}{\zeta}(u(x) - \varphi(x)) \leq \frac{1}{\zeta}(C + 1) \tag{5.9}$$

for some positive constant C, since u is bounded from above on M. Using the definition of h and the mean value theorem for integrals, we deduce

$$h(x) = f(y) + yf'(y)$$

for some $y = y(x) \in [u(x), \zeta v(x)]$. Since $u(x)$ and $v(x)$ are bounded form above on Ω_{-1},

$$h(x) \leq C \tag{5.10}$$

on Ω_{-1} for some constant $C > 0$.

Next we recall that $b(x) > 0$ on M to rewrite (5.8) in the form

$$\frac{1}{b(x)}L\varphi \geq \Big[\frac{a_-(x)}{b(x)} + h(x)\Big]\varphi + \zeta v\big[f(\zeta v) - f(v)\big].$$

Since $\varphi \leq 0$, (5.2) ii) and (5.10) imply

$$\Big[\frac{a_-(x)}{b(x)} + h(x)\Big]\varphi \geq C\varphi$$

for some appropriate constant $C > 0$ on Ω_{-1}. Thus

$$\frac{1}{b(x)}L\varphi \geq C\varphi + \zeta v\big[f(\zeta v) - f(v)\big]$$

on Ω_{-1}. Since f is ζ-increasing, there exists $A > 0$ such that

$$\zeta v\big[f(\zeta v) - f(v)\big] \geq \zeta A v f(v) \quad \text{on } \Omega_{-1}.$$

Now we use the fact that v, and hence $vf(v)$, is bounded from below by a positive constant to get

$$\frac{1}{b(x)}L\varphi \geq C\varphi + B \quad \text{on } \Omega_{-1},$$

for some positive constant B.

Finally, we choose $0 < \varepsilon < 1$ sufficiently small such that

$$C\varphi > -\frac{1}{2}B$$

on $\Omega_{-\varepsilon} = \{x \in M : \varphi(x) > -\varepsilon\} \subset \Omega_{-1}$. Therefore,

$$\frac{1}{b(x)}L\varphi \geq \frac{1}{2}B > 0 \quad \text{on } \Omega_{-\varepsilon}.$$

Having assumed the validity of the $1/b$-weak maximum principle for the operator L on M, we immediately get a contradiction, proving that $\zeta \leq 1$. □

6. A glimpse at the nonlinear case

In this section we will introduce an extension of Theorems A and B to the nonlinear case. Since solutions of PDEs involving the type of operators we shall consider are not, in general, of class $\mathcal{C}^2$ even for constant coefficients, it will be more appropriate to work, from the very beginning, in the weak setting. Think for instance of the p-Laplace operator with $p \neq 2$, $p > 1$.

We let $A : \mathbb{R}^+ \to \mathbb{R}$ and we define $\varphi(t) = tA(t)$. The following assumptions will be crucial for applying Theorems 5.3 and 5.6 of [26] and shall therefore be assumed throughout this section:

(A1) $A \in \mathcal{C}^1(\mathbb{R}^+)$.

(A2) i) $\varphi'(t) > 0$ on $\mathbb{R}^+$, ii) $\varphi(t) \to 0$ as $t \to 0^+$.

(A3) $\varphi(t) \leq Ct^\delta$ on $(0, \omega)$ for some $\omega, C, \delta > 0$.

(T1) T is a positive definite, symmetric, $(2,0)$-tensor field on M.

(T2) For every $x \in M$ and for every $\xi \in T_xM$, $\xi \neq 0$, the bilinear form

$$\frac{A'(|\xi|)}{|\xi|}\langle \xi, \cdot \rangle \odot T(\xi, \cdot) + A(|\xi|)T(\cdot, \cdot)$$

is symmetric and positive definite. Here $\odot$ denotes the symmetric tensor product.

Note that the above requirements are not mutually independent. Indeed the bilinear form in (T2) is automatically symmetric when T is. Furthermore, if (T2) is written it in terms of φ, it becomes the condition that, for every $x \in M$ and for every $\xi, v \in T_xM$, $\xi, v \neq 0$,

$$\frac{1}{|\xi|^2}\Big(\varphi'(|\xi|) - \frac{\varphi(|\xi|)}{|\xi|}\Big)\langle \xi, v\rangle T(\xi, v) + \frac{\varphi(|\xi|)}{|\xi|}T(v, v) > 0.$$

In particular, the choice $v = \xi$ shows that

$$\varphi'(t) > 0 \quad \text{on} \quad \mathbb{R}^+,$$

that is, requirement i) in (A2). Condition (T2) is in fact equivalent to i) in (A2) in case $T = t(x)\langle , \rangle$ is a “pointwise conformal” deformation of the metric for some smooth function $t(x) > 0$ on M. Indeed, in this case (T2) reduces to

$$\frac{1}{|\xi|^2}\,\varphi'(|\xi|)t(x)\langle\xi,v\rangle^2 + \frac{\varphi(|\xi|)}{|\xi|^3}\,t(x)\left(|v|^2|\xi|^2 - \langle\xi,v\rangle^2\right) > 0$$

for every $x \in M$ and for every $\xi, v \in T_xM$, $\xi, v \neq 0$.

Having fixed a vector field X on M, we define the operator $L = L_{A,T,X}$ acting on $\mathcal{C}^1(M)$ by

$$Lu = \operatorname{div}\left(A(|\nabla u|)T(\nabla u,\cdot)^\sharp\right) - \langle X, \nabla u\rangle$$

for each $u \in \mathcal{C}^1(M)$, where $^\sharp : T^*M \to TM$ denotes the musical isomorphism. Of course, the above operator L has to be understood in the appropriate weak sense.

Observe that sometimes we shall refer to ω, C and δ in (A3) as the *structure constants* of the operator L.

L gives rise to various familiar operators. For instance, choosing $T = \langle,\rangle$ and $X = 0$ we have:

1. For $\varphi(t) = t^{p-1}$, $p > 1$,

$$Lu = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

is the usual p-Laplacian. Note that for the structure constants we have $C = 1$, $\delta = p - 1$ and $\omega = +\infty$. Of course the case $p = 2$ yields the usual Laplace–Beltrami operator.

2. For $\varphi(t) = t/\sqrt{1+t^2}$ the operator

$$Lu = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

is the usual mean curvature operator. Here $C = 1$, $\delta = 1$ and $\omega = +\infty$;

and so on.

We let, as in the linear case, $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, be such that, for some compact $K \subset M$, $q(x) > 0$ on $M \setminus K$. However, since our setting now is that of solutions in the weak sense, for technical reasons (see for instance (6.3) in the proof of Theorem A″ below) we need the local integrability of $1/q$ also inside K. Thus, from now on we suppose

$$\text{(Q)} \qquad \frac{1}{q} \in L^1_{\mathrm{loc}}(M).$$

This fact was also pointed out in Remark 3.1 for the linear case whenever we deal with functions u on M which are merely of class $\mathcal{C}^1$.

Next, we introduce the following Khas'minskiĭ type condition.

Definition 6.1. We say that the (q-SK) condition holds if there exists a telescoping exhaustion of relatively compact open sets $\{\Sigma_j\}_{j\in\mathbb{N}}$ such that $K \subset \Sigma_1$, $\overline{\Sigma}_j \subset \Sigma_{j+1}$ for every j and, for any pair $\Omega_1 = \Sigma_{j_1}$, $\Omega_2 = \Sigma_{j_2}$, with $j_1 < j_2$, and for each $\varepsilon > 0$, there exists $\gamma \in \mathcal{C}^0(M\setminus\Omega_1)\cap\mathcal{C}^1(M\setminus\overline{\Omega}_1)$ with the following properties:

i) $\gamma \equiv 0$ on $\partial\Omega_1$,

ii) $\gamma > 0$ on $M \setminus \Omega_1$,

iii) $\gamma \leq \varepsilon$ on $\Omega_2 \setminus \Omega_1$,

iv) $\gamma(x) \to +\infty$ when $x \to \infty$,

v) $q(x)L\gamma \leq \varepsilon$ on $M \setminus \overline{\Omega}_1$.

Since property v) has to be understood in the weak sense, we mean that

$$L\gamma \leq \frac{\varepsilon}{q(x)} \quad \text{weakly on } M \setminus \overline{\Omega}_1.$$

That is, for all $\psi \in \mathcal{C}_0^\infty(M \setminus \overline{\Omega}_1)$, $\psi \geq 0$,

$$\int_{M\setminus\overline{\Omega}_1} \Big(A(|\nabla\gamma|)T(\nabla\gamma, \nabla\psi) + \langle X, \nabla\gamma\rangle\psi + \frac{\varepsilon}{q}\psi\Big) \geq 0.$$

Of course we expect the (q-SK) condition in Definition 6.1 to be equivalent in the linear case to the weak form of (Γ) of Theorem A, which obviously reads as follows:

Definition 6.2. We say that the (q-KL) condition holds if there exist a compact set $H \supset K$ and a function $\tilde{\gamma} \in \mathcal{C}^1(M)$ with the following properties:

j) $\tilde{\gamma}(x) \to +\infty$ when $x \to \infty$,

jj) $q(x)L\tilde{\gamma} \leq B$ on $M \setminus H$ for some constant B, in the weak sense.

Obviously, the (q-SK) condition implies the (q-KL) condition simply by choosing $H = \overline{\Omega}_2$, setting $\tilde{\gamma} = \gamma$ on $M\setminus\Omega_2$ and extending it on Ω_2 to be of class $\mathcal{C}^1$ on M. We shall prove the equivalence of the two conditions in the linear case after the proof of Theorem A$''$. The point is that in the form (q-SK) the Khas'minskiĭ type condition is not only sufficient for the validity of the q-weak maximum principle but indeed equivalent in many cases (see [20]). For a certain class of operators this happens also in the nonlinear case as is shown in [20].

Before stating Theorem A$''$ we recall that for an operator L, a function $q(x) > 0$ on an open set $\Omega \subset M$ and $u \in \mathcal{C}^1(\Omega)$ the inequality

$$\inf_{\Omega} \big\{q(x)Lu(x)\big\} \leq 0$$

holds in the weak sense if for each $\varepsilon > 0$

$$-\int_\Omega \big(A(|\nabla u|)T(\nabla u, \nabla\psi) + \langle X, \nabla u\rangle\psi\big) \leq \int_\Omega \frac{\varepsilon}{q}\psi$$

for each $\psi \in \mathcal{C}_0^\infty(\Omega)$, $\psi \geq 0$.

We are now ready to state the nonlinear version of Theorem A.

Theorem A$''$. *Let $(M, \langle,\rangle)$ be a Riemannian manifold and let L be as above. Let $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, and suppose that $q(x) > 0$ outside some compact set $K \subset M$ and $1/q \in L^1_{\mathrm{loc}}(M)$. Assume the validity of* (q-SK). *If $u \in \mathcal{C}^1(M)$ and $u^* = \sup_M u < +\infty$ then, for each $\eta > 0$,*

$$\inf_{A_\eta} \big\{q(x)Lu(x)\big\} \leq 0 \tag{6.1}$$

holds in the weak sense, where

$$A_\eta = \big\{x \in M : u(x) > u^* - \eta\big\}. \tag{6.2}$$

Proof. We argue by contradiction and suppose that for some $\eta > 0$ there exists $\varepsilon_0 > 0$ such that

$$Lu \geq \frac{\varepsilon_0}{q(x)}$$

holds weakly on A_η, that is, for each $\psi \in \mathcal{C}_0^\infty(A_\eta)$, $\psi \geq 0$,

$$\int_{A_\eta} \left(A(|\nabla u|)T(\nabla u, \nabla\psi) + \langle X, \nabla u\rangle\psi + \frac{\varepsilon_0}{q}\psi\right) \leq 0. \tag{6.3}$$

Note that, since in general $A_\eta \not\subset M \setminus K$, the assumption (Q) is essential.

First we observe that u^* cannot be attained at any point $x_0 \in M$. Otherwise $x_0 \in A_\eta$ and, because of (6.3), on the open set A_η there holds weakly

$$Lu \geq 0. \tag{6.4}$$

Since, with our assumptions, the strong maximum principle given in Theorem 5.6 of [26] holds, we deduce that $u \equiv u^*$ on the connected component of A_η containing x_0, which contradicts (6.3).

Next we let Σ_j be the telescoping sequence of relatively compact open domains of condition (q-SK). Given $u^* - \frac{\eta}{2}$, there exists Σ_{j_1} such that

$$u^*_{j_1} = \sup_{\overline{\Sigma}_{j_1}} u > u^* - \frac{\eta}{2}.$$

We set $\Omega_1 = \Sigma_{j_1}$ and define

$$u_1^* = u^*_{j_1}.$$

Note that, since u^* is not attained on M,

$$u^* - \frac{\eta}{2} < u_1^* < u^*. \tag{6.5}$$

We can therefore fix α so that

$$u_1^* < \alpha < u^*. \tag{6.6}$$

Since $\alpha > u_1^*$, there exists Σ_{j_2} with $j_2 > j_1$ such that, setting $\Omega_2 = \Sigma_{j_2}$, $u_2^* = \sup_{\Omega_2} u = \max_{\bar{\Omega}_2} u$, we have

$$\overline{\Omega}_1 \subset \Omega_2$$

and furthermore

$$u_1^* < \alpha < u_2^* < u^*. \tag{6.7}$$

We fix $\bar{\eta} > 0$ so small that

$$\alpha + \bar{\eta} < u_2^* \tag{6.8}$$

and

$$\bar{\eta} < \varepsilon_0. \tag{6.9}$$

We apply the (q-SK) condition with the choice $\varepsilon = \bar{\eta}$ and Ω_1 and Ω_2 as above to obtain the existence of $\gamma \in \mathcal{C}^0(M \setminus \Omega_1) \cap \mathcal{C}^1(M \setminus \overline{\Omega}_1)$ satisfying the properties listed in Definition 6.1. Construct

$$\sigma(x) = \alpha + \gamma(x). \tag{6.10}$$

Then

$$\sigma(x) = \alpha \text{ on } \partial\Omega_1, \tag{6.11}$$

$$\alpha < \sigma(x) \leq \alpha + \bar{\eta} \text{ on } \Omega_2 \setminus \bar{\Omega}_1, \tag{6.12}$$

$$\sigma(x) \to +\infty \text{ as } x \to \infty, \tag{6.13}$$

and, since $\nabla\sigma = \nabla\gamma$, $L\sigma = L\gamma$ and by v) of Definition 6.1

$$q(x)L\sigma \leq \bar{\eta} \text{ in the weak sense on } M \setminus \bar{\Omega}_1. \tag{6.14}$$

Next, we consider the function $u - \sigma$. Because of (6.11) and (6.6), we have for every $x \in \partial\Omega_1$,

$$(u - \sigma)(x) = u(x) - \alpha \leq u_1^* - \alpha < 0. \tag{6.15}$$

Since $u_2^* = \max_{\bar{\Omega}_2} u$ and $\bar{\Omega}_2$ is compact, u_2^* is attained at some $\bar{x} \in \bar{\Omega}_2$. Note that $\bar{x} \notin \bar{\Omega}_1$, because otherwise

$$u_1^* \geq u(\bar{x}) = u_2^*,$$

contradicting (6.7). Thus $\bar{x} \in \bar{\Omega}_2 \setminus \bar{\Omega}_1$. By (6.8) we have

$$u(\bar{x}) > \alpha + \bar{\eta}.$$

Thus, by (6.12) and (6.8), we have

$$(u - \sigma)(\bar{x}) = u_2^* - \sigma(\bar{x}) \geq u_2^* - \alpha - \bar{\eta} > 0. \tag{6.16}$$

Finally, because of (6.13), there exists Σ_ℓ, $\ell > j_2$, such that

$$(u - \sigma)(x) < 0 \text{ on } M \setminus \Sigma_\ell. \tag{6.17}$$

Because of (6.15), (6.16) and (6.17) the function $u - \sigma$ attains an absolute maximum $m > 0$ at a certain point $z_0 \in \Sigma_\ell \setminus \bar{\Omega}_1 \subset M \setminus \bar{\Omega}_1$. At z_0, by (6.6) and (6.5), we have

$$u(z_0) = \sigma(z_0) + m > \sigma(z_0) = \alpha + \gamma(z_0) \geq \alpha > u_1^* > u^* - \frac{\eta}{2},$$

an hence $z_0 \in A_\eta$. It follows that

$$\Xi = \left\{x \in M \setminus \bar{\Omega}_1 : (u - \sigma)(x) = m\right\} \subset A_\eta. \tag{6.18}$$

Since A_η is open there exists a neighborhood U_Ξ of Ξ contained in A_η. Pick any $y \in \Xi$, fix $\beta \in (0, m)$ and call $\Xi_{\beta,y}$ the connected component of the set

$$\left\{x \in M \setminus \bar{\Omega}_1 : (u - \sigma)(x) > \beta\right\}$$

containing y. Since $\beta > 0$,

$$\Xi_{\beta,y} \subset \bar{\Sigma}_\ell \setminus \bar{\Omega}_1 \subset M \setminus \bar{\Omega}_1,$$

and we can also choose β sufficiently near to m so that $\bar{\Xi}_{\beta,y} \subset A_\eta$. Furthermore, $\bar{\Xi}_{\beta,y}$ is compact. Because of (6.14), (6.9) and (6.3), on $\Xi_{\beta,y}$ we have

$$q(x)Lu(x) \geq \varepsilon_0 > q(x)L\gamma(x)$$

in the weak sense. Furthermore,

$$u(x) = \sigma(x) + \beta \quad \text{on } \partial\Xi_{\beta,y}.$$

Hence by Theorem 5.3 of [26],

$$u(x) \leq \sigma(x) + \beta \quad \text{on } \Xi_{\beta,y}.$$

This contradicts the fact that $y \in \Xi_{\beta,y}$. Indeed,

$$u(y) = \sigma(y) + m > \sigma(y) + \beta$$

since $m > \beta$. This completes the proof of Theorem A$''$. □

Suppose now that L is linear, that is, $A(t) = 1$ (and hence $\varphi(t) = t$). Once (T1) is satisfied, assumptions (A1), (A2), (A3) and (T2) are also satisfied. Let $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, be such that, for some compact $K \subset M$, $q(x) > 0$ on $M \setminus K$ and $1/q \in L^1_{\mathrm{loc}}(M)$. Observe that in this case the (q-KL) condition and the linearity of L imply the (q-SK) condition. Indeed, fix a strictly increasing divergent sequence $\{T_j\} \nearrow +\infty$ and let

$$\Sigma_j = \big\{x \in M : \tilde{\gamma}(x) < T_j\big\}.$$

Obviously, each Σ_j is open and because of j) in the (q-KL) condition one immediately verifies that $\bar{\Sigma}_j = \{x \in M : \tilde{\gamma}(x) \leq T_j\}$ is compact. For the same reason we can suppose T_1 chosen so large that $K \subset H \subset \Sigma_1$. Furthermore $\bar{\Sigma}_j \subset \Sigma_{j+1}$ and again by j) in the (q-KL) condition, $\{\Sigma_j\}$ is a telescoping exhaustion. Consider any pair

$$\Omega_1 = \Sigma_{j_1} = \big\{x \in M : \tilde{\gamma}(x) < T_{j_1}\big\} \quad \text{and} \quad \Omega_2 = \Sigma_{j_2} = \big\{x \in M : \tilde{\gamma}(x) < T_{j_2}\big\}$$

with $j_2 > j_1$, and choose $\varepsilon > 0$. Let $\sigma \in (0, \sigma_0)$ and define $\gamma : M \setminus \Omega_1 \to \mathbb{R}^+_0$ by

$$\gamma(x) = \sigma(\tilde{\gamma}(x) - T_{j_1}).$$

Then

i) $\gamma(x) = 0$ for every $x \in \partial\Omega_1$,

ii) $\gamma(x) > 0$ if $x \in M \setminus \overline{\Omega}_1 = \{x \in M : \tilde{\gamma}(x) > T_{j_1}\}$,

iii) on $\Omega_2 \setminus \Omega_1 = \{x \in M : T_{j_1} \leq \tilde{\gamma}(x) < T_{j_2}\}$ we have $\gamma(x) < \sigma(T_{j_2} - T_{j_1})$ and hence, up to having chosen σ_0 sufficiently small, $\gamma(x) \leq \varepsilon$ on $\Omega_2 \setminus \Omega_1$,

iv) $\gamma(x)\to+\infty$ when $x\to\infty$, because of j), and

v) on $M\setminus\overline{\Omega}_1$ and by linearity of L,

$$q(x)L\gamma = q(x)L(\sigma(\tilde\gamma - T_{j_1})) = q(x)\sigma L\tilde\gamma \le \sigma B \le \varepsilon$$

because of jj) and up to having chosen σ_0 sufficiently small.

Remark 6.3. It is worth giving some examples where the (q-SK) condition is satisfied. For the sake of simplicity we limit ourselves to the case $T=\langle,\rangle$ and $X\equiv 0$. Let $(M,\langle,\rangle)$ be a complete, noncompact Riemannian manifold of dimension $m\ge 2$. Let $o\in M$ be a fixed reference point, denote by $r(x)$ the Riemannian distance from o and suppose that

$$\mathrm{Ric}(\nabla r,\nabla r)\ge -(m-1)G(r)^2 \tag{6.19}$$

for some positive non-decreasing function $G(r)\in\mathcal{C}^0(\mathbb{R}_0^+)$, $G(r)>0$, with $1/G\notin L^1(+\infty)$. Similarly to what has been done in Section 4 and for the same ψ defined there (see (4.4)), by the Laplacian comparison theorem we have

$$\Delta r\le (m-1)\frac{\psi'}{\psi}(r) \tag{6.20}$$

weakly on M for $r\ge R_0>0$ sufficiently large.

Suppose now that the function $q(x)\in\mathcal{C}^0(M)$, $q(x)\ge 0$, satisfies

$$q(x)\le\Theta(r(x)) \tag{6.21}$$

outside a compact set $K\subset M$, for some non-increasing continuous function $\Theta:\mathbb{R}_0^+\to\mathbb{R}^+$ with the property that

$$\Theta(t)\le BG^{\delta-1}(t) \tag{6.22}$$

for $t\gg 1$ and some constant $B>0$ (here δ is as in (A3)). Note that if $\delta\ge 1$, (6.22) is automatically satisfied.

Fix $\sigma>0$ and $R\ge R_0$ such that $K\subset B_R$, B_R being the geodesic ball of radius R, and define the function

$$\chi_\sigma(r)=\int_R^r \varphi^{-1}(\sigma h(t))\,dt,\qquad r\in[R,+\infty), \tag{6.23}$$

where

$$h(t)=\psi^{1-m}(t)\int_R^t\frac{\psi^{m-1}(s)}{\Theta(s)}ds.$$

Since $\varphi:\mathbb{R}_0^+\to[0,\varphi(+\infty))=I\subseteq\mathbb{R}_0^+$ increasingly, $\varphi:I\to\mathbb{R}_0^+$. Therefore in order that χ_σ be well defined when $\varphi(+\infty)<+\infty$, we need that, for every $t\in[R,+\infty)$,

$$\sigma h(t)\in I. \tag{6.24}$$

To this end we note that

$$\frac{\psi'}{\psi}(t) = G(t)\frac{e^{\int_0^t G(s)ds}}{e^{\int_0^t G(s)ds}-1} \sim CG(t) \quad \text{as } t \to +\infty. \tag{6.25}$$

Then

$$h(t) \le \frac{1}{\Theta(t)}\,\psi^{1-m}(t)\int_R^t \psi^{m-1}(s)\,ds \le \frac{C}{\Theta(t)\,G(t)} \tag{6.26}$$

for $t \gg 1$ and some $C > 0$. The assumption

$$\limsup_{r\to+\infty} \frac{1}{\Theta(r)G(r)} < +\infty$$

is therefore enough to guarantee that $h(t)$ is bounded above. By choosing σ sufficiently small, say $0 < \sigma \le \sigma_0$, we obtain the validity of (6.24) so that (6.23) is well defined on $[R, +\infty)$.

Define $\gamma(x) = \chi_\sigma(r(x))$ for $x \in M \setminus B_R$ and note that

i) $\gamma \equiv 0$ on ∂B_R,

ii) $\gamma > 0$ on $M \setminus \overline{B_R}$,

Moreover, having fixed $\varepsilon > 0$ and a second geodesic ball $B_{\hat{R}}$ with $\hat{R} > R$, since $\varphi^{-1}(t) \to 0$ as $t \to 0^+$, up to choosing $\sigma > 0$ sufficiently small we have also $\chi_\sigma(r) \le \varepsilon$ if $R \le r < \hat{R}$, so that

iii) $\gamma \le \varepsilon$ on $B_{\hat{R}} \setminus B_R$,

On the other hand, since $1/G \notin L^1(+\infty)$, to prove that

iv) $\gamma(x) \to +\infty$ when $x \to \infty$

it suffices to show that

$$\varphi^{-1}(\sigma h(t)) \ge \frac{\hat{C}}{G(t)} \qquad \text{for } t \gg 1$$

for some constant $\hat{C} > 0$. Equivalently, that there exists a constant $\hat{C} > 0$ such that

$$\frac{h(t)}{\varphi\big(\frac{\hat{C}}{G(t)}\big)} \ge \frac{1}{\sigma} \qquad \text{for } t \gg 1. \tag{6.27}$$

Without loss of generality we can suppose $G(t) \to +\infty$ as $t \to +\infty$. By the structural condition (A3) on φ we have

$$\varphi\Big(\frac{\hat{C}}{G(t)}\Big) \le C\frac{\hat{C}^\delta}{G(t)^\delta},$$

so that

$$\frac{h(t)}{\varphi\big(\hat{C}/G(t)\big)} \ge \frac{A(t)}{B(t)}$$

with

$$A(t) = G(t)^{\delta} \int_R^t \frac{\psi^{m-1}(s)}{\Theta(s)} ds \quad \text{and} \quad B(t) = C\hat{C}^{\delta}\psi^{m-1}(t).$$

Note that both $A(t)$ and $B(t)$ diverge to $+\infty$ as $t \to +\infty$. Hence,

$$\liminf_{t\to+\infty} \frac{A(t)}{B(t)} \geq \liminf_{t\to+\infty} \frac{A'(t)}{B'(t)}.$$

A computation that uses $G' \geq 0$, $\Theta > 0$ and (6.22) shows that

$$\frac{A'(t)}{B'(t)} \geq \frac{G(t)}{BC\hat{C}^{\delta}(m-1)\frac{\psi'(t)}{\psi(t)}}, \quad t \gg 1,$$

and since $\psi'(t)/\psi(t) \sim G(t)$ as $t \to +\infty$, we can choose $\hat{C} > 0$ sufficiently small that

$$\liminf_{t\to+\infty} \frac{A'(t)}{B'(t)} \geq \frac{1}{\sigma},$$

proving the validity of (6.27)

Clearly, by definition, $\chi_\sigma(t)$ is non-decreasing and satisfies $\chi'_\sigma(t) = \varphi^{-1}(\sigma h(t))$, that is, $\varphi(\chi'_\sigma(t)) = \sigma h(t)$. Therefore

$$\nabla\gamma = \chi'_\sigma(r)\nabla r, \quad |\nabla\gamma| = \chi'_\sigma(r) \quad \text{and} \quad \varphi(|\nabla\gamma|) = \sigma h(r).$$

Since

$$h'(t) = \frac{1}{\Theta(t)} - (m-1)\frac{\psi'}{\psi}(t)h(t),$$

a computation using (6.20) and (6.21) gives

$$\begin{aligned} L\gamma &= \operatorname{div}\left(\frac{\varphi(|\nabla\gamma|)}{|\nabla\gamma|}\nabla\gamma\right) = \operatorname{div}\left(\sigma h(r)\nabla r\right) = \sigma h'(r)|\nabla r|^2 + \sigma h(r)\Delta r \\ &= \frac{\sigma}{\Theta(r)} + \sigma h(r)\Big(\Delta r - (m-1)\frac{\psi'}{\psi}(r)\Big) \leq \frac{\sigma}{\Theta(r)} \leq \frac{\sigma}{q(x)} \end{aligned} \tag{6.28}$$

if $r \geq R$. That is,

v) $q(x)L\gamma \leq \sigma$ on $M \setminus \overline{B_R}$

outside the cut locus and weakly on all of $M \setminus \overline{B_R}$ as can be proved easily.

It is now clear how to satisfy the requirements of the (q-SK) condition in Definition 6.1 by choosing a telescoping exhaustion $\{B_{R+j}\}_{j\in\mathbb{N}}$.

Remark 6.4. Here we give another example where the (q-SK) condition is satisfied with $T = \langle , \rangle$ and arbitrary X. Let $(M, \langle , \rangle)$ be a complete, noncompact Riemannian manifold of dimension $m \geq 2$. Let $o \in M$ be a fixed reference point, denote by $r(x)$ the Riemannian distance from o and suppose, as in the previous example, that

$$\operatorname{Ric}(\nabla r, \nabla r) \geq -(m-1)G(r)^2 \tag{6.29}$$

for some positive non-decreasing function $G(r) \in \mathcal{C}^0(\mathbb{R}_0^+)$, $G(r) > 0$, with $1/G \notin L^1(+\infty)$. We know that, for the same function ψ,

$$\Delta r \leq (m-1)\frac{\psi'}{\psi}(r) \leq CG(r) \tag{6.30}$$

weakly on M for $r \geq R_0 > 0$ sufficiently large and some $C > 0$.

Suppose now that the function $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, satisfies

$$q(x) \leq \frac{1}{G(r(x)) + |X(x)|} \tag{6.31}$$

outside a compact set $K \subset M$. Fix $\sigma > 0$ and $R \geq R_0$ such that $K \subset B_R$, B_R being the geodesic ball of radius R centered at o, and define the function

$$\gamma(x) = \sigma(r(x) - R) \quad \text{for } x \in M \setminus B_R. \tag{6.32}$$

Obviously,

i) $\gamma \equiv 0$ on ∂B_R,

ii) $\gamma > 0$ on $M \setminus \overline{B_R}$,

Moreover, having fixed $\varepsilon > 0$ and a second geodesic ball $B_{\hat{R}}$ with $\hat{R} > R$, up to choosing $\sigma > 0$ sufficiently small we also have

iii) $\gamma \leq \varepsilon$ on $B_{\hat{R}} \setminus B_R$,

On the other hand, since M is complete,

iv) $\gamma(x) \to +\infty$ when $x \to \infty$

Finally, a direct computation using (6.30) and (6.31) gives

$$\begin{aligned} L\gamma &= \operatorname{div}\Big(\frac{\varphi(|\nabla\gamma|)}{|\nabla\gamma|}\nabla\gamma\Big) - \langle X, \nabla\gamma\rangle = \operatorname{div}\big(\varphi(\sigma)\nabla r\big) - \sigma\langle X, \nabla r\rangle \\ &= \varphi(\sigma)\Delta r - \sigma\langle X, \nabla r\rangle \leq \varphi(\sigma)CG(r) + \sigma|X| \\ &\leq \varepsilon(G(r) + |X|) \leq \frac{\varepsilon}{q(x)} \end{aligned}$$

if $r \geq R$, up to choosing $\sigma > 0$ sufficiently small, since $\varphi(\sigma) \to 0$ as $\sigma \to 0^+$. That is,

v) $q(x)L\gamma \leq \varepsilon$ on $M \setminus \overline{B_R}$

outside the cut locus $\operatorname{cut}(o)$ and weakly on all of $M \setminus \overline{B_R}$ as can be proved easily. It is now clear how to satisfy the requirements of the (q-SK) condition in Definition 6.1 by choosing a telescoping exhaustion $\{B_{R+j}\}_{j\in\mathbb{N}}$.

For the next result we define the (q-SK∇) condition as the (q-SK) condition with the added requirement:

vi) $|\nabla\gamma| < \varepsilon$ on $M \setminus \Omega_1$.

Theorem B″. *Let* $(M, \langle,\rangle)$ *be a Riemannian manifold and let* L *be as above. Let* $q(x) \in \mathcal{C}^0(M)$, $q(x) \geq 0$, *satisfy* (Q). *Assume the validity of* (q-SK∇). *If* $u \in \mathcal{C}^1(M)$ *and* $u^* = \sup_M u < +\infty$ *then, for each* $\eta > 0$,

$$\inf_{B_\eta} \{q(x)Lu(x)\} \leq 0 \tag{6.33}$$

holds in the weak sense, where

$$B_\eta = \{x \in M : u(x) > u^* - \eta \quad \textit{and} \quad |\nabla u(x)| < \eta\}.$$

Proof. First of all note that the validity of (q-SK∇) implies, once we fix arbitrarily a pair $\Omega_1 \subset \Omega_2$, an $\varepsilon > 0$ and a corresponding γ, that the metric is geodesically complete. Indeed, let $\varsigma : [0, \ell) \to M$ be any divergent path parametrized by arc-length. Thus ς lies eventually outside any compact subset of M. From vi), $|\nabla\gamma| \leq \varepsilon$ outside the compact subset $\bar{\Omega}_1$. We set $h(t) = \gamma(\varsigma(t))$ on $[t_0, \ell)$, where t_0 has been chosen so that $\varsigma(t) \notin \bar{\Omega}_1$ for all $t_0 \leq t < \ell$. Then, for every $t \in [t_0, \ell)$ we have

$$|h(t) - h(t_0)| = \left| \int_{t_0}^{t} h'(s)ds \right| \leq \int_{t_0}^{t} \left|\nabla\gamma(\varsigma(s))\right| ds \leq \varepsilon(t - t_0).$$

Since ς is divergent, then $\varsigma(t) \to \infty$ as $t \to \ell^-$, so that $h(t) \to +\infty$ as $t \to \ell^-$ because of iv). Therefore, letting $t \to \ell^-$ in the inequality above, we conclude that $\ell = +\infty$. This shows that divergent paths in M have infinite length and in other words, that the metric is complete.

Since the metric is complete, we can apply the Ekeland quasi-minimum principle to deduce that $B_\eta \neq \emptyset$ and therefore that the infimum in (6.33) is meaningful.

Now we proceed as in the proof of Theorem A″ replacing, as in the linear case, the subset A_η with the smaller open set B_η. We need to show that the compact set Ξ defined in (6.18) satisfies $\Xi \subset B_\eta$. Because of (6.8) it is enough to prove that for every $z \in \Xi$,

$$|\nabla u(z)| < \eta. \tag{6.34}$$

But z is a point of absolute maximum for $(u - \sigma)$ and $z \in M \setminus \bar{\Omega}_1$. Hence, using vi) of (q-SK∇),

$$|\nabla u(z)| = |\nabla\sigma(z)| = |\nabla\gamma(z)| < \varepsilon.$$

Thus $\Xi \subset B_\eta$ and the rest of the proof is now exactly as at the end of Theorem A″. This finishes the proof of Theorem B″. □

Suppose now that L is linear; we get an analogue of condition (q-KL), called (q-KL∇), by adding the requirement

jjj) $|\nabla\tilde{\gamma}| \leq A$ on $M \setminus H$, for some constant $A > 0$.

It is immediate to show that this condition and linearity of L imply (q-KS∇).

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