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# A graph counterexample to Davies' conjecture

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**Abstract.** There exists a graph with two vertices $x$ and $y$ such that the ratio of the heat kernels $p(x,x;t)/p(y,y;t)$ does not converge as $t\to\infty$.

## 1. Introduction

This paper is concerned with a conjecture of Brian Davies from 1997 on the heat kernel of Riemannian manifolds, see [4], §5. We will not disprove the conjecture as stated, but rather transform it to the realm of graphs using a well-known (though informal) "dictionary" between these two categories, and build a graph that will serve as a counterexample. We will make some remarks on how the construction might be carried over back to the category of manifolds, but we will not give all the details. The bulk of this paper is about graphs.

We start by describing the conjecture in its original setting. Let $M$ be a connected Riemannian manifold, and let $p$ be the *heat kernel* associated with the *Laplace–Beltrami operator* on $M$. Then Davies' conjecture states that for any $M$ and any 3 points $x,y,z\in M$ the limit

$$\lim_{t\to\infty}\frac{p(x,y;t)}{p(z,z;t)} \tag{1}$$

exists and is positive. Here $p(x,y;t)$ is the value of the heat kernel at time $t$ and at points $x$ and $y$. This property is known as the "strong ratio limit property" (where the "weak" version is an averaged result due to Döblin, [5]) or SRLP for short. So Davies' conjecture is that in these settings SRLP always holds. SRLP holds for manifolds with one end [3], and for strongly Liouville manifolds (i.e., manifolds where any positive harmonic function is constant), see Corollary 2.7 of [1], which also makes interesting connections between these properties and the infinite Brownian loop.

Ratio limit properties were considered for Markov chains even earlier. If $M$ is any Markov chain on a countable state space, then we say that $M$ satisfies SRLP

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if (1) holds for any three states $x$, $y$ and $z$, where $p(x, y; t)$ is the probability that the Markov chain started at $x$ will be at $y$ at time $t$. For general Markov chains there are a few examples where SRLP does not hold. Clearly it does not hold when the Markov chain has some kind of periodicity. F. J. Dyson constructed an example of an aperiodic recurrent Markov chain which does not satisfy SRLP, see [2], part I, § 10. That example utilizes long chains of states with only one outgoing edge, which the walker must traverse sequentially. In particular it is not *reversible*.

Now, the Laplace–Beltrami operator is self-adjoint so a proper analog of Davies conjecture needs to assume that the Markov chain is reversible. Reversible Markov chains are also known as random walks on weighted graphs. The issue of periodicity can be dealt with by looking at random walk in continuous time or at lazy random walk. Lazy random walk is a walk where the walker, at every step, chooses with probability 1/2 to stay where it is, and with probability 1/2 moves to one of the neighbours (with probability proportional to the weights).

The main result in this paper is:

**Theorem.** *There exists a connected graph $G$ with bounded weights and vertices $x, y \in G$ such that the heat kernel of the lazy random walk satisfies*

$$\frac{p(x,x;t)}{p(y,y;t)} \nrightarrow \quad \text{as } t \to \infty. \tag{2}$$

Let us remark on the "bounded weights" clause. When making analogies between manifolds and graphs, it is often assumed that the manifold has bounded geometry and the graph has bounded weights. Davies, however, explicitly does not assume bounded geometry. Thus one might wonder what exactly the graph analog is. All this is moot, of course, since the counterexample does have bounded weights (and hence a manifold example constructed along the same line should have bounded geometry).

It is easy to see that in this setting (reversible, irreducible, lazy) this ratio must be bounded between two constants independent of $t$. Hence if it does not converge then it must fluctuate between two values. The proof constructs a graph with two halves, denoted $H^{\mathrm{e}}$ and $H^{\mathrm{o}}$ (e and o standing for even and odd, $H$ standing for half), which are connected by one edge, $(x, y)$. On the "odd scales", $H^{\mathrm{e}}$ will "look like $\mathbb{Z}^{22}$" while $H^{\mathrm{o}}$ will "look like $\mathbb{Z}^3$". This means that to get from $x$ to $x$ (where $x$ is on the $H^{\mathrm{e}}$ side), the most beneficial strategy is to move to $y$ as fast as possible, spend most of your time on the $H^{\mathrm{o}}$ side and return to the $H^{\mathrm{e}}$ side only at the last minute. Clearly this would mean that $p(x, x; t)$ is smaller than $p(y, y; t)$ as the random walk starting from $y$ can stay on its side at all time, not losing the constant that $x$ needs for the maneuver. For the "even scales" the picture is reversed and $y$ is at a disadvantage. See Figure 1 – drawing in 22 dimensions might have distracted the reader, so the figure demonstrates the construction in 1 and 2 dimensions. The smaller two braces in the figure are the first scale, in which $H^{\mathrm{o}}$ is really one dimensional and $H^{\mathrm{e}}$ is really two-dimensional. The larger two braces indicate the second scale. This time $H^{\mathrm{o}}$ is a network of lines so it should be thought of as two-dimensional, while $H^{\mathrm{e}}$ is a thick column, so it should be thought

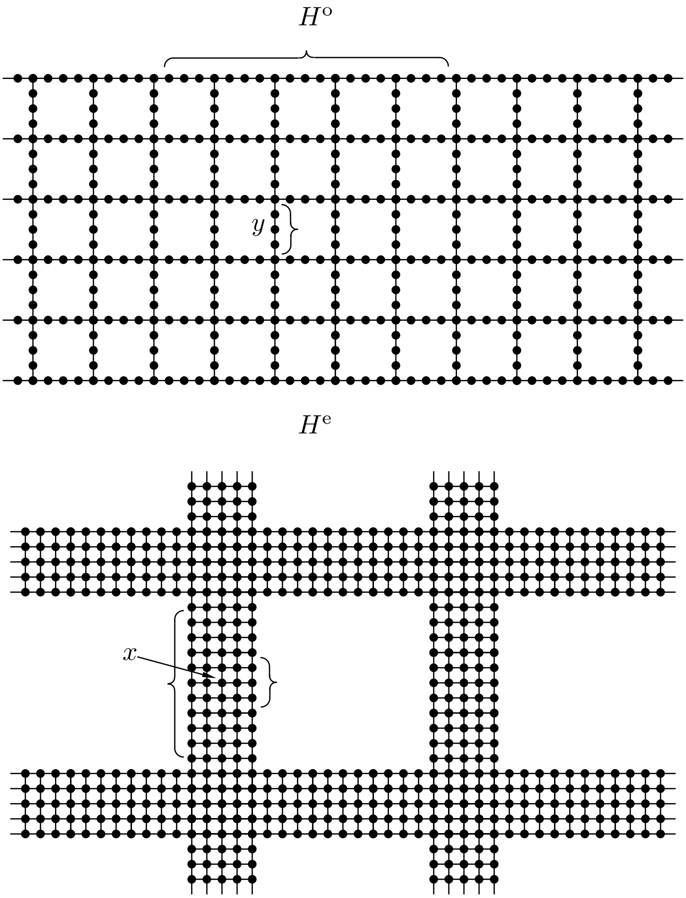


FIGURE 1. The graphs $H^{\mathrm{o}}$ and $H^{\mathrm{e}}$.

of as one-dimensional. The third scale is only hinted at in this figure, but one can imagine that $H^{\mathrm{o}}$ now becomes a thick band, so it is again one-dimensional, while $H^{\mathrm{e}}$ becomes a network of these thick columns and bands, so it is back to being two-dimensional.

As one might expect, the numbers 3 and 22 have no particular significance. They both must be greater than 2, since otherwise our graph would be recurrent and recurrent graphs always satisfy SRLP, see [10], Theorem 3. And of course they have to be different. We took here the large value 22 in order to be able to be wasteful at various points (sum over times and such stuff), but the proof could proceed with any value larger than 3.

This paper was first written in 2006. I wish to take this opportunity to apologize to all those who have had to wait so long for it to appear, with no real reason. My intentions were good but my time management was abysmal. I wish to thank Yehuda Pinchover for telling me about the problem and for reading early drafts.

## 2. Proof

The construction uses $\mathbb{Z}^d$-like graphs as building blocks, so we start by quoting a few results about these. We first recall the notion of rough isometry, [8].

**Definition.** Let $X$ and $Y$ be metric spaces. We say that $X$ and $Y$ are *roughly isometric* if there is a constant $C$ and a map $\varphi\colon X \to Y$ with the following properties:

i) For all $x$ and $y$ in $X$,

$$\frac{1}{C}\,d(x,y) - C \le d(\varphi(x),\varphi(y)) \le C\,d(x,y) + C$$

ii) The image of $\varphi$ is roughly dense, that is, for all $y \in Y$ there is an $x \in X$ such that $d(y,\varphi(x)) \le C$

If $G$ and $H$ are graphs we say that they are roughly isometric if they are roughly isometric when considered with the metric $d$ being the graph distance, namely $d(x,y)$ is the length of the shortest path between $x$ and $y$, or $\infty$ if no such path exists.

With this definition we can state the following standard result, essentially due to Delmotte.

**Lemma 1.** *Let $G$ be a graph roughly isometric to $\mathbb{Z}^d$. Then the heat kernel $p$ for the lazy walk on $G$ satisfies, for all $t \ge 1$,*

$$c\,t^{-d/2} \le p(x,x;t) \le C\,t^{-d/2}.$$

Here $G$ is a simple graph (we do not allow weights or multiple edges), and $C$ and $c$ are constants which do not depend on $t$. In general we will use $c$ for constants which are small enough and $C$ for constants which are large enough, and different appearances of $c$ and $C$ can refer to different constants.

*Proof.* By Delmotte's theorem, [7], any $G$ which satisfies volume doubling and the Poincaré inequality, satisfies

$$p(x,x;t) \approx \frac{1}{|B(x,\sqrt{t})|}\,,$$

where $B(x,r)$ is the ball around $x$ with radius $r$ (again with the graph distance), and $|B(x,r)|$ is the sum of the degrees of the vertices in $B$. The notation $X \approx Y$ is short for $cY \le X \le CY$. The fact that $G$ is roughly isometric to $\mathbb{Z}^d$ gives

$$|B(x,r)| \approx r^d\,, \tag{3}$$

so we would get $p(x,x;t) \approx t^{-d/2}$, as needed. So we need only show that $G$ satisfies volume doubling and the Poincaré inequality.

Now, the definition of volume doubling is that, for every vertex $x$ of $G$ and every $r \geq 1$,

$$|B(x, 2r)| \leq C|B(x, r)|,$$

and this follows immediately from (3). The Poincaré inequality is not much more complicated. The definition is that, for every vertex $x$, for every $r$ and for every function $f : B(x, 2r) \to \mathbb{R}$,

$$\sum_{y \in B(x,r)} \deg(y)\, |f(y) - \overline{f}|^2 \leq C\, r^2 \sum_{(y,z) \in E(B(x,2r))} (f(y) - f(z))^2, \tag{4}$$

where

$$\overline{f} = \frac{1}{|B(x, r)|} \sum_{y \in B(x,r)} \deg(y) f(y),$$

$\deg(y)$ is the degree of $y$, and $E(B)$ is the set of edges both whose vertices are in $B$. Now, $\mathbb{Z}^d$ satisfies the Poincaré inequality (see e.g. [11], §4.1.1). It is well known and not difficult to see that the Poincaré inequality is preserved by rough isometries (this uses the fact that $\sum \deg(y)|f(y) - a|^2$ is minimized when $a = \overline{f}$). This finishes the proof. □

**Lemma 2.** *Let $G$ be a graph roughly isometric to $\mathbb{Z}^d$, $d \geq 3$, and let $x$ be some vertex. Let $p$ be the probability that lazy random walk starting from $x$ returns to $x$ for the first time at time $t$. Then $p \geq c\, t^{-d/2}$.*

*Proof.* Let $p_1$ be the same probability but without the restriction that this is the first return to $x$. This is exactly $p(x, x; t)$ and by Lemma 1 we have $p_1 \geq ct^{-d/2}$. Fix some $K$ and examine the event that the random walk returns to $x$ at $t$ and also at some time $s \in [K, t - K]$. Let $p_2$ be its probability. Using the other direction in Lemma 1, we can write

$$p_2 \leq \sum_{s=K}^{t-K} p(x, x; s)\, p(x, x; t - s) \leq C \sum_{s=K}^{t-K} s^{-d/2}(t - s)^{-d/2} \leq C\, K^{1-d/2}\, t^{-d/2}.$$

Since $d \geq 3$ we can choose $K$ sufficiently large such that $p_2 \leq p_1/2$. So we know that with probability $p_1 - p_2 \geq ct^{-d/2}$ the walk does not return to $x$ between $K$ and $t - K$. If it does reach $x$ before time $K$, do some local modification so that it does not. For example, if the original walker reached $x$ at some time $s < K$ and on the next step went to some neighbour $y$ of $x$, modify it to walk to $y$ in the first step and stay there for $s$ steps (remember that our walk is lazy) and then continue like the original walker. Clearly this "costs" only a constant and ensures our walker does not visit $x$ in the interval $[1, K]$. Do the same for the interval $[t - K, t - 1]$, losing another constant. This finishes the proof. □

**Lemma 3.** *Let $G$ be a graph and let $p$ be the heat kernel for the lazy walk on $G$. Let $t$ and $s$ satisfy $|t - s| \leq \sqrt{t}$. Then,*

$$|p(x, x; t) - p(x, x; s)| \leq C\, \frac{|t - s| \log^3 t}{\sqrt{t}}\, p(x, x; t) + C\, e^{-c \log^2 t}.$$

*Proof.* Denote by $q(x,y;t)$ the heat kernel for the *simple* random walk on $G$. Then by definition,

$$p(x,x;t)=\sum_{i=0}^{t} q(x,x;i)\binom{t}{i}2^{-t}.$$

Writing the same formula for $p(x,x;s)$ and subtracting we get

$$|p(x,x;t)-p(x,x;s)|\le\sum_i q(x,x;i)\left(\binom{t}{i}2^{-t}-\binom{s}{i}2^{-s}\right)=\Sigma_1+\Sigma_2\,,$$

where $\Sigma_1$ is the sum over all $|t-2i|\le\sqrt{t}\log t$ and $\Sigma_2$ is the rest. A simple calculation with Stirling's formula shows that

$$2^{-t}\binom{t}{i}=\sqrt{\frac{2}{\pi t}}\exp\Big(-\frac{(t-2i)^2}{2t}\Big(1+O\Big(\frac{|t-2i|+1}{t}\Big)\Big)\Big).$$

We now bound the difference between these expressions at $t$ and at $s$ by the maximum of the derivative, and get, for $i$ such that $|t-2i|\le\sqrt{t}\log t$,

$$\begin{aligned}\left|2^{-t}\binom{t}{i}-2^{-s}\binom{s}{i}\right|&\le|t-s|\Big(\frac{C+C|t-2i|}{t^{3/2}}+\frac{C|t-2i|^3}{t^{5/2}}\Big)\exp\Big(-\frac{(t-2i)^2}{2t}\Big)\\&\le\frac{C\,|t-s|\log^3 t}{t}\exp\Big(-\frac{(t-2i)^2}{2t}\Big).\end{aligned}$$

Summing over $i$ now gives

$$\Sigma_1\le\sum_{|t-2i|\le\sqrt{t}\log t} q(x,x;i)\sqrt{\frac{2}{\pi t}}e^{-(t-2i)^2/2t}\frac{C|t-s|\log^3 t}{\sqrt{t}}\le C\frac{|t-s|\log^3 t}{\sqrt{t}}p(x,x;t),$$

while

$$\Sigma_2\le C\sum_{|t-2i|>\sqrt{t}\log t} e^{-c(t-2i)^2/t}\le Ce^{-c\log^2 t},$$

proving the lemma. □

*Proof of the Theorem.* Abusing notation, for subsets $H\subset\mathbb{Z}^d$ we will not distinguish between $H$ as a set and as an induced subgraph of $\mathbb{Z}^d$ ($d$ will be 22). For the construction we need a sufficiently fast increasing sequence $a_1<a_2<\cdots$. We further assume that $a_k$ are all even and that $a_{k-1}$ divides $a_k/2$. It would have probably been enough to choose $a_k=2^{a_{k-1}}$, but it turns out simpler to choose the $a_k$ inductively, and we do this as follows. Let $a_1=2$. Assume $a_1,\dots,a_{k-1}$ have been defined. Define, for integers $m<l/2$ and $i\in\{1,\dots,22\}$,

$$\begin{aligned}Q_{l,m,i}&:=\big\{\vec{n}\in\mathbb{Z}^{22}:|n_i \bmod l|\le m\big\}\\Q_{l,m}&:=\bigcup_{\substack{I\subset\{1,\dots,22\}\\|I|=19}}\ \bigcap_{i\in I}Q_{l,m,i}.\end{aligned}$$

Here $n \bmod l \in \{-\lfloor(l-1)/2\rfloor, \dots, \lfloor l/2\rfloor\}$. In words, $Q_{l,m,i}$ is a 21-dimensional subspace of $\mathbb{Z}^{22}$ orthogonal to one of the axes, fattened up by $2m+1$ (a "slab") and repeated periodically with period $l$. $Q_{l,m}$ is the collection of all 3-dimensional subspaces, fattened and repeated similarly. The particular point $\vec{0}$ is in fact contained in all $\binom{22}{3}$ of these 3-dimensional slabs which will be a little inconvenient, so let us shift $Q_{l,m}$ by

$$v(m) = \Big(\underbrace{\tfrac{1}{2}m, \dots, \tfrac{1}{2}m}_{3 \text{ times}}, \underbrace{0, \dots, 0}_{19 \text{ times}}\Big).$$

In the shifted set $Q_{l,m} + v(m)$, the geometry of the neighbourhood of $\vec{0}$ is simpler; it is contained in just one slab. Compare with the figure on page 3. The point $x$ is in the *middle* of a fat column and not at the intersection of a column and a band.

We want to use these graphs with $l = a_j$ and $m$ a little larger than $a_{j-1}$. Precisely, define

$$b_j = \sum_{k=1}^{j} a_k.$$

With this choice of $b_j$, $Q(a_j, b_{j-1}, i)$ contains only complete components of $Q(a_l, b_{l-1}, i)$ for each $l < j$. Each such component is either contained in $Q(a_j, b_{j-1}, i)$ or disjoint from it. The same holds for the translations $Q(a_l, b_{l-1}, i) + v(a_l)$ (we need here that $a_l > 4a_{l-1}$ so let us assume this from now on). For brevity, define $v_j = v(a_j)$.

We may now define two graphs, denoted by $H^{\mathrm{e}}_{k-1}$ and $H^{\mathrm{o}}_{k-1}$ ("e" and "o" standing for even and odd) by

$$H^{\mathrm{e/o}}_{k-1} := \bigcap_{\substack{2 \le j \le k-1 \\ j \text{ even/odd}}} (Q_{a_j, b_{j-1}} + v_j).$$

We shall usually suppress the $k-1$ from the notation. It is not difficult to check that $H^{\mathrm{e/o}}$ are both roughly isometric to $\mathbb{Z}^{22}$ (the rough isometry constant depends on the "past" $a_1, \dots, a_{k-1}$). Therefore by Lemma 1 we see that there exists an $\alpha$ (again, depending on the past) such that

$$p_{H^{\mathrm{e/o}}}(x, x; t) \le \alpha\, t^{-11}. \tag{5}$$

Examine now the graphs

$$F^{\mathrm{e/o}}_{k-1} := H^{\mathrm{e/o}}_{k-1} \cap \left\{ \vec{n} \in \mathbb{Z}^{22} : |n_i| \le b_{k-1}\ \forall i = 4, \dots, 22 \right\}.$$

$F^{\mathrm{e/o}}$ are both roughly isometric to $\mathbb{Z}^3$ so by Lemma 2 there exists some $\beta$ such that

$$\mathbb{P}_{F^{\mathrm{e/o}}}(\text{the walk returns to } \vec{0} \text{ for the first time at } t) \ge \frac{1}{\beta} t^{-3/2}. \tag{6}$$

Define $\gamma_k := \lceil \max\{\alpha, \beta\} \rceil$ (as usual, $\lceil\cdot\rceil$ stands for the upper integer value). With these we can define $a_k$ to be any even number satisfying $a_k > 2\gamma_k^4 + 4a_{k-1}$ and such that $a_{k-1}$ divides $a_k/2$. This completes the description of the induction, and we define

$$H_\infty^{\mathrm{e/o}} := \bigcap_{\substack{2\le j \\ j \text{ even/odd}}} (Q_{a_j,a_{j-1}} + v_j).$$

These graphs will be the two halves of our target graph $G$.

Before continuing, let us collect some simple facts about $H_\infty^{\mathrm{e/o}}$:

i) $H_\infty^{\mathrm{e/o}}$ is connected. In fact we used this indirectly when we claimed $H_k^{\mathrm{e/o}}$ are roughly isometric to $\mathbb{Z}^{22}$.

ii) $H_\infty^{\mathrm{e/o}}$ are transient. This follows because each contains a copy of $\mathbb{Z}^3$ (namely $\{n_4 = \cdots = n_{22} = 0\}$) and transience is preserved upon adding edges. This last fact follows from conductance arguments, see e.g. [6].

Define therefore the escape probabilities

$$\varepsilon^{\mathrm{e/o}} := \mathbb{P}^{\vec{0}}_{H_\infty^{\mathrm{e/o}}}(R(t) \neq \vec{0}\,\forall t > 0)$$

($R$ being the random walk on the graph) and let $\delta := \frac{1}{2}\min\{\varepsilon^{\mathrm{e}}, \varepsilon^{\mathrm{o}}\}$. Define the graph $G$ by connecting $H_\infty^{\mathrm{e}}$ to $H_\infty^{\mathrm{o}}$ with a single edge with weight $\delta$ between $\vec{0}^{\mathrm{e}}$ and $\vec{0}^{\mathrm{o}}$. Define $x := \vec{0}^{\mathrm{e}}$ and $y = \vec{0}^{\mathrm{o}}$. This is our construction and we need to show (2), which will follow if we show that, for $k$ sufficiently large,

$$\left.\begin{aligned} p(x,x;t_{2k}) &\ge 3\,p(y,y;t_{2k}) \\ p(x,x;t_{2k+1}) &\le \tfrac{1}{3}\,p(y,y;t_{2k+1}) \end{aligned}\right\} \quad t_k := \gamma_k^4. \tag{7}$$

We will only prove the even case. The proof of the odd case is similar.

Examine $p(x,x;t_{2k})$. Since $a_{2k} > t_{2k}$ we get that

$$H_\infty^{\mathrm{e/o}} \cap [-t_{2k}, t_{2k}]^{22} = H_{2k}^{\mathrm{e/o}} \cap [-t_{2k}, t_{2k}]^{22}$$

or in other words, the steps after $2k$ have no effect whatsoever. Similarly it is possible to simplify the last stage. Namely,

$$\begin{aligned} H_{2k}^{\mathrm{e}} \cap [-t_{2k}, t_{2k}]^{22} &= H_{2k-1}^{\mathrm{e}} \cap (Q_{a_{2k},b_{2k-1}} + v_{2k}) \cap [-t_{2k}, t_{2k}]^{22} = \\ &= H_{2k-1}^{\mathrm{e}} \cap \left\{\vec{n} \in \mathbb{Z}^{22} : |n_i| \le b_{2k-1}\ \forall i = 4, \ldots, 22\right\} = F_{2k-1}^{\mathrm{e}} \end{aligned}$$

(here is where the translations by $v_j$ are used). By (6),

$$\begin{aligned} p_G(x,x;t_{2k}) &\ge \frac{1}{2}\,\mathbb{P}_{H_{2k}^{\mathrm{e}}}(R \text{ returns to } x \text{ for the first time at } t) \ge \\ &\overset{(6)}{\ge} \frac{1}{2\gamma_{2k}}\,t_{2k}^{-3/2} = \frac{1}{2}\,t_{2k}^{-7/4} \end{aligned} \tag{8}$$

(the $1/2$ comes from the first step).

To estimate $p(y,y;t_{2k})$ we divide the event $\{R(t_{2k}) = y\}$ according to whether $R$ "essentially goes through $x$" or not. Formally, denote by $T_1$ and $T_2$ the first and last time before $t_{2k}$ when $R(T) = x$ (if this does not happen, denote $T_1 = \infty$ and $T_2 = -\infty$). Then we define

$$\begin{aligned} p_1 &:= \mathbb{P}(T_1 > \gamma_{2k},\, R(t_{2k}) = y) \\ p_2 &:= \mathbb{P}(T_2 < t_{2k} - \gamma_{2k},\, R(t_{2k}) = y) \\ p_3 &:= \mathbb{P}(T_1 \leq \gamma_{2k},\, T_2 \geq t_{2k} - \gamma_{2k},\, R(t_{2k}) = y), \end{aligned}$$

so that $p(y, y; t_{2k}) \leq p_1 + p_2 + p_3$.

Now, $p_1$ and $p_2$ are easy to estimate. As above we have

$$H^{\circ}_{2k} \cap [-t_{2k}, t_{2k}]^{22} = H^{\circ}_{2k-1} \cap [-t_{2k}, t_{2k}]^{22},$$

so (5) applies and we get

$$\mathbb{P}^y_{H^{\circ}_{\infty}}(R(t) = y) \leq \gamma_{2k} t^{-11} \quad \forall t \leq t_{2k}. \tag{9}$$

Therefore,

$$\begin{aligned} p_1 &\leq \sum_{t=\gamma_{2k}}^{t_{2k}-1} \mathbb{P}(T_1 = t,\, R(t_{2k}) = y) + \mathbb{P}(T_1 = \infty,\, R(t_{2k}) = y) \\ &\leq \sum_{t=\gamma_{2k}-1}^{t_{2k}-2} \mathbb{P}_{H^{\circ}_{\infty}}(R(t) = y) + \mathbb{P}_{H^{\circ}_{\infty}}(R(t_{2k}) = y) \\ &\overset{(9)}{\leq} \sum_{t=\gamma_{2k}-1}^{t_{2k}-2} \gamma_{2k} \cdot t^{-11} + \gamma_{2k} \cdot t_{2k}^{-11}\, C\, \gamma_{2k}^{-9} = C\, t_{2k}^{-9/4} \overset{(8)}{=} o(p(x, x; t)), \end{aligned} \tag{10}$$

and similarly for $p_2$. As for $p_3$, we have

$$\mathbb{P}(T_1 \leq \gamma_{2k}) \leq \Big( \sum_{i=0}^{\infty} \mathbb{P}_{H^{\circ}_{\infty}}(r \text{ visits } y \ i \text{ times before } \gamma_{2k}) \Big) \cdot \delta \leq \frac{\delta}{\epsilon^{\text{o}}} \leq \frac{1}{2},$$

and similarly (using time reversal) for $\mathbb{P}(T_2 \geq t_{2k} - \gamma_{2k})$. Hence we get

$$p_3 \leq \frac{1}{4} \max_{t_{2k} - 2\gamma_{2k} \leq s \leq t_{2k}} p(x, x; s),$$

and, by Lemma 3,

$$\begin{aligned} p_3 &\leq \frac{1}{4}\, p(x, x; t_{2k}) \Big( 1 + O\Big( \frac{\gamma_{2k} \log^3 t_{2k}}{\sqrt{t_{2k}}} \Big) \Big) + O(e^{-c \log^2 t_{2k}}) \\ &\overset{(8)}{\leq} \frac{1}{4}\, p(x, x; t_{2k})\, (1 + o(1)). \end{aligned}$$

With the estimate (10) for $p_1$ and the corresponding estimate for $p_2$ we get

$$p(y, y; t_{2k}) \leq p(x, x; t_{2k}) \Big( \frac{1}{4} + o(1) \Big).$$

A completely symmetric argument shows that at $t_{2k+1}$ the opposite occurs:

$$p(x,x;t_{2k+1}) \le p(y,y;t_{2k+1})\Big(\frac{1}{4}+o(1)\Big)$$

proving the theorem. □

**Remark.** If one wants an example with unweighted graphs, this is not a problem; $H^{\mathrm{e}}$ and $H^{\mathrm{o}}$ are already unweighted, so the only thing needed is to connect them, instead of with an edge of weight $\delta$, with a segment sufficiently long such that the probability to traverse it is $\le \delta$. The proof remains essentially the same.

## 3. Manifolds

We would like to exhibit a manifold $M$ and two points $x, y \in M$ such that the heat kernel on $M$ satisfies

$$\frac{p(x,x;t)}{p(y,y;t)} \not\to$$

as $t \to \infty$. Here is how one might translate the construction of our theorem to the setting of manifolds. The dimension of the manifold plays little role, so we might as well construct a surface.

To a subset $H \subset \mathbb{Z}^{22}$ one can associate a manifold $H^*$ by replacing each vertex $v \in H$ with a sphere $v^*$ and every edge with a empty, baseless cylinder. Since the degree of every vertex in $H$ is $\le 44$, we may simply designate 44 disjoint circles on $\mathbb{S}^2$ and attach the cylinders to the spheres at these circles. This is reminiscent of the well-known "infinite jungle gym" construction, see some lovely pictures in [9]. The exact method of doing so is unimportant since anyway the manifold that we get is roughly isometric to $H$, considered as an induced subgraph of $\mathbb{Z}^{22}$ (one of the nice features of rough isometry is that continuous and discrete objects may be roughly isometric, as rough isometry inspects only the large scale geometry). Clearly $H^*$ can be made to be $C^\infty$.

One can then construct a (possibly different) sequence $a_k$ and two manifolds $\big(H_\infty^{\mathrm{e/o}}\big)^*$ with the only difference being that the $\alpha$ and $\beta$ must satisfy (5) and (6) for our choice of the $^*$ operation. This should be possible since $\big(H_k^{\mathrm{e/o}}\big)^*$ and $\big(F_k^{\mathrm{e/o}}\big)^*$ are roughly isometric to $\mathbb{Z}^{22}$ and $\mathbb{Z}^3$ respectively. Instead of Delmotte's theorem [7] one can use the manifold version [12] (or rather, Delmotte's theorem is the graph version of earlier results for manifolds, see [12] for historical remarks).

The argument for the transience of $\big(H_\infty^{\mathrm{e/o}}\big)^*$ should also be direct translation. Each contains a submanifold (with boundary) which is roughly isometric to $\mathbb{Z}^3$ and therefore is transient. Since transience is equivalent to the fact that for some $c > 0$ every function which is 1 at $x$ and 0 at infinity satisfies that the Dirichlet form $\langle \nabla f, \nabla f\rangle > c$, and since restricting to a submanifold only decreases the Dirichlet form, we see that $\big(H_\infty^{\mathrm{e/o}}\big)^*$ are transient. Write

$$\epsilon^{\mathrm{e/o}} = \inf_{x\in v^*, v\sim\vec{0}^{\mathrm{e/o}}} \mathbb{P}^x\big(W[0,\infty)\cap\big(\vec{0}^{\mathrm{e/o}}\big)^* = \emptyset\big),$$

where $W$ here is the Brownian motion on the manifold $\left(H_\infty^{\mathrm{e}/\mathrm{o}}\right)^*$; and where the infimum is taken over all $x$ belonging to a sphere $v^*$ where $v$ is some neighbour of $\vec{0}^{\mathrm{e}/\mathrm{o}}$ in $H_\infty^{\mathrm{e}/\mathrm{o}}$. One can now define $\delta = \frac{1}{2}\min(\varepsilon^{\mathrm{e}}, \varepsilon^{\mathrm{o}})$ and connect $\vec{0}^{\mathrm{e}}$ to $\vec{0}^{\mathrm{o}}$ by a cylinder sufficiently thin (or sufficiently long) such that the probability to traverse it in either direction before reaching a neighbouring sphere is $\le \delta$. This concludes a possible construction of a manifold $M$, and one may take $x$ to be an arbitrary point in $\left(\vec{0}^{\mathrm{e}}\right)^*$ and $y$ and arbitrary point in $\left(\vec{0}^{\mathrm{o}}\right)^*$.

The proof that $p(x,x;t)/p(y,y;t)$ does not converge should not require significant changes. We note that in our case it is possible for a Brownian motion at time $t$ to exit the box $[-t,t]^{22}$, but it is exponentially difficult to do so. Hence, for example, instead of (8) we get

$$p(x,x;t_{2K}) \ge \frac{1}{2\gamma_{2k}^2}\, t_{2k}^{-3/2} - C\, e^{-c\, t_{2k}} \ge \frac{1}{4}\, t_{2k}^{-7/4}$$

for $k$ sufficiently large. Another point to note is that Lemma 3 needs to be replaced with an appropriate analog.

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