Rev. Mat. Iberoam. **30** (2014), no. 1, 65–78
DOI 10.4171/RMI/769

Restriction spaces of A^∞

Dietmar Vogt

Abstract. In the present paper it is shown that for certain totally disconnected Carleson sets E the restriction space $A_\infty(E) = \{f|_E : f \in A^\infty\}$ has a basis. Its isomorphism type is determined. The result disproves a claim of S.R. Patel in [12]. To prove our result we analyze restriction spaces $C_\infty(E) = \{f|_E : f \in C^\infty(\mathbb{R})\}$ and then, using a result of Alexander, Taylor and Williams, we show that $A_\infty(E) = C_\infty(E)$. Among our examples there are the classical Cantor set and sets of type $E = \{x_n : n \in \mathbb{N}\} \cup \{0\}$, where $(x_n)_{n\in\mathbb{N}}$ is a null sequence in $\mathbb{R}$ with certain properties.

1. Introduction

In his paper [12] Patel claims the following result: if $E \subset [0, 2\pi[$ is a compact, totally disconnected Carleson set, then the space of restrictions of A^∞ to E in its natural locally convex topology fails to have a Schauder basis. This result, if true, would have provided us with a wealth of quite natural counterexamples for the basis problem for nuclear Fréchet spaces. This problem has, of course, been solved in the negative a long time ago by Mityagin and Zobin [7], [8], [9]. Further counterexamples have been given by Djakov and Mityagin [5], Djakov [4] and Moscatelli [10]. Quite recently the author of this note has given a very simple counterexample, which is a Fréchet algebra of C^∞-functions on $\mathbb{R}^2$ [19]. That the proof of Patel's result has a gap has been widely noted. However it remained an interesting question whether the result is correct or not. Unfortunately it is not. We present examples of sets E fulfilling all the above mentioned assumptions and for which the restriction space $A_\infty(E)$ has a basis.

In this paper A^∞ will be considered as the space of all 2π-periodic C^∞-functions on $\mathbb{R}$ for which all negative Fourier coefficients vanish. E will always denote a compact subset of $\mathbb{R}$ and when it comes to considerations about A^∞ we will always assume that $E \subset [0, 2\pi[$.

We recall that the sets E which are zero sets of an A^∞-function have been characterized by Taylor and Williams [13] and Novinger [11] as those satisfying

Mathematics Subject Classification (2010): Primary 46E10; Secondary 46A04, 46A11, 46A35, 30E05.
Keywords: Nuclear Fréchet space, basis, analytic functions, C^∞-functions.

the Carleson condition

$$\int_0^{2\pi} \log \frac{1}{d(x,E)}\, dx < \infty.$$

The sets E with the property that for any (periodic) C^∞-function f on $\mathbb{R}$ there is $g \in A^\infty$ such that f and g and all their derivatives coincide on E have been characterized by Alexander, Taylor and Williams [2] as those satisfying the strong Carleson condition (ATW-condition): there are constants C_1 and C_2 such that

$$\frac{1}{b-a}\int_a^b \log \frac{1}{d(x,E)}\, dx \le C_1 \log \frac{1}{b-a} + C_2$$

for all $0 \le a < b \le 2\pi$.

For functional analytic terminology and results we refer to [6]. For all notation concerning power series spaces, invariants like diametral dimension, (DN), (Ω), etc., and related results we refer also to the survey article [18].

2. Restriction spaces of $C^\infty(\mathbb{R})$

Let $E \subset \mathbb{R}$ be a closed set and 0 an accumulation point of E. We set

$$C_\infty(E) = \big\{f|_E \,:\, f \in C^\infty(\mathbb{R})\big\} \quad \text{and} \quad J(E) = \big\{f \in C^\infty(\mathbb{R}) \,:\, f|_E = 0\big\}.$$

Then we have in a natural way

$$C_\infty(E) \cong C^\infty(\mathbb{R})/J(E)$$

and this makes $C_\infty(E)$ a nuclear Fréchet space.

We want to characterize the functions $\varphi \in C_\infty(E)$. Of course such a characterization in terms of divided differences has been given by Whitney a long time ago, see [21] and there is a vast literature on this problem. We will give a description in this special case which fits our purposes.

Lemma 2.1. *If $\varphi \in C_\infty(E)$ and $\varphi = f|_E$ for $f \in C^\infty(\mathbb{R})$, then $f^{(p)}(0)$ is uniquely determined by φ for all $p \in \mathbb{N}_0$.*

Proof. We proceed by induction. First, $f^{(0)}(0) = f(0) = \varphi(0)$. Assume that $f^{(0)}(0), \ldots, f^{(p)}(0)$ have been shown to be uniquely determined. We obtain for $x \in E$, $x \neq 0$,

$$f^{(p+1)}(\xi) = \frac{(p+1)!}{x^{p+1}}\Big(f(x) - \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j\,!}\, x^j\Big)$$

with a suitable ξ between 0 and x.

For $x \to 0$ we have $f^{(p+1)}(\xi) \to f^{(p+1)}(0)$, hence we have

$$f^{(p+1)}(0) = \lim_{x\to 0, x\in E} \frac{(p+1)!}{x^{p+1}}\Big(\varphi(x) - \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j\,!}\, x^j\Big).$$

In particular, this limit exists. □

Definition 2.2. We set $\varphi^{(p)}(0) := f^{(p)}(0)$ for some $f \in C^\infty(\mathbb{R})$ with $f|_E = \varphi$.

Corollary 2.3. *If $f \in J(E)$ then f is flat at 0, that is, $f^{(p)}(0) = 0$ for all p.*

Proof. This follows from Lemma 2.1 because f is an extension of 0 and $g \equiv 0$ is another one. □

Lemma 2.4. *$\delta_p : \varphi \mapsto \varphi^{(p)}(0)$ is a continuous linear form on $C_\infty(E)$.*

Proof. If δ_p^∞ is the same map considered on $C^\infty(\mathbb{R})$ and $\rho : C^\infty(\mathbb{R}) \to C_\infty(E)$ the restriction map then $\delta_p^\infty = \delta_p \circ \rho$, hence δ_p is continuous, by the definition of the topology on $C_\infty(E)$. □

Lemma 2.5. *$\Delta(\varphi) := (\delta_p)_{p\in\mathbb{N}_0}$ defines a continuous, linear surjective map Δ from $C_\infty(E)$ onto the space ω of all scalar sequences.*

Proof. Continuity follows from Lemma 2.4, and surjectivity from a theorem of E. Borel (see 26.29 in [6]). □

We set $J^\infty(0) := \{f \in C^\infty(\mathbb{R}) : f^{(p)}(0) = 0 \text{ for all } p\}$ and $J_\infty(0) := \{\varphi \in C_\infty(E) : \varphi^{(p)}(0) = 0 \text{ for all } p\} = \{f|E : f \in J^\infty(0)\}$.

If $f \in J^\infty(0)$ we have, for all $x \in \mathbb{R}$ and $p \in \mathbb{N}$,

$$f(x) = \frac{f^{(p)}(\xi)}{p\,!}\, x^p$$

where ξ is between 0 and x. Therefore for any $0 \le x \le R$ and $p \in \mathbb{N}_0$ we get, setting $\|f\|_M := \sup\big\{|f(t)| : t \in M\big\}$ for any function on a set M,

$$\big|f(x)\big| \le \big\|f^{(p)}\big\|_{[0,R]} \frac{|x|^p}{p\,!}. \tag{2.1}$$

From now on we assume that $E = \{x_1, x_2, \dots\} \cup \{0\}$, where $x_n \searrow 0$. We set $\varepsilon_n = x_n - x_{n+1}$ and assume that $\varepsilon_n \ge \varepsilon_{n+1} > 0$ for all n.

Let χ be an even C^∞-function with support in $[-1/2, +1/2]$ such that $\chi \equiv 1$ in a neighborhood of 0. We set $\chi_\varepsilon(x) := \chi(x/\varepsilon)$. For any sequence $\xi \in \omega$ the function

$$f(x) = \sum_{n=1}^\infty \xi_n\, \chi_{\varepsilon_n}(x - x_n)$$

is in $C^\infty(\mathbb{R} \setminus \{0\})$ and $f(x_n) = \xi_n$ for all $n \in \mathbb{N}$.

Lemma 2.6. *Let f be as above. Then $f \in J^\infty(0)$ if and only if*

$$\lim_{n\to\infty} \frac{|\xi_n|}{\varepsilon_n^p} = 0 \quad \text{for all } p \in \mathbb{N}_0.$$

Proof. For all p we have

$$\sup_{0<|x|\le x_N} \big|f^{(p)}(x)\big| = \sup_{n\ge N} \big|\xi_n\big|\, \big\|\chi_{\varepsilon_n}^{(p)}\big\|_{\mathbb{R}} = \big\|\chi^{(p)}\big\|_{\mathbb{R}} \sup_{n\ge N} \frac{|\xi_n|}{\varepsilon_n^p}. \tag{2.2}$$

This proves the result. □

We assume now that there is $q \in \mathbb{N}$ such that

$$\sup_n \frac{x_n^q}{\varepsilon_n} < \infty. \tag{2.3}$$

Remark 2.7. If condition (2.3) is fulfilled then for any scalar sequence ξ the following are equivalent:

1. $\lim_{n\to\infty} |\xi_n|/\varepsilon_n^p = 0$ for all $p \in \mathbb{N}_0$.
2. $\lim_{n\to\infty} |\xi_n|/x_n^p = 0$ for all $p \in \mathbb{N}_0$.

We set $\alpha_n := -\log x_n$. Because $\sum_n x_n^q \leq C \sum_n \varepsilon_n < \infty$, the space

$$\Lambda_\infty(\alpha) := \left\{\xi = (\xi_1, \xi_2 \dots) \,:\, |\xi|_p = \sup_n |\xi_n|\, e^{p\alpha_n} < \infty \text{ for all } p\right\}$$

is nuclear, by the Grothendieck–Pietsch criterion (see 28.15 in [6]). We obtain:

Proposition 2.8. *If condition (2.3) is fulfilled then $\Phi : \varphi \mapsto (\varphi(x_n))_{n\in\mathbb{N}}$ maps $J_\infty(0)$ isomorphically onto $\Lambda_\infty(\alpha)$.*

Proof. If $\varphi \in J_\infty(0)$ and $f \in C^\infty(\mathbb{R})$ is any extension of φ then $f \in J^\infty(0)$ and, by inequality (2.1), we have

$$|\varphi(x_n)| \leq \frac{\|f^{(p)}\|_{[0,x_1]}}{p\,!}\, e^{-p\alpha_n}.$$

Since this holds for every extension f of φ we have

$$\sup_n |\varphi(x_n)|\, e^{p\alpha_n} \leq s(\varphi),$$

where s is a continuous seminorm on $J_\infty(0)$.

Obviously Φ is injective. Surjectivity of Φ follows from Lemma 2.6. We have, using the notation of Lemma 2.6,

$$\Phi^{-1}(\xi) = \sum_{n=1}^{\infty} \xi_n\, \chi_{\varepsilon_n}(x - x_n).$$

Continuity of Φ^{-1} follows from equation (2.2) with $N = 1$ or from the open mapping theorem. □

We will now investigate the structure of $C_\infty(E)$.

Theorem 2.9. *Let $\varphi \in C(E)$. Then $\varphi \in C_\infty(E)$ if and only if the following holds: there are numbers A_p, $p \in \mathbb{N}_0$, such that $A_0 = \varphi(0)$ and for all $p \in \mathbb{N}_0$ we have*

$$A_{p+1} = \lim_{n\to\infty} \frac{(p+1)!}{x_n^{p+1}} \Big(\varphi(x_n) - \sum_{j=0}^{p} \frac{A_j}{j\,!}\, x_n^j\Big). \tag{2.4}$$

In this case $A_p = \varphi^{(p)}(0)$ for all $p \in \mathbb{N}_0$.

Proof. Necessity follows from Lemma 2.1. From this lemma it follows also that necessarily $A_p = \varphi^{(p)}(0)$ for all $p \in \mathbb{N}_0$. We have to show that the condition is also sufficient.

Given the sequence A_p, $p \in \mathbb{N}_0$, there exists, by the E. Borel theorem, a function $g \in C^\infty(\mathbb{R})$ such that $g^{(p)}(0) = A_p$ for all $p \in \mathbb{N}_0$.

We consider the function $h = \varphi - g|_E$. For $n \in \mathbb{N}$ we have

$$h(x_n) = \varphi(x_n) - g(x_n) = \sum_{j=0}^{p} \frac{A_j}{j\,!}\, x_n^j + \frac{\tilde{A}_{p+1}}{(p+1)!}\, x_n^{p+1} - g(x_n)$$

$$= \frac{\tilde{A}_{p+1}}{(p+1)!}\, x_n^{p+1} - \frac{g^{(p+1)}(\xi)}{(p+1)!}\, x_n^{p+1} = \big(\tilde{A}_{p+1} - g^{(p+1)}(\xi)\big)\frac{x_n^{p+1}}{(p+1)!}\,.$$

By (2.4), $\tilde{A}_{p+1}$ depends on n and converges to A_{p+1} for large n, and $\xi \in]0, x_n[$ comes from Taylor's formula with the Lagrange remainder. Hence we have

$$\lim_{n\to\infty} \frac{|h(x_n)|}{x_n^{p+1}} = \lim_{n\to\infty} \frac{1}{(p+1)!}\, \big|\tilde{A}_{p+1} - g^{(p+1)}(\xi)\big| = 0$$

for all $p \in \mathbb{N}_0$.

By Lemma 2.6 and condition (2.2) there is a function $H \in J^\infty(0)$ such that $H(x_n) = h(x_n)$ for all $n \in \mathbb{N}$; that is $H|_E = h$. We set $f := g + H$. Then $f \in C^\infty(\mathbb{R})$ and $f|_E = \varphi$. □

On $C_\infty(E)$ we consider for $p = 0, 1, \ldots$ the seminorms

$$|\varphi|_p = \sup_n \Big| \frac{p\,!}{x_n^p} \Big(\varphi(x_n) - \sum_{j=0}^{p-1} \frac{\varphi^{(j)}(0)}{j\,!}\, x_n^j \Big) \Big|.$$

We fix p. For every n the function $\varphi \mapsto |\ldots|$ is a continuous seminorm, since $\varphi \to \varphi(x_n)$ and δ_j, $j = 0, \ldots, p-1$, are continuous linear forms on $C_\infty(E)$. The supremum exists for all φ, hence, by the Banach–Steinhaus Theorem, the $|\cdot|_p$ are continuous seminorms on $C_\infty(E)$.

Theorem 2.10. *The family of norms $\{|\cdot|_p,\, p \in \mathbb{N}_0\}$, is a fundamental system of seminorms in $C_\infty(E)$.*

Proof. It suffices to show that $C_\infty(E)$ is complete in the topology generated by the norms $|\cdot|_p$. Let φ_k, $k \in \mathbb{N}$, be a Cauchy sequence with respect to $|\cdot|_p$, $p \in \mathbb{N}_0$.

Since $|\varphi|_0 = \sup_{n\in\mathbb{N}} |\varphi(x_n)| = \sup\{|\varphi(x)| : x \in E\}$ the sequence φ_k converges uniformly on E to a function $\varphi \in C(E)$.

For every p the sequence

$$\frac{p\,!}{x_n^p} \Big(\varphi_k(x_n) - \sum_{j=0}^{p-1} \frac{\varphi_k^{(j)}(0)}{j\,!}\, x_n^j \Big), \quad k = 1, 2, \ldots$$

converges uniformly in n. Therefore the right-hand side of

$$\varphi_k^{(p+1)}(0) = \lim_{n\to\infty} \frac{(p+1)!}{x_n^{p+1}} \Big(\varphi_k(x_n) - \sum_{j=0}^{p} \frac{\varphi_k^{(j)}(0)}{j\,!}\, x_n^j \Big)$$

converges for all $p \in \mathbb{N}_0$. We set for $p \in \mathbb{N}_0$

$$A_{p+1} = \lim_{k\to\infty} \varphi_k^{(p+1)}(0)$$

and arrive, by induction, at the condition (2.4) for $\varphi \in C(E)$. By Theorem 2.9 we get that $\varphi \in C_\infty(E)$.

The proof that $\lim_{k\to\infty} |\varphi_k - \varphi|_p = 0$ for all p is now standard. □

Remark 2.11. The system of seminorms $|\cdot|_p$, $p \in \mathbb{N}_0$, is not increasing. To see this we choose $\varphi = P|_E$ where P is a polynomial of degree m. Then $|\varphi|_p = 0$ for $p > m$. In this case, fundamental system of seminorms means that every continuous seminorm s on $C_\infty(E)$ satisfies an estimate of the form $s(\varphi) \le C \max_{p=0,\dots,p_0} |\varphi|_p$.

3. $C_\infty(E)$ and $A_\infty(E)$

Lemma 3.1. *Condition (2.3) implies that $E = \{x_1, x_2, \dots\}$ is a Carleson set.*

Proof. We may assume that $0 < x_1 \le 1$ and obtain, with suitable $s > 0$,

$$\sum_{n=1}^{\infty} \varepsilon_n \log \frac{1}{\varepsilon_n} \le s + q \sum_{n=1}^{\infty} \varepsilon_n \log \frac{1}{x_n} \le s + q \int_0^1 \log \frac{1}{x}\, dx = s + q.$$

The second sum is a lower Riemann sum for the integral whence the second estimate. □

We will now carefully study the Carleson condition and also the strong Carleson condition of Alexander–Taylor–Williams (ATW-condition), see [2]. We start with a simple calculation. For $0 \le a < b$ we obtain

$$\int_a^b \log \frac{1}{d(x,\{a,b\})}\, dx = (b-a) \log \frac{1}{b-a} + (1 + \log 2)(b-a). \tag{3.1}$$

For $A < B$ and $a \in [(A+B)/2, B]$ we have

$$\int_a^B \log \frac{1}{d(x,\{A,B\})}\, dx = \int_a^B \log \frac{1}{B-x}\, dx = (B-a) \log \frac{1}{B-a} + (B-a).$$

For $a \in [A, (A+B)/2]$ we get

$$\begin{aligned}\int_a^B \log \frac{1}{d(x,\{A,B\})}\, dx &\le \int_A^B \log \frac{1}{d(x,\{A,B\})}\, dx \\ &= (B-A) \log \frac{1}{B-A} + (1+\log 2)(B-A) \\ &\le 2(B-a) \log \frac{1}{B-a} + 2(1+\log 2)(B-a),\end{aligned}$$

since $B - a \le B - A \le 2(B-a)$. Therefore we have in both cases

$$\int_a^B \log \frac{1}{d(x,\{A,B\})}\, dx \le 2\,(B-a) \log \frac{1}{B-a} + 4\,(B-a). \tag{3.2}$$

In the same way we get, for $b \in [A, B]$,

$$\int_A^b \log \frac{1}{d(x, \{A, B\})}\, dx \le 2\,(b - A) \log \frac{1}{b - A} + 4\,(b - A). \tag{3.3}$$

We need another elementary inequality. For $0 < a \le b$ we have, using the mean value theorem, with $a < \xi < a + b$,

$$\begin{aligned}(a + b) \log(a + b) - a \log a &= b\,(\log \xi + 1)\\ &\le b\,(\log(a + b) + 1) \le b \log b + b \log 2 + b,\end{aligned} \tag{3.4}$$

and therefore

$$a \log \frac{1}{a} + b \log \frac{1}{b} \le (a + b) \log \frac{1}{a + b} + 2\,b. \tag{3.5}$$

Assume now that we have numbers $0 < a_1 \le a_2 \le \cdots \le a_m$ such that

$$\sum_{j=1}^{k} a_j \le a_{k+1}$$

for $k = 1, \ldots, m - 1$. We set $a = \sum_{j=1}^m a_j$ and we obtain, by inductive use of the estimate (3.5),

$$\sum_{j=1}^{m} a_j \log \frac{1}{a_j} \le a \log \frac{1}{a} + 2\,a. \tag{3.6}$$

We return to our previous setting and we have shown:

Lemma 3.2. *If $x_{k+1} \le \varepsilon_k$ for all $k \in \mathbb{N}$ then, for $0 \le a < b \le x_1$,*

$$\frac{1}{b - a} \int_a^b \log \frac{1}{d(x, E)}\, dx \le 2 \log \frac{1}{b - a} + 16\,. \tag{3.7}$$

Proof. First we apply for any j the formulas (3.1), (3.2) or (3.3), respectively, to the interval $[\alpha_{j+1}, \alpha_j] = [x_{j+1}, x_j] \cap [a, b]$ and obtain in all cases

$$\int_{\alpha_{j+1}}^{\alpha_j} \log \frac{1}{d(x, E)}\, dx \le 2(\alpha_j - \alpha_{j+1}) \log \frac{1}{\alpha_j - \alpha_{j+1}} + 4\,(\alpha_j - \alpha_{j+1}). \tag{3.8}$$

If $b \in [\alpha_{m+1}, \alpha_m]$ then we obtain, by use of the formula (3.6),

$$\int_a^{\alpha_{m+1}} \log \frac{1}{d(x, E)}\, dx \le 2(\alpha_{m+1} - a) \log \frac{1}{\alpha_{m+1} - a} + 8\,(\alpha_{m+1} - a). \tag{3.9}$$

Applying the formula (3.5) to (3.8), with $j = m$, and to (3.9), we arrive at

$$\int_a^b \log \frac{1}{d(x, E)}\, dx \le 2(b - a) \log \frac{1}{b - a} + 16\,(b - a),$$

which is equivalent to (3.7). □

Now, for $E \subset [0, 2\pi[$, we set

$$A_\infty(E) = \{f|_E \,:\, f \in A^\infty\}.$$

From the result of Alexander, Taylor and Williams (Theorem 1.1 in [2]) we obtain:

Theorem 3.3. *If $x_{n+1} \leq \varepsilon_n$ for all $n \in \mathbb{N}$ we have $C_\infty(E) = A_\infty(E)$.*

4. Structure of $C_\infty(E)$

We will now investigate the linear topological structure of $C_\infty(E)$. Clearly it is nuclear and, being a quotient of $C^\infty(\mathbb{R})$, it has property (Ω). We will show that for suitable sequences $(x_n)_{n\in\mathbb{N}}$ it has also property (DN). The argument we will use is due to Tidten. In fact the proof of the following theorem is an easy adaptation of the proof of Tidten (Satz 1 in [15]) where we have Whitney jets and E is 1-perfect.

First we will define an increasing fundamental system of seminorms for $C_\infty(E)$. We let

$$R^p\varphi(x_n) = \varphi(x_n) - \sum_{j=0}^{p} \frac{\varphi^{(j)}(0)}{j\,!}\, x_n^j$$

and define

$$\|\varphi\|_k := \max_{p=0,\ldots,k} \Big\{ |\varphi^{(p)}(0)| + \sup_{n\in\mathbb{N}} \frac{|R^p\varphi(x_n)|}{x_n^p} \Big\}.$$

Since $|\varphi^{(p)}(0)| \leq |\varphi|_p$ and

$$\sup_{n\in\mathbb{N}} \frac{|R^p\varphi(x_n)|}{x_n^p} \leq x_1\, |\varphi|_{p+1}$$

for all p, the $\|\cdot\|_k$ are continuous seminorms on $C_\infty(E)$. Because

$$\frac{p\,!}{x_n^p}\, R^{p-1}\varphi(x_n) = \frac{p\,!}{x_n^p}\, R^p\varphi(x_n) + \varphi^{(p)}(0),$$

we have

$$|\varphi|_p \leq p\,!\, \|\varphi\|_p$$

for all p. Therefore the $\|\cdot\|_k$ are a fundamental system of seminorms in $C_\infty(E)$.

Theorem 4.1. *If there is a constant C such that $x_n \leq C\, x_{n+1}$ for all $n \in \mathbb{N}$, then $C_\infty(E)$ has property* (DN).

Proof. We follow the proof of Tidten (Satz 1 in [15]). We present it here, with the necessary changes (in fact, simplifications), for the convenience of the reader.

i) We want to show that there is a constant C_1, such that for $M > 1$, $k \in \mathbb{N}$, and $\varphi \in C_\infty(E)$ with $\|\varphi\|_{k-1} \leq 1$ and $\|\varphi\|_{k+1} \leq M$ we have

$$\frac{|R^{k-1}\varphi(x_n)|}{x_n^k} \leq C_1 M^{1/2} \quad \text{for all } n \in \mathbb{N}.$$

We set

$$Q := \frac{R^{k-1}\varphi(x_n)}{x_n^k}.$$

For $M \le x_1^{-2}$ we obtain i) with any $C_1 \ge x_1^{-2}$:

$$|Q| \le \frac{|R^k\varphi(x_n)|}{x_n^k} + \frac{1}{k\,!}\,|\varphi^{(k)}(0)| \le \|\varphi\|_k \le \|\varphi\|_{k+1} \le M \le x_1^{-2} \le C_1 \le C_1 M^{1/2}.$$

Now let $M > x_1^{-2}$. We consider two cases.
In the case $M^{1/2} \ge 1/x_n$ we obtain i) for any $C_1 \ge 1$,

$$|Q| = \frac{1}{x_n}\,\frac{|R^{k-1}\varphi(x_n)|}{x_n^{k-1}} \le \frac{1}{x_n}\,\|\varphi\|_{k-1} \le \frac{1}{x_n} \le M^{1/2} \le C_1 M^{1/2}.$$

There remains the case $1/x_1 < M^{1/2} < 1/x_n$. Because $x_1 > M^{-1/2}$, there is a maximal $m \in \mathbb{N}$ such that $x_m > M^{-1/2}$. For that m we have

$$x_{m+1} \le M^{-1/2} < x_m \le C\,x_{m+1}.$$

We set $\tilde{x} := x_{m+1}$ and we have

$$\tilde{x} \le M^{-1/2}, \quad \frac{1}{\tilde{x}} < C\,M^{1/2}, \quad x_n < M^{-1/2} < C\tilde{x}.$$

We obtain

$$\begin{aligned}(4.1)\qquad \Big|Q - \frac{1}{k\,!}\,\varphi^{(k)}(0)\Big| = \frac{|R^k\varphi(x_n)|}{x_n^k} &= x_n\,\Big|\frac{R^{k+1}\varphi(x_n)}{x_n^{k+1}} + \frac{1}{(k+1)\,!}\varphi^{(k+1)}(0)\Big| \\ &\le x_n\,\|\varphi\|_{k+1} \le x_n\,M.\end{aligned}$$

We set

$$\widetilde{Q} := \frac{R^{k-1}\varphi(\tilde{x})}{\tilde{x}^k}$$

and obtain, replacing x_n in (4.1) with $\tilde{x} = x_{m+1}$,

$$(4.2)\qquad \Big|\widetilde{Q} - \frac{1}{k\,!}\,\varphi^{(k)}(0)\Big| \le \tilde{x}\,M.$$

From (4.1) and (4.2) we obtain

$$(4.3)\qquad |Q - \widetilde{Q}| \le (x_n + \tilde{x})M \le 2M^{1/2}.$$

Because of $\|\varphi\|_{k-1} \le 1$ we have

$$(4.4)\qquad |\widetilde{Q}| = \frac{1}{\tilde{x}}\,\frac{|R^{k-1}\varphi(\tilde{x})|}{\tilde{x}^{k-1}} \le \frac{1}{\tilde{x}} \le C\,M^{1/2}.$$

From (4.3) and (4.4) we get

$$|Q| \le |Q - \widetilde{Q}| + |\widetilde{Q}| < (C+2)\,M^{1/2}.$$

So, finally, we have shown claim i) with $C_1 = \max\{x_1^{-2}, 2C+1\}$.

ii) Let φ be as in i). From (2.4) we know that

$$\varphi^{(k)}(0) = \lim_{n\to\infty} k!\,\frac{R^{k-1}\varphi(x_n)}{x_n^k}.$$

Therefore i) implies $|\varphi^{(k)}(0)| \le k!\,C_1 M^{1/2}$.

We obtain

$$\frac{|R^k\varphi(x_n)|}{x_n^k} \le \frac{|R^{k-1}\varphi(x_n)|}{x_n^k} + \frac{1}{k!}\,|\varphi^{(k)}(0)| \le 2\,C_1 M^{1/2},$$

and therefore

$$\begin{aligned}\|\varphi\|_k &= \max\left\{\|\varphi\|_{k-1}, |\varphi^{(k)}(0)| + \sup_{n\in\mathbb{N}} \frac{|R^k\varphi(x_n)|}{x_n^k}\right\}\\ &\le \max\left\{1, k!\,C_1M^{1/2} + 2\,C_1M^{1/2}\right\} \le C_2 M^{1/2},\end{aligned}$$

with $C_2 = (k!+2)C_1$. This implies easily that $\|\varphi\|_k \le C_2\|\varphi\|_{k-1}^{1/2}\|\varphi\|_{k+1}^{1/2}$ for all $k\in\mathbb{N}$. □

5. Sets with one accumulation point

We made assumptions on the sequence $(x_n)_{n\in\mathbb{N}}$ in (2.3), in Lemma 3.2 and in Theorem 4.1. They all are fulfilled if we have, with suitable $C>0$,

$$2x_{n+1} \le x_n \le Cx_{n+1} \tag{5.1}$$

because this implies $x_{n+1} \le \varepsilon_n$ and therefore also $x_n = x_{n+1} + \varepsilon_n \le 2\varepsilon_n$.

Theorem 5.1. *If (5.1) is fulfilled then $A_\infty(E) = C_\infty(E) \cong \Lambda_\infty(\alpha)$ where $\alpha_n = -\log x_n$.*

Proof. By Theorem 3.3 we have $A_\infty(E) = C_\infty(E)$. Since (5.1) implies (2.3) we obtain from Proposition 2.8 that $J_\infty(0) \cong \Lambda_\infty(\alpha)$. Therefore we have an exact sequence

$$0 \longrightarrow \Lambda_\infty(\alpha) \longrightarrow C_\infty(E) \longrightarrow \omega \longrightarrow 0$$

where ω denotes the space of all scalar sequences. Because of (5.1) the space $\Lambda_\infty(\alpha)$ is stable. For the diametral dimensions we get $\Delta(\Lambda_\infty(\alpha)) \cap \Delta(\omega) = \Delta(\Lambda_\infty(\alpha))$ and this is stable. So we obtain from Proposition 4.2 of [17] that $\Delta(C_\infty(E)) = \Delta(\Lambda_\infty(\alpha))$ and this is stable.

Clearly $C_\infty(E)$ has property (Ω) since it is a quotient of $C^\infty(\mathbb{R})$, by Theorem 4.1 it has also property (DN) and, of course it is nuclear. By Aytuna–Krone–Terzioğlu Theorem 2.2 of [1], we get $C_\infty(E) \cong \Lambda_\infty(\alpha)$. □

Example 5.2. Let $x_n = 2^{-n}$. Then (5.1) is fulfilled and $C_\infty(E) = A_\infty(E) \cong H(\mathbb{C})$.

We remark that, because $\alpha_n = n\log 2$, it is easily seen that the space $\Lambda_\infty(\alpha)$ is isomorphic to the space $H(\mathbb{C})$ of entire functions on $\mathbb{C}$.

6. The Cantor set

Now let E be the classical Cantor set. It has been known for a long time that E is a Carleson set (see Beurling [3]). We will show that it also fulfills the ATW-condition.

For this we will use that $(3^k E) \cap [0,1] = E$ for all $k \in \mathbb{N}$. We will again need an elementary formula: for that let $M \subset [0,1]$ be a compact subset. We have, for $a > 0$,

$$\int_0^a \log \frac{1}{d(x, aM)}\, dx = a \log \frac{1}{a} + a \int_0^1 \log \frac{1}{d(t, M)}\, dt. \tag{6.1}$$

Let now $0 \le a < b < 1$ be given. We set $b - a := \gamma = 0, \gamma_1 \gamma_2 \ldots$, where the last expression denotes the triadic expansion of γ, finite if possible. In the first step we restrict to the case of γ with a finite expansion, say $\gamma = 0, \gamma_1 \ldots \gamma_m$. We set $a_0 = a$ and $a_k = a + 0, \gamma_1 \ldots \gamma_k$, so that $a_{k+1} = a_k + \gamma_{k+1} 3^{-k-1}$. We obtain

$$\int_a^b \log \frac{1}{d(x, E)}\, dx = \sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx.$$

Since γ_k assumes only the values $0, 1, 2$ we have to estimate the integrals from above over intervals of length 3^{-k-1} or $2 \cdot 3^{-k-1}$.

Now we consider the subdivision of $[0,1]$ into 3^k intervals of length 3^{-k} and refer to the classical stepwise construction of the Cantor set. Some of the intervals, we call them windows, have already been excluded from the Cantor set, and we call them white, while some await for treatment, and we call them black.

We restrict now to the nontrivial case of $\gamma_{k+1} \neq 0$. Our interval of length 3^{-k-1} or $2 \cdot 3^{-k-1}$ hits at most two of the windows. If it is of length 3^{-k-1}, and completely in a white window, the worst case is (see equation (3.1))

$$\int_0^{3^{-k-1}} \log \frac{1}{x}\, dx = 3^{-k-1} \log \frac{1}{3^{-k-1}} + 3^{-k-1}.$$

If it is of length $2 \cdot 3^{-k-1}$ and completely in a white window we estimate roughly by 2 times the previous case and obtain for both cases

$$\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx \le 2 \int_0^{3^{-k-1}} \log \frac{1}{x}\, dx \le 3^{-k} \log \frac{1}{3^{-k-1}} + 3^{-k}. \tag{6.2}$$

If it is completely in a black window we take into account that, by shifting the lower end of the window to zero and multiplying by 3^k we obtain E. The interval $[a_k, a_{k+1}]$, if nontrivial, extends to an interval of length $1/3$ or $2/3$. Therefore we have, estimating by the integral over the whole window and using (6.1),

$$\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx \le 3^{-k} \log \frac{1}{3^{-k-1}} + D_0\, 3^{-k}, \tag{6.3}$$

where

$$D_0 = \log 3 + \int_0^1 \log \frac{1}{d(x, E)}\, dx.$$

Therefore we have in all cases, estimating crudely by the sum of estimate (6.2) and estimate (6.3),

$$\begin{aligned}\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x,E)}\, dx &\le 2 \cdot 3^{-k} \log \frac{1}{3^{-k-1}} + (D_0+1)\, 3^{-k} \\ &\le 6\, \gamma_{k+1} 3^{-k-1} \log \frac{1}{3^{-k-1}} + 3(D_0+1)\, \gamma_{k+1} 3^{-k-1} \\ &\le 6\, \gamma_{k+1} 3^{-k-1} \log \frac{1}{\gamma_{k+1} 3^{-k-1}} + D\, \gamma_{k+1} 3^{-k-1},\end{aligned}$$

where $D = 6 \log 2 + 3(D_0 + 1)$.

Therefore

$$\int_a^b \log \frac{1}{d(x,E)}\, dx \le 6 \sum_{k=0}^{m-1} \gamma_{k+1} 3^{-k-1} \log \frac{1}{\gamma_{k+1} 3^{-k-1}} + D\,(b-a). \tag{6.4}$$

To apply the estimate (3.6), counting reversely, we need the following:

$$\sum_{k=n}^{m-1} \gamma_{k+1} 3^{-k-1} \le 2 \sum_{k=n}^{\infty} 3^{-k-1} = 3^{-n} \le \gamma_\nu 3^{-\nu},$$

where ν is the biggest number $\le n$ with $\gamma_\nu \neq 0$. If there is no such ν we are done, as we need add no further summand.

From (6.4) and (3.6) we get now

$$\int_a^b \log \frac{1}{d(x,E)}\, dx \le 6\,(b-a) \log \frac{1}{b-a} + (D+15)\,(b-a)$$

for all triadic numbers in $[0,1[$. Since we know that E is Carleson, that is $\log \frac{1}{d(x,E)}$ is integrable over $[0,1]$, the left and the right-hand sides depend continuously on a and b. Therefore the estimate is true for all $0 \le a < b \le 1$.

Applying the result of Alexander, Taylor and Williams (Theorem 1.1 in [2]) we have shown:

Proposition 6.1. *If E is the classical Cantor set we have $A_\infty(E) = C_\infty(E)$.*

Remark 6.2. Because of Corollary 2.3 the functions $f \in J(E)$ and all all their derivatives vanish on E. This means $C_\infty(E) = \mathscr{E}(E)$, where $\mathscr{E}(E)$ is the space of Whitney jets on E.

From Tidten ([15], Folgerung, p. 76) we know that $\mathscr{E}(E)$ is isomorphic to a complemented subspace of s. Clearly $C_\infty(E)$ is stable, because

$$C_\infty(E) \cong C_\infty(E \cap [0, 1/3]) \oplus C_\infty(E \cap [2/3, 1]) \cong C_\infty(E)^2.$$

Again, using Aytuna–Krone–Terzioğlu ([1], Theorem 2.2) (or Wagner, [20], Theorem 1), we obtain:

Theorem 6.3. *If E is the classical Cantor set then $A_\infty(E) = C_\infty(E)$ and $A_\infty(E)$ has a basis. In fact, it is isomorphic to a power series space of infinite type.*

7. Final remarks

We return to the notation of Section 2 and assume that (2.3) holds. We define, for $f \in J^\infty(0)$,

$$Pf(x) = f(x) - \sum_{n=1}^{\infty} f(x_n)\chi_{\varepsilon_n}(x - x_n).$$

Then, by (2.1) and Lemma 2.6, P is a linear map from $J^\infty(0)$ to $J(E)$ which is continuous by the estimates (2.1) and (2.2). We have shown:

Lemma 7.1. *If (2.3) is fulfilled, then P is a continuous projection from $J^\infty(0)$ onto $J(E)$.*

Corollary 7.2. *If (2.3) is fulfilled, then $J(E)$ has property (Ω).*

Proof. $J^\infty(0)$ has property (Ω) by Tidten ([16], Satz 2.2) and (Ω) is inherited by complemented subspaces. □

We obtain:

Theorem 7.3. *If there are $q \in \mathbb{N}$ and $C > 0$ such that $x_n^q \le C\varepsilon_n$ and $x_n \le Cx_{n+1}$ for all $n \in \mathbb{N}$, then there is a continuous linear extension operator from $C_\infty(E)$ to $C^\infty(\mathbb{R})$.*

Proof. We have the natural exact sequence

$$0 \longrightarrow J(E) \longrightarrow C^\infty(\mathbb{R}) \stackrel{\rho}{\longrightarrow} C_\infty(E) \longrightarrow 0$$

where ρ is the restriction map. Then $J(E)$ has property (Ω) by Corollary 7.2, $C_\infty(E)$ has property (DN) by Theorem 4.1, and all the spaces appearing are nuclear. By the (DN)-(Ω)-splitting theorem (see 30.1 in [6]) the sequence splits, hence ρ has a continuous linear right inverse, that is, there is a continuous linear extension operator. □

Examples of this include not only exponentially decreasing sequences x_n, but also, for example, $x_n = 1/n$, $n \in \mathbb{N}$.

Finally, let us remark that for the classical Cantor set E there is a continuous linear extension operator from $C_\infty(E) = \mathscr{E}(E)$ to $C^\infty(\mathbb{R})$ by Tidten ([15], Folgerung, p. 76).

References

[1] AYTUNA, A., KRONE, J. AND TERZIOĞLU, T.: Complemented infinite type power series subspaces of nuclear Fréchet spaces. *Math. Ann.* **283** (1989), no. 2, 193–202.

[2] ALEXANDER, H., TAYLOR, B. A. AND WILLIAMS, D. L.: The interpolating sets for A^∞. *J. Math. Anal. Appl.* **36** (1971), 556–566.

[3] BEURLING, A.: Ensembles exceptionnels. *Acta Math.* **72** (1940), 1–13.

[4] Djakov, P.: Several remarks on nuclear Fréchet spaces without a basis. *Serdica* **2** (1976), no. 2, 171–176.

[5] Djakov, P. and Mityagin, B. S.: Modified construction of nuclear Fréchet spaces without basis. *J. Funct. Analysis* **23** (1976), no. 4, 415–433.

[6] Meise, R. and Vogt, D.: *Introduction to functional analysis.* Oxford Graduate Texts in Mathematics 2, The Clarendon Press, Oxford University Press, New York, 1997.

[7] Mitjagin, B. S. and Zobin, N. M.: Contre exemple à l'existence d'une base dans un espace de Fréchet nucléaire. *C. R. Acad. Sci. Paris Sér. A* **279** (1974), 255–256.

[8] Mitjagin, B. S. and Zobin, N. M.: Contre-exemple à l'existence d'une base dans un espace de Fréchet nucléaire. *C. R. Acad. Sci. Paris Sér. A* **279** (1974), 325–327.

[9] Mitjagin, B. S. and Zobin, N. M.: Examples of nuclear metric linear spaces without a basis. *Funkcional. Anal. i Priložen.* **8** (1974), no. 4, 35–47.

[10] Moscatelli, V. B.: Fréchet spaces without continuous norms and without bases. *Bull. London Math. Soc.* **12** (1980), no. 1, 63–66.

[11] Novinger, W. P.: Holomorphic functions with infinitely differentiable boundary values. *Illinois J. Math.* **15** (1971), 80–90.

[12] Patel, S. R.: Closed ideals of A^∞ and a famous problem of Grothendieck. *Rev. Mat. Iberoam.* **27** (2011), no. 3, 977–995.

[13] Taylor, B. A. and Williams, D. L.: Zeros of Lipschitz functions analytic in the unit disc. *Michigan Math. J.* **18** (1971), 129–139.

[14] Terzioğlu, T.: On the diametral dimension of some classes of F-spaces. *J. Karadeniz Uni. Fac. Arts Sci. Ser. Math.-Phys.* **8** (1985), 1–13.

[15] Tidten, M.: Kriterien für die Existenz von Ausdehnungsoperatoren zu $\mathfrak{E}(K)$ für kompakte Teilmengen K von $\mathbb{R}$. *Arch. Math. (Basel)* **40** (1983), no. 1, 73–81.

[16] Tidten, M.: Fortsetzungen von C^∞-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}^n$ definiert sind. *Manuscripta Math.* **27** (1979), no. 3, 291–312.

[17] Vogt, D.: Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen. *Manuscripta Math.* **37** (1982), no. 3, 269–301.

[18] Vogt, D.: Structure theory of power series spaces of infinite type. *RACSAM. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **97** (2003), no. 2, 339–363.

[19] Vogt, D.: A nuclear Fréchet space of C^∞-functions which has no basis. *Note Mat.* **25** (2005/06), no. 2, 187–190.

[20] Wagner, M. J.: Some new methods in the structure theory of nuclear Fréchet spaces. In *Advances in the theory of Fréchet spaces (Istanbul, 1988),* 333–354. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 287, Kluwer Acad. Publ., Dordrecht, 1989.

[21] Whitney, H.: Differentiable functions defined in closed sets. I. *Trans. Amer. Math. Soc.* **36** (1934), no. 2, 369–387.

Received March 14, 2012.

Dietmar Vogt: Bergische Universität Wuppertal. FB Math.-Nat. Gauss-Str. 20, D-42097 Wuppertal, Germany.

E-mail: dvogt@math.uni-wuppertal.de