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Bouligand–Severi tangents in MV-algebras

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Abstract. In their important recent paper published in the Annals of Pure and Applied Logic, Dubuc and Poveda call an MV-algebra A *strongly semisimple* if all principal quotients of A are semisimple. All boolean algebras are strongly semisimple, and so are all finitely presented MV-algebras. We show that for any 1-generator MV-algebra, semisimplicity is equivalent to strong semisimplicity. Further, a semisimple 2-generator MV-algebra A is strongly semisimple if and only if its maximal spectral space $\mu(A) \subseteq [0,1]^2$ does not have any rational Bouligand–Severi tangents at its rational points. In general, when A is finitely generated and $\mu(A) \subseteq [0,1]^n$ has a Bouligand–Severi tangent then A is not strongly semisimple. An MV-algebra A is strongly semisimple if and only if so is every 2-generator subalgebra of A.

1. Introduction

We refer to [4] and [8] for background on MV-algebras. Following Dubuc and Poveda [5], we say that an MV-algebra A is *strongly semisimple* if for every principal ideal I of A the quotient A/I is semisimple. Since $\{0\}$ is a principal ideal of A, every strongly semisimple MV-algebra is semisimple. The definition of "logically complete" MV-algebras in [1] is a variant of this notion, where one further assumes $I \neq \{0\}$. The paper [7] is devoted to the frame-theoretic variant of strongly semisimple MV-algebras, called "Yosida frames". These papers, together with the results of the present paper, show that strong semisimplicity is a very interesting purely algebraic counterpart of the simplicial, topological, and differential structure of MV-algebras. Further, from the logical viewpoint, 4.3 in [9] shows that strongly semisimple MV-algebras coincide with Lindenbaum algebras of theories Θ in infinite-valued Łukasiewicz logic having the following property: for any formula ψ, the set of syntactic consequences of $\Theta \cup \{\psi\}$ coincides with the set of (Bolzano–Tarski) semantic consequences of $\Theta \cup \{\psi\}$.

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From a classical result of Hay [6] and Wójcicki [14] (see also 4.6.7 in [4] and 1.6 in [8]), it follows that every finitely presented MV-algebra is strongly semisimple. Trivially, all hyperarchimedean MV-algebras, hence in particular all boolean algebras, are strongly semisimple, and so are all simple and all finite MV-algebras (see 3.5 and 3.6.5 in [4]).

For any real-valued function g we will write $Zg = g^{-1}(0)$ for its zero set.

Our paper is devoted to n-generator strongly semisimple MV-algebras. When $n = 1$, strong semisimplicity is equivalent to semisimplicity (Theorem 5.1). To deal with the general case, we first recall that the free n-generator MV-algebra is the MV-algebra $\mathcal{M}([0,1]^n)$ of all McNaughton functions $f\colon [0,1]^n \to [0,1]$, with pointwise operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. See 9.1.5 in [4].

For any nonempty closed set $X \subseteq [0,1]^n$ we let $\mathcal{M}(X)$ denote the MV-algebra of restrictions to X of the functions in $\mathcal{M}([0,1]^n)$. In symbols,

$$\mathcal{M}(X) = \{f \restriction X \mid f \in \mathcal{M}([0,1]^n)\}.$$

By 3.6.7 in [4], $\mathcal{M}(X)$ is a semisimple MV-algebra; actually, up to isomorphism, $\mathcal{M}(X)$ is the most general possible n-generator semisimple MV-algebra A. To see this, pick generators $\{a_1, \ldots, a_n\}$ of A. Let $\pi_i\colon [0,1]^n \to [0,1]$ be the projection functions in the free MV-algebra $\mathcal{M}([0,1]^n)$ for $i = 1, \ldots, n$. Then the assignment that maps $\pi_i \mapsto a_i$ for each $i = 1, \ldots, n$, uniquely extends to a homomorphism $\eta_a\colon \mathcal{M}([0,1]^n) \to A$ of the free n-generator MV-algebra onto A. Let $\mathfrak{h}_a = \ker(\eta_a)$ be the kernel of this homomorphism and let

$$\mathcal{Z}_a = \bigcap \{Zf \mid f \in \mathfrak{h}_a\} \tag{1.1}$$

be the intersection of the zero sets of the McNaughton functions in $\mathfrak{h}_a$. Then

$$A \cong \mathcal{M}(\mathcal{Z}_a). \tag{1.2}$$

A point $x \in \mathbb{R}^n$ is said to be *rational* if so are all its coordinates. By a *rational vector* we mean a nonzero vector $w \in \mathbb{R}^n$ such that the line $\mathbb{R}w \subseteq \mathbb{R}^n$ contains at least two rational points. An MV-algebra A is strongly semisimple if and only if so is every 2-generator subalgebra of A (Proposition 4.1). A 2-generator MV-algebra $A = \mathcal{M}(X)$, with nonempty closed $X \subseteq [0,1]^2$, is strongly semisimple if and only if X has no rational outgoing Bouligand–Severi tangent vector at any of its rational points, [2], [12], and [10]. See Theorem 3.1. As proved in Theorem 2.3, for any closed $X \subseteq [0,1]^n$, having such a tangent is a condition sufficient for $\mathcal{M}(X)$ not to be strongly semisimple.

Notation. Following p. 33 in [4] or p. 21 in [8], for $k \in \mathbb{N}$, $k.g$ stands for the k-fold pointwise truncated addition of g.

2. Strong semisimplicity and Bouligand–Severi tangents

Severi (see §53, p. 59 and p. 392 of [11], as well as §1, p. 99 of [12]) and independently, Bouligand (p. 32 in [2]) called a half-line $H \subseteq \mathbb{R}^n$ *tangent* to a set $X \subseteq \mathbb{R}^n$ at an accumulation point x of X if for all $\epsilon, \delta > 0$ there is $y \in X$ different from x

such that $||y - x|| < \epsilon$, and the angle between H and the half-line through y originating at x is $< \delta$. Here as usual, $||v||$ is the length of the vector $v \in \mathbb{R}^n$.

On §2, p. 100 and §4, p. 102 of [12], Severi noted that for any accumulation point x of a closed set X there is a half-line H tangent to X at x.

Today (see, e.g., p. 16 in [3], or p. 1376 in [10]), Bouligand–Severi tangents are routinely defined as follows.

Definition 2.1. Let x be an element of a closed subset X of $\mathbb{R}^n$, and u a unit vector in $\mathbb{R}^n$. We then say that u is a *Bouligand–Severi tangent* (*unit*) *vector to* X *at* x if X contains a sequence $x_0, x_1, \dots$ of elements, all different from x, such that

$$\lim_{i\to\infty} x_i = x \quad \text{and} \quad \lim_{i\to\infty} (x_i - x)/||x_i - x|| = u.$$

Observe that x is an accumulation point of X. We further say that u is *outgoing* if for some $\lambda > 0$ the segment $\mathrm{conv}(x, x + \lambda u)$ intersects X only at x.

Already Severi noted that his definition of tangent half-line $H = x + \mathbb{R}_{\geq 0}u$ is equivalent to Definition 2.1. More precisely:

Proposition 2.2. (§5, p. 103 of [12]). *For any nonempty closed subset X of $\mathbb{R}^n$, point $x \in X$, and unit vector $u \in \mathbb{R}^n$ the following conditions are equivalent:*

(i) *For all $\epsilon, \delta > 0$, the cone $\mathrm{cone}_{x,u,\epsilon,\delta}$ with apex x, axis parallel to u, vertex angle 2δ and height ϵ contains infinitely many points of X.*

(ii) *u is a Bouligand–Severi tangent vector to X at x.*

When $n = 1$, $\mathrm{cone}_{x,u,\epsilon,\delta}$ is the segment $\mathrm{conv}(x, x + \epsilon u)$. When $n = 2$, $\mathrm{cone}_{x,u,\epsilon,\delta}$ is the isosceles triangle $\mathrm{conv}(x, a, b)$ with vertex x, basis $\mathrm{conv}(a, b)$, height equal to ϵ (and parallel to u), and vertex angle $\widehat{axb} = 2\delta$.

The next two results provide necessary and sufficient geometric conditions on X for the semisimple MV-algebra $\mathcal{M}(X)$ to be strongly semisimple. These conditions are stated in terms of the nonexistence of Bouligand–Severi tangent vectors having certain rationality properties.

Theorem 2.3. *Let X be a nonempty closed set in $[0, 1]^n$. Suppose X has a Bouligand–Severi rational outgoing tangent vector u at some rational point $x \in X$. Then $\mathcal{M}(X)$ is not strongly semisimple.*

Proof. Since u is outgoing, let $\lambda > 0$ satisfy $X \cap \mathrm{conv}(x, x + \lambda u) = \{x\}$. Without loss of generality $x + \lambda u \in \mathbb{Q}^n$. By Definition 2.1, our hypothesis yields a sequence $w_1, w_2, \dots$ of distinct points of X, all distinct from x, accumulating at x, at strictly decreasing distances from x, in such a way that the sequence of unit vectors u_i given by $(w_i - x)/||w_i - x||$ tends to u as i tends to ∞. Let $y = x + \lambda u$. Since $X \cap \mathrm{conv}(x, y) = \{x\}$, no point w_i lies on the segment $\mathrm{conv}(x, y)$, and we can further assume that the sequence of angles $\widehat{w_i x y}$ is strictly decreasing and tends to zero as i tends to ∞.

Since both points x and y are rational, by 2.10 in [8], for some $g \in \mathcal{M}([0, 1]^n)$ the zero set

$$Zg = \{z \in [0, 1]^n \mid g(z) = 0\}$$

coincides with the segment $\mathrm{conv}(x, y)$. Thus,

$$\frac{\partial g(x)}{\partial(u)} = 0.$$

Let J be the ideal of $\mathcal{M}([0,1]^n)$ generated by g,

$$J = \{f \in \mathcal{M}([0,1]^n) \mid f \leq k \cdot g \text{ for some } k = 0, 1, 2, \ldots\}.$$

Then for each $f \in J$,

$$\frac{\partial f(x)}{\partial(u)} = 0.$$

Since the directional derivatives of f at x are continuous (meaning that the map $t \mapsto \partial f(x)/\partial t$ is continuous), it follows that

$$\lim_{t \to u} \frac{\partial f(x)}{\partial(t)} = \frac{\partial f(x)}{\partial(u)} = 0. \tag{2.1}$$

Let $g' = g \restriction X$ and let

$$J' = \{f' \in \mathcal{M}(X) \mid \ f' \leq k \cdot g' \text{ for some } k = 0, 1, 2, \ldots\}$$

be the ideal of $\mathcal{M}(X)$ generated by g'. A moment's reflection shows that

$$J' = \{l \restriction X \mid l \in J\}. \tag{2.2}$$

One inclusion is trivial. For the converse inclusion, if $f \restriction X \leq (k \cdot g) \restriction X$ then letting $l = f \wedge k \cdot g$ we get $l \leq k \cdot g$. So $l \in J$ and $l \restriction X = f \restriction X$, whence $f \restriction X$ is extendible to some $l \in J$.

For any $f \in \mathcal{M}([0,1]^n)$, the piecewise linearity of f ensures that for all large i the value of the incremental ratio $(f(w_i) - f(x))/||w_i - x||$ coincides with the directional derivative $\partial f(x)/\partial u_i$ along the unit vector $u_i = (w_i - x)/||w_i - x||$. Thus in particular, if $f \restriction X = f' \in J'$, from (2.1)–(2.2) it follows that

$$\lim_{i \to \infty} \frac{f'(w_i) - f'(x)}{||w_i - x||} = 0.$$

Since x is rational, again by 2.10 in [8] there is $j \in \mathcal{M}([0,1]^n)$ with $Zj = \{x\}$. For some $\omega > 0$ we have $\partial j(x)/\partial(u) = \omega$, whence

$$\lim_{i \to \infty} \frac{j'(w_i) - j'(x)}{||w_i - x||} = \omega.$$

Therefore, $j' \notin J'$. Since $Zg \cap X = \{x\}$, recalling 4.19 in [8] we see that the only maximal ideal of $\mathcal{M}(X)$ containing J' is the set of all functions in $\mathcal{M}(X)$ that vanish at x. Thus, j' belongs to all maximal ideals of $\mathcal{M}(X)$ containing J'. By 3.6.6 in [4], $\mathcal{M}(X)$ is not strongly semisimple; specifically, j'/J' is infinitesimal in the principal quotient $\mathcal{M}(X)/J'$. □

3. A partial converse of Theorem 2.3

Theorem 3.1. *Let $X \subseteq [0,1]^n$ be a nonempty closed set. Suppose the MV-algebra $\mathcal{M}(X)$ is not strongly semisimple.*

(i) *Then X has a Bouligand–Severi tangent vector u at some point $x \in X$ satisfying the following* nonalignment *condition: there is a sequence of distinct $w_i \in X$, all distinct from x such that*

$$\lim_{i\to\infty} w_i = x, \quad \lim_{i\to\infty} \frac{w_i - x}{||w_i - x||} = u, \quad w_i \notin \operatorname{conv}(x, x+u) \text{ for all } i.$$

(ii) *In particular, if $n = 2$, then X has a Bouligand–Severi* outgoing rational *tangent vector u at some* rational *point $x \in X$.*

Proof. (i) The hypothesis yields a function $g \in \mathcal{M}([0,1]^n)$, with its restriction $g' = g \restriction X \in \mathcal{M}(X)$, in such a way that the principal ideal J' of $\mathcal{M}(X)$ generated by g',

$$J' = \{l' \in \mathcal{M}(X) \mid l' \leq k \centerdot g' \text{ for some } k = 1, 2, \ldots\}$$

is strictly contained in the intersection I of all maximal ideals of $\mathcal{M}(X)$ containing J'. Thus for some $j \in \mathcal{M}([0,1]^n)$ letting $j' = j \restriction X$ we have $j' \in I \setminus J'$. By 3.6.6 in [4] and 4.19 in [8],

$$j' = 0 \text{ on } Zg', \text{ i.e., } X \cap Zj \supseteq X \cap Zg \tag{3.1}$$

and

$$\forall m = 0, 1, \ldots, \ \exists z_m \in X, \ j'(z_m) > m \centerdot g'(z_m). \tag{3.2}$$

There is a sequence of integers $0 < m_0 < m_1 < \ldots$ and a subsequence $y_0, y_1, \ldots$ of $\{z_i, z_2, \ldots\}$ such that $y_i \neq y_l$ for $i \neq l$ and

$$\forall t = 0, 1, \ldots, \ j'(y_t) > m_t \centerdot g'(y_t). \tag{3.3}$$

The compactness of X yields an accumulation point $x \in X$ of the y_t. Without loss of generality (taking a subsequence, if necessary) we can further assume

$$||y_0 - x|| > ||y_1 - x|| > \cdots, \text{ whence } \lim_{i\to\infty} y_i = x. \tag{3.4}$$

By (3.3), for all t, $j'(y_t) > 0$. Then by (3.1), $g'(y_t) > 0$. For each $i = 0, 1, \ldots$, defining the unit vector $u_i \in \mathbb{R}^n$ by $u_i = (y_i - x)/||y_i - x||$, we obtain a sequence of (possibly repeated) unit vectors $u_i \in \mathbb{R}^n$. Since the boundary of the unit ball in $\mathbb{R}^n$ is compact, some unit vector $u \in \mathbb{R}^n$ satisfies

$$\forall \epsilon > 0 \text{ there are infinitely many } i \text{ such that } ||u_i - u|| < \epsilon.$$

Some subsequence $w_0, w_1, \ldots$ of the y_i will satisfy the condition

$$\forall \epsilon, \delta > 0 \text{ there is } k \text{ such that for all } i > k, \quad w_i \in \operatorname{cone}_{x,u,\epsilon,\delta}. \tag{3.5}$$

Correspondingly, the sequence $v_0, v_1, \ldots$ given by $v_k = (w_k - x)/||w_k - x||$ will satisfy

$$\lim_{i\to\infty} v_i = u. \tag{3.6}$$

We have just proved that u is a Bouligand–Severi tangent to X at x.

To complete the proof of (i) we need the following:

Fact 1. $g^{\shortmid}(x) = 0$.

Otherwise, from the continuity of g, for some real $\rho > 0$ and suitably small $\epsilon > 0$, we have the inequality $g(z) > \rho$ for all z in the open ball $B_{x,\epsilon}$ of radius ϵ centered at x. By (3.5), $B_{x,\epsilon}$ contains infinitely many w_i. There is a fixed integer $\bar{m} > 0$ such that $1 = \bar{m} \mathbin{\scriptstyle\blacksquare} g^{\shortmid} \geq j^{\shortmid}$ for all these w_i, which contradicts (3.3).

Fact 2. $j^{\shortmid}(x) = 0$.

This immediately follows from (3.1) and Fact 1.

Fact 3. $\partial g(x)/\partial u = 0$.

Aiming at a contradiction, suppose $\partial g(x)/\partial u = \theta > 0$. In view of the continuity of the map $t \mapsto \partial g(x)/\partial t$, let $\delta > 0$ be such that $\partial g(x)/\partial r > \theta/2$, for any unit vector r such that $\widehat{ru} < \delta$. Since, by Fact 2, $j(x) = 0$ and both g and j are piecewise linear, there is an $\epsilon > 0$ together with an integer $\bar{k} > 0$ such that $\bar{k} \mathbin{\scriptstyle\blacksquare} g \geq j$ over the cone $C = \mathrm{cone}_{x,u,\epsilon,\delta}$. By (3.5), C contains infinitely many w_i, in contradiction with (3.3).

To conclude the proof of the nonalignment condition in (i), it is sufficient to show the following:

Fact 4. *There is $\lambda > 0$ such that for all large i the segment $\mathrm{conv}(x, x+\lambda u)$ contains no w_i.*

For otherwise, from Fact 3, $\partial g(x)/\partial(u) = 0$, whence the piecewise linearity of g ensures that g vanishes on infinitely many w_i of $\mathrm{conv}(x, x + \lambda u)$ arbitrarily near x. Any such w_i belongs to X. Hence, by (3.1), $j(w_i) = 0$, in contradiction with (3.3).

The proof of (i) is now complete.

(ii) Let $H^{\pm}$ be the two closed half-spaces of $\mathbb{R}^2$ determined by the line passing through x and $x+u$. By (3.5), infinitely many w_i lie in the same closed half-space, say, H^+. Without loss of generality, $H^+ \cap \mathrm{int}([0,1]^2) \neq \emptyset$. Let $u^{\perp}$ be the vector orthogonal to u such that $x + u^{\perp} \in H^+$.

Fact 5. *For all small $\epsilon > 0$,*

$$\frac{\partial g(x + \epsilon u)}{\partial u^{\perp}} > 0.$$

Aiming at a contradiction, assume $\partial g(x+\epsilon u)/\partial u^{\perp}=0$. Since g is piecewise linear, by Facts 1 and 3, for suitably small $\eta,\omega>0$, the function g vanishes over the triangle $T=\mathrm{conv}(x,x+\eta u,x+\eta u+\omega u^{\perp})$. By (3.5), T contains infinitely many w_i. By (3.1), $g(w_i)=j(w_i)=0$, contradicting (3.3).

Fact 6.

$$\frac{\partial j(x)}{\partial u}>0.$$

Otherwise, $\partial j(x)/\partial u=0$. Fact 5 yields a fixed integer $\bar{h}$ such that, on a suitably small triangle of the form $T=\mathrm{conv}(x,x+\epsilon u,x+\epsilon u+\omega u^{\perp})$, we have $\bar{h}\,\text{.}\,g\geq j$. By (3.5), T contains infinitely many w_i, again contradicting (3.3).

We now prove a strong form of Fact 4, showing that u is an *outgoing* tangent vector:

Fact 7. *For some* $\lambda>0$ *the segment* $\mathrm{conv}(x,x+\lambda u)$ *intersects* X *only at* x*.*

Otherwise, from Facts 1 and 3 it follows that g vanishes on infinitely many points of $X\cap\mathrm{conv}(x,x+\lambda u)$ converging to x. By (3.1), $j^{\prime}$ vanishes on all these points. Since j is piecewise linear, $\partial j(x)/\partial u=0$, contradicting Fact 6.

By a *rational* line in $\mathbb{R}^n$ we mean a line passing through at least two distinct rational points.

Fact 8. *x is a rational point, and u is a rational vector.*

As a matter of fact, Facts 6 and 2 yield a rational line L through x. On the other hand, Facts 3 and 5 show that the line passing through x and $x+u$ is rational and different from L. Thus x is rational, hence so is the vector u.

We conclude that X has u as a Bouligand–Severi *outgoing rational* tangent vector at the rational point x. □

Figure 1 is a sketch of the functions g and j in the foregoing proof.

Recalling Theorem 2.3 we now obtain:

Corollary 3.2. *Let $X\subseteq[0,1]^2$ be a nonempty closed set. Then $\mathcal{M}(X)$ is not strongly semisimple iff X has a Bouligand–Severi outgoing rational tangent vector u at some rational point $x\in X$.*

Examples. The above corollary provides many examples of 2-generator strongly semisimple MV-algebras:

(i) Let $\kappa\in[0,1]$ be irrational. Let W be the arc of parabola $\{(x,y)\in[0,1]^2\mid y=\kappa x^2\}$. Then $\mathcal{M}(W)$ is strongly semisimple – for want of rational points in W. One can similarly construct 2-generator strongly semisimple MV-algebras of the form $\mathcal{M}(V)$, by letting V be a closed subset of $[0,1]^2$ without rational points, or else, without outgoing rational tangents.

(ii) Following [13], let $Q\subseteq[0,1]^2$ be a *polyhedron* in $[0,1]^2$, i.e., a finite union of m-simplexes $(m=0,1,2)$ in $[0,1]^2$. Then Q does not have any outgoing Bouligand–Severi tangent, whence $\mathcal{M}(Q)$ is strongly semisimple.

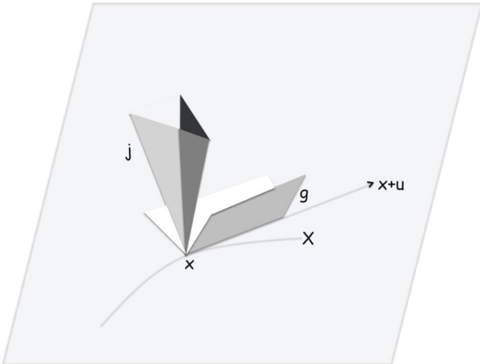

Figure 1. A Bouligand–Severi outgoing tangent vector u to X at x, and two functions g and j. The restriction $g \upharpoonright X$ generates a principal ideal J' of $\mathcal{M}(X)$. The restriction $j \upharpoonright X$ does not belong to J', but belongs to the only maximal ideal I' of $\mathcal{M}(X)$ containing J', namely the set of all functions in $\mathcal{M}(X)$ vanishing at x. So the principal quotient $\mathcal{M}(X)/J'$ is not semisimple.

(iii) (Generalizing (ii)). Let A be a 2-generator subalgebra of a semisimple tensor product (see §9.4 in [8]) of the form $[0,1] \otimes D$, where D is a finitely presented MV-algebra. Using Lemma 3.6 and Theorem 6.3 in [8], one sees that A is isomorphic to an MV-algebra of the form $\mathcal{M}(Q)$ for some polyhedron $Q \subseteq [0,1]^2$. Thus A is strongly semisimple.

4. The general case

The central role of finitely generated, and especially of 2-generator strongly semisimple MV-algebras among all strongly semisimple MV-algebras, is shown by the following result:

Proposition 4.1. *For any MV-algebra A the following conditions are equivalent:*

(i) *A is strongly semisimple;*

(ii) *A is the direct limit of a direct system $\mathcal{S} = \{A_i, \phi_{ij}\}$ of finitely generated strongly semisimple algebras A_i, where all the homomorphisms $\phi_{ij} \colon A_i \to A_j$ are embeddings;*

(iii) *each 2-generator subalgebra of A is strongly semisimple.*

Proof. Recall that an MV-algebra is semisimple if and only if it has no infinitesimals. For any MV-algebras C and D, and embedding $\phi \colon C \to D$, letting, for any $y \in C$, $\langle y \rangle_C$ denote the ideal generated by y in C, we first make the following elementary observations:

(I) For each $c \in C$, the map $\bar{\phi}\colon C/\langle c\rangle_C \to D/\langle \phi(c)\rangle_D$ defined by $x/\langle c\rangle_C \mapsto \phi(x)/\langle \phi(c)\rangle_D$ is an embedding. This immediately follows by observing that $\phi(\langle c\rangle_C) = \langle \phi(c)\rangle_D \cap \phi(C)$.

(II) $c \in C$ is an infinitesimal of C if and only if $\phi(c)$ is an infinitesimal of D.

(III) If D is strongly semisimple then so is C. As a matter of fact, for any $c \in C$, the map $\bar{\phi}\colon C/\langle c\rangle_C \to D/\langle \phi(c)\rangle_D$ of (I) is an embedding. By hypothesis, $D/\langle \phi(c)\rangle_D$ is semisimple, whence so is $C/\langle c\rangle_C$ by (II).

We are now ready to prove the proposition.

(i)⇒(ii). Let $\mathcal{A} = \{A_i \subseteq A \mid A_i$ is a finitely generated subalgebra of $A\}$, and let $\phi_{ij}\colon A_i \to A_j$ be the inclusion map whenever $A_i \subseteq A_j$. Then $\mathcal{A}$ together with the homomorphisms ϕ_{ij} is a direct system of MV-algebras, having A as its direct limit. By (III), each A_i is strongly semisimple.

(ii)⇒(i). Let $\mathcal{S} = \{A_i, \phi_{ij}\}$ be a directed system of strongly semisimple MV-algebras, indexed by the directed partially ordered set I, where each ϕ_{ij} is an embedding of A_i into A_j. Let A be the direct limit of $\mathcal{S}$ with the telescopic maps $\phi_{i\infty}\colon A_i \to A$. Each $\phi_{i\infty}$ is an embedding. Suppose that A is not strongly semisimple, (absurdum hypothesis), and let $g \in A$ be such that $A/\langle g\rangle_A$ is not semisimple. Then there is an element $e \in A$ such that $e/\langle g\rangle_A$ is an infinitesimal of $A/\langle g\rangle_A$. Since the partial order of the index set I is directed, for some $i \in I$ there are $g_i, e_i \in A_i$ with $\phi_{i\infty}(g_i) = g$ and $\phi_{i\infty}(e_i) = e$. The map $\bar{\phi}_{i\infty}\colon A_i/\langle g_i\rangle_{A_i} \to A/\langle g\rangle_A$ of (I) is an embedding. By (II), $e_i/\langle g_i\rangle_{A_i}$ is an infinitesimal element of $A_i/\langle g_i\rangle_{A_i}$, contrary to the hypothesis that A_i is strongly semisimple.

(i)⇒(iii). Immediate from (III).

(iii)⇒(i). If A is not strongly semisimple there are elements $g, e \in A$ such that $e/\langle g\rangle_A$ is an infinitesimal in $A/\langle g\rangle_A$. Let $B \subseteq A$ be the subalgebra of A generated by g and e. By (I) and (II), $e/\langle g\rangle_B$ is an infinitesimal element of $B/\langle g\rangle_B$, and B is not strongly semisimple. □

5. Coda: one-generator MV-algebras

The following result is an easy consequence of Theorem 3.1. We include the elementary proof because it provides a technique for dealing with strong semisimplicity independently of Bouligand–Severi tangents.

Theorem 5.1. *Every one-generator semisimple MV-algebra A is strongly semisimple.*

Proof. As in (1.1)–(1.2), let $X \subseteq [0,1]$ be a nonempty closed set such that $A \cong \mathcal{M}(X)$. For some $g \in \mathcal{M}([0,1])$ let J be the principal ideal of $\mathcal{M}([0,1])$ generated by g, and let J' be the principal ideal of $\mathcal{M}(X)$ generated by $g' = g \restriction X$.

The short argument immediately following (2.2) shows that $J' = \{l \restriction X \mid l \in J\}$. For every $f \in \mathcal{M}([0,1])$, letting $f' = f \restriction X$ we must prove: if f' belongs to all maximal ideals of $\mathcal{M}(X)$ to which g' belongs, then f' belongs to J'. By 3.6.6 in [4] and 4.19 in [8], this amounts to proving

$$\text{if } f = 0 \text{ on } Zg \cap X, \text{ then } f \restriction X \in J'. \tag{5.1}$$

Let Δ be a triangulation of $[0,1]$ such that f and g are linear over every simplex of Δ. The existence of Δ follows from the piecewise linearity of f and g, [13]. In view of the compactness of X and $[0,1]$, it is sufficient to settle the following:

Claim. Suppose $f \in \mathcal{M}([0,1])$ vanishes over $Zg \cap X$. Then for all $x \in X$ there is an open neighbourhood $\mathcal{N}_x \ni x$ in $[0,1]$ together with an integer $m_x \geq 0$ such that $m_x \centerdot g \geq f$ on $\mathcal{N}_x \cap X$.

We proceed by cases.

Case 1. $g(x) > 0$. Then for some integer r and open neighbourhood $\mathcal{N}_x \ni x$ we have $g > 1/r$ on $\mathcal{N}_x$. Letting $m_x = r$ we have $1 = m_x \centerdot g \geq f$ on $\mathcal{N}_x$, whence a fortiori, $m_x \centerdot g \geq f$ on $\mathcal{N}_x \cap X$.

Case 2. $g(x) = 0$. Since f vanishes on $Zg \cap X$, then $f(x) = 0$. Let T be a 1-simplex of Δ such that $x \in T$. Let T_x be the smallest face of T containing x.

Subcase 2.1. $T_x = T$. Then $x \in \text{int}(T)$. Since g is linear over T g vanishes on T. By our hypotheses on f and Δ, f vanishes on T, whence $0 = g \geq f = 0$ on T. Letting $\mathcal{N}_x = \text{int}(T)$ and $m_x = 1$, we get $m_x \centerdot g \geq f$ on $\mathcal{N}_x$ whence a fortiori, the inequality holds on $\mathcal{N}_x \cap X$.

Subcase 2.2. $T_x = \{x\}$. Then $T = \text{conv}(x, y)$ for some $y \neq x$. Without loss of generality, $y > x$. We will exhibit a right open neighbourhood $\mathcal{R}_x \ni x$ and an integer $r_x \geq 0$ such that $r_x \centerdot g \geq f$ on $\mathcal{R}_x \cap X$. The same argument yields a left neighbourhood $\mathcal{L}_x \ni x$ and an integer $l_x \geq 0$ such that $l_x \centerdot g \geq f$ on $\mathcal{L}_x \cap X$. One then takes $\mathcal{N}_x = \mathcal{R}_x \cup \mathcal{L}_x$ and $m_x = \max(r_x, l_x)$.

Subsubcase 2.2.1. If both g and f vanish at y, then they vanish on T (because they are linear on T). Defining $\mathcal{R}_x = \text{int}(T) \cup \{x\}$ and $r_x = 1$, we get $r_x \centerdot g \geq f$ on $\mathcal{R}_x$, whence, in particular, on $\mathcal{R}_x \cap X$.

Subsubcase 2.2.2. If both g and f are positive at y, then for all suitably large m we have $m \centerdot g \geq f$ on T because $f(x) = 0$ and both f and g are linear on T. Letting r_x be the smallest such m and letting $\mathcal{R}_x = \text{int}(T) \cup \{x\}$, we have the desired inequality on $\mathcal{R}_x$, and a fortiori on $\mathcal{R}_x \cap X$.

Subsubcase 2.2.3. $g(y) = 0, f(y) > 0$. By our hypotheses on Δ, g is linear on T and hence $g = 0$ on T. It follows that $X \cap T = \{x\}$; for otherwise, our assumption $Zf \cap X \supseteq Zg \cap X$ together with the linearity of f on T would imply $f(y) = 0$, contrary to our current hypothesis. Letting $\mathcal{R}_x = \text{int}(T) \cup \{x\}$ and $r_x = 1$ we have $r_x \centerdot g \geq f$ on $\mathcal{R}_x \cap X$. □

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