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Global regularity for minimal sets near a $\mathbb{T}$-set and counterexamples

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Abstract. We discuss the global regularity of 2-dimensional minimal sets that are near a $\mathbb{T}$-set (i.e., the cone over the 1-skeleton of a regular tetrahedron centered at the origin), that is, whether every global minimal set in $\mathbb{R}^n$ that looks like a $\mathbb{T}$-set at infinity is a $\mathbb{T}$-set or not. The main point is to use the topological properties of a minimal set at a large scale to control its topology at smaller scales. This is how one proves that all 1-dimensional Almgren-minimal sets in $\mathbb{R}^n$ and all 2-dimensional Mumford–Shah-minimal sets in $\mathbb{R}^3$ are cones. In this article we discuss two types of 2-dimensional minimal sets: Almgren-minimal sets in $\mathbb{R}^3$ whose blow-in limits are $\mathbb{T}$-sets, and topological minimal sets in $\mathbb{R}^4$ whose blow-in limits are $\mathbb{T}$-sets. For the former we eliminate a potential counterexample that was proposed by several people, and show that a genuine counterexample should have a more complicated topological structure; for the latter we construct a potential example using a Klein bottle.

0. Introduction

This paper deals with the global regularity of 2-dimensional minimal sets in $\mathbb{R}^3$ and $\mathbb{R}^4$ that look like a $\mathbb{T}$-set at infinity in $\mathbb{R}^3$ and $\mathbb{R}^4$. The motivation is that we want to decide whether all global minimal sets in $\mathbb{R}^n$ are cones.

This Bernstein type of problem is of interest for all kinds of minimizing problems in geometric measure theory and calculus of variations. It is natural to ask what a global minimizer looks like, once we know local regularity for minimizers. Well known examples are global regularity for complete 2-dimensional minimal surfaces in $\mathbb{R}^3$, area or size minimizing currents in $\mathbb{R}^n$, or global minimizers for the Mumford–Shah functional. Some of them admit very nice descriptions. See [2], [16], [15], and [3] for further information.

Now let us say something more precise about minimal sets. Briefly, a minimal set is a closed set which minimizes the Hausdorff measure among a certain class

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of competitors. Different choices of classes of competitors give different kinds of minimal sets. So we have the following general definition.

Definition 0.1 (Minimal sets). Let $0 < d < n$ be integers. A closed set E in $\mathbb{R}^n$ is said to be minimal of dimension d in $\mathbb{R}^n$ if

$$H^d(E \cap B) < \infty \quad \text{for every compact ball } B \subset \mathbb{R}^n, \tag{0.1}$$

and

$$H^d(E \backslash F) \leq H^d(F \backslash E) \quad \text{for any competitor } F \text{ for } E. \tag{0.2}$$

Remark 0.2. We can of course give the definition of locally minimal sets (and the definitions of Almgren and topological competitors that will appear later) where we replace $\mathbb{R}^n$ in Definition 0.1 by any open set $U \subset \mathbb{R}^n$. This makes no difference when we discuss local regularity, but for global regularity, the ambient space $\mathbb{R}^n$ always plays an important role.

In this paper we will discuss the following two kinds of minimal sets, that is, sets that minimize the Hausdorff measure among two classes of competitors.

Definition 0.3 (Almgren competitor, Al-competitor for short). Let E be a closed set in $\mathbb{R}^n$. An Almgren competitor for E is a closed set $F \subset \mathbb{R}^n$ that can be written as $F = \varphi(E)$, where $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map such that there exists a compact ball $B \subset \mathbb{R}^n$ such that

$$\varphi|_{B^C} = \text{id} \quad \text{and} \quad \varphi(B) \subset B. \tag{0.3}$$

Such a φ is called a deformation in B, and F is also called a deformation of E in B.

Roughly speaking, we say that E is Almgren-minimal when there is no deformation $F = \varphi(E)$, where φ is Lipschitz and $\varphi(x) - x$ is compactly supported, for which the Hausdorff measure $H^d(F)$ is smaller than $H^d(E)$ in large balls. The definition of Almgren minimal sets was invented by Almgren [1] to describe the behaviors of physical objects that span a given boundary with as little surface area as possible, such as soap films.

The second type of competitors was introduced by the author in [10] and [12], where she tried to generalize the definition of Mumford–Shah minimal sets (MS-minimal for short) to higher codimensions. In both definitions, for MS competitors and topological competitors, we ask that a competitor has certain topological properties of the initial set. Sometimes this condition is easier to handle than the deformation condition that is imposed for Al-competitors.

Definition 0.4 (Topological competitor). Let E be a closed set in $\mathbb{R}^n$. We say that a closed set F is a topological competitor of dimension d ($d < n$) of E, if there exists a ball $B \subset \mathbb{R}^n$ such that

1) $F \backslash B = E \backslash B$;

2) For every Euclidean $(n-d-1)$-sphere $S \subset \mathbb{R}^n \backslash (B \cup E)$, if S represents a nonzero element in the singular homology group $H_{n-d-1}(\mathbb{R}^n \backslash E; \mathbb{Z})$, then it is also nonzero in $H_{n-d-1}(\mathbb{R}^n \backslash F; \mathbb{Z})$.

Remark 0.5. When $d = n-1$, this is the definition of a MS-competitor, where we impose a separation condition on the complement of the set.

The so-defined class of topological minimizers is contained in the class of Almgren minimal sets (see [12], Corollary 3.17), and admits some good properties that we are not able to prove for Almgren minimal sets.

Our goal is to show that a minimal set in $\mathbb{R}^n$ is a cone. Topological minimal sets are automatically Almgren minimal, hence we start with Almgren minimal sets, knowing that what we shall say below will also hold for topological minimal sets.

Let E be a d-dimensional reduced Almgren minimal set in $\mathbb{R}^n$. Reduced means that there are no unnecessary points. More precisely, we say that E is reduced if

$$H^d(E \cap B(x,r)) > 0 \quad \text{for } x \in E \text{ and } r > 0. \tag{0.4}$$

Recall that the definition of minimal set is invariant modulo sets of measure zero, and it is not hard to see that for each Almgren (resp. topological) minimal set E, its closed support E^* (the reduced set $E^* \subset E$ with $H^2(E \backslash E^*) = 0$) is a reduced Almgren (resp. topological) minimal set. Hence we can restrict ourselves to discussing only reduced minimal sets.

Now fix any $x \in E$, and set

$$\theta_x(r) = r^{-d} H^d(E \cap B(x,r)). \tag{0.5}$$

This density function θ_x is nondecreasing for $r \in (0, \infty)$ (see Proposition 5.16 in [4]). In particular the two values

$$\theta(x) = \lim_{t \to 0^+} \theta_x(t) \quad \text{and} \quad \theta_\infty(x) = \lim_{t \to \infty} \theta_x(t) \tag{0.6}$$

exist, and are called the density of E at x, and the density of E at infinity, respectively. It is easy to see that $\theta_\infty(x)$ does not depend on x, hence we shall denote it by θ_∞.

Theorem 6.2 of [4] says that if E is a minimal set, $x \in E$, and $\theta_x(r)$ is a constant function of r, then E is a minimal cone centered on x. Thus, by the monotonicity of the density functions $\theta_x(r)$ for any $x \in E$, if we can find a point $x \in E$ such that $\theta(x) = \theta_\infty$, then E is a cone and we are done.

On the other hand, the possible values for $\theta(x)$ and θ_∞ for any E and $x \in E$ are not arbitrary. By Proposition 7.31 of [4], for each x, $\theta(x)$ is equal to the density at the origin of a d-dimensional Al-minimal cone in $\mathbb{R}^n$. The argument given near equation (18.33) of [4], which is similar to the proof of Proposition 7.31 of [4], gives that $\theta(x)$ is also equal to the density at the origin of a d-dimensional Al-minimal cone in $\mathbb{R}^n$. In other words, if we denote by $\Theta_{d,n}$ the set of all possible numbers that can be the density at the origin of a d-dimensional Almgren-minimal cone in $\mathbb{R}^n$, then $\theta_\infty \in \Theta_{d,n}$, and, for any $x \in E$, $\theta(x) \in \Theta_{d,n}$.

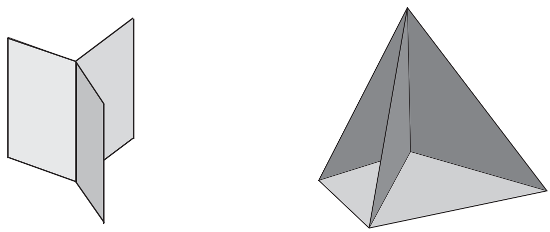

FIGURE 1. A $\mathbb{Y}$-set (left); a $\mathbb{T}$-set (right).

Thus we restrict the range of θ_∞ and $\theta(x)$. Recall that the set $\Theta_{d,n}$ is possibly very small for any d and n. For example, $\Theta_{2,3}$ contains only three values: 1 (the density of a plane), 1.5 (the density of a $\mathbb{Y}$-set, which is the union of three closed half planes with a common boundary L, and that meet along the line L with 120° angles), and d_T (the density of a $\mathbb{T}$-set, i.e., the cone over the 1-skeleton of a regular tetrahedron centered at 0). See Figure 1.

Recall that the reason why θ_∞ has to lie in $\Theta_{d,n}$ is that, for any Al-minimal set E, all its blow-in limits have to be Al-minimal cones (see the argument near equation (18.33) of [4]). A blow-in limit of E is the limit of any converging (for the Hausdorff distance) subsequence of

$$E_r = r^{-1}E, \quad r \to \infty. \tag{0.7}$$

Hence the value of θ_∞ implies that at sufficiently large scales, E looks like an Al-minimal cone of density θ_∞.

This is the same reason why $\theta(x) \in \Theta_{d,n}$. Here we look at the behavior of E_r when $r \to 0$, and the limit of any converging subsequence is called a blow-up limit (this might not be unique!). Such a limit is also an Al-minimal cone C ([4], Proposition 7.31). This means that, at some very small scales around each x, E looks like an Al-minimal cone C of density $\theta(x)$. In this case we call x a C type point of E.

After the discussion above, our problem will be solved if we can prove that every minimal cone C satisfies the following property:

(0.8) there exists $\epsilon = \epsilon_C > 0$, such that for every minimal set E, if $d_{0,1}(C, E) < \epsilon$, then there exists $x \in E \cap B(0,1)$ whose density $\theta(x)$ is the same as that of C at the origin.

Here $d_{x,r}$ stands for the relative distance in the ball $B(x,r)$: for any closed sets E and F,

$$\begin{aligned} & d_{x,r}(E,F) \\ &= \frac{1}{r}\max\big\{\sup\{d(y,F) : y \in E\cap B(x,r)\}, \sup\{d(y,E) : y \in F \cap B(x,r)\}\big\}. \end{aligned} \tag{0.9}$$

The discussion above uses only the values of densities at small scales and at infinity. A geometric interpretation is: there exists $x \in E \cap B(0,1)$ such that a blow-up limit C_x of E at x admits the same density as C at the origin.

Remark 0.6. Note that C_x should also be a minimal cone. It is natural to ask whether two minimal cones that admit the same density at the origin should be the same cone (modulo isometry). This is too much to hope, because in [11] the author gave a continuous family of minimal cones having the same density at the origin, but for which any two cones in the family are nonisometric. However, the cones in this family admit the same topology. We do not know whether two minimal cones with the same density at the origin must admit the same topology.

Besides the global regularity, the property (0.8) helps also to control the relative distances $d_{x,r}$ between a minimal set and minimal cones in the balls $B(x,r)$ and the local speed of decay of the density function $\theta_x(r)$, because this property gives a lower bound on $\theta_x(r)$. When we prove (0.8) for a minimal cone C, we can get nicer local regularity results, that is, if a minimal set is very near C in a ball, then it should be equivalent to C in a smaller ball via a bi-Hölder homeomorphism (a C^1-diffeomorphism in good cases). See [5] for details.

So far we know many minimal cones that satisfy the property (0.8). For a plane it is easily derived from the rectifiability of minimal sets; for a $\mathbb{Y}$-set, the proof is based on a topological argument (see [4], Proposition 16.24); for the unions of two almost orthogonal planes in $\mathbb{R}^4$, the author proved in [14] the property (0.8) for them, by constructing competitors with minimal graphs and using some regularity results for solutions of elliptic systems.

We do not know any minimal cone that does not satisfy the property (0.8), but there are at least two minimal cones for which we do not know whether (0.8) holds: the $\mathbb{T}$-set, and the sets $Y \times Y \in \mathbb{R}^4$, whose minimality has recently been proved in [13]. The topology of the set $Y \times Y$ is more complicated than that of $\mathbb{T}$-sets, but as we will see soon, the situation of $\mathbb{T}$-sets is already tricky.

In this paper we will treat the property (0.8) for the minimal cones $\mathbb{T}$ under the two types of definitions for "minimal".

In Section 1, we discuss (0.8) for $\mathbb{T}$-sets in $\mathbb{R}^3$, where the set E in (0.8) is an Almgren-minimal set. Notice that for the $\mathbb{T}$-set, no topological argument is enough. There is an example $E_0 \subset \overline{B}(0,1)$ proposed by several people (see [17], page 110, or [4], section 19), which is such that

$$E_0 \cap \partial B(0,1) = T \cap \partial B(0,1), \tag{0.10}$$

where T is a $\mathbb{T}$-set centered at the origin, and E_0 satisfies all the known local regularity properties for Al-minimal sets, but E_0 contains no $\mathbb{T}$-point (see [4], Section 19, for a description for E_0). We will eliminate this potential counterexample E_0, and give some descriptions for real potential counterexamples if they exist. In fact, the topological structure of E_0 is too simple; the set of its $\mathbb{Y}$ type points is a union of unknotted C^1 curves. In this case we can easily deform E along one of these unknotted curves to another set with less measure. Notice that, for every minimal set, the set of type $\mathbb{Y}$ points is a union of C^1 curves, hence a potential counterexample should contain a knotted curve in the set of its type $\mathbb{Y}$ points. See Proposition 1.2 and Corollary 1.4 for details.

From the author's point of view, this complicated topology condition contradicts the spirit of minimal sets. However, we will still give an example of a set that admits this complicated topology. Hence we are still not able to prove (0.8) for $\mathbb{T}$-sets and Almgren minimal sets.

In Section 2 we discuss the property (0.8) for $\mathbb{T}$-sets in $\mathbb{R}^4$, with topological minimal sets. Recall that in $\mathbb{R}^3$ this property has already been proved in [4]). Topological minimality seems to be stronger than Al-minimality, and it is proved in Section 18 of [4] that in $\mathbb{R}^3$ the property (0.8) holds for $\mathbb{T}$-sets. In particular Theorem 1.9 in [4] says that all the 2-dimensional topological minimal sets in $\mathbb{R}^3$ are cones. However, in $\mathbb{R}^4$, when the codimension is 2, things are complicated. We do not know the list of minimal cones in this case. And even if we make some additional assumption (see (2.1), which says that in $\mathbb{R}^4$ there is no 2-dimensional minimal cone whose density is less than the density of a $\mathbb{T}$-set other than a plane or a $\mathbb{Y}$-set), we still end up with a topological counterexample that satisfies all the known local regularity properties.

Some of the results attributed in the present article to [4] can be found in other (earlier) references, e.g. [19], but here, for simplicity, [4] is cited systematically.

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Some useful notation

In all that follows, minimal set means Almgren minimal set;
$[a, b]$ is the line segment with end points a and b;
$[a, b)$ is the half-line with initial point a and passing through b;
$B(x, r)$ is the open ball with radius r and centered on x;
$\overline{B}(x, r)$ is the closed ball with radius r and center x;
$\overrightarrow{ab}$ is the vector $b - a$;
H^d is the Hausdorff measure of dimension d;
$d_H(E, F) = \max\{\sup\{d(y, F) : y \in E, \sup\{d(y, E) : y \in F\}\}$ is the Hausdorff distance between two sets E and F.
$d_{x,r}$, the relative distance with respect to the ball $B(x, r)$, is defined by

$$d_{x,r}(E, F) = \frac{1}{r}\max\{\sup\{d(y, F) : y \in E\cap B(x, r)\}, \sup\{d(y, E) : y \in F\cap B(x, r)\}\}.$$

1. Existence of a point of type $\mathbb{T}$ for a 2-dimensional Al-minimal set in $\mathbb{R}^3$

1.1. Introduction

In this section we treat the old problem of the characterization of 2-dimensional Al-minimal sets in $\mathbb{R}^3$, and restrict the class of potential Al-minimal sets that are not cones.

Recall that this problem for 2-dimensional topological minimal sets in $\mathbb{R}^3$ (which coincide with MS-minimal sets in this case) has been solved positively in [4], where Theorem 1.9 says that all 2-dimensional MS-minimal sets in $\mathbb{R}^3$ are cones.

The proof of this theorem consists essentially in proving the property (0.8) for all 2-dimensional MS-minimal cones in $\mathbb{R}^3$. There are only three types of minimal cones in this case, namely, planes, $\mathbb{Y}$-sets, and $\mathbb{T}$-sets. In [4], (0.8) has been proved for planes and $\mathbb{Y}$-sets, only under the assumption of Almgren minimality. MS-minimality is used to prove (0.8) for $\mathbb{T}$-sets, for which Al-minimality seems to be less powerful.

However, in Subsection 1.2 we are going to eliminate the well-known potential counterexample (see [4], Section 19). Topologically this example satisfies all known local regularity properties for Al-minimal sets, but we will still manage to give another topological criterion (Proposition 1.2 and Corollary 1.4) for minimal sets that look like a $\mathbb{T}$-set at infinity, and use this property to prove that the potential counterexample, as well as some other similar sets, cannot be Almgren-minimal. This topological criterion seems to be really strange, and, intuitively, cannot be satisfied by any global minimal set. However, topologically, sets that admit such a property exist, and we construct such an example in Subsection 1.3.

In Subsection 1.4 we will treat another similar problem, that is, for a $\mathbb{T}$-set T, is the set $T\cap B(0,1)$ the only minimal set E in $\overline{B}(0,1)$ such that $E\cap\partial B(0,1)=T\cap\partial B(0,1)$? While all the above arguments give some methods for controlling the measure of a set by topology, in Subsection 1.4 we will give some way to control the topology of a set by its measure.

1.2. A topological criterion for potential counterexamples

In this section we given a topological condition that must be satisfied by any 2-dimensional non-conical Almgren minimal set in R^3.

First let us recall some facts about such sets. Let E be such a set. We look at the sets

$$E(r,x)=\frac{1}{r}(E-x) \tag{1.1}$$

where r tends to infinity.

For every sequence $\{t_k\}_{k\in\mathbb{N}}$ which tends to infinity and such that $E(t_k,x)$ converges (in all compact sets, for the Hausdorff distance), the limit (called a blow-in limit) should be a minimal cone C (see the arguments in [4] near equation (18.33)). Now by the classification of singularities in [19], C should be a plane, a $\mathbb{Y}$-set, or a $\mathbb{T}$-set. By [4], C cannot be a plane or a $\mathbb{Y}$-set. Hence C is a $\mathbb{T}$-set. Thus there exists a $\mathbb{T}$-set T centered at the origin, and a sequence $\{t_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}t_k=\infty \quad\text{and}\quad \lim_{t_k\to\infty}d_{0,t_k}(E,T)=0. \tag{1.2}$$

Denote the unit ball by $B=B(0,1)$. Denote by y_i, $1\le i\le 4$, the 4 points of type $\mathbb{Y}$ of $T\cap\partial B$. Denote by C the convex hull of $\{y_i,1\le i\le 4\}$, which is a regular tetrahedron inscribed in B. Set $T_C=T\cap C$. A simple calculation gives

$$\frac{1}{2}H^2(\partial C)=\frac{4}{3}\sqrt{3}<2\sqrt{2}=H^2(T_C). \tag{1.3}$$

Set $\delta=\frac{1}{4}(H^2(T_C)-\frac{1}{2}H^2(\partial C))$. Then a minor modification of the proof of Lemma 16.43 of [4] gives:

Lemma 1.1. *There exists $\epsilon_1 > 0$ such that, if $d_{0,2}(E,T) < \epsilon_1$, then*

$$H^2(E \cap C) > H^2(T_C) - \delta. \tag{1.4}$$

On the other hand, there exists $\epsilon_2 > 0$ such that if $d_{0,1}(E,T) < \epsilon_2$, then in the annulus $B(0,3/2)\backslash B(0,1/2)$, E is a C^1 version of T (see [4], Section 18). More precisely, in $B(0,3/2)\backslash B(0,1/2)$, the set E_Y of points of type $\mathbb{Y}$ in E is the union of four C^1 curves $\eta_i, 1 \le i \le 4$. Each η_i is very near the half-line $[o, y_i)$, and around each η_i, there exists a tubular neighborhood $\mathcal{T}_i$ of η_i, which contains $B([0,y_i), r)$ for some $r > 0$, such that E is a C^1 version of a $\mathbb{Y}$-set in $\mathcal{T}_i$. And for the part of $E\backslash E_Y$, $E \cap B(0,3/2)\backslash B(0,1/2)$ is composed of 6 flat surfaces $E_{ij}, 1 \le i < j \le 4$. Each E_{ij} is very near T_{ij}, where T_{ij} is the cone over the great arc l_{ij}, which is the great arc on ∂B that connects y_i and y_j. Thus each E_{ij} is a locally minimal set that is near a plane. Then by an argument similar to the proof of Proposition 6.14 of [11], outside $\cup_{1\le i\le 4}\mathcal{T}_i$, E_{ij} is the graph of a C^1 function of T_{ij}. Hence, in $B(0,3/2)\backslash B(0,1/2)$, E is the image of T by a C^1 diffeomorphism φ, whose derivative is very near the identity.

Thus by (1.2), and possibly modulo a dilation, we can suppose that for $t_k = 2$,

$$d_{0,2}(E,T) < \min\{\epsilon_1, \epsilon_2\}, \tag{1.5}$$

which gives (1.4), and that in $B(0,3/2)\backslash B(0,1/2)$, E is a C^1 version of T.

In particular, on the boundary of C, $E \cap \partial C$ admits the same topology as $T \cap \partial C$. That is, $E \cap \partial C$ is composed of six piecewise C^1 curves $w_{ij}, 1 \le i < j \le 4$. The end points of each w_{ij} are b_i and b_j. In other words, for each $1 \le i \le 4$, the three curves $w_{ij}, j \ne i$ meet at their common end point b_i. Each b_i is very near y_i, where $y_i, 1 \le i \le 4$ are the 4 points of type $\mathbb{Y}$ of $T \cap \partial C$. Then w_{ij} is very near $[b_i, b_j]$. Moreover, if we denote by Ω_i, $1 \le i \le 4$, the connected component of $\partial C\backslash E$ which is opposite to b_i, bounded by the $w_{kl}, k, l \ne i$, then we can ask that ϵ is small enough so that

$$\text{for each } 1 \le i \le 4, H^2(\Omega_i) > \frac{1}{4}H^2(\partial C) - \delta, \tag{1.6}$$

where $\frac{1}{4}H^2(\partial C)$ is the measure of a face of ∂C (see Figure 2).

Now suppose that there is no point of type $\mathbb{T}$ in $E \cap C$. Recall that E_Y is the set of all points of type $\mathbb{Y}$ in E. Then by the C^1 regularity around points of type $\mathbb{Y}$ (see [5], Theorem 1.15 and Lemma 14.6), $E_Y \cap C$ is composed of C^1 curves, whose endpoints are $b_i, 1 \le i \le 4$. Then there exists two curves $\gamma_1, \gamma_2 \subset E_Y$ whose endpoints are the b_i. Suppose, for example, that $\gamma_1 \cap \partial C = \{b_1, b_2\}$, and $\gamma_2 \cap \partial D = \{b_3, b_4\}$.

Now by the C^1 regularity for points of type $\mathbb{Y}$ (see [5], Theorem 1.15 and Lemma 14.6), for each $x \in \gamma_1$, there exists a neighborhood $B(x,r)$ such that in $B(x,r)$, E is a C^1 version of $Y + x$ which cuts $B(x,r)$ into 3 connected components. Then by the compactness of γ_1, there exists $r > 0$ such that in the tubular neighborhood $B(\gamma_1, r)$ of γ_1, E is a distorted Y set, whose singular set is γ_1, and E divides $B(\gamma_1, r)$ into three connected components. Each component is a long tube that joins one of the three Ω_i near b_1 to one of the three Ω_i near b_2. Notice that if,

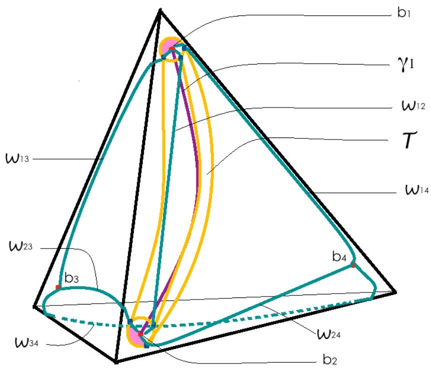

FIGURE 2.

for $i \neq j$, Ω_i and Ω_j are connected by one of these long tubes, then they lie in the same connected component of $B\backslash E$. As a result, there exist $1 \leq i, j \leq 4, i \neq j$ such that Ω_i and Ω_j are in the same connected component of $B\backslash E$, and there exists a long tube $\mathcal{T}$ along γ_1 which connects Ω_i and Ω_j.

Now suppose that there exists a deformation f in C (see Definition 0.3), two indices $1 \leq i \neq j \leq 4$, and two points $x \in \Omega_i, y \in \Omega_j$, such that

$$f(E) \subset C\backslash[x, y]. \tag{1.7}$$

It is then not hard to find a Lipschitz deformation $g : C\backslash B([x, y], r) \to G := \partial C\backslash(\Omega_i \cup \Omega_j)$ such that $g = \mathrm{id}$ on G. To construct of such a g, we can imagine that we enlarge the "hole" $B([x, y], r)$ and push every point in $C\backslash B([x, y], r)$ towards the set G. For example we give, in Figure 3, a sketch illustrating what happens when $E \cap \partial C = T \cap \partial C$. For any set E we have only to make some tiny modification, since $E \cap \partial C$ is a C^1 version of $T \cap \partial C$. Here for each half-plane D that is bounded by the line containing $[x, y]$, we just map $D \cap C\backslash B([x, y], r)$ to the boundary $G \cap D$ (the thicker segments or point in the figure).

Then the function $h := g \circ f$ sends E to a subset of G for t large, and moreover, g does not move $E \cap C = \cup_{k=1}^4 \partial\Omega_k$.

The above argument implies that in C we can deform E to a subset of G.

Now by (1.4) and (1.6),

$$\begin{aligned} H^2(h(E)) &\leq H^2(G) = H^2(\partial C) - H^2(\Omega_i) - H^2(\Omega_j) \\ &< \frac{1}{2}H^2(\partial C) + 2\delta = H^2(T_C) - 2\delta < H^2(E \cap C), \end{aligned} \tag{1.8}$$

which contradicts the fact that E is minimal. As a result, if E does not contain any $\mathbb{T}$ type point, then there is no deformation f of E in C such that $C\backslash f(E)$ contains a segment that connects two different Ω_i. On the other hand, if E contains

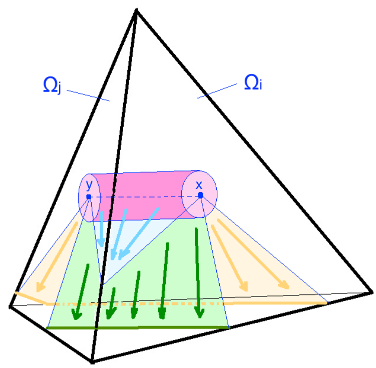

FIGURE 3.

a $\mathbb{T}$-point, then by the argument surrounding (0.6), E is in fact the T centered at this $\mathbb{T}$-point. In this case there is no such deformation f, either. We have therefore:

Proposition 1.2. *Let E be a 2-dimensional Almgren-minimal set in $\mathbb{R}^3$ such that*

1) $d_{0,2}(E,T) < \min\{\epsilon_1, \epsilon_2\}$;
2) *E does not contain any $\mathbb{T}$-point.*

Let C and Ω_i be as above. Then there exists no deformation f of E in C such that $C\backslash f(E)$ contains a segment that connects two different $\Omega_i, 1 \le i \le 4$.

Remark 1.3. By Proposition 1.2, the tube $\mathcal{T}$ along γ_1 cannot be too simple. For example if there exists a Lipschitz homeomorphism f which is a deformation in C such that

$$f(\gamma_1) = [b_1, b_2], \tag{1.9}$$

(in this case we say that γ_1 is not "knotted"), then

$$C\backslash f(E) = f(C\backslash E) \supset f(\gamma_1) = [b_1, b_2], \tag{1.10}$$

which contradicts Proposition 1.2. Thus we get the following:

Corollary 1.4. *If E contains no $\mathbb{T}$-point, then γ_1 and γ_2 are "knotted".*

Because of this corollary, the potential counterexample E_0 proposed in [4] is not a real counterexample, since neither $\gamma_i, i = 1, 2$ in this example is knotted (in the next section we shall explain what E_0 looks like topologically). Thus we have:

Corollary 1.5. *The set E_0 given in Section 19 of [4] is not Almgren-minimal.*

To sum up, if a minimal set E satisfies (1.2), then both γ_1 and γ_2 are knotted. It is not easy to imagine how to knot a $\mathbb{Y}$-set without producing new singularities. However this kind of set does exist. We will construct an example in the next section.

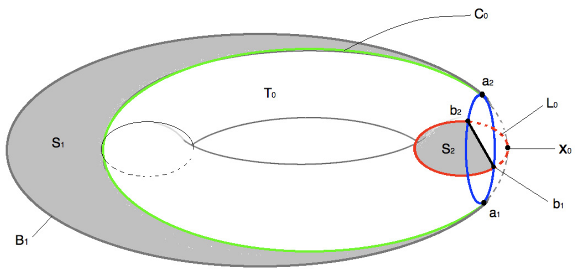

FIGURE 4.

1.3. A set that admits two knotted $\mathbb{Y}$-curves

The purpose of this subsection is to give a topological example of an E for which both γ_1 and γ_2 are knotted. First we look at the well-known example E_0, because such an example already requires a certain imagination. In this example both $\gamma_i, i = 1, 2$ are not knotted.

We take a torus T_0 (see Figure 4). Denote by C_0 (the green circle in the figure) the longest horizontal circle (the equator), and fix any vertical circle L_0 in T_0 (the red circle in the figure). Denote by x_0 their intersection. Take $r_0 > 0$ such that $B_0 = B(x_0, r_0) \cap T_0$ (the blue circle) is a non-degenerate topological disc. Denote by a_1 and a_2 the intersection of ∂B_0 and C_0, and by b_1 and b_2 the intersection of ∂B_0 and L_0.

Denote by $\widetilde{a_1 a_2} = C_0 \backslash B_0$ the arc between a_1 and a_2, and by $\widetilde{b_1 b_2} = L_0 \backslash B_0$ the arc between b_1 and b_2. Next denote by S_2 the vertical planar domain bounded by $[b_1, b_2] \cup \widetilde{b_1 b_2}$. On the other hand, denote by P the plane containing C_0, and take a closed disk $\overline{B}_1 \subset P$ which contains $\widetilde{a_1 a_2}$ and whose boundary contains a_1 and a_2. Now denote by $\widehat{a_1 a_2} = \partial B_1 \backslash B_0$ the larger arc of ∂B_1 between a_1 and a_2, and denote by $S_1 \subset P$ the part between $\widetilde{a_1 a_2}$ and $\widehat{a_1 a_2}$.

Now we happily claim that the set $(T_0 \backslash B_0) \cup S_1 \cup S_2$ is topologically the example E_0 given in Section 19 of [4]. Here a_1, a_2, b_1, b_2 are the four $\mathbb{Y}$-points which correspond to the four $\mathbb{Y}$-points in $E_0 \cap \partial B(0, 1)$, $\widetilde{a_1 a_2}$ and $\widetilde{b_1 b_2}$ correspond to γ_1 and γ_2 respectively; $\widehat{a_1 a_2}$ and $[b_1, b_2]$, together with the four arcs on ∂B_0 between a_i and b_j, $i, j = 1, 2$, correspond to the six curves of $E_0 \cap \partial B(0, 1)$ (but of course we need to deform this picture a lot). If we were to modify our topological example so that the surfaces meet each other with $120°$ angles along the curves γ_i, then we would get E_0.

After the above discussion, we are now ready to construct (in $\mathbb{R}^3$) our example E_1, where γ_1 and γ_2 are both knotted. Moreover, in $\mathbb{R}^3 \backslash E_1$ there is no non-knotted curve that connects Ω_1 to Ω_2, or Ω_3 to Ω_4. The idea is to replace the γ_1 and γ_2 in E_0, which are a pair of cogenerators of $\pi_1(T_0)$, by another pair of knotted representatives of cogenerators of $\pi_1(T_0)$ of the torus T_0.

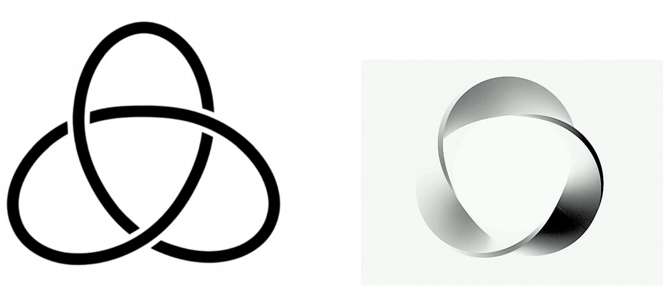

FIGURE 5.

Let us first point out that the following example E_1 is just a topological one, and it is not very likely that E_1 is minimal.

Still, take our torus T_0. Let the notation be as before; that is, C_0 denotes the longest horizontal circle (the equator) and L_0 is a vertical circle in T_0. Denote by x_0 their intersection. Take $r_0 > 0$ such that $B_0 = B(x_0, r_0) \cap T_0$ (the blue circle) is a non-degenerate topological disc. Denote by a_1 and a_2 the intersection of ∂B_0 and C_0, and by b_1 and b_2 the intersection of ∂B_0 and L_0.

Denote by $\Gamma = \mathbb{Z}^2$ the integer lattice in $\mathbb{R}^2$. We identify T_0 with the image of $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. For any two integers $m, n \in \mathbb{Z}$, denote by $d(m,n) = [(0,0),(m,n)]$ the segments with endpoints $(0,0)$ and (m,n). Denote by $K(m,n) = \pi(d(m,n))$. Then $K(m,n)$ is a simple closed curve (that is, π is injective on $d(m,n)$) if and only if the greatest common divisor (m,n) of m and n equals 1. For any integers m, n, a, b with $(m,n) = (a,b) = 1$, $K(m,n)$ and $K(a,b)$ represent a pair of cogenerators of $\pi_1(T_0)$ if and only if $\left|\det\begin{pmatrix} m & n \\ a & b \end{pmatrix}\right| = 1$ (see [18]). Without loss of generality, suppose that $K(1,0) = L_0$, and $K(0,1) = C_0$.

Take a pair of knotted curves $K(2,3)$ and $K(3,4)$ which represent a pair of cogenerators of $\pi_1(\mathbb{T}_0)$. Then the two curves intersect at one point. Without loss of generality, suppose this point of intersection is x_0. Denote by $\mathrm{Int}(T_0)$ and $\mathrm{Ext}(T_0)$ the two connected components of $S^3 \backslash T_0$.

First we want to construct two surfaces S_1 and S_2, such that $S_1 \subset \mathrm{Ext}(T_0)$, $S_2 \subset \mathrm{Int}(T_0)$, $\partial S_1 = K(2,3)$, and $\partial S_2 = K(3,4)$.

Notice that the torus knot $K(3,2)$ is a trefoil knot (see Figure 5, left), which bounds a nonorientable surface $S_1' \subset \mathrm{Int}(T_0)$ (see Figure 5, right). The pair of topological spaces $(\mathrm{Int}(T_0) \cup T_0, T_0)$ is homeomorphic to $(\mathrm{Ext}(T_0) \cup T_0, T_0)$, by some homeomorphism φ or S^3 that sends the point ∞ to a point in $\mathrm{Int}(T_0)$, and $\varphi(K(3,2)) = K(2,3)$. Thus $K(2,3)$ bounds a surface $S_1 = \varphi(S_1') \subset \mathrm{Ext}(T_0)$.

The curve $K(3,4)$ intersects with the vertical circle L_0 at four points p_0, p_1, p_2 and p_3 in clockwise order. Denote by $s_0 \subset \mathrm{Int}(T_0)$ the vertical planar disk whose boundary is L_0, and for $\theta \in [0, 2\pi]$, let s_θ denote the vertical section disk of $\mathrm{Int}(T_0)$ with polar angle θ (see Figure 6). Then s_θ also intersects $K(3,4)$ at four points, and when $\theta < 2\pi$, the intersection of $K(3,4)$ with the tube $\cup_{0 \le \alpha \le \theta} s_\alpha$ is the disjoint union of four curves. Then for each $0 \le i \le 3$, there is a point on $K(3,4) \cap s_\theta$ that is connected to p_i by one of these four curves. Denote by this point $p_i(\theta)$. Notice that at the angle 2π, we have $s_0 = s_{2\pi}$, hence for each $0 \le i \le 3$, $p_i(2\pi)$ is one of the two points on s_0 that is adjacent to p_i, and $\{p_i(2\pi), 0 \le i \le 3\} = \{p_i(0), 0 \le i \le 3\}$. Moreover, since $K(3,4) = \cup_{0\le i\le 3} p_i([0,2\pi))$ is connected, if $p_i(2\pi) = p_j$, then $p_j(2\pi) \ne p_i$. Under all these conditions, there are only two possibilities: either $p_i(2\pi) = p_{i+1}$ for $i = 0,1,2$ and $p_3(2\pi) = p_0$, or $p_i(2\pi) = p_{i-1}$ for $i = 1,2,3$ and $p_0(2\pi) = p_3$. Without loss of generality, suppose that $p_i(2\pi) = p_{i+1}$ for $i = 0,1,2$ and $p_3(2\pi) = p_0$.

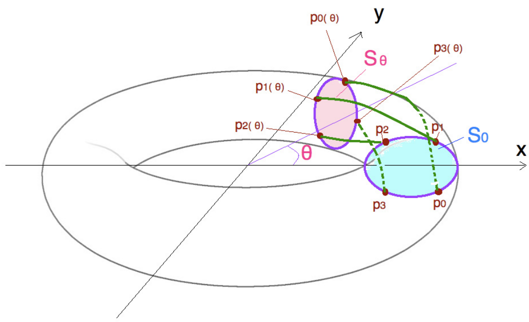

FIGURE 6.

Take an isotopy $f : [0,2\pi] \times s_0 \to \text{Int}(T_0) \cup T_0$ such that $f_0 = id$, $f_\theta(s_0) = f(\theta, s_0) = s_\theta$, $f_\theta(\partial s_0) = \partial s_\theta$, and $f_\theta(p_i) = p_i(\theta)$. Then the image $S_2' = f([0,3\pi/2]\times ([p_0,p_1] \cup [p_2,p_3]))$ is a surface inside T_0, whose boundary is the curve

$$\{K(3,4) \cap (\textstyle\bigcup_{0\le\theta\le 3\pi/2} s_\theta)\} \cup ([p_0,p_1] \cup [p_2,p_3]) \cup f_{3\pi/2}([p_0,p_1] \cup [p_2,p_3]).$$

Now we have to find a surface in the remainder, i.e., in $\cup_{3\pi/2\le\theta\le 2\pi} s_\theta$, whose boundary is $\{K(3,4)\cap(\cup_{3\pi/2\le\theta\le 2\pi} s_\theta)\}\cup([p_0,p_1]\cup[p_2,p_3])\cup f_{3\pi/2}([p_0,p_1]\cup[p_2,p_3])$. Notice that we cannot continue to use the image under f_t, $3\pi/2 \le \theta \le 2\pi$, because $f_{2\pi}([p_0,p_1])$ will be something that connects p_1 and p_2, rather than a curve that connects p_0 to p_1 or p_2 to p_3. However, we can find the solution by a saddle surface S_2''. Refer to Figure 7, where a_i denotes $f_{3\pi/2}(p_i)$.

Denote by $S_2 = S_2' \cup S_2'' \subset \text{Int}(T_0)$, then $\partial S_2 = K(3,4)$.

Now to sum up, we have found two surfaces $S_1 \in \text{Ext}(T_0)$ and $S_2 \in \text{Int}(T_0)$, with $\partial S_1 = K(2,3)$ and $\partial S_2 = K(3,4)$.

Now we take a diffeomorphism of S^3, which maps T_0 to T_0, $\text{Int}(T_0)$ to $\text{Int}(T_0)$, and $\text{Ext}(T_0)$ to $\text{Ext}(T_0)$. Moreover we ask that the images l_1 of $K(2,3)$ and l_2 of $K(3,4)$ satisfy that $l_1 \cap l_2 = x_0$; $l_1 \cap B_0 = C_0 \cap B_0$, the shorter arc of C_0 between a_1 and a_2; and $l_2 \cap B_0 = L_0 \cap B_0$ the arc of L_0 between b_1 and b_2 that passes through x_0. Then the images of S_1 and S_2 are still two surfaces S_3 and S_4, with $\partial S_3 = l_1$, $\partial S_4 = l_2$, $S_3 \subset \text{Ext}(T_0)$ and $S_4 \subset \text{Int}(T_0)$.

We still need to make some modifications, because the two surfaces S_3 and S_4 meet each other at the boundary. So we take a homeomorphism φ of S^3, which fixes $T_0 \backslash B_0$, and satisfies $\varphi(l_1 \cap B_0) = \widehat{a_1, a_2}$ and $\varphi(l_2 \cap B_0) = [b_1, b_2]$. Then

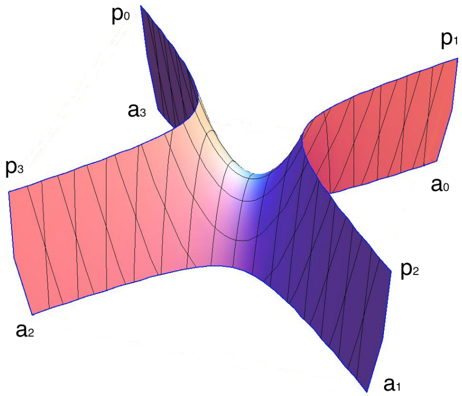

FIGURE 7.

$S_5 = \varphi(S_3) \subset \text{Ext}(T_0)$, and $S_6 = \varphi(S_4)$ is contained in $\text{Int}(T_0)\backslash C$, where C denotes the convex hull of $\{a_1, a_2, b_1, b_2, x_0\}$.

Let $E_1 = (T_0\backslash B_0) \cup S_5 \cup S_6$. Let $\gamma_1 = l_1\backslash B_0$, and $\gamma_2 = l_2\backslash B_0$. These are two knotted $\mathbb{Y}$ curves of E_1, because as in E_0, we have two surfaces, S_5 and S_6, such that $\partial S_5 = \gamma_1 \cup \widehat{a_1, a_2}$ and $\partial S_6 = \gamma_2 \cup [b_1, b_2]$. We can deform E_1 into $B(0,1)$, such that $E_1 \cap \partial B(0,1) = T \cap \partial B(0,1)$ for some $\mathbb{T}$-set T. Here a_1, a_2, b_1, b_2 are the four $\mathbb{Y}$-points which correspond to the four $\mathbb{Y}$-points in $E_1 \cap \partial B(0,1)$, while $\widehat{a_1 a_2}$ and $[b_1, b_2]$, together with the four arcs on ∂B_0 between a_i and $b_j, i, j = 1, 2$ correspond to the six curves of $E_1 \cap \partial B(0,1)$.

Thus we have constructed an example whose set of $\mathbb{Y}$-points is the union of two knotted curves. Moreover, we cannot find any non-knotted curve in $B(0,1)\backslash E_1$ that connects a_1 to a_2, or b_1 to b_2. That is, there is no deformation f of E_1 in $B(0,1)$ such that $B(0,1)\backslash f(E_1)$ contains a segment that connects two different Ω_i. While this example E_1 seems too complicated to be minimal, we do not know how to prove this.

However, for another closely related problem, we can prove that a minimal set does not admit a knotted $\mathbb{Y}$ curve. See the next section.

1.4. Another related problem

We take a $\mathbb{T}$-set T centered at the origin. That is, T is the cone over the 1-skeleton of a regular tetrahedron C centered at the origin and inscribed in the unit ball.

In this section we will discuss whether there exists a set $E \subset \overline{B}(0,1)$ different from $T \cap \overline{B}(0,1)$, that is minimal in $B(0,1)$, and such that $E \cap \partial B(0,1) = T \cap \partial B(0,1)$.

Denote by $B = B(0,1) \subset \mathbb{R}^3$, and by $\overline{B}$ its closure. Then T divides the sphere ∂B into four equal triangular open regions $\{S_i\}_{1 \le i \le 4}$, with

$$\cup_{i=1}^4 \overline{S_i} = \partial B \quad \text{and} \quad \cup_{i=1}^4 S_i = \partial B\backslash T. \tag{1.11}$$

Recall that T divides ∂C into four equal open planar triangles $\{\Omega_i\}_{1\le i\le 4}$. For notational convenience we ask that for each i, S_i and Ω_i share the same three vertices.

Denote by a_j, $1\le j\le 4$, the four vertices of $T\cap\partial B$, where $a_j=\cap_{i\neq j}\overline{S_j}\cap\partial B$ is the point opposite to S_j.

Proposition 1.6. *Let $E\subset\overline{B}\cap\mathbb{R}^3$ be a closed, 2-rectifiable, locally Ahlfors regular set with*

$$E\cap\partial B=T\cap\partial B. \tag{1.12}$$

Then:

1) *If $H^2(E)<H^2(T\cap B)$,*

$$\text{there exists } 1\le i<j\le 4, \text{ and four points } a,\ b,\ c \text{ and } d \text{ that lie in a common plane such that } a\in S_i,\ d\in S_j,\ b,c\in B\backslash E,\ \angle abc>\pi/2,\ \angle bcd>\pi/2 \text{ and } [a,b]\cup[b,c]\cup[c,d]\subset\overline{B}\backslash E. \tag{1.13}$$

Here $[x,y]$ denotes the segment with endpoints x and y, and $\angle abc\in[0,\pi]$ denotes the angle of the smaller sector bounded by $\overline{ba}$ and $\overline{bc}$.

2) *If E is a reduced minimal set in B and satisfies (1.12), then*

$$\text{either } E=T\cap\overline{B}, \text{ or (1.13) is true.} \tag{1.14}$$

Before we prove Proposition 1.6, we first give a corollary.

Corollary 1.7. *Let $E\subset\overline{B}$ be a reduced minimal set in B and satisfying (1.12). Then if $E\neq T\cap\overline{B}$, we have*

$$H^2(E)\le H^2((T\cap\overline{B}\backslash C)\cup G)=H^2(T\cap\overline{B})-(2\sqrt{2}-4\sqrt{3}/3)\ (\approx 0.519). \tag{1.15}$$

Proof. Let E be such a set. Then by (1.14), (1.13) is true. Since (1.13) gives the existence of a deformation f in B such that $f(E)\subset B\backslash[a,d]$, we can deform E on a subset of $(T\cap B\backslash C^\circ)\cup G$ (C° denotes the interior of C), where $G=\partial C\backslash(\Omega_i\cup\Omega_j)$ (recall that Ω_i and Ω_j are the two faces of C corresponding to the two faces S_i and S_j of ∂B, where S_i and S_j contain the points a and d).

Thus by (2.4),

$$H^2(E)\le H^2((T\cap\overline{B}\backslash C)\cup G)=H^2(T\cap\overline{B})-(2\sqrt{2}-4\sqrt{3}/3). \tag{1.16}$$ □

Proof of Proposition 1.6. We are going to prove 1) by contraposition. Suppose that (1.13) is not true.

Denote by P_j the plane orthogonal to $\overrightarrow{oa_j}$ and tangent to the unit sphere, and denote by p_j the orthogonal projection to P_j. Set $R_j=p_j(\overline{S}_j)\subset p_j(\cup_{i\neq j}\overline{S}_i)\subset P_j$. Then for each $1\le j\le 4$ and each $x\in R_j$,

$$p_j^{-1}(x)\cap E\neq\emptyset. \tag{1.17}$$

In fact if (1.17) is not true for some j, that is, $R_j\backslash p_j(E) \neq \emptyset$. As the projection of a compact set, $p_j(E)$ is compact. Thus $R_j\backslash p_j(E)$ is a nonempty open set. Note that $R_j\backslash(\cup_{i\neq j} p_j(S_i))$ has measure zero, therefore

$$(R_j\backslash p_j(E)) \cap (\cup_{i\neq j} p_j(S_i)) \neq \emptyset. \tag{1.18}$$

Take $x \in (R_j\backslash p_j(E)) \cap (\cup_{i\neq j} p_j(S_i))$. Then $x \notin \partial R_j$, because $\partial S_j \subset E$ and hence $\partial R_j = \partial p_j(S_j) = p_j(\partial S_j) \subset p_j(E)$. As a result, $p_j^{-1}(x) \cap B$ is a segment $[a, d]$ perpendicular to P_j, with $a \neq d$, $a \in S_j^\circ$ and $d \in \cup_{i\neq j} S_i^\circ$. Take $b, c \in [a, d]$ such that a, b, c and d are different. Then (1.13) holds, which contradicts our hypothesis.

Hence (1.17) holds. Now for each $x \in R_j$, denote by $f_j(x)$ the point in $p_j^{-1}(x) \cap E$ which is the nearest to R_j. In other words, $f_j(x)$ is the first point in E whose projection is x. This point exists by (1.17), and is unique, since $p_j^{-1}(x)$ is a line orthogonal to R_j.

Let $A_j = f_j(R_j)$. Then A_j is measurable. In fact,

$$\begin{aligned} A_j &= \{x \in E : \forall y \in E \text{ such that } d(y, P_j) < d(x, P_j), |p_j(y) - p_j(x)| > 0\} \\ &= \bigcap_{p,q} \{x \in E : \forall y \in E \text{ such that } d(y, P_j) < d(x, P_j) - 2^{-p}, |p_j(y) - p_j(x)| > 2^{-q}\}. \end{aligned} \tag{1.19}$$

Now E is rectifiable, hence $A_j \subset E$ is also rectifiable. Therefore for almost all $x \in A_j$, the approximate tangent plane $T_x A_j$ of A_j at x exists. Denote by $v_j = \overrightarrow{oa_j}/|oa_j|$ the unit exterior normal vector of P_j, and denote by $w_j(x)$ the unit vector orthogonal to $T_x A_j$ such that $\langle v_j, w_j(x)\rangle \geq 0$. Then $w_j(x)$ is well defined for every $x \in A_j$ with $T_x A_j \not\perp P_j$.

Denote by

$$E_j = \{x \in A_j : T_x A_j \not\perp P_j\}. \tag{1.20}$$

Then w_j is a measurable vector field on E_j. On the other hand, by Sard's theorem, $H^2(p_j(A_j\backslash E_j)) = 0$. Since p_j is injective on A_j, $p_j(A_j\backslash E_j) = p_j(A_j)\backslash p_j(E_j) = R_j\backslash p_j(E_j)$, and thus

$$H^2(R_j\backslash p_j(E_j)) = 0. \tag{1.21}$$

Moreover, for almost all $x \in E_j, T_x A_j = T_x E_j$.

We are going to show that

$$\int_{E_j} \langle v_j, w_j(x)\rangle\, dx = H^2(R_j). \tag{1.22}$$

First, we apply the area formula for Lipschitz maps between rectifiable sets (see 3.2.20 of [7]), with $m = \nu = 2$, $W = E_j$, $f = p_j, g = 1_{R_j}$, and we get

$$\int_{E_j} || \wedge_2 apDp_j(x)||\, dH^2 x = \int_{R_j} N(p_j, z)\, dH^2 z. \tag{1.23}$$

Moreover, by (1.17) and (1.21), $N(p_j, z) \geq 1$ for almost all $z \in R_j$. On the other hand, $N(p_j, z) \leq 1$ since E_j is contained in the set A_j on which p_j is injective. Hence $N(p_j, z) = 1$ for almost all $z \in R_j$. Therefore

$$\int_{R_j} N(p_j, z)\, dH^2 z = H^2(R_j). \tag{1.24}$$

For the left side of (1.23), take $w_j^1(x)$ to be a unit vector in $T_x E_j$ such that $w_j^1(x)//R_j$, and $w_j^2(x)$ to be the unit vector in $T_x E_j$ orthogonal to $w_j^1(x)$. Then $p_j(w_j^1(x)) \perp p_j(w_j^2(x))$, by elementary geometry in $\mathbb{R}^3$. Therefore

$$\begin{aligned} || \wedge_2 apDp_j(x)|| &= ||p_j(w_j^1(x)) \wedge p_j^j(w_2(x))|| = |p_j(w_j^1(x))|\, |p_j(w_j^2(x))| \\ &= |p_j(w_j^2(x))|. \end{aligned} \tag{1.25}$$

The first inequality is because p_j is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, the second inequality is because $p_j(w_j^1(x)) \perp p_j(w_j^2(x))$, and the last is because $w_j^1(x)//S_j$.

Now set $v_j^2(x) = p_j(w_j^2(x))/|p_j(w_j^2(x))| \in P_j$. This is well defined because $T_x E_j \not\perp P_j$ and hence $|p_j(w_j^2(x))| > 0$. Then $w_j(x), w_j^2(x), v_j, v_j^2(x)$ are all orthogonal to $w_j^1(x)$, and hence belong to a single plane, with $w_j(x) \perp w_j^2(x), v_j \perp v_j^2(x)$. Therefore

$$|\langle w_j(x), v_j\rangle| = |\langle w_j^2(x), v_j^2(x)\rangle| = |p_j(w_j^2(x))|. \tag{1.26}$$

Since by definition $\langle w_j(x), v_j\rangle \geq 0$,

$$\langle w_j(x), v_j\rangle = |p_j(w_j^2(x))| = || \wedge_2 apDp_j(x)||, \tag{1.27}$$

by (1.25). Combining (1.23), (1.24) and (1.27), we get (1.22). Note that v_j does not depend on E.

Now for $x \in A_j \backslash E_j$, we define a measurable vector field $w_j(x)$ such that $w_j(x) \perp T_x A_j$. Then $\langle w_j(x), v_j\rangle = 0$ for almost all $x \in A_j \backslash E_j$. Hence we have

$$\int_{A_j} \langle v_j, w_j(x)\rangle dx = H^2(R_j). \tag{1.28}$$

We sum over j, and get

$$\sum_{j=1}^4 \int_{A_j} \langle v_j, w_j(x)\rangle\, dx = \sum_{j=1}^4 \int_{E_j} \langle v_j, w_j(x)\rangle\, dx = \sum_{i=1}^4 H^2(R_j). \tag{1.29}$$

Next, set $E_j^0 = E_j \backslash \cup_{i\neq j} E_i$, $E_{ij} = (E_i \cap E_j) \backslash \cup_{k\neq i,j} E_k$ for $i \neq j$. We claim that

$$E_j \backslash (E_j^0 \cup \textstyle\bigcup_{i\neq j} E_{ij}) \text{ is of measure zero for all } j. \tag{1.30}$$

Suppose (1.30) is not true. Then there exist three different i, j and k such that $E_i \cap E_j \cap E_k$ has positive measure. Suppose for example that $i = 1$, $j = 2$ and $k = 3$, and set $E_{123} = E_1 \cap E_2 \cap E_3$. Now since E_{123} is a measurable rectifiable set of positive measure, and $E_{123} \subset E$, for almost all $x \in E_{123}$, the approximate tangent plane T_xE_{123} of E_{123} at x exists and equals T_xE. Moreover, since E is locally Ahlfors regular, T_xE is a real tangent plane (see, for example, [3], Exercise 41.21, page 277). We choose and fix such an $x \in E_{123}$.

By definition of A_j, for $j = 1, 2, 3$, the segment $[x, p_j(x)] \cap E = \{x\}$. And by definition of E_j, $T_xE \not\perp P_j$, and hence $[x, p_j(x)] \cap (T_xE + x) = \{x\}$, since $[x, p_j(x)] \perp P_j$. The affine subspace $T_xE + x$ separates $\mathbb{R}^3$ into two half-spaces, and since for $j = 1, 2, 3$, $(x, p_j(x)] \cap (T_xE + x) = \emptyset$, there exist $1 \le i < j \le 3$ such that $(x, p_i(x)]$ and $(x, p_j(x)]$ are on the same side of $T_xE + x$. Suppose for example that $i = 1$ and $j = 2$.

For $i = 1, 2$, denote by α_i the angle between $[x, p_i(x)]$ and $T_xE + x$. Set $\alpha = \min\{\alpha_1, \alpha_2\}$. Then since T_xE is a real tangent plane, there exists $r > 0$ such that for all $y \in E \cap B(x, r)$,

$$d(y, T_xE + x) < \frac{r}{2} \sin\alpha. \tag{1.31}$$

Set $b = [x, p_1(x)] \cap \partial B(x, r)$ and $c = [x, p_2(x)] \cap \partial B(x, r)$. Then by definition of α, $d(b, T_xE + x) \ge r \sin\alpha$, and $d(c, T_xE + x) \ge r \sin\alpha$. Since b and c are on the same side of $T_xE + x$, for all $y \in [b, c]$, $d(y, T_xE + x) \ge r \sin\alpha$, and hence $[b, c] \cap E = \emptyset$, because of (1.31).

Now set $a = p_1(x)$ and $d = p_2(x)$. Note that in the triangle Δ_{xbc}, $|xb| = |xc|$, which gives that $\angle xbc = \angle xcb$. But $\angle xbc + \angle xcb + \angle bxc = \pi$, $\angle bxc > 0$, Hence $\angle xbc = \angle xcb < \pi/2$. As a result, $\angle abc = \pi - \angle xbc > \pi/2$ and $\angle bcd = \pi - \angle xcb > \pi/2$. Thus we have found four points a, b, c and d such that (1.13) is true, which contradicts our hypothesis.

Thus we get (1.30). And consequently we have

$$H^2(\cup_{j=1}^4 E_j) = \sum_{j=1}^4 H^2(E_j^0) + \sum_{1 \le i < j \le 4} H^2(E_{ij}). \tag{1.32}$$

For estimating the measure, we are going to use the paired calibration method (introduced in [9]). Recall that v_j is the unit exterior normal vector of P_j. Thus by (1.29),

$$\begin{aligned}\sum_{i=1}^4 H^2(R_j) &= \sum_{j=1}^4 \int_{E_j} \langle v_j, w_j(x) \rangle \, dx \\ &= \sum_{j=1}^4 \int_{E_j^0} \langle v_j, w_j(x) \rangle \, dx + \sum_{1 \le i < j \le 4} \int_{E_{ij}} \langle v_i, w_i(x) \rangle + \langle v_j, w_j(x) \rangle \, dx. \end{aligned} \tag{1.33}$$

For the first term,

$$\Big| \int_{E_j^0} \langle v_j, w_j(x) \rangle \, dx \Big| \le \int_{E_j^0} \big| \langle v_j, w_j(x) \rangle \big| \, dx \le \int_{E_j^0} |v_j| \, |w_j(x)| \, dx = H^2(E_j^0), \tag{1.34}$$

and hence

$$\Big| \sum_{j=1}^4 \int_{E_j^0} \langle v_j, w_j(x) \rangle \, dx \Big| \le \sum_{j=1}^4 \Big| \int_{E_j^0} \langle v_j, w_j(x) \rangle \, dx \Big| \le \sum_{j=1}^4 H^2(E_j^0). \tag{1.35}$$

For the second term, observe that $w_i(x) = \pm w_j(x)$ for $x \in E_{ij}$, hence we set $\epsilon_x = w_i(x)/w_j(x)$. Then

$$\begin{aligned}\big|\langle v_i, w_i(x)\rangle + \langle v_j, w_j(x)\rangle\big| &= \big|\langle v_i + \epsilon(x)v_j, w_i(x)\rangle\big| \\ &\leq |v_i + \epsilon(x)v_j|\,|w_i(x)| = |v_i + \epsilon(x)v_j| \leq \max\{|v_i + v_j|, |v_i - v_j|\}.\end{aligned} \tag{1.36}$$

By definition of v_j, the angle between v_i and v_j is the supplementary angle of the angle θ_{ij} between P_i and P_j. A simple calculus gives

$$|v_i + v_j| = \frac{2}{\sqrt{3}} < 1, \quad |v_i - v_j| = \frac{2\sqrt{2}}{\sqrt{3}} > 1. \tag{1.37}$$

Hence $\max\{|v_i + v_j|, |v_i - v_j|\} = |v_i - v_j| > 1$. Denote by D this value. By (1.36),

$$\begin{aligned}&\Big|\int_{E_{ij}} \langle v_i, w_i(x)\rangle + \langle v_j, w_j(x)\rangle\, dx\Big| \\ &\qquad \leq \int_{E_{ij}} \big|\langle v_i, w_i(x)\rangle + \langle v_j, w_j(x)\rangle\big|\, dx \leq DH^2(E_{ij}),\end{aligned} \tag{1.38}$$

and hence

$$\begin{aligned}&\Big|\sum_{1\leq i<j\leq 4} \int_{E_{ij}} \langle v_i, w_i(x)\rangle + \langle v_j, w_j(x)\rangle\, dx\Big| \\ &\qquad \leq \sum_{1\leq i<j\leq 4} \Big|\int_{E_{ij}} \langle v_i, w_i(x)\rangle + \langle v_j, w_j(x)\rangle\, dx\Big| = D \sum_{1\leq i<j\leq 4} H^2(E_{ij}).\end{aligned} \tag{1.39}$$

Combining (1.33), (1.35), and (1.39), we get

$$\begin{aligned}\sum_{i=1}^4 H^2(R_j) &\leq \sum_{j=1}^4 H^2(E_j^0) + D \sum_{1\leq i<j\leq 4} H^2(E_{ij}) \\ &\leq D\Big[\sum_{j=1}^4 H^2(E_j^0) + \sum_{1\leq i<j\leq 4} H^2(E_{ij})\Big] \quad (\text{since } D > 1) \\ &= DH^2(\cup_{j=1}^4 E_j) \leq DH^2(E).\end{aligned} \tag{1.40}$$

On the other hand, we can do the same thing for T, the cone over the 1-skeleton of the regular tetrahedron C. Since T separates the four faces of C, (1.13) is automatically false for T. Then, by the foregoing, we can see that $T_i^0 = \emptyset$ for all i, $\epsilon_{ij} = -1$ for all $i \neq j$, and $(v_i - v_j) \perp T_x T$ for almost all $x \in T_{ij}$, which implies that

$$\langle v_i, w_i(x)\rangle + \langle v_j, w_j(x)\rangle = D \tag{1.41}$$

for all $x \in T_{ij}$. So briefly, the inequalities throughout the argument above are equalities for $T \cap \overline{B}$. As a result,

$$DH^2(E) \geq \sum_{i=1}^4 H^2(R_j) = DH^2(T \cap \overline{B}), \tag{1.42}$$

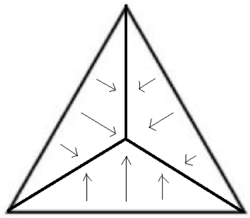

FIGURE 8.

and hence

$$H^2(E) \geq H^2(T \cap \overline{B}) \tag{1.43}$$

for all E that do not satisfy (1.13).

Now we prove 2). Let E be a reduced minimal set. Then it is rectifiable and locally Ahlfors regular in B ([6]).

First note that $H^2(E) \leq H^2(T \cap \overline{B})$. In fact, for each $x \in \overline{B} \backslash T$, there exists $1 \leq i \leq 4$, such that x and S_i belong to the same connected component of $B \backslash T$. Denote by $f(x)$ the first intersection of $x + [0, a_i)$ with T. Then $f : \overline{B} \to T$ is a 2-Lipschitz retraction (see Figure 8). Now if E is a minimal set that satisfies (1.12), for each $\epsilon > 0$, we define $g_\epsilon : \partial B \cup E \to T \cup \partial B$ by $g(x) = f(x)$ for $x \in E \cap \overline{B}(0, 1-\epsilon)$, $g(x) = x$ for $x \in \partial B$. Then we can extend g_ϵ to a 2-Lipschitz map which sends $\overline{B}$ to $\overline{B}$, by Kirszbraun's Theorem ([7], Thm. 2.10.43), see Figure 8. Thus g_ϵ deforms $E \cap \overline{B}(0, 1-\epsilon)$ to a subset of $T \cap \overline{B}(0, 1-\epsilon)$. Thus we have

$$\begin{aligned} H^2(g_\epsilon(E)) &= H^2(g_\epsilon(E \cap \overline{B}(0, 1-\epsilon))) + H^2(g_\epsilon(E \backslash \overline{B}(0, 1-\epsilon))) \\ &\leq H^2(g_\epsilon(T \cap \overline{B}(0, 1-\epsilon))) + \mathrm{Lip}(g_\epsilon)^2 H^2(E \backslash \overline{B}(0, 1-\epsilon) \\ &= H^2(T \cap \overline{B}(0, 1-\epsilon)) + 4H^2(E \backslash \overline{B}(0, 1-\epsilon)) \\ &< H^2(T \cap \overline{B}) + 4H^2(E \backslash \overline{B}(0, 1-\epsilon) \end{aligned} \tag{1.44}$$

The second term tends to 0 when ϵ tends to 0. That is, for any $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that

$$H^2(g_\epsilon(E)) < H^2(T \cap \overline{B}) + \delta. \tag{1.45}$$

Now E is minimal, hence for any $\delta > 0$,

$$H^2(E) \leq H^2(g_\epsilon(\delta)(E)) \leq H^2(T \cap \overline{B}) + \delta, \tag{1.46}$$

therefore

$$H^2(E) \leq H^2(T \cap \overline{B}). \tag{1.47}$$

Hence to prove 2), it is enough to prove that if (1.13) does not hold, and $H^2(E) = H^2(T \cap \overline{B})$, then $E = T \cap \overline{B}$. In particular E contains a point of type $\mathbb{T}$.

By the arguments in 1), if (1.13) is not true, and $H^2(E) = H^2(T \cap \overline{B})$, then the inequalities (1.34)–(1.36) and (1.38)–(1.40) are all equalities. Thus we have

1) For almost all $x \in E_{ij}$, $T_xE_{ij} \perp v_i - v_j$. Denote by P_{ij} the plane perpendicular to $v_i - v_j$. Then for almost all $x \in E_{ij}$, $T_xE_{ij} = P_{ij}$.
2) For all j, $H^2(E_j^0) = 0$, since $D > 1$.
3) For all j, $H^2(A_j \backslash E_j) = 0$.
4) For all j, $p_j(E) = p_j(E_j) = R_j$.

Thus for almost all $x \in E$, T_xE exists and is one of the P_{ij}. If x is a point such that T_xE exists, by the C^1 regularity ([5], Theorem 1.15 and Lemma 14.4), there exists $r = r(x) > 0$ such that in $B(x, r)$, E is the graph of a C^1 function from T_xE to $T_xE^{\perp}$, which implies that in $B(x, r)$, the function $f : E \cap B(x, r) \to G(3, 2)$, $f(y) = T_yE$ is continuous. But for T_yE we have only six choices $P_{ij}, 1 \le i < j \le 4$, which are isolated points in $G(3, 2)$, and so $T_yE = T_xE$ for all $y \in B(x, r) \cap E$. As a result $E \cap B(x, r) = (T_xE + x) \cap B(x, r)$, a disk parallel to P_{ij}.

Still by the C^1 regularity, the set $E_P = \{x \in E \cap B : T_xE \text{ exists}\}$ is a C^1 manifold, and is open in E. Thus we deduce that

(1.48) each connected component of E_P is part of a plane that is parallel to one of the P_{ij}.

Set $E_Y = \{x \in E : x \text{ is of type } \mathbb{Y}\}$. Then $E_Y \neq \emptyset$, because otherwise by (1.48), $E \cap B$ is the intersection of B with a translation of one of the P_{ij}, but then $E \cap \partial B$ is surely not $T \cap \partial B$.

Now if $x \in E_Y$, by the C^1 regularity around points of type $\mathbb{Y}$ ([5], Theorem 1.15 and Lemma 14.6), there exists $r = r(x) > 0$ such that in $B(x, r)$, E is C^1 equivalent to a $\mathbb{Y}$-set Y. Denote by L_Y the spine of Y, and by S_1, S_2 and S_3 the three open half-planes of Y. Then if we denote by φ the C^1 diffeomorphism which sends Y onto E in $B(x, r)$, the $\varphi(S_i) \cap B(x, r)$, $1 \le i \le 3$, are connected C^1 manifolds, and hence each of them is a part of a plane parallel to P_{ij}. Consequently, $\varphi(L_y) \cap B(x, r)$ is an open segment passing through x and parallel to one of the $D_j, 1 \le j \le 4$, where $D_j = P_{ij} \cap P_{jk}$.

Hence $E_Y \cap B$ is a union of open segments $I_1, I_2, \dots$, each of which is parallel to one of the D_j, and every endpoint is either a point in the sphere ∂B, or a point of type T. Moreover,

(1.49) for each $x \in E_Y$ such that $T_xE_Y = D_j$, there exists $r > 0$ such that, in $B(x, r), E$ is a $\mathbb{Y}$-set whose spine is $x + D_j$.

Now if $x \in E$ is a $\mathbb{T}$-point, then by the arguments above, the blow-up limit C_xE of E at x is the set T (the set T defined at the very beginning of this section). As a result, for each segment I_i, at least one of its endpoints is in the unit sphere. In fact, if both of the endpoints x and y of I_i are of type $\mathbb{T}$, then at least one of the two blow-up limits C_xE and C_yE is not the set T, because two parallel $\mathbb{T}$-sets cannot be connected by a common spine.

Hence all the segments I_i touch the boundary.

Lemma 1.8. *If x is a $\mathbb{T}$-point (and hence $C_xE = T$), then $(T+x)\cap B \subset E$.*

Proof. By the C^1 regularity around points of type T, there exists $r>0$ such that in $B(x,r)$, E is a C^1 version of $T+x$. Then by (1.48) and (1.49), $E\cap B(x,r) = (T+x)\cap B(x,r)$. Denote by L_i, $1\le i\le 4$, the four spines of $T+x$. Then $L_i\cap B\subset E_Y$, because $L_i\cap B(x,r)$ is part of a segment $I_j\subset E_Y$, which has already an endpoint x that does not belong to the unit sphere, hence the other endpoint must be in the sphere, which yields $I_j = L_i\cap B(0,1)$.

Now we take a one parameter family of open balls B_s with radii $r\le s\le 1$, with $B_r = B(x,r)$ and $B_1 = B(0,1)$, such that

1) $B_s\subset B_{s'}$ for all $s<s'$;
2) $\cap_{1>t>s}B_t = \overline{B}_s$ and $\cup_{t<s}B_t = B_s$ for all $r\le s\le 1$.

Set $R = \inf\{s>r, (T+x)\cap B_s \not\subset E\}$. We claim that $R=1$.

Suppose this is not true. By definition of B_s, the four spines and the six faces of $T+x$ are never tangent to ∂B_s, $r<s<1$, since $B(x,r)\subset B_s$.

Now for each $y\in\partial B_R\cap(T+x)$, y is not a $\mathbb{T}$-point. In fact, if y belongs to one of the L_i, then y is a $\mathbb{Y}$ point, since $L_i\backslash\{x\}\subset E_Y$ and $L_i\cap B\subset E_Y$; if y is not a $\mathbb{Y}$ point, then there exist i and j such that $y\in x+P_{ij}$. Thus there exists $r_y>0$ such that $B(y,r_y)\cap(x+T)$ is a disk D_y centered at y. Now by definition of R, for all $s<R$, $B_s\cap(T+x)\subset E$, and hence $B_R\cap(T+x)\subset E$. Hence $D\cap B_R\cap B(y,r_y)\subset E$, which means that y cannot be a point of type $\mathbb{T}$.

If y is a point of type $\mathbb{P}$ (i.e., a planar point), suppose for example that $y\in P_{ij}+x$. Then $T_yE = P_{ij}$. By (1.48), and since $R<1$, there exists $r_y>0$ such that $E\cap B(y,r_y) = (P_{ij}+y)\cap B(y,r_y)$. In other words,

$$\text{there exists } r_y>0 \text{ such that } E \text{ coincides with } T+x \text{ in } B_R\cup B(y,r_y). \tag{1.50}$$

If y is a point of type $\mathbb{Y}$, then it is in one of the L_i. By the same argument as above, using (1.49), we get also (1.50).

Hence (1.50) is true for all $y\in\partial B_R\cap(T+x)$. As $\partial B_R\cap(T+x)$ is compact, we have thus a uniform $r>0$ such that for each y, (1.50) is true if we set $r_y=r$. However, this contradicts the definition of R.

Hence $R=1$, but $B_1\subset B$ is of radius 1, so $B_1 = B$. Then by definition of R we get the conclusion of Lemma 1.8. □

By Lemma 1.8, we know that if x is a $\mathbb{T}$-point, then x has to be the origin, because of (1.12). Hence $T\cap B\subset E$. In this case, we have $E = T\cap\overline{B}$, because $H^2(E) = H^2(T\cap\overline{B})$.

We still have to discuss the case when there is no point of type $\mathbb{T}$. In this case, the same kind of argument as in Lemma 1.8 gives the following.

Lemma 1.9. *Let x be a $\mathbb{Y}$ point in E, and $T_xE_Y = D_j$. Denote by Y_j the Y whose spine is D_j. Then*

$$(Y_j+x)\cap B\subset E. \tag{1.51}$$

But this is impossible, because $E\cap\partial B = T\cap\partial B$ contains no full part of $(Y_j+x)\cap\partial B(0,1)$ for any x and j.

Hence we have $E = T\cap\overline{B}$, and thus (1.14). □

2. Existence of a point of type $\mathbb{T}$ for a 2-dimensional topological minimal set in $\mathbb{R}^4$

2.1. Introduction

In this section we discuss the property (0.8) for 2-dimensional topological minimal sets in $\mathbb{R}^4$ whose blow-in limits are $\mathbb{T}$-sets. This kind of set exists trivially because a $\mathbb{T}$-cone is topological minimal in $\mathbb{R}^3$, and by Proposition 3.18 of [11], it is topological minimal in any $\mathbb{R}^n$ for $n \geq 3$.

One wonders whether there is any other type of topological minimal sets in $\mathbb{R}^4$ that look at infinity like $\mathbb{T}$-sets without themselves being $\mathbb{T}$-sets. Recall that in $\mathbb{R}^3$ there are no such sets (see Proposition 18.1 of [4]). An important and useful property of $\mathbb{R}^3$ is that in $\mathbb{R}^3$ there are only two kinds of minimal cones whose densities are less than that of a $\mathbb{T}$-set: the planes and the $\mathbb{Y}$-sets. Hence if the blow-in limits of a non-conical minimal set are $\mathbb{T}$-sets, then by the monotonicity of density, in this minimal set all points are of type $\mathbb{P}$ or $\mathbb{Y}$, and hence there can hold the same properties as those stated in (1.4)-(1.6) and Figure 2, and by the same argument.

We do not know if there exists a minimal cone in $\mathbb{R}^4$ whose density is between those of $\mathbb{Y}$-sets and $\mathbb{T}$-sets. However $\mathbb{T}$-sets are the only minimal cones that admit the simplest topology except for planes and $\mathbb{Y}$-sets. Hence it is likely that in $\mathbb{R}^4$ there are no minimal cones between $\mathbb{Y}$-sets and $\mathbb{T}$-sets.

Consequently we make the following additional assumption. Denote by d_T the density of $\mathbb{T}$-sets, and suppose that

(2.1) the only minimal cones in $\mathbb{R}^4$ whose densities are less than d_T are the planes and the $\mathbb{Y}$-sets.

We are going to discuss, under the assumption (2.1), the Bernstein type property for topological minimal sets in $\mathbb{R}^4$ that look like a $\mathbb{T}$-set at infinity.

2.2. A topological criterion for potential counterexamples

Throughout this subsection, we assume that (2.1) is true.

Let E be a 2-dimensional topological minimal set in $\mathbb{R}^4$ that looks like a $\mathbb{T}$-set at infinity. That is, there exists a $\mathbb{T}$-set T centered at the origin, and a sequence $\{r_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} r_k \to \infty \quad \text{and} \quad \lim_{k\to\infty} d_{0,r_k}(E,T) = 0. \tag{2.2}$$

We want to find a type $\mathbb{T}$ point in the set E.

Now the set E is of codimension 2, hence the topological condition is imposed on the group $H_1(\mathbb{R}^4\backslash E, \mathbb{Z})$.

Denote by $\{y_i\}_{1\leq i\leq 4}$ the four $\mathbb{Y}$-points in $T\cap\partial B(0,1)$. Denote by $l_{ij}\subset T\cap\partial B(0,1)$ the great arc on the sphere that connects y_i and y_j. The cone T is composed of 6 closed sectors $\{T_{ij}\}_{1\leq i\neq j\leq 4}$, where T_{ij} is the cone over l_{ij}. Denote by x_{ij}, $1\leq i\neq j\leq 4$, the middle point of l_{ij}. Denote by P_{ij} the 2-plane orthogonal

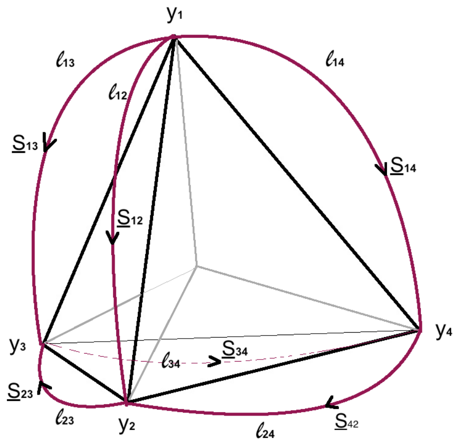

FIGURE 9.

to T_{ij} and passing through x_{ij}. Set $B_{ij} = B(x_{ij}, 1/10) \cap P_{ij}$, and denote by s_{ij} the boundary of B_{ij}. Then s_{ij} is a circle, that does not touch T, and $B_{ij} \cap T = B_{ij} \cap T_{ij} = x_{ij}$.

Fix an orthonormal basis $\{e_i\}_{1\leq i\leq 4}$ of $\mathbb{R}^4$. We are going to give an orientation to each s_{ij}, and denote these oriented circles by $\vec{s}_{ij}$.

For each B_{ij}, there are two orientations $\sigma_1 = x \wedge y$ and $\sigma_2 = -x \wedge y$, where x and y are two mutually orthogonal unit vectors that belong to the plane containing B_{ij}. Take the $k \in \{1, 2\}$ such that $\det_{\{e_i\}_{1\leq i\leq 4}} \overrightarrow{ox_{ij}} \wedge \overrightarrow{y_i y_j} \wedge \sigma_k > 0$, and denote by $\overrightarrow{B}_{ij}$ the oriented disk B_{ij} with this orientation. Denote by $\vec{s}_{ij} = \partial \overrightarrow{B}_{ij}$ the oriented circle, and by $[\vec{s}_{ij}]$ the element in $H_1(\mathbb{R}^4 \backslash T; \mathbb{Z})$ represented by $\vec{s}_{ij}$. The six $[\vec{s}_{ij}]$, $1 \leq i < j \leq 4$, are all different, however they are algebraically dependent.

Figure 9 gives an idea of the above definition (although it is drawn in $\mathbb{R}^3$). Since T is contained in $\mathbb{R}^3$, if we fix an orientation of the complementary dimension in $\mathbb{R}^4$, the orientation of B_{ij} defined before corresponds to one of the orientation of the line orthogonal to T_{ij} in $\mathbb{R}^3$. And this orientation of the line corresponds to the orientations of l_{ij} by the right-hand rule. Hence in Figure 9 we indicate the orientation of l_{ij} with arrows to express the orientation of $[\vec{s}_{ij}]$. In the figure, the orientation $\underline{S}_{ij}$ means $[\vec{s}_{ij}]$.

Thus we have

$$\vec{s}_{ij} = -\vec{s}_{ji}, [\vec{s}_{ij}] = -[\vec{s}_{ji}]. \tag{2.3}$$

Note that $\{[\vec{s}_{ij}], 1 \leq i, j \leq 4\}$ is a set of generators of the group $H_1(\mathbb{R}^4 \backslash T; \mathbb{Z})$.

We say that $[s_{ij}]$ and $[s_{kl}]$ (without the vector arrows) are different (in a homology group) if

$$[\vec{s}_{ij}] \neq \pm[\vec{s}_{kl}], \tag{2.4}$$

and write $s_{ij} \sim s_{kl}$ if $[\vec{s}_{ij}] = \pm[\vec{s}_{kl}]$.

Return to the set E. Without loss of generality, we can suppose (modulo replacing E by E/r_k for some k large) that $d_{0,3}(E,T)$ is small enough (for example less than a certain ϵ_0). Then (by the argument between (1.4) and (1.5)) in $B(0,5/2)\backslash B(0,1/2)$, E is a C^1 version of T. Therefore in $B(0,5/2)\backslash B(0,1/2)$, E is composed of six C^1 faces E_{ij}, that are very close to the T_{ij}. The E_{ij} meet in threes, on four C^1 curves η_i, $1 \leq i \leq 4$, each η_i is very near the half-line $[o, y_i)$, and near each η_i, there exists a tubular neighborhood $\mathcal{T}_i$ of η_i, which contains $B([oy_i), r)$ for some $r > 0$, in which E is a C^1 version of a $\mathbb{Y}$-set. See Section 18 of [4] for details. In total, there is a C^1 diffeomorphism φ, which is very near the identity, such that in $B(0,5/2)\backslash B(0,1/2)$, E coincides with $\varphi(T)$, E_{ij} corresponds to $\varphi(T_{ij})$, and η_i corresponds to $\varphi([0, y_i))$.

In particular, since E is very near T in $B(0,5/2)\backslash B(0,1/2)$, $s_{ij} \cap E = \emptyset$, and $B_{ij} \cap E = B_{ij} \cap E_{ij}$ is also a one point set, so that locally each s_{ij} links E_{ij}, and hence is an element (possibly zero) in $H_1(\mathbb{R}^4\backslash E, \mathbb{Z})$, too.

Now we discuss the values in $H_1(\mathbb{R}^4\backslash E, \mathbb{Z})$ for these s_{ij}.

Lemma 2.1. *Let E be an Al-minimal set that satisfies (2.2). Let the notation be as above. Then if*

(2.5) $$\text{for all } 1 \leq i < j \leq 4, \quad [\vec{s_{ij}}] \neq 0 \text{ in } H_1(\mathbb{R}^4\backslash E, \mathbb{Z}),$$

and

(2.6) $$\text{at least 5 of the } [s_{ij}] \text{ are mutually different in } H_1(\mathbb{R}^4\backslash E, \mathbb{Z}),$$

then E contains at least one point of type other than $\mathbb{P}$ and $\mathbb{Y}$.

Proof. We prove this by contradiction. Suppose that there are only $\mathbb{P}$ and $\mathbb{Y}$-points. Then for all $x \in E$, the density $\theta(x) = \lim_{r\to 0} H^2(B(x,r) \cap E)/r^2$ of E at x is either $3/2$ or 1. In other words, all singular points in E are of type $\mathbb{Y}$.

Denote by E_Y the set of all the $\mathbb{Y}$-points of E. Then $E_Y \cap B(0,2)$ are composed of C^1 curves, whose endpoints belong to $\partial B(0,2)$ (see [4], Lemma 18.11, and for the C^1 regularity around $\mathbb{Y}$-points, see [5] Theorem 1.15 and Lemma 14.6).

The following argument is the same as following Lemma 18.11 in Section 18 of [4] (where the reader can find more details). Here we only sketch the argument.

Since E looks very much like T in $B(0,5/2)\backslash B(0,1/2)$, we have $E_Y \cap \partial B(0,2) = \{a_1, a_2, a_3, a_4\}$, $E_Y \cap \partial B(0,1) = \{b_1, b_2, b_3, b_4\}$, where b_i is the point among the b_j, $1 \leq j \leq 4$, nearest to a_i. Then through each a_i there passes a curve in E_Y, and hence locally a_i lies in the intersection of three half surfaces $E_{ij}, j \neq i, 1 \leq j \leq 4$.

But on the sphere $\partial B(0,2)$ we have four $\mathbb{Y}$-points, hence, without loss of generality, we can suppose that there is a curve γ_1 of E_Y that enters the ball $B(0,2)$ at a_1 and leaves the ball at a_2, and another curve γ_2 which enters the ball at a_3 and leaves it at a_4 (see Figure 10, where the green curves represent the $\gamma_i, i = 1,2$, and we do not know much about the structure of E in $B_{1/2} = B(0,1/2)$). Near each point x of γ_1, there exists a C^1-ball $B(x, r_x)$ of x, in which E is the image of a $\mathbb{Y}$ set under a C^1 diffeomorphism. By compactness of the curve γ_1, there exists a tubular neighborhood I_1 of γ_1 such that $E \cap I_1$ is composed exactly of three surfaces that meet along the curve γ_1.

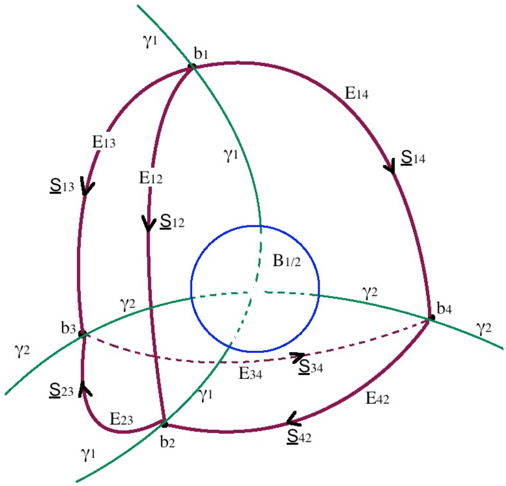

FIGURE 10.

Since γ_1 connects a_1 and a_2, it passes through b_1 and b_2. Near each b_i, the set E is composed of three half surfaces $E_{ij}, j \neq i$. Then since E is locally composed of three half surfaces all along γ_1, these three half surfaces connect E_{12}, E_{13} and E_{14} to E_{21}, E_{23} and E_{24}. Hence we know that s_{12}, s_{13} and s_{14} are homotopic to s_{21}, s_{23} and s_{24} (but we do not know which is homotopic to which). A similar argument gives also that s_{31}, s_{32} and s_{34} are homotopic to s_{41}, s_{42} and s_{43}.

For the part γ_1 we have the following six cases:

$$\begin{aligned} &s_{12} \sim s_{21}, \quad s_{13} \sim s_{23}, \quad s_{14} \sim s_{24};\\ &s_{12} \sim s_{21}, \quad s_{13} \sim s_{24}, \quad s_{14} \sim s_{23};\\ &s_{12} \sim s_{23}, \quad s_{13} \sim s_{21}, \quad s_{14} \sim s_{24};\\ &s_{12} \sim s_{23}, \quad s_{13} \sim s_{24}, \quad s_{14} \sim s_{21};\\ &s_{12} \sim s_{24}, \quad s_{13} \sim s_{21}, \quad s_{14} \sim s_{23};\\ &s_{12} \sim s_{24}, \quad s_{13} \sim s_{23}, \quad s_{14} \sim s_{21}. \end{aligned} \tag{2.7}$$

Note also that automatically $s_{12} \sim s_{21}$, hence the six cases reduce to the following four (modulo the symmetry between the indices 3 and 4):

$$\begin{aligned} &s_{13} \sim s_{23}, s_{14} \sim s_{24};\\ &s_{13} \sim s_{24}, s_{14} \sim s_{23};\\ &s_{12} \sim s_{23} \sim s_{13}, s_{14} \sim s_{24};\\ &s_{12} \sim s_{23} \sim s_{14}, s_{13} \sim s_{24}. \end{aligned} \tag{2.8}$$

Similarly, for the part γ_2, we have the following four cases:

$$\begin{aligned} &s_{31} \sim s_{41}, s_{32} \sim s_{42};\\ &s_{31} \sim s_{42}, s_{32} \sim s_{41};\\ &s_{34} \sim s_{41} \sim s_{31}, s_{32} \sim s_{42};\\ &s_{34} \sim s_{41} \sim s_{32}, s_{31} \sim s_{42}. \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9), we have the eight cases

$$
\begin{aligned}
&s_{13} \sim s_{23} \sim s_{42} \sim s_{14};\\
&s_{13} \sim s_{23} \sim s_{42} \sim s_{14} \sim s_{43};\\
&s_{13} \sim s_{24}, s_{14} \sim s_{23};\\
&s_{34} \sim s_{41} \sim s_{32}, s_{13} \sim s_{24};\\
&s_{13} \sim s_{23} \sim s_{42} \sim s_{14} \sim s_{12};\\
&s_{13} \sim s_{24}, s_{12} \sim s_{23} \sim s_{14};\\
&s_{12} \sim s_{13} \sim s_{23} \sim s_{42} \sim s_{14} \sim s_{43};\\
&s_{13} \sim s_{24}, s_{12} \sim s_{14} \sim s_{23} \sim s_{34}.
\end{aligned} \tag{2.10}
$$

In particular, at most four of the $[s_{ij}], 1 \leq i < j \leq 4$, are different, which contradicts our hypothesis that at least five of the $\{[s_{ij}], 1 \leq i < j \leq 4\}$ are different in $H^1(\mathbb{R}^4 \backslash E; \mathbb{Z})$. □

Corollary 2.2. *Let E be a 2-dimensional reduced Almgren-minimal set in $\mathbb{R}^4$ such that (2.2), (2.5) and (2.6) hold. Suppose also that (2.1) holds. Then E is a $\mathbb{T}$-set parallel to T.*

Proof. By Lemma 2.1, E contains a point x of type other than $\mathbb{P}$ and $\mathbb{Y}$, hence by (2.1), the density $\theta(x)$ of E at x is larger than or equal to d_T. Define $\theta(t) = t^{-2}H^2(E \cap B(x,t))$ the density function of E at x. By Proposition 5.16 of [4], $\theta(t)$ is nondecreasing on t. Then (2.2) and Lemma 16.43 of [4] give that $\lim_{t\to\infty} \theta_t = d_T$. Since we already know that $\theta(x) = \lim_{t\to 0} \theta(t) \geq d_T$, the monotonicity of θ yields that $\theta(t) = d_T$ for all $t > 0$. By Theorem 6.2 of [4], the set E is a minimal cone centered at x, with density d_T. Thus, by (2.2), E is a $\mathbb{T}$-set centered at x and parallel to the set T. □

After Corollary 2.2, there remains only to discuss the case where E is topologically minimal, and no more than 4 of the $[s_{ij}]$ are different.

First we prove some properties of these s_{ij}.

Lemma 2.3. 1)

$$\sum_{j\neq i} [\vec{s}_{ij}] = 0 \text{ for all } 1 \leq i \leq 4. \tag{2.11}$$

2) *For each $i \neq j \neq k$,*

$$[\vec{s}_{ij}] \neq 0, \tag{2.12}$$

and

$$[\vec{s}_{ij}] \neq [\vec{s}_{jk}]. \tag{2.13}$$

Proof. 1) Fix $1 \leq i \leq 4$. We write $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where $T \subset \mathbb{R}^3$.

Recall that y_i, $1 \leq i \leq 4$, are the four $\mathbb{Y}$-points of $T \cap \partial B(0,1)$; T_{ij} is the sector of T passing through the origin and y_i, y_j; x_{ij} is the middle point of the great arc passing through y_i, y_j; P_{ij} is the plane passing containing x_{ij} and orthogonal to T_{ij}; and $s_{ij} = \partial B_{ij}$, where $B_{ij} = B(x_{ij}, 1/10) \cap P_{ij}$.

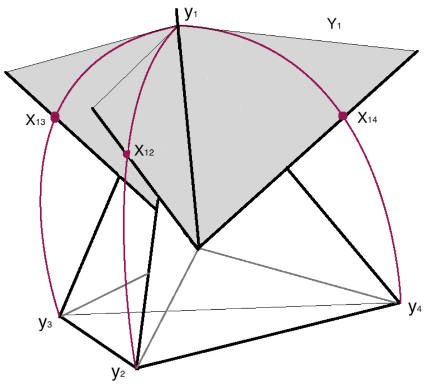

FIGURE 11.

Denote by Y_i the cone over $Z_i := \cup_{j\neq i}\widehat{y_i x_{ij}}$, where $\widehat{y_i x_{ij}}$ denotes the great arc connecting y_i and x_{ij} (see Figure 11 of $Y_1 \subset \mathbb{R}^3$ below), and by C_T the convex hull of Y_i. Set $C = C_T \times \mathbb{R}$. Since C is a cone, $C\backslash T$ is also a cone. Note that $Z_i \subset S^3 \cap C$ is a spherical $\mathbb{Y}$-set of dimension 1. We want to show that $\sum_{j\neq i}[\vec{s}_{ij}] = 0$ in $H_1(C\backslash T, \mathbb{Z})$.

Note that $\vec{s}_{ij}$ is homotopic in $C\backslash T$ to its radial projection $\vec{s}'_{ij}$ on S^3 (the orientation of $\vec{s}'_{ij}$ is induced by $\vec{s}_{ij}$ on the sphere S^3). In fact, denoting by π_S the radial projection of $\mathbb{R}^4\backslash\{0\}$ to S^3, for each $x \in s_{ij}$, the segment $[x, \pi_S(x)]$ belongs to a radial half-line that does not meet any other radial half-lines. In particular, since $x \in \mathbb{R}^4\backslash T$, where T is a union of radial half-lines, $[x, \pi_X(x)] \cap T = \emptyset$. Hence if we set $f_t(x) = (1-t)x + t\pi_S(x), 0 \le t \le 1$, then f_t is a homotopy between $\vec{s}_{ij}$ and $\vec{s}'_{ij} = \pi_S(\vec{s}_{ij})$.

Therefore on the sphere, in $C \cap S^3$, the s_{ij}, $j \neq i$, are topologically three circles that link respectively the three branches of Z_i. Recall that the pair of topological spaces $(C \cap S^3, Z_i)$ is homotopic to $(\mathbb{R}^3, Y)$ where Y is a 1-dimensional $\mathbb{Y}$-set. However in $(\mathbb{R}^3, Y)$, the union of the three oriented circles that link the three branches of Y is the boundary of an oriented manifold with boundary contained in $\mathbb{R}^3\backslash Y$. Hence, similarly, there exists an oriented manifold with two-dimensional boundary $\Sigma \subset C \cap S^3\backslash Z_i$ such that $\partial\Sigma = \cup_{j\neq i}\vec{s}'_{ij}$ (see Figure 12, where s_{ij} denotes the oriented circle $\vec{s}_{ij}$, and the orientation of Σ is indicated by the exterior normal vector $\vec{n}$). Therefore, after a smooth triangulation under which Γ and s_{ij} are all smooth chains, we have $\partial[\Sigma] = \cup_{j\neq i}[\vec{s}'_{ij}]$. Since $\Sigma \subset C \cap S^3\backslash Z_i \subset \mathbb{R}^4\backslash T$, $\sum_{j\neq i}[\vec{s}'_{ij}] = 0$ in $H_1(\mathbb{R}^4\backslash T, \mathbb{Z})$. Then since $\vec{s}'_{ij}$ is homotopic to $\vec{s}_{ij}$,

$$\sum_{j\neq i}[\vec{s}_{ij}] = 0 \quad \text{in } H_1(\mathbb{R}^4\backslash T, \mathbb{Z}). \tag{2.14}$$

Now since E is as near as we like to T, we can suppose that Σ and the $f_t(s_{ij})$ do not touch E. Thus we get (2.11).

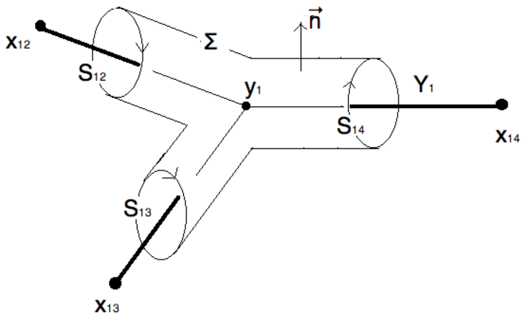

Figure 12.

2) Without loss of generality, suppose for example that $i = 1$, $j = 2$ and $k = 3$. If $[\vec{s}_{12}] = [\vec{s}_{23}]$, then, by (2.11),

$$[\vec{s}_{24}] = 0. \tag{2.15}$$

Hence we have only to prove (2.12).

Suppose for example that $i = 2$ and $j = 4$. Then (2.15) means that there exists a smooth simplicial 2-chain Γ in $\mathbb{R}^4\backslash E$ such that $\partial\Gamma = \vec{s}_{24}$. Since E is closed, there exists a neighborhood U of Γ such that $U \cap E = \emptyset$. In particular, $s_{24} \subset U$.

Set $D = E \cap \overline{B}(x_{24}, 1/10)$. Then by the regularity of the minimal set E which is very near T, in $B(x_{24}, 1/8)$, E is a piece of very flat surface that is almost a disc. Therefore D is a surface with positive measure.

Set $F = E\backslash D$, so $F\backslash B(0, 2) = E\backslash B(0, 2)$. We want to show that F is a topological competitor of E with respect to the ball $B(0, 2)$.

Suppose that $\gamma \subset \mathbb{R}^4\backslash(B(0, 2) \cup E)$ is an oriented circle. We have to show that if $[\gamma]$ is zero in $H_1(\mathbb{R}^4\backslash F, \mathbb{Z})$, then it is zero in $H^1(\mathbb{R}^4\backslash E, \mathbb{Z})$.

Now if $[\gamma]$ is zero in $H_1(\mathbb{R}^4\backslash F, \mathbb{Z})$, then there exists a smooth simplicial 2-chain $\Sigma \subset \mathbb{R}^4\backslash F$ such that $\partial\Sigma = \gamma$. By the transversality theorem (see, for example, Theorem 2.1 in Chapter 3 of [8]), we can require that Σ is transversal to $\partial B(x_{24}, 1/10)$.

If $\Sigma \cap B(x_{24}, 1/10) = \emptyset$, then $\Sigma \subset \mathbb{R}^4\backslash E$ too, and hence $[\gamma] = 0 \in H_1(\mathbb{R}^4\backslash E, \mathbb{Z})$. If $\Sigma \cap B(x_{24}, 1/10) \neq \emptyset$, then, by the transversality of Σ and $\partial B(x_{24}, 1/10)$, and by Proposition 2.36 of [12], their intersection is a closed smooth simplicial 1-chain $s \subset \partial B(x_{24}, 1/10)$.

Now we work in $\overline{B}_1 := \overline{B}(x_{24}, 1/10)$. Since D is a very flat topological disc,

$$H_1(\overline{B}_1\backslash D) = \mathbb{Z}, \tag{2.16}$$

whose generator is $[\vec{s}_{24}]$. As a result, there exists $n \in \mathbb{Z}$ such that $[s] = n[\vec{s}_{24}]$. Hence there exists a smooth simplicial 1-chain $R \subset \overline{B}_1\backslash D$ such that $\partial R = s - n\vec{s}_{24}$.

Recall that $\Gamma \subset \mathbb{R}^4\backslash E$ is such that $\partial\Gamma = \vec{s}_{24}$. As a result, $\Sigma' = \Sigma\backslash\overline{B}_1 + n\Gamma + R$ is a 2-chain satisfying $\partial[\Sigma'] = [\gamma]$. Moreover $\Sigma' \subset \mathbb{R}^4\backslash E$. Hence $[\gamma]$ is also zero in $H^1(\mathbb{R}^4\backslash E, \mathbb{Z})$.

Thus, $F = E\backslash D$ is a topological competitor of E.

However, D has positive measure, hence

$$H^2(F) < H^2(E), \tag{2.17}$$

which contradicts the fact that E is topologically minimal.

Thus we have proved (2.12), and hence (2.13), and so we have completed the proof of Lemma 2.3. □

Now we return to our discussion of the case where E is very near a $\mathbb{T}$-set T at scale 1, but contains no point of type other than $\mathbb{P}$ and $\mathbb{T}$. After a consideration (see Section 19 of [10] for details) of the eight cases in (2.10), using Lemma 2.3, the only possibility for $[\vec{s}_{ij}]$ in $H_1(\mathbb{R}^4\backslash E, \mathbb{Z})$ is:

$$[\vec{s}_{13}] = -[\vec{s}_{24}] = \alpha, [\vec{s}_{14}] = -[\vec{s}_{23}] = \beta, [\vec{s}_{34}] = \alpha - \beta, [\vec{s}_{12}] = -\alpha - \beta. \tag{2.18}$$

Thus we get the following proposition.

Proposition 2.4. *Let E be a reduced topological minimal set of dimension 2 in $\mathbb{R}^4$, that verifies (2.2). Let the conventions and notation be as at the beginning of this section, and suppose also that γ_1 connects a_1 and a_2, while γ_2 connects a_3 and a_4. Then if there exists $r > 0$ such that $d_{0,3r}(E, T) < \epsilon_0$ (where ϵ_0 is the one defined in the paragraph below (2.4)), but the s_{ij} do not satisfy (2.18) with respect to $\frac{1}{r}E$, then E is a $\mathbb{T}$-set parallel to T.*

2.3. An example

In this section, the notation and conventions are as in Subsection 2.2. We give an example of a set that satisfies (2.18).

Set $w_{ij} = E_{ij} \cap \partial B(0, 1)$ (see Figure 13, where $\underline{w}_{ij}$ is denoted $\vec{w}_{ij}$). Then the w_{ij} are C^1 curves. Denote also by $\vec{w}_{ij}$ the oriented curve from b_i to b_j.

Now suppose that $E_Y = \gamma_1 \cup \gamma_2$. In other words, all points in E are $\mathbb{P}$ points, except for the two curves. (For the case where $E_Y \neq \gamma_1 \cup \gamma_2$, we know that $E_Y\backslash(\gamma_1 \cup \gamma_2)$ is a union of closed curves, because the only endpoints of $E_Y \cap \partial B(0, 1)$ are $\{b_i\}_{1\leq i\leq 4}$. This is thus a more complicated case.)

Lemma 2.5. *$\gamma_1 \cup \gamma_2 \cup w_{12} \cup w_{34}$ is the boundary of a C^1 surface $S_0 \subset E$, and S_0 contains only points of type $\mathbb{P}$.*

Proof. By the C^1 regularity of minimal sets, the part of E in $B(0, 1)$ is composed of C^1 manifolds $S_1, S_2, \ldots$ whose boundaries are unions of curves in the set $Bd = \{w_{ij}, \gamma_1, \gamma_2\}$. Thus there exists $k \in \mathbb{N}$ such that w_{12} is part of the boundary of the manifold S_k. But ∂S_k is a union of several closed curves, while w_{12} is not closed. Hence there exists a curve $\gamma \in Bd$ that touches w_{12} and such that γ is also part of ∂S_k. If one of the w_{1i} (resp. one of the w_{2j}) is part of ∂S_k and touches w_{12}, we have $[\vec{s}_{12}] = [\vec{s}_{i1}]$ (with orientation) (resp. $[\vec{s}_{12}] = [\vec{s}_{2i}]$), which contradicts (2.13).

Hence the only possibility for γ is γ_1. This means the union of w_{12} and γ_1 is part of the boundary of a manifold S_k, and except for w_{34}, the boundary of S_k contains no other w_{ij}. A similar argument gives also that the union of w_{34} and γ_2

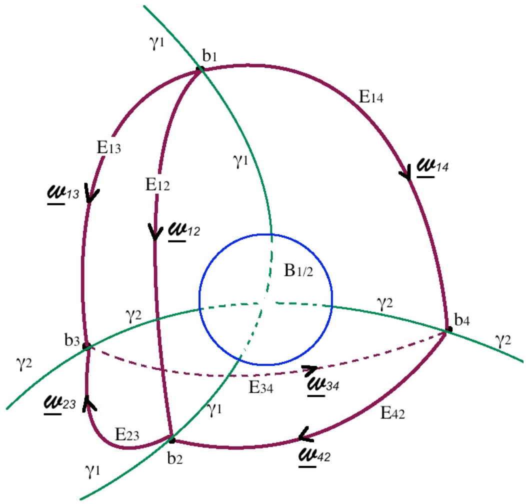

Figure 13.

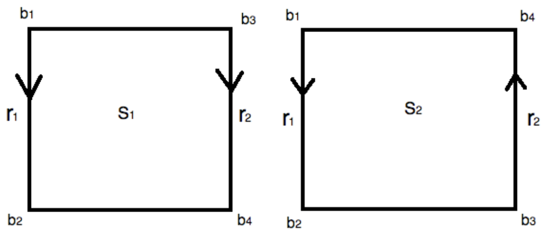

Figure 14.

is part of the boundary of a manifold S_l, and the boundary of S_l contains no other w_{ij}, except perhaps for w_{12}.

Thus, either the union of these four curves w_{12}, w_{34}, γ_1 and γ_2 is the boundary of a surface S_k, or the union of w_{12} and γ_1 and the union of w_{34} and γ_2 are the boundaries of two surfaces S_k and S_l. In any case, the union of the four curves is the boundary of a C^1 surface $S_0 \subset E$, which is not necessarily connected. □

By Lemma 2.5, if we excise the surface S_0 from E, then $E \backslash S$ is composed of a union of C^1 surfaces, whose boundaries are unions of curves belonging to $Bd' = \{w_{13}, w_{14}, w_{23}, w_{24}, \gamma_1, \gamma_2\}$. By the same argument above, there are two surfaces S_1 and S_2, with $\partial S_1 = w_{13} \cup \gamma_2 \cup w_{24} \cup \gamma_1$, and the other one $\partial S_2 = w_{23} \cup \gamma_2 \cup w_{14} \cup \gamma_1$. Moreover $S_1 \cup S_2$ is also a connected topological manifold, for which we can define a local orientation, even near ∂S_1 and ∂S_2.

Thus topologically the boundaries of the two surfaces S_1 and S_2 are like the boundaries of two squares, one with the four vertices b_1, b_3, b_4 and b_2 (we write them in the order of adjacency), the other with four vertices b_1, b_4, b_3 and b_2. Moreover, we have to glue the two $\overrightarrow{b_1b_2}$ in S_1 and S_2 together, and the same for $\overrightarrow{b_3b_4}$. Note that these two gluings have different directions (see Figure 14).

Notice also that after the gluing, $S_1 \cup S_2$ cannot be orientable.

Remark 2.6. Since $S_1 \cup S_2$ is not orientable, $[\vec{s}_{13}]$, $[\vec{s}_{14}]$, $[\vec{s}_{24}]$ and $[\vec{s}_{23}]$ are all of order 2 in $H_1(\mathbb{R}^4 \backslash E, \mathbb{Z})$.

In fact for a connected surface S, the non-orientability means that for each point $x \in S$ we can find a path $\gamma : [0,1] \to S$ such that $\gamma(0) = \gamma(1) = x$, and if we denote by $n(t) = x(t) \wedge y(t) \in \wedge_2 N_{\gamma(t)} S$ a continuous unit normal 2-vector field on γ, where $x(t), y(t) \in N_{\gamma(t)} S$ are unit normal vector fields, and n, x and y are continuous with respect to t, then $n(0) = -n(1)$. Note that $n(t)$ can also represent the oriented plane in $\mathbb{R}^4$. Define, for each $r > 0$, $s_r(t) : T = \mathbb{R}/\mathbb{Z} \to P_t = P(x(t) \wedge y(t))$ and $\theta \mapsto r[\cos(2\pi\theta)x(t) + \sin(2\pi\theta)y(t)]$. Then the images of $s_r(0)$ and $s_r(1)$ are the same circle, but with opposite orientations: $s_r(0)(t) = s_r(1)(-t)$.

Let $Q_t = \gamma(t) + P(t)$. Fix $r > 0$ sufficiently small, such that for each $t \in [0,1]$, $B(\gamma(t), r) \cap Q_t \cap S = \{\gamma(t)\}$.

Define $G : T \times [0,1] \to \mathbb{R}^4$ by $G(\theta, t) = s_r(t)(\theta) + \gamma(t)$. This is a continuous map, with $G(T \times \{0\}) = s_r(0)$ and $G(T \times \{1\}) = s_r(1) = -s_r(0)$. As a result, the oriented circle $s_r(0)$ is homotopic to $-s_r(0)$, and is hence of order 2.

Now for each $s \in \{[\vec{s}_{13}], [\vec{s}_{14}], [\vec{s}_{24}], [\vec{s}_{23}]\}$, we can first find a circle s' homotopic to s, such that there exists x and γ as before, and that there exists $R > 0$ such that $s_R(0) = s'$. We can find $r > 0$ as above. Then $s_r(0)$ is homotopic to s', and hence s. Therefore $[s] = [s_r(0)]$ is of order 2.

We will construct a set $E \subset \mathbb{R}^4$, with all the above properties. That is, in $B(0,1)$, the set E is the union of S_0 and $S_1 \cup S_2$ as above, $S_1 \cup S_2$ is a non-orientable topological manifold, and S_0 has two connected components, that meet $S_1 \cup S_2$ at γ_1 and γ_2 respectively. Outside the ball $B(0,1)$, E is a C^1 version of T, and it looks like T at infinity. Moreover, $H_1(\mathbb{R}^4 \backslash E)$ and the $[\vec{s}_{ij}]$ satisfy (2.18).

Take two copies of squares (see Figure 14), one with vertices (written in the clockwise order) b_1, b_3, b_4 and b_2, the other with vertices b_1, b_4, b_3 and b_2. We glue the two sides $\overrightarrow{b_3b_4}$ in S_1 and S_2 together, and we do the same for $\overrightarrow{b_1b_2}$. Thus we get a Möbius band in $\mathbb{R}^3$ (see Figure 15).

Next, take a very big regular tetrahedron centered at the origin (with vertices y_i, $1 \leq i \leq 4$) which contains the Möbius band constructed before. For each i, take a smooth curve L_i issuing from b_i and going to infinity, such that L_i tends to $[0, y_i)$ (see Figure 15).

Then take, for each $1 \leq i \neq j \leq 4$, a C^1 surface E_{ij}, homeomorphic to $\mathbb{R}^2$, whose boundary is $L_i \cup L_j \cup [b_i b_j]$. Note that all E_{ij} go to infinity, hence in $\mathbb{R}^3$, E_{23} and E_{14}, or E_{13} and E_{24} must meet each other. We work in $\mathbb{R}^4$ to avoid this.

Thus we get a set that looks like a T at infinity, and we cannot give any simple reason why a set with such a topology cannot be topologically minimal.

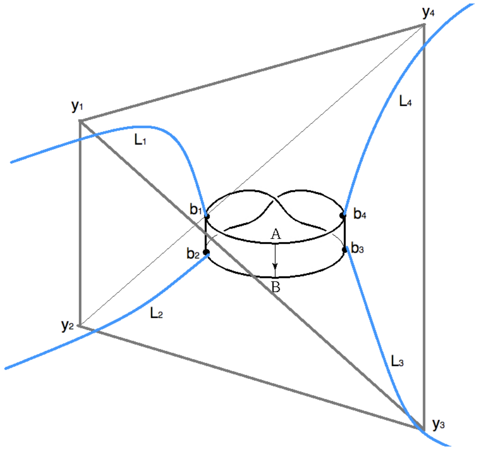

Figure 15.

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