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Finite C^∞-actions are described by a single vector field

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Abstract. In this work it is shown that given a connected C^∞-manifold M of dimension ≥ 2 and a finite subgroup $G \subset \mathrm{Diff}(M)$, there exists a complete vector field X on M such that its automorphism group equals $G \times \mathbb{R}$, where the factor $\mathbb{R}$ comes from the flow of X.

1. Introduction

This work fits within the framework of the so called *inverse Galois problem*: working in a category $\mathcal{C}$ and given a group G, decide whether or not there exists an object X in $\mathcal{C}$ such that $\mathrm{Aut}_{\mathcal{C}}(X) \cong G$.

This metaproblem has been addressed by researchers in a wide range of situations from algebra [2] and combinatorics [4], to topology [3]. In the setting of differential geometry, Kojima shows that any finite group occurs as $\pi_0(\mathrm{Diff}(M))$ for some closed 3-manifold M (see Corollary on page 297 of [8]), and more recently Belolipetsky and Lubotzky [1] have proved that for every $m \geq 2$, every finite group is realized as the full isometry group of some compact hyperbolic m-manifold, so extending previous results of Kojima [8] and Greenberg [5].

Here we consider automorphisms of vector fields. Although it is obvious that the automorphism group of a vector field is never finite, we show that every finite group of diffeomorphisms can be determined by a vector field. More precisely:

Theorem. *Consider a connected C^∞ manifold M of dimension $m \geq 2$ and a finite subgroup G of diffeomorphisms of M. Then there exists a complete G-invariant vector field X on M such that the map*

$$\begin{array}{rcl} G \times \mathbb{R} & \longrightarrow & \mathrm{Aut}(X) \\ (g,t) & \longmapsto & g \circ \Phi_t \end{array}$$

is a group isomorphism, where Φ and $\mathrm{Aut}(X)$ denote the flow and the group of automorphisms of X, respectively.

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The vector field X above is not unique, even from the geometrical point of view (see Remark 3.8). The study of the set $\mathfrak{X}(M,G)$ of vector fields satisfying the hypothesis of the theorem seems quite interesting. Of course, $\mathfrak{X}(M,G)$ is a subset of the Lie subalgebra of G-invariant vector fields on M, but we do not know if all its elements can be constructed by our method. Therefore, a first step in the study of $\mathfrak{X}(M,G)$ would be to construct its elements in a way different from ours.

Recall that, for any $m \geq 2$, every finite group G is a quotient of the fundamental group of some compact, connected C^∞-manifold M' of dimension m. Therefore G can be regarded as the group of desk transformations of a connected covering $\pi : M \to M'$ and $G \leq \mathrm{Diff}(M)$. Consequently the result above solves the inverse Galois problem for vector fields. Thus:

Corollary 1.1. *Let G be a finite group and $m \geq 2$. Then there exist a connected C^∞-manifold M of dimension m and a vector field X on M such that $\pi_0(\mathrm{Aut}(X)) \cong G$.*

Our results fit into the C^∞ setting, but it seems interesting to study the same problem for other kind of manifolds, in particular for topological manifolds. Namely, given a finite group $\tilde{G}$ of homeomorphisms of a connected topological manifold $\tilde{M}$ prove, or disprove, the existence of a continuous action $\tilde{\Phi}\colon \mathbb{R} \times \tilde{M} \to \tilde{M}$ such that:

(1) $\tilde{\Phi}_t \circ g = g \circ \tilde{\Phi}_t$ for any $g \in \tilde{G}$ and $t \in \mathbb{R}$;

(2) if f is a homeomorphism of $\tilde{M}$ and $\tilde{\Phi}_s \circ f = f \circ \tilde{\Phi}_s$ for every $s \in \mathbb{R}$, then $f = g \circ \tilde{\Phi}_t$ for some $g \in \tilde{G}$ and $t \in \mathbb{R}$ that are unique.

This paper, reasonably self-contained, is organized as follows. In Section 2 some general definitions and classical results are given. Section 3 is devoted to the main result of this work (Theorem 3.1) and its proof. The extension of Theorem 3.1 to manifolds with nonempty boundary is addressed in Section 4. In Section 5 we illustrate Theorem 3.1 with examples. The manuscript ends with an Appendix where a technical result needed in Section 4 is proved.

For general questions on differential geometry the reader is referred to [7], and for those on differential topology to [6].

2. Preliminary notions

Henceforth all structures and objects considered are real C^∞ and manifolds without boundary, unless otherwise stated. Given a vector field Z on an m-manifold M, the group $\mathrm{Aut}(Z)$ of automorphisms of Z is the subgroup of diffeomorphisms of M that preserve Z, that is

$$\mathrm{Aut}(Z) = \{f \in \mathrm{Diff}(M) : f_*(Z(p)) = Z(f(p)) \text{ for all } p \in M\}.$$

On the other hand, recall that a *regular trajectory* is the trace of a nonconstant maximal integral curve. Thus any regular trajectory is oriented by time in the obvious way and, if it is not periodic, its points are completely ordered. As usual, a *singular trajectory* is a singular point of Z.

If $Z(p) = 0$ and Z' is another vector field defined near p then $[Z', Z](p)$ only depends on $Z'(p)$; thus the formula $Z'(p) \to [Z', Z](p)$ defines an endomorphism of T_pM called *the linear part of Z at p*. For the purpose of this paper, we will say that $p \in M$ is a *source* (respectively a *sink*) of Z if $Z(p) = 0$ and its linear part at p is the product of a positive (negative) real number by the identity on T_pM.

A point $q \in M$ is called a *rivet* if

(a) q is an isolated singularity of Z,

(b) around q one has $Z = \psi\tilde{Z}$, where ψ is a function and $\tilde{Z}$ a vector field with $\tilde{Z}(q) \neq 0$.

Note that by (b), a rivet is the ω-limit of exactly one regular trajectory, the α-limit of another one and an isolated singularity of index zero.

Consider a singularity p of Z; let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of the linear part of Z at p and let $\mu_1, \dots, \mu_k$ be the same eigenvalues but taking each of them into account only once regardless of its multiplicity. Assume that $\mu_1, \dots, \mu_k$ are rationally independent; then $\lambda_j - \sum_{\ell=1}^m i_\ell\lambda_\ell \neq 0$ for any $j = 1, \dots, m$ and for any nonnegative integers $i_1, \dots, i_m$ such that $\sum_{\ell=1}^m i_\ell \geq 2$; and a linearization theorem by Sternberg (see [10] and [9]) shows the existence of coordinates $(x_1, \dots, x_m)$ such that $p \equiv 0$ and $Z = \sum_{j=1}^m \lambda_j x_j \partial/\partial x_j$. This is the case for sources ($\lambda_1 = \cdots = \lambda_m > 0$) and sinks ($\lambda_1 = \cdots = \lambda_m < 0$).

By definition, the *outset (or unstable manifold) R_p of a source p* will be the set of all points $q \in M$ such that the α-limit of its Z-trajectory equals p. One has:

Proposition 2.1. *Let p be a source of a complete vector field Z. Then R_p is open and there exists a diffeomorphism from R_p to $\mathbb{R}^m$ that maps p to the origin and Z to $a\sum_{j=1}^m x_j\partial/\partial x_j$ for some $a \in \mathbb{R}^+$. In other words, there exist coordinates $(x_1, \dots, x_m)$, whose domain R_p is identified with $\mathbb{R}^m$, such that $p \equiv 0$ and $Z = a\sum_{j=1}^m x_j\partial/\partial x_j$, $a \in \mathbb{R}^+$.*

Indeed, let Φ_t be the flow of Z. Consider coordinates $(y_1, \dots, y_m)$ such that $p \equiv 0$ and $Z = a\sum_{j=1}^m y_j\partial/\partial y_j$. Up to dilation and with the obvious identifications, one may suppose that S^{m-1} is included in the domain of these coordinates. Then $R_p = \{\Phi_t(y) \mid t \in \mathbb{R}, y \in S^{m-1}\} \cup \{0\}$ and it suffices to send the origin to the origin and each $\Phi_t(y)$ to $e^{at}y$ to construct the required diffeomorphism.

Remark 2.2. Observe that $R_p \cap R_q = \emptyset$ when p and q are different sources of Z.

Given a regular trajectory τ of Z with α-limit a source p, by the *linear α-limit of τ* one means the (open and starting at the origin) half-line in the vector space T_pM that is the limit, when $q \in \tau$ tends to p, of the half-line in T_qM spanned by $Z(q)$. From the local model around p, the existence of this limit follows. Moreover, if Z is multiplied by a positive function the linear α-limit does not change.

By definition, a *chain* of Z is a finite and ordered sequence of two or more different regular trajectories, each of them called a *link*, such that:

(a) the α-limit of the first link is a source;

(b) the ω-limit of the last link is not a rivet;

(c) between two consecutive links the ω-limit of the first equals the α-limit of the second. Moreover, this set consists in a rivet.

The *order of a chain* is the number of its links and its *α-limit* and *linear α-limit* those of its first link.

For the sake of simplicity, here countable includes the finite case as well. One says that a subset Q of M *does not exceed dimension* ℓ, or it *can be enclosed in dimension* ℓ, if there exists a countable collection $\{N_\lambda\}_{\lambda\in L}$ of submanifolds of M, all of them of dimension $\leq \ell$, such that $Q \subset \bigcup_{\lambda\in L} N_\lambda$. Note that the countable union of sets whose dimensions do not exceed dimension ℓ does not exceed dimension ℓ too. On the other hand, if $\ell < m$ then Q has measure zero so empty interior.

Given an m-dimensional real vector space V, a family $\mathcal{L} = \{L_1, \dots, L_s\}$, $s \geq m$, of half-lines of V is said to be *in general position* if any subfamily of $\mathcal{L}$ with m elements spans V.

Now consider a finite group $H \subset GL(V)$ of order k. A family $\mathcal{L}$ of half-lines of V is called a *control family with respect to* H if:

(a) $h(L) \in \mathcal{L}$ for any $h \in H$ and $L \in \mathcal{L}$.

(b) There exists a family $\mathcal{L}'$ of $\mathcal{L}$ with $km+1$ elements, which is in general position, such that $H \cdot \mathcal{L}' = \{h(L) \mid h \in H, L \in \mathcal{L}'\}$ equals $\mathcal{L}$.

Lemma 2.3. *Let $\mathcal{L}$ be a control family with respect to H and let φ be an element of $GL(V)$. If φ sends each orbit of the action of H on $\mathcal{L}$ into itself, then $\varphi = ah$ for some $a \in \mathbb{R}^+$ and $h \in H$.*

Indeed, as for every $L \in \mathcal{L}'$ there is $h' \in H$ such that $\varphi(L) = h'(L)$, there exist a subfamily $\mathcal{L}'' = \{L_1, \dots, L_{m+1}\}$ of $\mathcal{L}'$ and an $h \in H$ such that $\varphi(L_j) = h(L_j)$, $j = 1, \dots, m+1$. Therefore $h^{-1} \circ \varphi$ sends L_j into L_j, $j = 1, \dots, m+1$, and because $\mathcal{L}''$ is in general position $h^{-1} \circ \varphi$ has to be a multiple of the identity. Since every L_j is a half-line, this multiple is positive.

3. The main result

This section is devoted to prove the following result on finite groups of diffeomorphisms of a connected manifold.

Theorem 3.1. *Consider a connected manifold M of dimension $m \geq 2$ and a finite group $G \subset \mathrm{Diff}(M)$. Then there exists a complete vector field X on M, which is G-invariant, such that the map*

$$(g, t) \in G \times \mathbb{R} \to g \circ \Phi_t \in \mathrm{Aut}(X)$$

is a group isomorphism, where Φ denotes the flow of X.

Consider a Morse function $\mu \colon M \to \mathbb{R}$ that is G-invariant, proper and non-negative, whose existence is assured by a result of Wasserman (see the remark of page 150 and the proof of Corollary 4.10 in [11]). Denote by C the set of its critical points, which is closed, discrete (that is without accumulation points in M), so countable. As M is paracompact, there exists a locally finite family $\{A_p\}_{p\in C}$ of disjoint open set such that $p \in A_p$ for every $p \in C$.

Lemma 3.2. *There exists a G-invariant Riemannian metric $\tilde{g}$ on M such that if $J(p)\colon T_pM \to T_pM$, $p \in C$, is defined by $H(\mu)(p)(v,w) = \tilde{g}(p)(J(p)v,w)$, where $H(\mu)(p)$ is the Hessian of μ at p, then:*

(1) *if p is a maximum or a minimum then $J(p)$ is a multiple of the identity;*

(2) *if p is a saddle, that is $H(\mu)(p)$ is not definite, then the eigenvalues of $J(p)$, omitting repetitions due to multiplicity, are rationally independent.*

Proof. We start by constructing a 'good' scalar product on each T_pM, $p \in C$. If p is a minimum [respectively maximum] one takes $H(\mu)(p)$ (respectively $-H(\mu)(p)$). When p is a saddle consider a scalar product $\langle\, ,\rangle$ on T_pM invariant under the linear action of the isotropy group G_p of G at p. In this case as $J(p)$ is G_p-invariant (of course here $J(p)$ is defined with respect to $\langle\, ,\rangle$), $T_pM = \oplus_{j=1}^k E_j$ and $J(p)_{|E_j} = a_j \mathrm{Id}_{|E_j}$ where each E_j is G_p-invariant, $a_j \neq 0$, $\langle E_j, E_\ell\rangle = 0$ and $a_j \neq a_\ell$ if $j \neq \ell$.

Moreover, one may suppose that $a_1,\dots,a_k$ are rationally independent by taking, if necessary, a new scalar product $\langle\, ,\rangle'$ such that $\langle E_j, E_\ell\rangle' = 0$ when $j \neq \ell$ and $\langle\, ,\rangle'_{|E_j} = b_j\langle\, ,\rangle_{|E_j}$ for suitable scalars $b_1,\dots,b_k$.

In turn, this family of scalar products on $\{T_pM\}_{p\in C}$ can be made G-invariant. Indeed, this is obvious for maxima and minima since μ is G-invariant. On the other hand, if $C' \subset C$ is a G-orbit consisting of saddles, take a point p in C', endow T_pM with a 'good' scalar product, and extend to C' by means of the action of G.

It is easily seen, through the family $\{A_p\}_{p\in C}$, that all these scalar products on $\{T_pM\}_{p\in C}$ extend to a Riemannian metric $\tilde{g}$ on M. Finally, if $\tilde{g}$ is not G-invariant consider $\sum_{g\in G} g^*(\tilde{g})$. □

Let Y be the gradient vector field of μ with respect to some Riemannian metric $\tilde{g}$ as in Lemma 3.2. We will assume that Y is complete by multiplying $\tilde{g}$, if necessary, by a suitable G-invariant positive function (more exactly by $\exp((Y\cdot\rho)^2)$ where ρ is a G-invariant proper function). Since μ is nonnegative and proper, the α-limit of any regular trajectory of Y is a local minimum or a saddle of μ, whereas its ω-limit is empty, a local maximum or a saddle of μ.

Now $Y^{-1}(0) = C$ and, by the Sternberg's theorem, around each $p \in C$ (note that the linear part of Y at p equals $J(p)\colon T_pM \to T_pM$ defined in Lemma 3.2), there exist a natural $1 \leq k \leq m-1$ and coordinates $(x_1,\dots,x_m)$ such that $p \equiv 0$ and $Y = \sum_{j=1}^m \lambda_j x_j \partial/\partial x_j$ where $\lambda_1,\dots,\lambda_k > 0$ and $\lambda_{k+1},\dots,\lambda_m < 0$, or $Y = a\sum_{j=1}^m x_j\partial/\partial x_j$ where $a > 0$ if p is a source (that is a minimum of μ) and $a < 0$ if p is a sink (a maximum of μ.)

Let I be the set of local minima of μ, that is the set of sources of Y, and let S_i, $i \in I$, be the outset of i relative to Y. Obviously G acts on the set I.

Lemma 3.3. *In M, the family $\{S_i\}_{i\in I}$ is locally finite and the set $\bigcup_{i\in I} S_i$ dense.*

Proof. First notice that $\mu(S_i)$ is bounded from below by $\mu(i)$. However, I is a discrete set and μ a nonnegative proper Morse function, so in every compact set $\mu^{-1}((-\infty,a])$ there are only a finite number of elements of I. Therefore $\mu^{-1}((-\infty,a])$ and of course $\mu^{-1}(-\infty,a)$ only intersect a finite number of S_i. Finally, observe that $M=\bigcup_{a\in\mathbb{R}}\mu^{-1}(-\infty,a)$.

If the α-limit of the Y-trajectory of q is a saddle s, with the local model given above, there exists $t\in\mathbb{Q}$ such that $\Phi_t(q)$ is close to s and $x_{k+1}(\Phi_t(q))=\cdots=x_m(\Phi_t(q))=0$. Since the submanifold given by the equations $x_{k+1}=\cdots=x_m=0$ has dimension $\leq m-1$ and $\mathbb{Q}$ and the set of saddles are countable, it follows that the set of points coming from a saddle may be enclosed in dimension $m-1$ and its complement, that is $\bigcup_{i\in I}S_i$, must be dense. □

The vector field Y has no rivets since all its singularities are isolated with indices ±1. Therefore it has no chain; moreover, the regular trajectories are not periodic.

For each $i\in I$, let $\mathcal{L}_i$ be a control family on T_iM with respect to the action of the isotropy group G_i of G at i, such that if $g(i)=i'$ then g maps $\mathcal{L}_i$ to $\mathcal{L}_{i'}$. These families can be constructed as follows: for every orbit of the action of G on I choose a point i and k_im+1 different half-lines in general position, where k_i is the order of G_i. Now G_i-saturate this first family to obtain $\mathcal{L}_i$. For other points i' in the same orbit, choose $g\in G$ such that $g(i)=i'$ and move $\mathcal{L}_i$ to i' by means of g.

Let $\mathcal{L}$ be the set of all elements of $\mathcal{L}_i$, $i\in I$. By Proposition 2.1 each element of $\mathcal{L}$ is the linear α-limit of just one trajectory of Y. Let $\mathcal{T}$ be the set of such trajectories. Clearly G acts on $\mathcal{T}$, since Y and $\mathcal{L}$ are G-invariant, and the set of orbits of this action is countable. Therefore this last one can be regarded as a family $\{P_n\}_{n\in\mathbb{N}'}$ where $\mathbb{N}'\subset\mathbb{N}-\{0,1\}$, each P_n is a G-orbit and $P_n\neq P_{n'}$ if $n\neq n'$.

In each $T\in P_n$ one may choose $n-1$ different points in such a way that if $T'=g(T)$ then g sends the points considered in T to those of T'. Denote by W_n the set of all the points chosen in the trajectories of P_n.

Since $\{S_i\}_{i\in I}$ is locally finite (Lemma 3.3), the set $W=\bigcup_{n\in\mathbb{N}'}W_n$ is discrete, countable, closed and G-invariant. Therefore there exists a G-invariant function $\psi\colon M\longrightarrow\mathbb{R}$, which is non negative and bounded, such that $\psi^{-1}(0)=W$. Set $X=\psi Y$. One has:

(a) G is a subgroup of $\mathrm{Aut}(X)$.

(b) $X^{-1}(0)=Y^{-1}(0)\cup W$, the rivets of X are just the points of W and X has no periodic regular trajectories.

(c) X and Y have the same sources, sinks and saddles. Moreover if R_i , $i\in I$, is the X-outset of i , then $R_i\subset S_i$ and $\bigcup_{i\in I}(S_i-R_i)\subset\bigcup_{T\in P_n,n\in\mathbb{N}'}T$, so $\{R_i\}_{i\in I}$ is locally finite and $\bigcup_{i\in I}R_i$ is dense.

(d) Let C_T, $T\in P_n$, $n\in\mathbb{N}'$, be the family of X-trajectories of $T-W$ endowed with the order induced by that of T as Y-trajectory. Then C_T is a chain of X of order n whose rivets are the points of $T\cap W$ and whose α-limit and linear α-limit are those of T. Besides C_T, $T\in P_n$, are the only chains of X having order n.

As each P_n is a G-orbit in $\mathcal{T}$, the group G acts on the set of chains of X and every $\{C_T \mid T \in P_n\}$ is an orbit. Thus G acts transitively on the set of α-limit and on that of linear α-limits of the chains C_T, $T \in P_n$. Recall that:

Lemma 3.4. *Any map $\varphi : \mathbb{R}^k \to \mathbb{R}^s$ such that $\varphi(ay) = a\varphi(y)$, for all $(a, y) \in \mathbb{R}^+ \times \mathbb{R}^k$, is linear.*

Remark 3.5. As is well known, the foregoing lemma does not hold for continuous maps (in this paper maps are C^∞ unless otherwise stated).

Proposition 3.6. *Given $f \in \mathrm{Aut}(X)$ and $i \in I$, there exists $(g, t) \in G \times \mathbb{R}$ such that $f = g \circ \Phi_t$ on R_i.*

Proof. Consider $n \in \mathbb{N}'$ such that i is the α-limit of some chain of order n. Then $f(i)$ is the α-limit of some chain of order n and there exists $g \in G$ such that $g(i) = f(i)$. Therefore $(g^{-1} \circ f)(i) = i$, which reduces the problem, up to change of notation, to consideration of the case where $f(i) = i$.

Note that every $L \in \mathcal{L}_i$ is the linear α-limit of some $T \in \mathcal{T}$, so the linear α-limit of C_T. Moreover, $\mathcal{L}_i$ is the family of linear α-limit of all chains starting at i. As f sends chains starting at i into chains starting at i because f is an automorphism of X, it follows that $f_*(i)$ sends $\mathcal{L}_i$ into itself.

On the other hand, since for any $T \in P_n$ one has $f(C_T) = C_{T'}$ where T' belongs to P_n as well, there must exist $h \in G$ that sends the linear α-limit of C_T to the linear α-limit of $C_{T'}$. However, both chains start at i, so $h \in G_i$, which implies that $f_*(i)$ preserves each orbit of the action of G_i on $\mathcal{L}_i$. From Lemma 2.3 it follows that $f_*(i) = ch_*(i)$ with $c > 0$ and $h \in G_i$. Therefore considering $h^{-1} \circ f$ we may suppose, up to a further change of notation, that $f_*(i) = c\,\mathrm{Id}$, $c > 0$.

Now Proposition 2.1 allows us to regard f on R_i as a map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ that preserves the vector field $X = a\sum_{j=1}^m x_j \partial/\partial x_j$, $a \in \mathbb{R}^+$. But this last property implies that $\varphi(bx) = b\varphi(x)$ for any $b \in \mathbb{R}^+$ and $x \in \mathbb{R}^m$; therefore φ is linear (Lemma 3.4). Since $f_*(i) = c\,\mathrm{Id}$ one has $\varphi = c\,\mathrm{Id}$, $c > 0$; that is to say φ and $f_{|R_i}$ equal Φ_t for some $t \in \mathbb{R}$. □

Given $f \in \mathrm{Aut}(X)$, consider a family $\{(g_i, t_i)\}_{i\in}$ of elements of $G \times \mathbb{R}$ such that $f = g_i \circ \Phi_{t_i}$ on each R_i. We will show that $f = g \circ \Phi_t$ for some $g \in G$ and $t \in \mathbb{R}$.

Lemma 3.7. *If all g_i are equal, then all t_i are equal too.*

Proof. The proof reduces to the case where all $g_i = e_G$ (the identity of G) by composing f on the left with a suitable element of G. Obviously $f = \Phi_{t_i}$ on $\overline{R}_i$.

Assume that the set of these t_i has more than one element. Fix one of them, say t, let D_1 be the union of all $\overline{R}_i$ such that $t_i = t$, and let D_2 be the union of all $\overline{R}_i$ such that $t_i \neq t$. Since $\{R_i\}_{i\in I}$ is locally finite and $\bigcup_{i\in I} R_i$ is dense, the family $\{\overline{R}_i\}_{i\in I}$ is locally finite too and $\bigcup_{i\in I} \overline{R}_i = M$. Thus D_1 and D_2 are closed and $M = D_1 \cup D_2$. On the other hand if $p \in D_1 \cap D_2$ then $\Phi_t(p) = \Phi_{t_i}(p)$ for some $t \neq t_i$, so $\Phi_{t-t_i}(p) = p$ and $X(p) = 0$ since X has no periodic regular trajectories, which implies that $D_1 \cap D_2$ is countable. Consequently $M - D_1 \cap D_2$ is connected. But $M - D_1 \cap D_2 = (D_1 - D_1 \cap D_2) \cup (D_2 - D_1 \cap D_2)$ where the terms of this union are nonempty, disjoint and closed in $M - D_1 \cap D_2$, *contradiction.* □

Choose a $i_0 \in I$. Composing f on the left with a suitable element of G we may assume $g_{i_0} = e_G$. On the other hand, f sends each orbit of the action of G on I into itself because the points of every orbit are just the starting points of the chains of order n for some $n \in \mathbb{N}'$. Thus f equals a permutation on each orbit of G in I and there exists $\ell > 0$ such that f^ℓ is the identity on these orbits; for instance $\ell = r!$ where r is the order of G.

Now suppose that $f^\ell = h_i \circ \Phi_{s_i}$ on R_i, $i \in I$. Then $h_i \in G_i$. Since the order of G_i divides that of G, one has $f^{r\ell} = \Phi_{rs_i}$ on R_i. In short, there exists a natural number $k > 0$ such that $f^k = \Phi_{u_i}$ on R_i, and by Lemma 3.7 one has $f^k = \Phi_u$ on every R_i for some $u \in \mathbb{R}$.

In turn, composing f with $\Phi_{-u/k}$ we may assume, without lost of generality, that $f^k = \mathrm{Id}$ on M.

On R_{i_0} one has $f^k = \Phi_{kt_{i_0}}$, so $t_{i_0} = 0$ and $f = \mathrm{Id}$. But f spans a finite group of diffeomorphisms of M, which ensures us that f is an isometry of some Riemannian metric $\hat{g}$ on M. Recall that an isometry on connected manifolds is determined by its 1-jet at any point. Therefore from $f = \mathrm{Id}$ on R_{i_0} there follows $f = \mathrm{Id}$ on M.

In other words the map $(g, t) \in G \times \mathbb{R} \to g \circ \Phi_t \in \mathrm{Aut}(X)$ is an epimorphism. Now the proof of Theorem 3.1 will be finished once it is shown that it is an injection.

Assume that $g \circ \Phi_t = \mathrm{Id}$ on M. From $g^r = e_G$ there follows $\Phi_{rt} = \mathrm{Id}$, whence $t = 0$ because X has no periodic regular trajectories. Thus $g = e_G$.

Remark 3.8. From the proof of Theorem 3.1 above, it is clear that the vector field X above is not unique. It follows that the theorem holds for $X' = \rho X$ where $\rho\colon M \to \mathbb{R}$ is any G-invariant positive bounded function (use $(\rho\psi)Y$ instead of ψY). Although X and X' may not be not equivalent by a G-invariant diffeomorphism, they are equivalent from the geometric viewpoint, that is the structure of their trajectories is the same.

Nevertheless there exist infinitely many vector fields as in Theorem 3.1 which are not geometrically equivalent. Indeed, in our construction of the vector field X from the gradient vector field Y we make use of a set $\mathbb{N}' \subset \mathbb{N} - \{0, 1\}$. Note that $\mathbb{N}'$ is the set of the orders of the chains of X. Therefore, different sets $\mathbb{N}'$ (the only essential property of $\mathbb{N}'$ is its cardinality) give rise to different families of geometrically inequivalent vector fields, since they have different chains. When the set of sources of Y is finite the foregoing family is countable; otherwise it has the cardinality of $\mathbb{R}$.

Another way for constructing geometrically inequivalent vector fields is to consider Morse functions with different numbers of minima.

4. Actions on manifolds with boundary

Let P be an m-manifold with nonempty boundary ∂P. Set $M = P - \partial P$. First recall that there always exist a manifold $\tilde{P}$ without boundary and a function $\tilde{\varphi} : \tilde{P} \longrightarrow \mathbb{R}$ such that zero is a regular value of $\tilde{\varphi}$ and P is diffeomorphic to $\tilde{\varphi}^{-1}((-\infty, 0])$. We identify P and $\tilde{\varphi}^{-1}((-\infty, 0])$.

Now assume that G is a finite subgroup of $\mathrm{Diff}(P)$, P is connected and $m \geq 2$. Then G sends ∂P to ∂P and M to M; thus by restriction G becomes a finite subgroup of $\mathrm{Diff}(M)$.

Let X' be a vector field as in the proof of Theorem 3.1 with respect to M and let $G \subset \mathrm{Diff}(M)$. By Proposition 5.5 in the Appendix applied to M and X', there exists a bounded function $\varphi\colon \tilde{P} \to \mathbb{R}$, which is positive on M and vanishes elsewhere, such that the vector field $\varphi X'$ on M prolongs by zero to a (differentiable) vector field on $\tilde{P}$.

Lemma 4.1. *For every $g \in G$ the vector field X_g equal to $(\varphi \circ g)X'$ on M and vanishing elsewhere is differentiable.*

Proof. Obviously X_g is smooth on $\tilde{P} - \partial P$. Now consider any $p \in \partial P$. As $g\colon P \to P$ is a diffeomorphism, there exist an open neighborhood A of p on $\tilde{P}$ and a map $\hat{g}\colon A \to \tilde{P}$ such that $\hat{g} = g$ in $A \cap P$. Shrinking A, it can be assumed that $B = \hat{g}(A)$ is open, $\hat{g}\colon A \to B$ is a diffeomorphism, and $A - \partial P$ has two connected components A_1 and A_2 with $A_1 \subset M$ and $A_2 \subset \tilde{P} - P$; note that $\hat{g}(A_1) \subset M$, $\hat{g}(A_2) \subset \tilde{P} - P$ and $\hat{g}(A \cap \partial P) \subset \partial P$.

Thus $(X_g)_{|A} = \hat{g}_*^{-1}(X_\varphi)_{|B}$ since X' is G-invariant. □

On P, set $X = \sum_{g \in G} X_g$. Then $X_{|\partial P} = 0$ and $X_{|M} = \rho X'$, where $\rho = \sum_{g \in G} (\varphi_{|M}) \circ g$. Clearly $\rho\colon M \to \mathbb{R}$ is positive, bounded, and G-invariant; so, by Remark 3.8, Theorem 3.1 also holds for $X_{|M}$. Moreover X is complete on P.

If $f\colon P \to P$ belongs to $\mathrm{Aut}(X)$ then $f_{|M}$ belongs to $\mathrm{Aut}(X_{|M})$ and $f = g \circ \Phi_t$ on M and by continuity on P. In other words, Theorem 3.1 also holds for any connected manifold P, of dimension ≥ 2, with nonempty boundary.

5. Examples

In this section we illustrate Theorem 3.1 with two examples. The first example is worked out for the symmetric group, but the reader can easily modify the arguments so they apply to any permutation group.

Example 5.1. On $\mathbb{R}^m$, $m \geq 2$, consider the vector field $Y = \sum_{j=1}^m x_j \partial/\partial x_j$ and the function

$$h(x) = \Big(\sum_{j=1}^m (x_j - 1)^2\Big)\Big(\sum_{j=1}^m (x_j - 2)^2\Big) \prod_{j=1}^m \Big((x_j - 1)^2 + \sum_{k=1, k\neq j}^m x_k^2\Big).$$

Denote by e_j the point of $\mathbb{R}^m$ all of whose coordinates are 0 except for the jth coordinate, that equals 1. As the function $\psi = h(h+1)^{-1}$ is bounded, the vector field $X = \psi Y$ is complete.

Let G be the symmetric group of $\{1, \ldots, m\}$ regarded as the subgroup of the linear group of $\mathbb{R}^m$ consisting of automorphisms L such that, for some permutation σ, $L(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$. Clearly each element of G is an automorphism of X. Moreover $\mathrm{Aut}(X)$ equals $G \times \mathbb{R}$ where the factor $\mathbb{R}$ comes from the flow Φ_t of X.

Indeed, $X^{-1}(0) = \{0\} \cup h^{-1}(0)$ where the origin is a source and the elements of $h^{-1}(0)$ rivets. Let f be an automorphism of X; then $f(0) = 0$ and $f(h^{-1}(0)) = h^{-1}(0)$. Observe that X only has a chain of order three, whose rivets are $(1, \dots, 1)$ and $(2, \dots, 2)$. So $f(1, \dots, 1) = (1, \dots, 1)$ and $f(2, \dots, 2) = (2, \dots, 2)$. Thus $f(e_j) = e_{\sigma(j)}$, $j = 1, \dots, m$, for some permutation of $\{1, \dots, m\}$. Now composing on the left with a suitable element of G allows us to assume $f(e_j) = e_j$, $j = 1, \dots, m$

This implies that each regular trajectory linking the origin to $e_1, \dots, e_m$, and $(1, \dots, 1)$ respectively is sent into itself. So $e_1, \dots, e_m$, and $(1, \dots, 1)$ regarded as vectors are eigenvectors of the differential $f_*(0)$, with positive eigenvalues since X orients any regular trajectory. As $e_1, \dots, e_m$, and $(1, \dots, 1)$ are in general position, it follows that $f_*(0) = c\,\mathrm{Id}$, with $c > 0$.

Now following the lines of the last paragraph of the proof of Proposition 3.6, there holds that $f = \Phi_t$, for some $t \in \mathbb{R}$, on the outset R_0 of the origin. However, $\overline{R}_0 = \mathbb{R}^m$.

This kind of vector fields can be extended to the projective space $\mathbb{R}P^m$, regarded as $\mathbb{R}^m$ plus the hyperplane at infinity, as follows. First consider a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ h \geq 0$, $\varphi \circ h = 1$ on $\mathbb{R}^m - B_r(0)$ for r big enough and $(\varphi \circ h)^{-1}(0) = h^{-1}(0)$. Set $X_1 = (\varphi \circ h)Y$. Reasoning as above shows that $\mathrm{Aut}(X_1)$ is $G \times \mathbb{R}$.

Since $X_1 = \sum_{j=1}^m x_j \partial/\partial x_j$ outside $B_r(0)$, this vector field can be extended to a vector field X_P on $\mathbb{R}P^m$ (any linear vector field on $\mathbb{R}^m$ extends to $\mathbb{R}P^m$). Note that X_P vanishes on $\mathbb{R}P^m - \mathbb{R}^m$.

On the other hand G may be seen as a subgroup of the group of projective transformations. Since the points of $\mathbb{R}P^m - \mathbb{R}^m$ are non-isolated singularities, any automorphism of X_P has to transform $\mathbb{R}P^m - \mathbb{R}^m$ in $\mathbb{R}P^m - \mathbb{R}^m$ and $\mathbb{R}^m$ in itself. Thus $\mathrm{Aut}(X_P)$ equals $G \times \mathbb{R}$.

Set $S^m = \{y \in \mathbb{R}^{m+1} \mid \ \| y \| = 1\}$. The vector field X_1 gives rise to a vector field on S^m as well. Indeed, identify $S^m - \{(0, \dots, 0, 1)\}$ to $\mathbb{R}^m$ by means of the stereographic projection and pull X_1 back. Since the map $F\colon \mathbb{R}^m - \{0\} \to \mathbb{R}^m - \{0\}$ defined by $F(x) = \| x \|^{-2} x$ transforms X_1 outside $\overline{B}_r(0)$ in $-\sum_{j=1}^m x_j \partial/\partial x_j$ on $B_{1/r}(0) - \{0\}$, our vector field prolongs to a vector field X_S on S^m.

Moreover, in an obvious way, G becomes a group of diffeomorphisms of S^m (if $g \in G$ is associated to the permutation σ, consider the diffeomorphism of S^m given by $y \to (y_{\sigma(1)}, \dots, y_{\sigma(m)}, y_{m+1})$). Note that $(0, \dots, 0, 1)$ is the only sink of X_S, so it is a fixed point of any automorphism of X_S. Thus $\mathrm{Aut}(X_S)$ equals $G \times \mathbb{R}$.

Example 5.2. On $S^2 \subset \mathbb{R}^3$ consider the function $h(x) = x_1 x_2 x_3$. Let X be the gradient vector field of h with respect to the canonical Riemannian metric on the sphere S^2, that is, the metric induced by the scalar product of $\mathbb{R}^3$. Then

$$X = (1 - 3x_1^2)\, x_2 x_3 \frac{\partial}{\partial x_1} + (1 - 3x_2^2)\, x_1 x_3 \frac{\partial}{\partial x_2} + (1 - 3x_3^2)\, x_1 x_2 \frac{\partial}{\partial x_3}$$

(recall that X is the orthogonal projection on S^2 of the gradient of h on $\mathbb{R}^3$.)

Let G be the order 24 subgroup of $O(3)$ consisting of isometries $L(x) = (a_1 x_{\sigma(1)}, a_2 x_{\sigma(2)}, a_3 x_{\sigma(3)})$, where σ is a permutation of $\{1, 2, 3\}$, $|a_1| = |a_2| = |a_3| = 1$ and $a_1 a_2 a_3 = 1$. Obviously G acts on S^2 and preserves h so X too. We will show that $\mathrm{Aut}(X)$ equals the product $G \times \mathbb{R}$, where the factor $\mathbb{R}$ comes from the flow Φ_t of X.

First note that the zeros of X, that is the singularities of $h\colon S^2 \to \mathbb{R}$, are the points:

(a) $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$,

(b) $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$.

The points of (b) are just an orbit of the action of G, and those of (a) constitute two orbits depending on the sign of the product of their coordinates.

Considering (x_1, x_2) as coordinates on S^2 near $(0, 0, 1)$ shows that this point is a saddle of h; the same happens with any point of type (b). Thus topologically each singularity of X of type (b) is equivalent to $y_1\partial/\partial y_1 - y_2\partial/\partial y_2$ at the origin of $\mathbb{R}^2$. Therefore, there are two regular trajectories of X with ω-limit this singularity.

The Hessian of $h\colon S^2 \to \mathbb{R}$ at $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$ equals $2/\sqrt{3}$ times the canonical metric of S^2 at this point. Indeed, it suffices to compute this Hessian by means of vector fields $-x_2\partial/\partial x_1 + x_1\partial/\partial x_2$ and $-x_3\partial/\partial x_1 + x_1\partial/\partial x_3$. The same happens at every point of type (a) whose product of coordinates is negative. Analogously, for remaining points of (a), the factor is $-2/\sqrt{3}$.

In short $h\colon S^2 \to \mathbb{R}$ is a Morse function with four minima, four maxima and six saddles. Moreover at each minimum of this function the linear part of X equals $(2/\sqrt{3})\mathrm{Id}$, therefore this point is a source with our definition while every maximum is a sink.

Let I be the set of minima of $h\colon S^2 \to \mathbb{R}$. Since I consists of four elements and is an orbit of the action of G, the isotropy group G_i of G at i, $i \in I$, has six elements and acts on T_iS^2 as a group of isometries. It is easily checked that every G_i is isomorphic to the group of permutations of three elements (for instance consider the point $(-1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$), which implies that G_i, as isometries of T_iS^2, equals the group of motions of an equilateral triangle.

Since X is the gradient of the Morse function h, which vanishes at each saddle, every regular trajectory with ω-limit a saddle has a source as α-limit. On the other hand I is an orbit of G and there are $2 \times 6 = 12$ regular trajectories with ω-limit a saddle; therefore each $i \in I$ is the α-limit of three such trajectories. Hence the linear α-limits of these trajectories constitute a set $\mathcal{L}_i$ of three open half-lines in T_iS^2 starting at the origin, which is a geometric invariant of X. Thus the set $\mathcal{L}_i$ is invariant under the action of G_i.

On the other hand, when G_i is regarded as the motion group of an equilateral triangle, necessarily $\mathcal{L}_i$ becomes the set of half-bisectors (from the center) of this triangle and G_i acting on $\mathcal{L}_i$ its permutation group.

Consider $f \in \mathrm{Aut}(X)$ and $i \in I$. Then there exists $(g, t) \in G \times \mathbb{R}$ such that $f = g \circ \Phi_t$ on the outset R_i of i (that is, the analogue of Proposition 3.6 holds). Indeed, $f(i)$ has to be a source as well and by composing on the left with a suitable element of G we may suppose $f(i) = i$. In this case, $f_*(i)\colon T_iS^2 \to T_iS^2$ gives rise to a permutation of $\mathcal{L}_i$ and composing on the left with a suitable element of G_i allows us to assume that this permutation is the identity. But the elements of $\mathcal{L}_i$ are in general position so $f_*(i) = c\,\mathrm{Id}$ with $c > 0$. Now reasoning as in the last paragraph of the proof of Proposition 3.6, it follows that $f = \Phi_t$ on R_i for some $t \in \mathbb{R}$.

Henceforth it suffices to copy the remainder of the proof of Theorem 3.1, from Lemma 3.7 up to Remark 3.8, to conclude that $\mathrm{Aut}(X) = G \times \mathbb{R}$.

To finish this section, notice that in the general case (any finite group and any action) rivets and chains are needed for controlling the differential of an automorphism at a source. However in some particular cases, for instance the second example, this differential may be controlled by other means.

Appendix

In this appendix we prove Proposition 5.5, which was needed in Section 4. First consider a family of compact sets $\{K_r\}_{r\in\mathbb{N}}$ in an open set $A \subset \mathbb{R}^n$, such that

$$K_r \subset \overset{\circ}{K}_{r+1}, \quad r \in \mathbb{N}, \quad \text{and} \quad \bigcup_{r\in\mathbb{N}} K_r = A.$$

Lemma 5.3. *Given a family of positive continuous functions $\{f_r \colon A \to \mathbb{R}\}_{r\in\mathbb{N}}$ there exists a function $f \colon A \to \mathbb{R}$ vanishing on $\mathbb{R}^n - A$ and positive on A such that, whenever $r \in \mathbb{N}$, one has $f \le f_j$, $0 \le j \le r$, on $A - K_r$.*

Proof. One may assume $f_0 \ge f_1 \ge \cdots \ge f_r \ge \cdots$ by taking $\min\{f_0, \ldots, f_r\}$ instead of f_r if necessary. Consider functions $\varphi_r : \mathbb{R}^n \to [0,1] \subset \mathbb{R}$, $r \in \mathbb{N}$, such that each

$$\varphi_r^{-1}(0) = K_{r-1} \cup (\mathbb{R}^n - \overset{\circ}{K}_{r+1})$$

(as usual, $K_j = \emptyset$ if $j \le -1$).

Let D be a partial derivative operator. Multiplying each f_r by some small enough $\varepsilon_r > 0$ allows to suppose, without loss of generality, that $\varphi_r \le f_r/2$ on A and $\mid D\varphi_r \mid \le 2^{-r}$ on $\mathbb{R}^n$ for any D of order $\le r$.

Set

$$f = \sum_{r\in\mathbb{N}} \varphi_r.$$

By the second condition on functions φ_r, whenever $\tilde{D}$ is a partial derivative operator the series $\sum_{r\in\mathbb{N}} \tilde{D}\varphi_r$ converges uniformly on $\mathbb{R}^n$, which implies that f is differentiable. On the other hand it is easily checked that $f(\mathbb{R}^n - A) = 0$, $f > 0$ on A and $f \le f_r \le \cdots \le f_0$ on $A - K_r$. □

One says that a function defined around a point p of a manifold is *flat at p* if its ∞-jet at this point vanishes. Note that given a function ψ on a manifold and a function $\tau : \mathbb{R} \to [0,1] \subset \mathbb{R}$ flat at the origin and positive on $\mathbb{R} - \{0\}$ (for instance $\tau(t) = e^{-1/t^2}$ if $t \ne 0$ and $\tau(0) = 0$), then $\tau \circ \psi$ is flat at every point of $(\tau \circ \psi)^{-1}(0) = \psi^{-1}(0)$ and $\mathrm{Im}(\tau \circ \psi) \subset [0,1]$.

Lemma 5.4. *Consider an open set A of a manifold M and a function $f \colon A \to \mathbb{R}$. Then there exists a function $\varphi \colon M \to \mathbb{R}$ vanishing on $M - A$ and positive on A, such that the function $\hat{f} \colon M \to \mathbb{R}$ given by $\hat{f} = \varphi f$ on A and $\hat{f} = 0$ on $M - A$ is differentiable.*

Proof. The manifold M can be seen as a closed imbedded submanifold of some $\mathbb{R}^n$. Let $\pi : E \to M$ be a tubular neighborhood of M. If the result is true for $\pi^{-1}(A)$ and $f \circ \pi : \pi^{-1}(A) \to \mathbb{R}$, by restriction it is true for A and f. In other words, it suffices to consider the case of an open set A of $\mathbb{R}^n$.

We will say that a function $\psi : A \to \mathbb{R}$ is *neatly bounded* if, for each point p of the topological boundary of A and any partial derivative operator D, there exists an open neighborhood B of p such that $| D\psi |$ is bounded on $A \cap B$. First assume that f is neatly bounded. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a function that is positive on A and flat at every point of $\mathbb{R}^n - A$; then φ satisfies Lemma 5.4.

Indeed, only the points $p \in (\bar{A} - A)$ need to be examined. Consider a natural number $1 \leq j \leq n$; since $j_p^\infty \varphi = 0$ near p one has $\varphi(x) = \sum_{i=1}^n (x_i - p_i)\tilde{\varphi}_i(x)$, and from the definition of partial derivative follows that $(\partial \hat{f}/\partial x_j)(p) = 0$. Thus $\partial \hat{f}/\partial x_j = (\partial\varphi/\partial x_j)f + \varphi\partial f/\partial x_j$ on A and $\partial \hat{f}/\partial x_j = 0$ on $\mathbb{R}^n - A$, which shows that f is C^1.

Since obviously the function $\partial f/\partial x_j$ is neatly bounded and $\partial\varphi/\partial x_j$ is flat on $\mathbb{R}^n - A$, the same argument as before applied to $(\partial\varphi/\partial x_j)f$ and $\varphi\partial f/\partial x_j$ shows that f is C^2 and, by induction, shows the differentiability of f.

Let us consider the general case. On A the continuous functions $| Df | + 1$, where D is any partial derivative operator, give rise to a countable family of continuous positive functions $g_0, \ldots, g_r, \ldots$ Let $\{K_r\}_{r\in\mathbb{N}}$ be a collection of compact sets such that $K_r \subset \overset{\circ}{K}_{r+1}$, $r \in \mathbb{N}$, and $\bigcup_{r\in\mathbb{N}} K_r = A$. By Lemma 5.3 there exists a function $\rho : \mathbb{R}^n \to \mathbb{R}$ vanishing on $\mathbb{R}^n - A$ and positive on A such that $\rho \leq g_j^{-1}$, $0 \leq j \leq r$, on $A - K_r$, $r \in \mathbb{N}$.

For every $k \in \mathbb{N}$ let $\lambda_k : \mathbb{R} \to \mathbb{R}$ be the function defined by $\lambda_k(t) = t^{-k}e^{-1/t}$ if $t > 0$ and $\lambda_k(t) = 0$ elsewhere. Then the function $\tilde{f} = \lambda_0(\rho/2)f$ is neatly bounded on A. Indeed, consider any $p \in (\bar{A} - A)$ and any partial derivative operator D. Then $D\tilde{f}$ equals a linear combination, with constant coefficients, of products of some partial derivatives of ρ, a function $\rho^{-k}e^{-2/\rho} = \lambda_k(\rho)e^{-1/\rho}$ and some partial derivative $D'f$. On the other hand, there always exists a natural ℓ such that $g_\ell =| D'f | +1$. Near p one has $e^{-1/\rho}| D'f | \leq \rho | D'f |\leq \rho g_\ell \leq 1$; therefore $D\tilde{f}$ is bounded close to p.

Finally, take a function $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$ positive on A and flat at every point of $\mathbb{R}^n - A$ and set $\varphi = \tilde{\varphi}\lambda_0(\rho/2)$. □

Proposition 5.5. *Consider a vector field X on an open set A of a manifold M. Then there exists a bounded function $\varphi\colon M \to \mathbb{R}$, which is positive on A and vanishes on $M - A$, such that the vector field $\hat{X}$ on M defined by $\hat{X} = \varphi X$ on A and $\hat{X} = 0$ on $M - A$ is differentiable.*

Proof. Regard M as a closed imbedded submanifold of some $\mathbb{R}^n$. Let $\pi : E \to M$ be a tubular neighborhood of M. Then there exists a vector field X' on $\pi^{-1}(A)$ such that $X' = X$ on A and, by restriction of the function, it suffices to show our result for X' and $\pi^{-1}(A)$. That is to say, we may suppose, without loss of generality, that A is an open set of $\mathbb{R}^n$.

In this case, on A one has $X = \sum_{j=1}^{n} f_j \partial/\partial x_j$. Applying Lemma 5.4 to every function f_j yields a family of functions $\varphi_1, \ldots, \varphi_n$. Now it suffices to set $\varphi = \varphi_1 \cdots \varphi_n$.

Finally, if φ is not bounded take $\varphi/(\varphi + 1)$ instead of φ. $\square$

References

[1] Belolipetsky, M. and Lubotzky, A.: Finite groups and hyperbolic manifolds. *Invent. Math.* **162** (2005), 459–472.

[2] Bumagin, I. and Wise, D. T.: Every group is an outer automorphism group of a finitely generated group. *J. Pure Appl. Algebra* **200** (2005), 137–147.

[3] Costoya, C. and Viruel, A.: Every finite group is the group of self homotopy equivalences of an elliptic space. To appear in *Acta Math.*

[4] Frucht, R.: Herstellung von Graphen mit vorgegebener abstrakter Gruppe. *Compositio Math.* **6** (1939), 239–250.

[5] Greenberg, L.: Maximal groups and signatures. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, 207–226. Ann. of Math. Studies 79, Princeton University Press, Princeton, NJ, 1974.

[6] Hirsch, M. S.: *Differential topology.* Graduate Texts in Mathematics 33, Springer-Verlag, New York-Heidelberg, 1976.

[7] Kobayashi, S. and Nomizu, K.: *Foundations of differential geometry. Vol. I.* Interscience Publishers, John Wiley & Sons, New York-London, 1963.

[8] Kojima, S.: Isometry transformations of hyperbolic 3-manifolds. *Topology Appl.* **29** (1988), 297–307.

[9] Roussarie, R: *Modèles locaux de champs et de formes.* Astérisqûe 30, Société Mathématique de France, Paris, 1975.

[10] Sternberg, S.: On the structure of local homeomorphisms of euclidean n-spaces. II. *Amer. J. Math.* **80** (1958), 623–631.

[11] Wasserman, A.: Equivariant differential topology. *Topology* **8** (1969), 127–150.

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