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The structure of Sobolev extension operators

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Abstract. Let $L^{m,p}(\mathbb{R}^n)$ denote the Sobolev space of functions whose m-th derivatives lie in $L^p(\mathbb{R}^n)$, and assume that $p > n$. For $E \subseteq \mathbb{R}^n$, denote by $L^{m,p}(E)$ the space of restrictions to E of functions $F \in L^{m,p}(\mathbb{R}^n)$. It is known that there exist bounded linear maps $T : L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)$ such that $Tf = f$ on E for any $f \in L^{m,p}(E)$. We show that T cannot have a simple form called "bounded depth".

1. Introduction

Let $\mathbb{X}$ denote any of the following standard function spaces on $\mathbb{R}^n$:

- $\mathbb{X} = C^m(\mathbb{R}^n)$, the space of real-valued $F \in C^m_{\text{loc}}(\mathbb{R}^n)$ for which the norm
$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \max_{|\alpha|\le m} |\partial^\alpha F(x)| \text{ is finite;}$$
- $\mathbb{X} = C^{m,s}(\mathbb{R}^n)$, the space of all functions $F \in C^m(\mathbb{R}^n)$ for which the norm
$$\|F\|_{C^{m,s}(\mathbb{R}^n)} := \|F\|_{C^m(\mathbb{R}^n)} + \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \max_{|\alpha|=m} \frac{|\partial^\alpha F(x) - \partial^\alpha F(y)|}{|x-y|^s}$$
is finite (here $0 < s < 1$);
- $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$, the homogeneous Sobolev space of all real-valued functions F for which the seminorm
$$\|F\|_{L^{m,p}(\mathbb{R}^n)} := \|\nabla^m F\|_{L^p(\mathbb{R}^n)} \text{ is finite.}$$
(Here, we take $p > n$, so that $\mathbb{X} \subseteq C^{m-1,1-n/p}_{\text{loc}}(\mathbb{R}^n)$, by the Sobolev theorem.)

For $E \subseteq \mathbb{R}^n$, we set $\mathbb{X}(E) := \{F|_E : F \in \mathbb{X}\}$, equipped with the seminorm
$$\|f\|_{\mathbb{X}(E)} := \inf \big\{\|F\|_{\mathbb{X}} : F \in \mathbb{X},\ F = f \text{ on } E\big\}.$$

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Let $A \geq 1$ be a real number. An *extension operator* for $\mathbb{X}(E)$ with norm A is a linear map $T : \mathbb{X}(E) \to \mathbb{X}$ such that for all $f \in \mathbb{X}(E)$ we have

$$Tf = f \quad \text{on } E$$

and

$$\|Tf\|_{\mathbb{X}} \leq A \, \|f\|_{\mathbb{X}(E)}.$$

For $\mathbb{X} = C^m(\mathbb{R}^n)$ or $C^{m,s}(\mathbb{R}^n)$ and $E \subseteq \mathbb{R}^n$ arbitrary, there exists an extension operator whose norm depends only on m and n. Similarly, for $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$ and E arbitrary, there exists an extension operator whose norm depends only on m, n and p. See [1], [2], and [4].

We want to know whether such extension operators can be taken to have a simple form when E is finite. Recall that any linear map $T : \mathbb{X}(E) \to \mathbb{X}$ ($E \subseteq \mathbb{R}^n$ finite) has the form

$$Tf(x) = \sum_{y \in E} \lambda(x,y) f(y) \quad (\text{all } x \in \mathbb{R}^n),$$

with coefficients $\lambda(x,y)$ independent of f. Let D be a positive integer. We say that T has *depth* D if, for each fixed x, at most D of the coefficients $\lambda(x,y)$ are nonzero.

Let $\mathbb{X} = C^m(\mathbb{R}^n)$ or $C^{m,s}(\mathbb{R}^n)$, and let $E \subseteq \mathbb{R}^n$ be finite. Then there exists an extension operator for $\mathbb{X}(E)$, whose norm and depth depend only on m and n. See [1] and [3].

Thus, it is natural to ask the following:

Let $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$, and let $E \subseteq \mathbb{R}^n$ be finite. Does there exist an extension operator for $\mathbb{X}(E)$, whose norm and depth depend only on m, n and p?

Unfortunately, the answer is NO. In this paper, we establish the following result.

Theorem 1. *Let $p > 2$, $A \geq 1$ and $D \geq 1$ be given. Then there exists a finite set $E \subseteq \mathbb{R}^2$ such that $L^{2,p}(E)$ has no extension operator of norm A and depth D.*

More precisely, for $N \geq 2$, let

$$E_N := \big\{(2^{-k}, (2^{-k})^{2-2/p}) : k = 2, \ldots, N\big\} \cup \big\{(0,0)\big\} \subseteq \mathbb{R}^2. \tag{1.1}$$

Theorem 2. *Let $p > 2$, $A \geq 1$, $D \geq 1$, and let $0 < \epsilon < 3/p$. If $L^{2,p}(E_N)$ has an extension operator with norm A and depth D, then*

$$A \cdot D^{5/p} > c(\epsilon, p) \cdot N^{\epsilon}, \quad \textit{where } c(\epsilon,p) \textit{ depends only on } \epsilon \textit{ and } p.$$

Theorem 2 will be proven in the next section. Theorem 1 follows at once from Theorem 2.

We mention a few related results in the literature. For $\mathbb{X} = C^{m,s}(\mathbb{R}^n)$, Luli [6] constructed extension operators of bounded depth without the assumption that E is finite. The analogous result for $\mathbb{X} = C^m(\mathbb{R}^n)$ is false; however, there exist extension operators of "bounded breadth". (See [3].) For $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$ and E finite, an extension operator may be taken to have "assisted bounded depth"; see [4].

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2. Proof of Theorem 2

Fix $p > 2$ and $0 < \epsilon < 1/(3p)$, and let $\alpha := 1 - 2/p$. Unless stated otherwise, C, c, etc. denote constants depending only on p, which may change value from one occurrence to the next.

For any C^1 function $F : \mathbb{R}^2 \to \mathbb{R}$ and $y \in \mathbb{R}^2$, let $J_y F$ denote the first order Taylor polynomial of F at y:

$$(J_y F)(x) = F(y) + \nabla F(y) \cdot (x - y).$$

We require $p > 2$ so that the Sobolev theorem holds. In particular, after modification on some measure zero subset, each $F \in L^{2,p}(\mathbb{R}^2)$ belongs to $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$ and satisfies the inequalities:

$$\begin{aligned} |\nabla F(x) - \nabla F(y)| &\leq C \, \|F\|_{L^{2,p}(\mathbb{R}^2)} \, |x - y|^{\alpha} \\ |F(x) - J_y F(x)| &\leq C \, \|F\|_{L^{2,p}(\mathbb{R}^2)} \, |x - y|^{1+\alpha} \end{aligned} \qquad (\text{all } x, y \in \mathbb{R}^2). \tag{2.1}$$

We extend the $L^{2,p}$ norm to $\mathbb{R}^2$-valued functions by setting

$$\|\Psi\|_{L^{2,p}(\mathbb{R}^2)} := \|\Psi_1\|_{L^{2,p}(\mathbb{R}^2)} + \|\Psi_2\|_{L^{2,p}(\mathbb{R}^2)}, \quad \text{where } \Psi = (\Psi_1, \Psi_2) \text{ in coordinates.}$$

We define the curve $\gamma := \big\{ (s, s^{1+\alpha}) : s \in [0,1] \big\} \subseteq \mathbb{R}^2$. Let $N \geq 2$. We write E for the subset E_N defined in the introduction:

$$E := \big\{ (2^{-k}, (2^{-k})^{1+\alpha}) : k = 2, \ldots, N \big\} \cup \big\{ (0,0) \big\} \subseteq \gamma. \tag{2.2}$$

In proving Theorem 2, it suffices to assume that N is sufficiently large. More precisely, we henceforth assume that

$$N \geq Z, \quad \text{where } Z \geq 1 \text{ is some large constant that depends only on } p \text{ and } \epsilon. \tag{2.3}$$

We determine Z through Lemma 1 below.

Lemma 1. *There exists $Z \geq 1$ depending only on p and ϵ, such that the following holds. Assume (2.3). Then for any $G \in L^{2,p}(\mathbb{R}^2)$ with*

$$G = 0 \text{ on } E \quad \text{and} \quad \|G\|_{L^{2,p}(\mathbb{R}^2)} \leq 1,$$

we have $|\nabla G(0)| \leq N^{-\epsilon}$.

Lemma 2. *For any integer $D \geq 1$ and subset $S \subseteq \gamma$ with $\#S \leq D$, there exists $H \in L^{2,p}(\mathbb{R}^2)$ that satisfies*

$$H = 0 \text{ on } S, \quad |\nabla H(0)| \geq 1, \quad \text{and} \quad \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C_2 \, D^{5/p}, \tag{2.4}$$

where $C_2 = C_2(p)$ depends only on p.

We now prove Theorem 2, presuming the validity of Lemmas 1 and 2. These lemmas are proven later in the section.

In proving Theorem 2, it suffices to assume that (2.3) holds with Z determined by Lemma 1.

Let $A \geq 1$, $D \geq 1$, and let $T : L^{2,p}(E) \to L^{2,p}(\mathbb{R}^2)$ be an extension operator with norm A and depth D. In other terms, for any $f : E \to \mathbb{R}$,

$$Tf = f \text{ on } E, \tag{2.5}$$

$$\|Tf\|_{L^{2,p}(\mathbb{R}^2)} \leq A\|f\|_{L^{2,p}(E)}, \text{ and} \tag{2.6}$$

$$Tf(x) = \sum_{y\in E} \lambda(x,y)f(y) \quad \text{for all } x \in \mathbb{R}^2, \tag{2.7}$$

where the coefficients $\lambda(x, y)$ satisfy

$$\#\big\{y \in E : \lambda(x,y) \neq 0\big\} \leq D \quad \text{for all } x \in \mathbb{R}^2. \tag{2.8}$$

Note that $\lambda(x, y) = (T\delta_y)(x)$, where $\delta_y : E \to \mathbb{R}$ equals 1 at y, and equals 0 on $E \setminus \{y\}$. Thus, $\lambda(\cdot, y) \in L^{2,p}(\mathbb{R}^2)$ for each fixed $y \in E$. It follows from the Sobolev theorem that the function $x \mapsto \lambda(x, y)$ belongs to $C^1(\mathbb{R}^2)$ for each fixed $y \in E$.

Let

$$S := \big\{y \in E : \nabla_x\lambda(0,y) \neq 0\big\}. \tag{2.9}$$

We claim that $\#S \leq D$. Indeed, suppose for the sake of contradiction that there exist distinct $y_1, \ldots, y_{D+1} \in E$ such that $\nabla_x\lambda(0, y_k) \neq 0$ for each $k = 1, \ldots D+1$. Then, by the implicit function theorem, there exists $x \in \mathbb{R}^2$ such that $\lambda(x, y_k) \neq 0$ for each $k = 1, \ldots D + 1$. This contradicts (2.8), hence proving $\#S \leq D$.

Note that $S \subseteq \gamma$ (see (2.2), (2.9)). By Lemma 2 there exists $H \in L^{2,p}(\mathbb{R}^2)$ with

$$H = 0 \text{ on } S, \quad |\nabla H(0)| \geq 1, \quad \text{and} \quad \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C_2\, D^{5/p}. \tag{2.10}$$

Define $F = T(H|_E)$. From (2.7),

$$\nabla F(0) = \sum_{y\in E} \nabla_x\lambda(0,y)H(y),$$

For $y \in S$ the summand vanishes because $H = 0$ on S, while for $y \in E \setminus S$ the summand vanishes by definition of S (see (2.9)). Therefore, $\nabla F(0) = 0$. Finally, (2.5) implies that $F = H$ on E, while (2.6) and (2.10) imply that

$$\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq A\,\|H|_E\|_{L^{2,p}(E)} \leq A\,\|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C_2\,A\,D^{5/p}.$$

We define $F_0 := F - H$. From (2.10) and the above properties of F,

$$F_0 = 0 \text{ on } E, \quad |\nabla F_0(0)| = |\nabla H(0)| \geq 1, \quad \text{and} \quad \|F_0\|_{L^{2,p}(\mathbb{R}^2)} \leq (C_2+1)\,A\,D^{5/p}.$$

Taking $G = F_0 \cdot \big[(C_2 + 1)\,A\,D^{5/p}\big]^{-1}$ in Lemma 1, we obtain

$$N^{-\epsilon} \geq |\nabla G(0)| \geq \big[(C_2+1)\,A\,D^{5/p}\big]^{-1}.$$

This completes the proof of Theorem 2. In the following subsections we prove Lemmas 1 and 2.

2.1. Besov spaces

The Besov seminorm of a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ is

$$\|\varphi\|_{\dot{B}_p(\mathbb{R})} := \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\varphi'(s) - \varphi'(t)|^p}{|s-t|^p} \, ds \, dt \Big)^{1/p}.$$

The Besov space $\dot{B}_p(\mathbb{R})$ consists of functions with finite Besov seminorm.

The Besov and Sobolev spaces are related through the following trace/extension theorem (see [7], [8]).

Theorem 3. *Let $\mathcal{R}$ denote the restriction operator $\mathcal{R}(F) = F|_{\mathbb{R}\times\{0\}}$, defined for continuous functions $F : \mathbb{R}^2 \to \mathbb{R}$.*

- *The restriction operator $\mathcal{R} : L^{2,p}(\mathbb{R}^2) \to \dot{B}_p(\mathbb{R})$ is bounded. In other terms, $\|\mathcal{R}(G)\|_{\dot{B}_p(\mathbb{R})} \leq C_{\mathrm{SB}} \|G\|_{L^{2,p}(\mathbb{R}^2)}$ for every $G \in L^{2,p}(\mathbb{R}^2)$.*

- *There exists a bounded extension operator $\mathcal{E} \colon \dot{B}_p(\mathbb{R}) \to L^{2,p}(\mathbb{R}^2)$. In other terms, $\mathcal{E}(g)|_{\mathbb{R}\times\{0\}} = g$ and $\|\mathcal{E}(g)\|_{L^{2,p}(\mathbb{R}^2)} \leq C_{\mathrm{SB}} \|g\|_{\dot{B}_p(\mathbb{R})}$ for any $g \in \dot{B}_p(\mathbb{R})$.*

Given $\overline{E} = \{s_1, \dots, s_K\} \subseteq \mathbb{R}$ and $\phi : \overline{E} \to \mathbb{R}$, where $s_1 < \cdots < s_K$, we denote the Besov trace seminorm of ϕ by

$$\|\phi\|_{\dot{B}_p(\overline{E})} := \inf\big\{ \|\varphi\|_{\dot{B}_p(\mathbb{R})} : \varphi \in \dot{B}_p(\mathbb{R}), \ \varphi = \phi \text{ on } \overline{E} \big\}.$$

Let $s_0 := -\infty$ and $s_{K+1} := +\infty$. Define

$$A_{kl} := \int_{s_{k-1}}^{s_k} \int_{s_l}^{s_{l+1}} \frac{1}{|s-t|^p} \, ds \, dt \quad (\text{all } 1 \leq k < l \leq K). \tag{2.11}$$

For $1 \leq k \leq K$, let $n(k) \in \{1, \dots, K\}$ be such that $s_{n(k)} \in \overline{E}$ is a nearest neighbor of s_k, and let

$$m_k := \frac{\phi(s_k) - \phi(s_{n(k)})}{s_k - s_{n(k)}}.$$

For $1 \leq k \leq K-1$, let $\Delta_k := |s_k - s_{k+1}|$, and let

$$M_k := \frac{|m_k - m_{k+1}|}{\Delta_k} + \frac{|\phi(s_k) + m_k \cdot (s_{k+1} - s_k) - \phi(s_{k+1})|}{\Delta_k^2}.$$

The following expression for the Besov trace seminorm can be found in [5] (see Claims 1 and 3 in the proof of Proposition 3.2):

$$c \cdot \|\phi\|^p_{\dot{B}_p(\overline{E})} \leq \sum_{k=1}^{K-1} M_k^p \Delta_k^2 + \sum_{k=1}^{K-1} \sum_{l=k+1}^{K} |m_k - m_l|^p A_{kl} \leq C \cdot \|\phi\|^p_{\dot{B}_p(\overline{E})}. \tag{2.12}$$

2.2. Proof of Lemma 1

Recall that $0 < \epsilon < 1/(3p)$. Let $Z \geq 1$ be a parameter, determined before the end of the proof. We assume that (2.3) holds, that is, $N \geq Z$. In this subsection, constants written C, c, etc., may depend on p and ϵ, but are independent of other parameters.

For the sake of contradiction, suppose that $G \in L^{2,p}(\mathbb{R}^2)$ satisfies

$$\begin{aligned} &G = 0 \text{ on } E = \big\{(2^{-k}, (2^{-k})^{1+\alpha}) : k = 2, \dots, N\big\} \cup \big\{(0,0)\big\}, \\ &\|G\|_{L^{2,p}(\mathbb{R}^2)} \leq 1 \quad \text{and} \quad |\nabla G(0)| \geq N^{-\epsilon}. \end{aligned} \tag{2.13}$$

Furthermore, by renormalizing G we may assume

$$N^{-\epsilon} \leq |\nabla G(0)| \leq 1. \tag{2.14}$$

Let $\delta := N^{-1/\alpha}$, and let $\theta \in C_0^\infty(\mathbb{R}^2)$ satisfy

$$\begin{aligned} &\text{(a) } \operatorname{supp}(\theta) \subseteq B(0,\delta), \quad \text{(b) } \theta = 1 \text{ on } B(0, \delta/2), \quad \text{and} \\ &\text{(c) } |\partial^\beta \theta| \leq C\delta^{-|\beta|}, \quad \text{whenever } |\beta| \leq 2. \end{aligned} \tag{2.15}$$

Define $H = \theta G + (1-\theta) J_0 G$. First we use the Leibniz rule, (2.15.c) and the fact that H is affine on $\mathbb{R}^2 \setminus B(0,\delta)$ (this follows from (2.15.a)), and then we use the Sobolev theorem (see (2.1)) and $\|G\|_{L^{2,p}(\mathbb{R}^2)} \leq 1$, obtaining that

$$\begin{aligned} \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C \cdot \Big(&\|G\|_{L^{2,p}(\mathbb{R}^2)} + \delta^{-1} \|\nabla G - \nabla J_0 G\|_{L^p(B(0,\delta))} \\ &+ \delta^{-2} \|G - J_0 G\|_{L^p(B(0,\delta))} \Big) \leq C'. \end{aligned} \tag{2.16}$$

From (2.15.b) and $G = 0$ on E,

$$H = 0 \quad \text{on} \quad E \cap B(0, \delta/2). \tag{2.17}$$

Note that $\nabla H(0) = \nabla G(0)$, thanks to (2.15.b). Thus, for each $y \in B(0,\delta)$, applying the Sobolev theorem and (2.16) we obtain

$$|\nabla H(y) - \nabla G(0)| = |\nabla H(y) - \nabla H(0)| \leq C' \|H\|_{L^{2,p}(\mathbb{R}^2)} |y|^\alpha \leq C'' \delta^\alpha = C'' N^{-1}. \tag{2.18}$$

Note that (2.18) also holds for $y \in \mathbb{R}^2$, since H is affine on $\mathbb{R}^2 \setminus B(0,\delta)$. Since N is sufficiently large (see (2.3)) and $\epsilon < 1$, it follows from (2.14) and (2.18) that

$$c\, N^{-\epsilon} \leq |\nabla H(y)| \leq C \quad \text{for all } y \in \mathbb{R}^2. \tag{2.19}$$

Note that $H(y_0) = H(y_1) = 0$, where $y_0 := (0,0)$ and $y_1 := (2^{-N}, 2^{-N(1+\alpha)})$, for N sufficiently large. This follows from (2.17), since $y_1 \in B(0, N^{-1/\alpha}/2)$ when N is sufficiently large. Thus, for $v := (y_0 - y_1)/|y_0 - y_1|$, the mean value theorem implies that $v \cdot \nabla H(x^*) = 0$ for some $x^* \in B(0,\delta)$ on the line segment joining y_0 and y_1. By the Sobolev theorem and (2.16) it follows that

$$|v \cdot \nabla H| \leq C\, \delta^\alpha = C N^{-1} \quad \text{on } B(0,\delta).$$

Hence, $|\partial_1 H| \leq C' N^{-1}$ on $B(0,\delta)$, thanks to the upper bound from (2.19) and the fact $|v-(1,0)| \leq C\, 2^{-N\alpha}$. Since H is affine on $\mathbb{R}^2 \setminus B(0,\delta)$, we conclude that

$$|\partial_1 H(y)| \leq C' N^{-1} \quad \text{for all } y \in \mathbb{R}^2. \tag{2.20}$$

Thus, for N sufficiently large, the lower bound in (2.19) and $\epsilon < 1$ imply that

$$|\partial_2 H(y)| \geq c' N^{-\epsilon} \quad \text{for all } y \in \mathbb{R}^2. \tag{2.21}$$

We define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(s,t) = (s, H(s,t))$. The diffeomorphism Φ maps onto $\mathbb{R}^2$ because $|\partial_2 H|$ is bounded away from zero (see (2.21)). By (2.19)–(2.21), $\nabla\Phi(x)$ takes the form

$$\nabla\Phi(x) = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}, \quad \text{where} \quad |a| \leq C\, N^{-1} \quad \text{and} \quad c\, N^{-\epsilon} \leq |b| \leq C. \tag{2.22}$$

Thus, $\nabla\Phi(x)$ is invertible for each $x \in \mathbb{R}^2$ and

$$\big[\nabla\Phi(x)\big]^{-1} = \begin{pmatrix} 1 & 0 \\ \overline{a} & \overline{b} \end{pmatrix}, \quad \text{where} \quad |\overline{a}| \leq \overline{C}\, N^{\epsilon-1} \quad \text{and} \quad |\overline{b}| \leq \overline{C}\, N^{\epsilon}. \tag{2.23}$$

We now define $\Psi = \Phi^{-1}$, and write $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ in coordinates. Differentiating twice the identity $\Psi \circ \Phi = \mathrm{Id}$ shows that

$$\nabla\Phi(x)\cdot\nabla^2\Psi_j(\Phi(x))\cdot\nabla\Phi(x) = -\sum_{l=1}^{2} \nabla^2\Phi_l(x)\cdot\partial_l\Psi_j(\Phi(x)) \quad (\text{all } x \in \mathbb{R}^2,\ j \in \{1,2\}).$$

Now, perform the following operations on the above equation: multiply through twice by $[\nabla\Phi(x)]^{-1}$ (on the left and right), use the identity $\nabla\Psi(\Phi(x)) = [\nabla\Phi(x)]^{-1}$, substitute $x = \Phi^{-1}(y)$ on both sides, take p^{th} powers, sum over $j \in \{1,2\}$, integrate over $y \in \mathbb{R}^2$, and perform the change of variable $y = \Phi(x)$ on the right-hand side. Thus, we obtain

$$\|\Psi\|^p_{L^{2,p}(\mathbb{R}^2)} \leq C\, \|\Phi\|^p_{L^{2,p}(\mathbb{R}^2)}\, \|\det(\nabla\Phi)\|_{L^\infty}\, \|(\nabla\Phi)^{-1}\|^{3p}_{L^\infty}. \tag{2.24}$$

Next, insert into (2.24) the bounds $\|\det(\nabla\Phi)\|_{L^\infty} \leq C$, $\|(\nabla\Phi)^{-1}\|_{L^\infty} \leq CN^{\epsilon}$ and $\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} = \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C'$ obtained from (2.22), (2.23) and (2.16). Thus,

$$\|\Psi\|_{L^{2,p}(\mathbb{R}^2)} \leq CN^{3\epsilon}. \tag{2.25}$$

Define $\varphi = \Psi_2|_{\mathbb{R}\times\{0\}}$. By (2.25) and Theorem 3,

$$\|\varphi\|_{\dot{B}_p(\mathbb{R})} \leq C_{\mathrm{SB}}\, \|\Psi_2\|_{L^{2,p}(\mathbb{R}^2)} \leq C' N^{3\epsilon}. \tag{2.26}$$

It follows from (2.17) and the definition $\Phi(s,t) = (s, H(s,t))$ that

$$\Phi(E \cap B(0,\delta/2)) \subseteq \mathbb{R} \times \{0\}.$$

In coordinates, $\Psi = \Phi^{-1}$ takes the form $\Psi(u,v) = (u, \Psi_2(u,v))$. Applying Ψ to the previous set containment and using the definition of φ, we obtain

$$E \cap B(0,\delta/2) \subseteq \big\{(u,\varphi(u)) : u \in \mathbb{R}\big\}. \tag{2.27}$$

For some integer $K \geq 0$, we write

$$E \cap B(0,\delta/2) = \big\{(0,0), (2^{-N}, 2^{-N(1+\alpha)}), \ldots, (2^{K-N}, 2^{(K-N)(1+\alpha)})\big\}.$$

Thus, $2^{K-N} \geq c\delta$ for some $c > 0$. Since $\delta = N^{-1/\alpha}$, we obtain

$$K \geq N - C\log(N). \tag{2.28}$$

Let $s_k := 2^{k-N}$ for $k = 1, \ldots, K$, and let $\overline{E} := \{s_1, \ldots, s_K\}$. Define $\phi : \overline{E} \to \mathbb{R}$ by $\phi(2^{k-N}) = (2^{k-N})^{1+\alpha}$ for $k = 1, \ldots, K$.

Next, we apply (2.12) for the $\overline{E}$ and ϕ chosen above. The quantity A_{kl} defined in (2.11) satisfies, for all $1 \leq k < l \leq K$,

$$A_{kl} \geq \int_{2^{k-1-N}}^{2^{k-N}} \int_{2^{l-N}}^{2^{l+1-N}} \frac{1}{|s-t|^p}\, ds\, dt \geq c \cdot 2^{-(l-N)p}\, 2^{k-N}\, 2^{l-N}. \tag{2.29}$$

Thanks to (2.27), the function φ equals ϕ on $\overline{E}$. Thus, from (2.12) and (2.29),

$$\|\varphi\|^p_{\dot{B}_p(\mathbb{R})} \geq \|\phi\|^p_{\dot{B}_p(\overline{E})} \geq c \sum_{k=2}^{K-1} \sum_{l=k+1}^{K} |m_k - m_l|^p \cdot 2^{-(l-N)p}\, 2^{k-N}\, 2^{l-N},$$

where

$$m_i := \frac{\left(2^{i-N}\right)^{1+\alpha} - \left(2^{i-1-N}\right)^{1+\alpha}}{2^{i-N} - 2^{i-1-N}} = (2 - 2^{-\alpha}) \cdot 2^{(i-N)\alpha}.$$

Note that $|m_k - m_l| \geq c \cdot 2^{(l-N)\alpha}$ for $2 \leq k < l \leq K$. Inserting this inequality in the above equation, and using $\alpha p = p - 2$, we obtain

$$\|\varphi\|^p_{\dot{B}_p(\mathbb{R})} \geq c' \sum_{k=2}^{K-1} \sum_{l=k+1}^{K} 2^{(l-N)(p-2)} 2^{-(l-N)p}\, 2^{k-N}\, 2^{l-N} \geq c'' \sum_{k=2}^{K-1} 1 = c'' \cdot (K-2).$$

Finally, from (2.26) and (2.28), we obtain

$$c''N - C''\log(N) \leq (C')^p N^{3\epsilon p}.$$

Since $\epsilon < 1/(3p)$, the above inequality gives a contradiction when N is sufficiently large. Thus, (2.13) cannot hold, completing the proof by contradiction. We now take $Z = Z(\epsilon, p)$ sufficiently large, so that the previous arguments hold for $N \geq Z$. This completes the proof of Lemma 1. □

2.3. Proof of Lemma 2

Let $S \subseteq \gamma$ with $\#S \leq D$ be given. For ease of notation, we may assume that $\#S = D$. We must construct an $H \in L^{2,p}(\mathbb{R}^2)$ that satisfies (2.4). To start, write

$$S = \{(s_1, s_1^{1+\alpha}), \ldots, (s_D, s_D^{1+\alpha})\} \quad \text{with } 0 \leq s_1 < s_2 < \cdots < s_D \leq 1.$$

Let $\overline{S} := \{s_1, \ldots, s_D\}$, and define $\phi : \overline{S} \to \mathbb{R}$ by $\phi(s_k) = (s_k)^{1+\alpha}$ for $k = 1, \ldots, D$. Next, we apply (2.12) to this subset $\overline{S}$ and function ϕ.

We first obtain an estimate on the A_{kl} defined in (2.11):

$$A_{kl} \leq \int_{-\infty}^{s_k} \int_{s_l}^{\infty} \frac{1}{|s-t|^p}\, ds\, dt \leq C \cdot |s_k - s_l|^{2-p} \quad (\text{all } 1 \leq k < l \leq D). \tag{2.30}$$

Let $s_{n(k)} \in \overline{S}$ be a nearest neighbor to s_k, for each $1 \leq k \leq D$, and let

$$m_k := \frac{(s_k)^{1+\alpha} - (s_{n(k)})^{1+\alpha}}{s_k - s_{n(k)}}.$$

From (2.12), (2.30) and $\alpha p = p - 2$, there exists $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$S \subseteq \{(s, \varphi(s)) : s \in \mathbb{R}\}, \quad \text{and} \tag{2.31}$$

$$\|\varphi\|^p_{\dot{B}_p(\mathbb{R})} \leq C \sum_{k=1}^{D-1} \frac{|(s_k)^{1+\alpha} + m_k \cdot (s_{k+1} - s_k) - (s_{k+1})^{1+\alpha}|^p}{|s_{k+1} - s_k|^{(1+\alpha)p}} + C \sum_{k=1}^{D-1} \sum_{l=k+1}^{D} \frac{|m_k - m_l|^p}{|s_k - s_l|^{\alpha p}}. \tag{2.32}$$

By the mean value theorem, each m_k takes the form $(1+\alpha)t_k^\alpha$ for some t_k between s_k and $s_{n(k)}$. Thus, $|m_k - m_l| \leq C|t_k - t_l|^\alpha \leq C\, 3^\alpha |s_k - s_l|^\alpha$ for $k \neq l$. (Here, we use the inequalities $|t_k - s_k| \leq |s_k - s_{n(k)}| \leq |s_k - s_l|$ and $|t_l - s_l| \leq |s_l - s_{n(l)}| \leq |s_k - s_l|$.) Similarly, $|m_k - (1+\alpha)s_k^\alpha| \leq C|s_{k+1} - s_k|^\alpha$, hence Taylor's theorem provides uniform control on each term from the first sum in (2.32). Therefore,

$$\|\varphi\|^p_{\dot{B}_p(\mathbb{R})} \leq C D^2. \tag{2.33}$$

Applying the extension operator $\mathcal{E}$ from Theorem 3, the function $F = \mathcal{E}(\varphi)$ satisfies $F|_{\mathbb{R}\times\{0\}} = \varphi$ and $\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq C_{\mathrm{SB}}\, \|\varphi\|_{\dot{B}_p(\mathbb{R})}$. Thus, from (2.31),

$$S \subseteq \{(s, F(s,0)) : s \in \mathbb{R}\}, \tag{2.34}$$

while from (2.33) we obtain

$$\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq C' D^{2/p}. \tag{2.35}$$

We may assume that $\#S \geq 2$, for otherwise Lemma 2 is trivial. Note that $S \subseteq [0,1]^2$ lies on a Lipschitz graph. Thus, by (2.34), there exists $s^* \in [0,1]$ such that $|\partial_1 F(s^*, 0)| \leq C$. By (2.35) and the Sobolev theorem, $|\partial_1 F(0)| \leq C' D^{2/p}$.

Let

$$M := \max\{\|F\|_{L^{2,p}(\mathbb{R}^2)},\ |\partial_1 F(0)|,\ 1\}.$$

Without loss of generality, by adding to F some multiple of the coordinate function $(s,t) \mapsto t$, we may assume that $\partial_2 F(0) = RM$, where $R \geq 1$ shall be determined later. This does not affect statements from the previous two paragraphs. To summarize:

$$|\partial_1 F(0)| \leq M,\ \partial_2 F(0) = RM, \text{ and} \tag{2.36}$$

$$\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq M, \text{ where } 1 \leq M \leq C' D^{2/p}. \tag{2.37}$$

Pick $\widehat{\theta} \in C_0^\infty(\mathbb{R}^2)$ that satisfies

$$\begin{aligned} &\text{(a) } \operatorname{supp}(\widehat{\theta}) \subseteq [-1,2]^2, \quad \text{(b) } \widehat{\theta} = 1 \text{ on } [-1/2, 3/2]^2, \text{ and} \\ &\text{(c) } |\partial^\beta \widehat{\theta}| \leq C, \quad \text{whenever } |\beta| \leq 2. \end{aligned} \tag{2.38}$$

Define $\widehat{F} := \theta F + (1-\theta) J_0 F$.

Mimicking the proof of (2.16) with help from (2.37), (2.38.a), (2.38.c), we obtain

$$\|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \leq CM. \tag{2.39}$$

Mimicking the proof of (2.18) with help from (2.38.a), (2.38.b), (2.39), we obtain

$$|\nabla \widehat{F}(y) - \nabla F(0)| \leq C'M \quad (\text{all } y \in \mathbb{R}^2).$$

Now, choose R sufficiently large, determined by p, so that the previous inequality and (2.36) imply that

$$|\partial_1 \widehat{F}(y)| \leq CM \quad \text{and} \quad \frac{RM}{2} \leq |\partial_2 \widehat{F}(y)| \leq 2RM \quad (\text{all } y \in \mathbb{R}^2). \tag{2.40}$$

Finally, (2.34),(2.38.b) and $S \subseteq [0,1]^2$ imply that

$$S \subseteq \{(s, \widehat{F}(s,0)) : s \in \mathbb{R}\}. \tag{2.41}$$

We define $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\Phi(s,t) = (s, \widehat{F}(s,t))$. The diffeomorphism Φ maps onto $\mathbb{R}^2$ because $|\partial_2 \widehat{F}|$ is bounded away from zero (see (2.40)).

We define $\Psi = \Phi^{-1}$. We write $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ in coordinates. As in (2.24), we obtain

$$\|\Psi\|_{L^{2,p}(\mathbb{R}^2)} \leq C\, \|\Phi\|_{L^{2,p}(\mathbb{R}^2)} \cdot \|\det(\nabla\Phi)\|_{L^\infty}^{1/p} \cdot \|(\nabla\Phi)^{-1}\|_{L^\infty}^3.$$

It follows from (2.39) and (2.40) that

$$\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} = \|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \leq CM,\ \|\det(\nabla\Phi)\|_{L^\infty} \leq 2RM \text{ and } \|(\nabla\Phi)^{-1}\|_{L^\infty} \leq C'.$$

Therefore,

$$\|\Psi_2\|_{L^{2,p}(\mathbb{R}^2)} \leq \|\Psi\|_{L^{2,p}(\mathbb{R}^2)} \leq C'' M^{1+1/p} \leq C'' M^{3/2}. \tag{2.42}$$

In coordinates, $\Phi(s,t) = (s, \widehat{F}(s,t))$ and $\Psi(u,v) = (u, \Psi_2(u,v))$, where $\widehat{F}(u, \Psi_2(u,v)) = v$. Applying $\partial_2 = \partial/\partial v$, setting $u = v = 0$, and then using (2.40),

$$\partial_2 \Psi_2(0) = \left[\partial_2 \widehat{F}(\Psi(0))\right]^{-1} \geq C M^{-1}. \tag{2.43}$$

Finally, (2.41) implies that $S \subseteq \Phi(\mathbb{R} \times \{0\})$. Thus we obtain

$$\Psi(S) \subseteq \mathbb{R} \times \{0\}. \tag{2.44}$$

Let $H = \Psi_2/\partial_2\Psi_2(0)$. The bound $M \leq C \cdot D^{2/p}$ and (2.42)–(2.44) imply that H satisfies the conclusion of Lemma 2. This completes the proof of Lemma 2. □

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