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Size estimates for the EIT problem with one measurement: the complex case

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Abstract. In this paper we estimate the size of a measurable inclusion in terms of power measurements for a single applied boundary current. This problem arises in medical imaging for the screening of organs (see [17]). For this kind of problem one has to deal mathematically with the complex conductivity (admittivity) equation. In this case we are able to establish, for certain classes of admittivities, lower and upper bounds of the measure of the inclusion in terms of the power measurements. A novelty of our result is that we are also able to estimate the volume of an inclusion having part of its boundary in common with the reference body. Our analysis is based on the derivation of energy bounds and fine quantitative estimates of unique continuation for solutions to elliptic equations.

1. Introduction

In this paper we consider a mathematical problem arising in electrical impedance tomography (EIT), a nondestructive technique for determining electrical properties of a medium from measurements of voltages and currents at the boundary.

More precisely let Ω be the region occupied by a conducting medium and, at a fixed frequency ω, consider the complex-valued admittivity function

$$\gamma(x) = \sigma(x) + i\,\omega\,\epsilon(x),$$

where $\sigma(x)$ represents the electrical conductivity at the point $x \in \Omega$ and $\epsilon(x)$ the electrical permittivity at a point $x \in \Omega$.

EIT leads to the inverse problem of the determination of the admittivity γ from electrical measurements on $\partial\Omega$. This technique has several applications in medical imaging, nondestructive testing of materials and geophysical prospection of the underground. We refer to the review paper [9] and to [11] for an extensive bibliography comprising relevant examples of applications. For a variational approach of the admittivity equation see [12]. We point out that the admittivity equation also

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appears in the study of a model of electrical conduction in biological tissues as the asymptotic limit of an elliptic equation with memory subject to periodic Dirichlet boundary conditions (see [7] and [8]).

Relevant medical applications of EIT are for example breast cancer detection, (see for example [11]) and screening of organs in transplantation surgery ([17]). In these particular situations one can assume γ to have the form

$$\gamma = \gamma_0\,\chi_{\Omega\setminus D} + \gamma_1\,\chi_D,$$

where $D \subset \Omega$ is a measurable subset of Ω and $\gamma_0 \neq \gamma_1$. Here D represents the cancerous tissue or the degraded tissue which has a different admittivity than the surrounding healthy tissue represented by $\Omega\setminus D$. In particular in organ screening D represents a region occupied by the degraded tissue imbedded in the healthy tissue and an important test to decide the quality of the organ is to give an estimate of the size of D in terms of boundary observations ([17]).

We describe the mathematical problem: let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smooth, bounded domain and let $D \subset \Omega$ be a measurable subset of Ω. We denote by γ_0 and γ_1 the admittivities of $\Omega\setminus D$ and D, respectively, with

$$\gamma_0 = \sigma_0 + i\,\epsilon_0 \quad \text{and} \quad \gamma_1 = \sigma_1 + i\,\epsilon_1,$$

(for simplicity we set $\omega = 1$), we assume that

$$\sigma_0 \geq c_0 > 0, \quad \sigma_1 \geq c_0 > 0,$$

this last condition corresponding to the dissipation of energy, and we let

$$\gamma = \gamma_0\,\chi_{\Omega\setminus D} + \gamma_1\,\chi_D.$$

Let $h \in H^{-1/2}(\partial\Omega)$ be a complex-valued boundary current flux and consider the so-called *background* potential $u_0 \in H^1(\Omega)$ generated by the flux h, which is the solution of

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u_0) = 0 & \text{in } \Omega, \\ \gamma_0 \dfrac{\partial u_0}{\partial \nu} = h & \text{on } \partial\Omega, \end{cases}$$

and let $u_1 \in H^1(\Omega)$ be the *perturbed* potential generated by the flux h in the presence of the inclusion D, which is the solution to

$$\begin{cases} \operatorname{div}(\gamma \nabla u_1) = 0 & \text{in } \Omega, \\ \gamma \dfrac{\partial u_1}{\partial \nu} = h & \text{on } \partial\Omega, \end{cases}$$

where the systems defining u_0 and u_1 are subject to some combon normalization condition.

Consider now

$$W_1 = \int_{\partial\Omega} h\,\overline{u}_1,$$

which represents the power required to maintain the current h in the presence of the inclusion D and analogously define

$$W_0 = \int_{\partial\Omega} h\,\overline{u}_0,$$

the power required to maintain the current h in the unperturbed medium. Let

$$\delta W = W_1 - W_0$$

be the so called *power gap*.

We will show that, if the admittivities γ_0 and γ_1 are constant or if γ_0 and γ_1 are variable scalar admittivities with γ_0 satisfying $\Im\gamma_0 \equiv 0$ and some extra conditions, then the measure $|D|$ of D can be estimated in terms of $|\delta W|$. For, we follow the approach introduced in [5] and [6] where the authors derived estimates of $|D|$ in terms of the power gap for the real conductivity equation.

A different approach to deriving size estimates for real conductivity inclusions when D comprises several connected components each of small size has been introduced in [10]. There the authors use multiple boundary measurements of a particular form to derive optimal asymptotic estimates of D. Recently Kang et al. (see [18]) obtained sharp bounds of the size of two dimensional conductivity inclusions from a pair of boundary measurements using classical variational principles.

We want to point out that in the the screening of organs it seems to be crucial to consider complex admittivities since electrical permittivity plays an important role in discriminating between degraded and normal tissue ([17]).

To derive our main results, as mentioned above, we follow the approach of [5] and [6] making use of the following basic tools:

- Energy bounds.
- Quantitative estimates of unique continuation and A_p weights ([13]).

More precisely, the first step is to find energy bounds, i.e., lower and upper bounds for $\int_D |\nabla u_0|^2$ in terms of $|\delta W|$, and the second is to find lower and upper bounds for $\int_D |\nabla u_0|^2$ in terms of $|D|$ by using regularity and quantitative estimates of unique continuation of solutions to elliptic equations. Unfortunately, differently from the conductivity case, the first step in the complex case seems not to work for arbitrary admittivities but only for constant ones or for certain variable scalar admittivities (see assumption **(H3)** in Section 2).

On the other hand we would like to emphasize that, in [5] and [6], the authors make the following assumption

$$d(D, \partial\Omega) \geq d_0 > 0.$$

Clearly this hypothesis is rather restrictive in the medical applications we have in mind since regions of the degraded tissue might extend to the surface of the organ. In this paper we remove this assumption and prove size estimates also for an inclusion having part of its boundary in common with $\partial\Omega$. This is accomplished by deriving fine quantitative estimates of unique continuation (Lemma 4.4), using reflection principles and suitable changes of variables.

The paper is divided as follows. In Section 2 we state our main assumptions and our main results. In Section 3 we derive energy bounds of the form

$$K_1\,|\delta W| \le \int_D |\nabla u_0|^2 \le K_2\,|\delta W|.$$

In Section 4 we list some useful tools concerning quantitative estimates of unique continuation. Section 5 is devoted to the proof of our main results. In particular we derive lower and upper bounds for the measure of the inclusion in terms of the energy of the background potential on D. Finally, in the appendix (Section 6), we give, for the reader's convenience, the proof of the *doubling Inequality* stated in Section 4.

2. Main results

2.1. Notation and main assumptions

For every $x \in \mathbb{R}^n$ we set $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ for $n \ge 2$.

Let $x_0 \in \mathbb{R}^n$ and $r > 0$. We denote by $B_r(x_0)$ and $B'_r(x'_0)$ the open ball in $\mathbb{R}^n$ centered at x_0 of radius r and the open ball in $\mathbb{R}^{n-1}$ centered at x' of radius r, respectively. We denote by $Q_l(x_0) = \{x \in \mathbb{R}^n : |x_j - x_{0j}| \le l, j = 1,\dots,n\}$ the cube with center x_0 and side length $2l$.

Definition 2.1 ($C^{k,1}$ regularity). Let Ω be a bounded domain in $\mathbb{R}^n$. Given k, with $k = 0, 1$, we say that $\partial\Omega$ or Ω is of *class* $C^{k,1}$ with constants r_0 and M_0, if, for any $P \in \partial\Omega$, there exists a rigid transformation of coordinates under which $P = 0$ and

$$\Omega \cap \left\{B'_{r_0}(0) \times (-M_0 r_0, M_0 r_0)\right\} = \left\{x \in B'_{r_0}(0) \times (-M_0 r_0, M_0 r_0) : \, x_n > \psi(x')\right\},$$

where ψ is a $C^{k,1}$ function on $B'_{r_0}(0)$ such that

$$\psi(0) = 0, \quad |\nabla\psi(0)| = 0 \;\text{ when } k = 1, \quad \text{and} \quad \|\psi\|_{C^{k,1}(B'_{r_0})} \le M_0 r_0.$$

For $z, w \in \mathbb{C}^n$ we write by $z \cdot w = \sum_{j=1}^n z_j w_j$.

Remark 2.2. Our convention is to normalize all norms so that that their terms are dimensionally homogeneous with respect to their argument and they coincide with the standard definitions when the dimension parameter equals one. For instance, the norm appearing above is meant as follows when $k = 1$:

$$\|\psi\|_{C^{1,1}(B'_{r_0})} = \|\psi\|_{L^\infty(B'_{r_0})} + r_0\,\|\nabla\psi\|_{L^\infty(B'_{r_0})} + r_0^2\,\left|\nabla\psi\right|_{1,B'_{r_0}},$$

where

$$|\nabla\psi|_{1,B'_{r_0}} = \sup_{\substack{x,y\in B'_{r_0}\\ x\neq y}} \frac{|\nabla\psi(x) - \nabla\psi(y)|}{|x-y|}.$$

Similarly, given a function $u : \Omega \mapsto \mathbb{C}$,

$$\|u\|_{L^2(\Omega)} = r_0^{-1}\Big(\int_\Omega |u|^2\Big)^{1/2}, \quad \|u\|_{H^1(\Omega)} = r_0^{-1}\Big(\int_\Omega |u|^2 + r_0^2\int_\Omega |\nabla u|^2\Big)^{1/2},$$

and so on for boundary and trace norms such as $\|\cdot\|_{H^{1/2}(\partial\Omega)}$ or $\|\cdot\|_{H^{-1/2}(\partial\Omega)}$.

We denote by Ω_r, $r > 0$, the set

$$\Omega_r = \big\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > r\big\}.$$

Let now state our main assumptions.

(H1) *Assumptions on* Ω.

Let M_0, M_1 and r_0 be positive numbers such that $M_0 \geq 1$. We assume that

1. Ω is a bounded domain in $\mathbb{R}^n$ with connected boundary;
2. $\partial\Omega$ has $C^{0,1}$ regularity with constants r_0 and M_0;
3. $|\Omega| \leq M_1\, r_0^n$.

(H2) *Assumptions on* D.

D is a Lebesgue measurable subset of $\overline{\Omega}$ and

(H2a) there exists a positive constant d_0 such that $\mathrm{dist}(D, \partial\Omega) \geq d_0$,
or

(H2b) there exist $r_1 \in (0, r_0]$ and $P \in \partial\Omega$ such that $D \subset \overline{\Omega}\backslash B_{r_1}(P)$.

(H3) *Assumptions on the coefficients.*

Let $c_0 \in (0,1], \mu_0$, and L be positive numbers. We assume the reference medium and the inclusion have admittivities $\gamma_0 = \sigma_0 + i\epsilon_0$ and $\gamma_1 = \sigma_1 + i\epsilon_1$ satisfying

$$\sigma_j \geq c_0, \ |\gamma_j| \leq c_0^{-1} \ \text{ in } \Omega, \ \text{for } j = 0, 1,$$

and, moreover we assume that:

(H3i) γ_0 and γ_1 are constants, and we set $\mu_0 = |\gamma_0 - \gamma_1| > 0$,
or

(H3ii) $\epsilon_0(x) \equiv 0$ in Ω, and $|\sigma_0(x) - \sigma_0(y)| \leq \frac{L}{r_0}|x - y|$ for $x, y \in \Omega$, and

$$|\epsilon_1(x)| \geq \mu_0 \ \text{ or } \ \sigma_1(x) - \sigma_0(x) \geq \mu_0 \ \text{ in } \ \Omega.$$

(H4) *Assumptions on the boundary data.*

(H4a) Let $h \in H^{-1/2}(\partial\Omega)$ be a complex-valued nontrivial current density on $\partial\Omega$ satisfying

$$\int_{\partial\Omega} h = 0.$$

or

(H4b) Let $h \in H^{-1/2}(\partial\Omega)$ be a complex-valued nontrivial current density on $\partial\Omega$ satisfying

$$\int_{\partial\Omega} h = 0,$$

and such that

$$\operatorname{supp} h \subset \Gamma_0 := \partial\Omega \cap \overline{B}_{r_1/2}(P),$$

for the same r_1 and P as in assumption **(H2b)**.

We denote by $F(h)$ the frequency of h, that is

$$F(h) = \frac{\|h\|_{H^{-1/2}(\partial\Omega)}}{\|h\|_{H^{-1}(\partial\Omega)}}. \tag{2.1}$$

Let

$$\gamma = \gamma_0\, \chi_{\Omega\setminus D} + \gamma_1\, \chi_D$$

and consider the unique solution $u_1 \in H^1(\Omega)$ of the problem

$$\begin{cases} \operatorname{div}(\gamma\nabla u_1) = 0 & \text{in } \Omega, \\ \gamma\dfrac{\partial u_1}{\partial\nu} = h & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_1 = 0. \end{cases} \tag{2.2}$$

Analogously we define the background potential $u_0 \in H^1(\Omega)$ generated by the same current flux h, to be the unique solution to the problem

$$\begin{cases} \operatorname{div}(\gamma_0\nabla u_0) = 0 & \text{in } \Omega, \\ \gamma_0\dfrac{\partial u_0}{\partial\nu} = h & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_0 = 0. \end{cases} \tag{2.3}$$

We shall denote by W_1 and W_0 the power necessary to maintain the current h when D is present or absent, respectively, so that

$$W_1 = \int_{\partial\Omega} h\,\overline{u}_1 = \int_\Omega \gamma\,\nabla u_1\,\nabla\overline{u}_1,$$

and

$$W_0 = \int_{\partial\Omega} h\,\overline{u}_0 = \int_\Omega \gamma_0\,\nabla u_0\,\nabla\overline{u}_0.$$

Let $\delta W = W_1 - W_0$ be the *power gap*.

2.2. The main theorems

We first state our main result in the case of an inclusion D strictly contained in Ω.

Theorem 2.3. *Let Ω satisfy* **(H1)** *and let D be a measurable subset of Ω satisfying* **(H2a)**. *Let γ_0 and γ_1 satisfy* **(H3)** *and let h satisfy* **(H4a)**. *Then,*

$$C_1\Big|\frac{\delta W}{W_0}\Big| \leq \frac{|D|}{|\Omega|} \leq C_2\Big|\frac{\delta W}{W_0}\Big|^{1/p}$$

where C_1 depends on the parameters c_0, μ_0, M_0, M_1, d_0/r_0 and L, and the numbers $p > 1$ and C_2 depend on the same parameters and, in addition, on $F(h)$.

We now state our main result in the case of of an inclusion that might have part of its boundary in common with $\partial\Omega$.

Theorem 2.4. *Let Ω satisfy* **(H1)** *with $\partial\Omega \in C^{1,1}$ with constants r_0 and M_0 and let D be a measurable subset of Ω satisfying* **(H2b)**. *Let γ_0 and γ_1 satisfy* **(H3)** *and let h satisfy* **(H4b)**. *Then,*

$$C_1\Big|\frac{\delta W}{W_0}\Big| \leq \frac{|D|}{|\Omega|} \leq C_2\Big|\frac{\delta W}{W_0}\Big|^{1/p},$$

where C_1 depends on the parameters c_0, μ_0, M_0, M_1, r_1/r_0 and L,and the numbers $p > 1$ and C_2 depend on the same parameters and, in addition, on $F(h)$.

The proofs of Theorem 2.3 and 2.4 will be given in Section 5.

3. Energy bounds

3.1. Energy identities

In this section, following an idea first introduced in [19], we use energy identities in order to derive suitable energy bounds.

Let $\tilde{\gamma}$ be a complex admittivity and define the sesquilinear form

$$a_{\tilde{\gamma}}(u, v) = \int_\Omega \tilde{\gamma}\,\nabla u \cdot \nabla\overline{v}.$$

If $u_{\tilde{\gamma}}$ is a solution to

$$\begin{cases} \operatorname{div}(\tilde{\gamma}\nabla u_{\tilde{\gamma}}) = 0 & \text{in } \Omega, \\ \tilde{\gamma}\dfrac{\partial u_{\tilde{\gamma}}}{\partial\nu} = h & \text{on } \partial\Omega, \end{cases}$$

then

$$a_{\tilde{\gamma}}(u_{\tilde{\gamma}}, v) = \int_{\partial\Omega} h\overline{v}, \quad \forall v \in H^1(\Omega). \tag{3.1}$$

We observe that in general $a_{\tilde{\gamma}}$ is not complex symmetric:

$$a_{\tilde{\gamma}}(u, v) - a_{\tilde{\gamma}}(v, u) = \int_\Omega \tilde{\gamma}\left(\nabla u \cdot \nabla\overline{v} - \nabla v \cdot \nabla\overline{u}\right) = 2i\int_\Omega \tilde{\gamma}\,\Im(\nabla u \cdot \nabla\overline{v}).$$

Lemma 3.1. *Let γ_0 and γ_1 be in $L^\infty(\Omega)$, let $\gamma = \gamma_0\,\chi_{\Omega\setminus D} + \gamma_1\,\chi_D$, and let u_1 and u_0 be the solutions of* (2.2) *and* (2.3), *respectively. The following identities hold:*

$$\textbf{(id1)}\qquad \int_\Omega \gamma\,|\nabla(u_1-u_0)|^2 - \int_D (\gamma_1-\gamma_0)|\nabla u_0|^2 = \delta W + 2i\int_\Omega \gamma\Im(\nabla u_1\cdot\nabla\overline{u}_0),$$

$$\textbf{(id2)}\qquad \int_\Omega \gamma_0\,|\nabla(u_1-u_0)|^2 + \int_D (\gamma_1-\gamma_0)|\nabla u_1|^2 = -\delta W - 2i\int_\Omega \gamma_0\Im(\nabla u_1\cdot\nabla\overline{u}_0),$$

$$\textbf{(id3)}\qquad \int_D (\gamma_0-\gamma_1)\nabla u_1\cdot\nabla\overline{u}_0 = \delta W + 2i\int_\Omega \gamma_0\Im(\nabla u_1\cdot\nabla\overline{u}_0),$$

$$\textbf{(id4)}\qquad \int_D (\gamma_1-\gamma_0)\nabla u_0\cdot\nabla\overline{u}_1 = -\delta W - 2i\int_\Omega \gamma\Im(\nabla u_1\cdot\nabla\overline{u}_0).$$

Proof. We write $a_0(u,v) := a_{\gamma_0}(u,v)$ and $a_1(u,v) := a_\gamma(u,v)$ From (3.1) we have

$$a_0(u_0,v) = a_1(u_1,v) = \int_{\partial\Omega} h\overline{v}, \quad \forall v\in H^1(\Omega).$$

We compute

$$\begin{aligned} J_1 &:= a_1(u_1-u_0,u_1-u_0) - \big[a_1(u_0,u_0)-a_0(u_0,u_0)\big] \\ &= \int_{\partial\Omega} h\overline{u}_1 - \int_{\partial\Omega} h\overline{u}_0 + 2i\int_\Omega \gamma\,\Im(\nabla u_1\cdot\nabla\overline{u}_0). \end{aligned}\tag{3.2}$$

On the other hand,

$$\begin{aligned} J_1 &= \int_\Omega \gamma\,|\nabla(u_1-u_0)|^2 - \int_\Omega (\gamma-\gamma_0)|\nabla u_0|^2 \\ &= \int_\Omega \gamma\,|\nabla(u_1-u_0)|^2 - \int_D (\gamma_1-\gamma_0)|\nabla u_0|^2, \end{aligned}\tag{3.3}$$

and so, by (3.2) and (3.3) and the definition of δW, the identity (**id1**) follows.

Analogously we can compute

$$\begin{aligned} J_2 &:= a_0(u_0-u_1,u_0-u_1) - [a_0(u_1,u_1)-a_1(u_1,u_1)] \\ &= -\int_{\partial\Omega} h(\overline{u}_1-\overline{u}_0) - 2i\int_\Omega \gamma_0\,\Im(\nabla u_1\cdot\nabla\overline{u}_0). \end{aligned}$$

On the other hand,

$$J_2 = \int_\Omega \gamma_0\,|\nabla(u_1-u_0)|^2 + \int_\Omega (\gamma_1-\gamma_0)\,|\nabla u_1|^2$$

and, hence, (**id2**) follows.

Finally let us compute

$$a_0(u_1,u_0) - a_1(u_1,u_0) = \int_D (\gamma_0-\gamma_1)\nabla u_1\cdot\nabla\overline{u}_0,$$

and, observe that

$$\begin{aligned}a_0(u_1,u_0)-a_1(u_1,u_0)&=a_0(u_1,u_0)-a_1(u_1,u_0)+a_0(u_0,u_1)-a_0(u_0,u_1)\\&=\int_{\partial\Omega}h(\overline{u}_1-\overline{u}_0)+2i\int_\Omega\gamma_0\,\Im(\nabla u_1\cdot\nabla\overline{u}_0),\end{aligned}$$

so that (**id3**) follows.

By symmetry we have also shown (**id4**) □

Remark 3.2. Note that by combining (**id1**) and (**id4**), we get as an easy consequence of the definition of u_0 and u_1, that

$$\int_\Omega\gamma\,|\nabla(u_1-u_0)|^2=\int_D(\gamma_0-\gamma_1)\nabla(\overline{u}_1-\overline{u}_0)\nabla u_0. \tag{3.4}$$

3.2. The constant case

Proposition 3.3. *Assume γ_0 and γ_1 satisfy* (**H3i**) *and let u_0 and u_1 solve* (2.3) *and* (2.2)*. Then*

$$\frac{c_0}{(c_0+|\gamma_1-\gamma_0|)|\gamma_1-\gamma_0|}\,|\delta W|\le\int_D|\nabla u_0|^2\le\Big(\frac{1}{c_0}+\frac{2}{|\gamma_1-\gamma_0|}\Big)|\delta W|.$$

Proof. Since γ_0 is constant and not zero we can write

$$\begin{aligned}&\int_\Omega\Im(\nabla u_1\cdot\nabla\overline{u}_0)=-\int_\Omega\Im\big(\nabla u_0\cdot\nabla\overline{u}_1-\nabla u_0\cdot\nabla\overline{u}_0\big)\\&=-\int_\Omega\Im\Big(\gamma_0\big(\nabla u_0\cdot\nabla\overline{u}_1-\nabla u_0\cdot\nabla\overline{u}_0\big)\frac{1}{\gamma_0}\Big)\\&=-\Im\Big(\frac{1}{\gamma_0}\int_\Omega\gamma_0\big(\nabla u_0\cdot\nabla\overline{u}_1-\nabla u_0\cdot\nabla\overline{u}_0\big)\Big)=-\Im\Big(\frac{1}{\gamma_0}\int_{\partial\Omega}h(\overline{u}_1-\overline{u}_0)\Big)=-\Im\Big(\frac{\delta W}{\gamma_0}\Big),\end{aligned}$$

and, hence,

$$\int_\Omega\gamma_0\,\Im(\nabla u_1\cdot\nabla\overline{u}_0)=-\gamma_0\,\Im\Big(\frac{\delta W}{\gamma_0}\Big).$$

Then, if we set

$$\delta V=\delta W-2i\gamma_0\,\Im\Big(\frac{\delta W}{\gamma_0}\Big)=\delta W+2i\int_\Omega\gamma_0\,\Im(\nabla u_1\cdot\nabla\overline{u}_0) \tag{3.5}$$

we can write the identities of Lemma 3.1 as

$$\int_\Omega\gamma\,|\nabla(u_1-u_0)|^2-\int_D(\gamma_1-\gamma_0)|\nabla u_0|^2=2i\int_D(\gamma_1-\gamma_0)\Im\,(\nabla u_1\cdot\nabla\overline{u}_0)+\delta V, \tag{id1c}$$

$$\int_\Omega\gamma_0\,|\nabla(u_1-u_0)|^2+\int_D(\gamma_1-\gamma_0)|\nabla u_1|^2=-\delta V, \tag{id2c}$$

$$\int_D(\gamma_0-\gamma_1)\nabla u_1\cdot\nabla\overline{u}_0=\delta V, \tag{id3c}$$

$$\int_D(\gamma_1-\gamma_0)\nabla u_0\cdot\nabla\overline{u}_1=-2i\int_D(\gamma_1-\gamma_0)\Im\,(\nabla u_1\cdot\nabla\overline{u}_0)-\delta V. \tag{id4c}$$

We write

$$\begin{aligned}\int_D |\nabla u_0|^2 &= \int_D |\nabla(u_0-u_1)|^2 - \int_D |\nabla u_1|^2 + 2\int_D \Re(\nabla u_1\cdot\nabla\overline{u}_0)\\ &\le \int_\Omega |\nabla(u_0-u_1)|^2 - \int_D |\nabla u_1|^2 + 2\int_D \Re(\nabla u_1\cdot\nabla\overline{u}_0).\end{aligned} \tag{3.6}$$

By taking the real part of (**id2c**) we get

$$\int_\Omega \sigma_0|\nabla(u_0-u_1)|^2 + (\sigma_1-\sigma_0)\int_D |\nabla u_1|^2 = -\Re(\delta V).$$

By dividing by the positive constant σ_0 and using the fact that both σ_0 and σ_1 are positive we have

$$\int_\Omega |\nabla(u_0-u_1)|^2 - \int_D |\nabla u_1|^2 \le -\frac{\Re(\delta V)}{\sigma_0}. \tag{3.7}$$

Now we divide (**id3c**) by the constant $\gamma_0-\gamma_1\neq 0$ and take the real part. We get

$$\int_D \Re(\nabla u_1\cdot\nabla\overline{u}_0) = \Re\Big(\frac{\delta V}{\gamma_0-\gamma_1}\Big),$$

which, together with (3.7) and (3.6), gives

$$\int_D |\nabla u_0|^2 \le -\frac{\Re(\delta V)}{\sigma_0} + 2\Re\Big(\frac{\delta V}{\gamma_0-\gamma_1}\Big).$$

This leads to the upper bound

$$\int_D |\nabla u_0|^2 \le |\delta V|\Big(\frac{1}{c_0} + \frac{2}{|\gamma_0-\gamma_1|}\Big).$$

To prove the lower bound observe that, by (3.4) and since $\Re\gamma \ge c_0$, we have

$$\Big(\int_\Omega |\nabla(u_0-u_1)|^2\Big)^{1/2} \le \frac{|\gamma_0-\gamma_1|}{c_0}\Big(\int_D |\nabla u_0|^2\Big)^{1/2}. \tag{3.8}$$

Hence, using the identity (**id3c**), we have

$$\begin{aligned}|\delta V| &= \Big|\int_D (\gamma_0-\gamma_1)\nabla u_1\cdot\nabla\overline{u}_0\Big| = \Big|(\gamma_0-\gamma_1)\Big(\int_D \nabla(u_1-u_0)\cdot\nabla\bar{u}_0 + \int_D |\nabla u_0|^2\Big)\Big|\\ &\le |\gamma_0-\gamma_1|\Big(\Big(\int_D |\nabla(u_1-u_0)|^2\Big)^{1/2}\Big(\int_D |\nabla u_0|\Big)^{1/2} + \int_D |\nabla u_0|^2\Big)\\ &\le |\gamma_0-\gamma_1|\Big(\frac{|\gamma_0-\gamma_1|}{c_0}\int_D |\nabla u_0|^2 + \int_D |\nabla u_0|^2\Big),\end{aligned}$$

from which the lower bound

$$\int_D |\nabla u_0|^2 \ge \frac{1}{|\gamma_0-\gamma_1|\,(|\gamma_0-\gamma_1|/c_0+1)}\,|\delta V|$$

follows.

Now, by using (3.5), we can see that

$$\delta V = \frac{\gamma_0^2}{|\gamma_0|^2}\,\overline{\delta W}.$$

Hence, in particular,

$$|\delta V| = |\delta W|$$

and the claim follows. □

3.3. The variable case

Proposition 3.4. *Assume γ_0 and γ_1 satisfy* (**H3ii**) *and let u_0 be the solution of* (2.3). *Then*

$$K_1\,|\delta W| \le \int_D |\nabla u_0|^2 \le K_2\,|\delta W|, \tag{3.9}$$

where

$$K_1 = \frac{c_0^3}{2(2+c_0^2)} \qquad \text{and} \qquad K_2 = 2\Big(\frac{1}{\mu_0\,c_0^2} + \frac{1}{\mu_0} + \frac{1}{c_0}\Big).$$

Proof. If (**H3ii**) holds, then $\gamma_0 = \sigma_0$ and $\epsilon_0 = 0$. In this case, we have

$$\begin{aligned}\int_\Omega \sigma_0 \Im(\nabla u_1\cdot\nabla\overline{u}_0) &= \int_\Omega \sigma_0\Im\left(\nabla u_1\cdot\nabla\overline{u}_0 - \nabla u_0\cdot\nabla\overline{u}_0\right)\\ &= \Im\Big(\int_\Omega \sigma_0\nabla\overline{u}_0\cdot\nabla u_1 - \sigma_0\nabla\overline{u}_0\cdot\nabla u_0\Big)\\ &= \Im\Big(\int_{\partial\Omega}\overline{h}\,u_1 - \int_{\partial\Omega}\overline{h}\,u_0\Big) = \Im(\overline{\delta W}) = -\Im(\delta W),\end{aligned}$$

and the energy identities become

(**id1***) $$\int_\Omega \gamma|\nabla(u_0-u_1)|^2 - \int_D(\gamma_1-\gamma_0)|\nabla u_0|^2 = \overline{\delta W} + 2i\int_D(\gamma_1-\gamma_0)\Im(\nabla u_1\cdot\nabla\overline{u}_0),$$

(**id2***) $$\int_\Omega \gamma_0|\nabla(u_0-u_1)|^2 + \int_D(\gamma_1-\gamma_0)|\nabla u_1|^2 = -\overline{\delta W},$$

(**id3***) $$\int_D(\gamma_0-\gamma_1)\nabla u_1\cdot\nabla\overline{u}_0 = \overline{\delta W}.$$

By (**id3***) we have that

$$\begin{aligned}|\overline{\delta W}| &= \Big|\int_D(\gamma_0-\gamma_1)\nabla u_1\nabla\overline{u}_0\Big|\\ &= \Big|\int_D(\gamma_0-\gamma_1)\nabla(u_1-u_0)\nabla\overline{u}_0 + \int_D(\gamma_0-\gamma_1)|\nabla u_0|^2\Big|\\ &\le \sup_D|\gamma_0-\gamma_1|\Big(\Big(\int_D|\nabla(u_0-u_1)|^2\Big)^{1/2}\Big(\int_D|\nabla u_0|^2\Big)^{1/2} + \int_D|\nabla u_0|^2\Big).\end{aligned} \tag{3.10}$$

By (3.4), we have

$$\left(\int_\Omega |\nabla(u_1-u_0)|^2\right)^{1/2} \le \frac{\sup_D|\gamma_0-\gamma_1|}{c_0}\left(\int_D|\nabla u_0|^2\right)^{1/2},$$

and by combining this with (3.10) we get

$$|\delta W| \le \sup_D|\gamma_0-\gamma_1|\left(\frac{\sup_D|\gamma_0-\gamma_1|}{c_0}+1\right)\int_D|\nabla u_0|^2 \le \frac{2}{c_0}\left(\frac{2}{c_0^2}+1\right)\int_D|\nabla u_0|^2,$$

and one side of the estimate (3.9) follows.

To derive the upper bound, let us first assume

$$\sigma_1-\sigma_0\ge\mu_0. \tag{3.11}$$

From the real part of (**id2***) we get

$$\int_\Omega \sigma_0|\nabla(u_1-u_0)|^2+\int_D(\sigma_1-\sigma_0)|\nabla u_1|^2=-\Re(\delta W), \tag{3.12}$$

hence, by assumption (3.11),

$$\begin{aligned}
&\int_\Omega|\nabla(u_1-u_0)|^2\le-\frac{\Re(\delta W)}{c_0},\\
&\int_D|\nabla u_1|^2\le-\frac{\Re(\delta W)}{\mu_0},\\
&\int_D|\nabla u_0|^2\le 2\int_D|\nabla(u_0-u_1)|^2+2\int_D|\nabla u_1|^2\le-2\left(\frac{1}{c_0}+\frac{1}{\mu_0}\right)\Re(\delta W).
\end{aligned}$$

On the other hand, if $|\epsilon_1|\ge\mu_0$, then, from the imaginary part of (**id2***), we get

$$\int_D|\epsilon_1||\nabla u_1|^2=|\Im(\delta W)|,$$

and, hence,

$$\int_D|\nabla u_1|^2\le\frac{|\Im(\delta W)|}{\mu_0}. \tag{3.13}$$

From the real part of (**id2***) (see (3.12)) and from (3.13) we get

$$\begin{aligned}
\int_\Omega|\nabla(u_1-u_0)|^2 &\le \int_\Omega\sigma_0c_0^{-1}|\nabla(u_1-u_0)|^2=c_0^{-1}\int_D(\sigma_0-\sigma_1)|\nabla u_1|^2-c_0^{-1}\Re(\delta W)\\
&\le c_0^{-1}\sup_D|\sigma_0-\sigma_1|\int_D|\nabla u_1|^2-c_0^{-1}\Re(\delta W)\\
&\le\frac{1}{c_0\,\mu_0}\sup_D|\sigma_0-\sigma_1|\,|\Im(\delta W)|-c_0^{-1}\Re(\delta W) \qquad (3.14)\\
&\le\frac{1}{c_0^2\,\mu_0}|\Im(\delta W)|-\frac{1}{c_0}\Re(\delta W). \qquad (3.15)
\end{aligned}$$

By (3.13) and (3.14) we get the upper bound

$$\int_D|\nabla u_0|^2\le 2\left(\frac{1}{\mu_0\,c_0^2}+\frac{1}{c_0}+\frac{1}{\mu_0}\right)|\delta W|. \qquad \square$$

3.4. A one-dimensional example

We are not able to derive energy bounds and hence also estimates on the size of D for arbitrary variable admittivities. Although the lack of symmetry in condition **(H3ii)** may seem unnatural, it is in some sense optimal, as the following example shows.

On the other hand we have seen in Proposition 3.4 that assumption (**H3ii**) leads to energy estimates. The lack of symmetry of condition (**H3ii**), that seems not natural, is in some sense optimal as the following example shows.

Let $\Omega = (-1, 1)$ and let $D = [a, b] \subset (-1, 1)$. Consider the background solution u_0 of

$$\begin{cases} (\gamma_0 u_0')' = 0 \ \text{ in } \ (-1,1), \\ (\gamma_0 u_0')(-1) = (\gamma_0 u_0')(1) = K \in \mathbb{C}, \quad u_0(-1) + u_0(1) = 0. \end{cases}$$

Integrating the equation, $(\gamma_0 u_0')' = 0$, and using the normalization conditions one gets that

$$u_0(x) = F_0(x) + M, \quad \text{for } x \in (-1, 1),$$

where

$$F_0(x) = \int \frac{K}{\gamma_0(x)}\, dx,$$

and $M = (F_0(1) + F_0(-1))/2$.

Considering the perturbed solution u_1 of

$$\begin{cases} (\gamma u_1')' = 0 \ \text{ in } \ (-1,1), \\ (\gamma u_1')(-1) = (\gamma u_1')(1) = K \in \mathbb{C}, \quad u_1(-1) + u_1(1) = 0, \end{cases}$$

one gets

$$u_1(x) = \begin{cases} F_0(x) + M & \text{if } x \in (-1, a), \\ F_1(x) + M + \dfrac{F_0(a) + F_0(b)}{2} - \dfrac{F_1(a) + F_1(b)}{2} & \text{if } x \in (a, b), \\ F_0(x) + M + \dfrac{F_1(b) - F_1(a)}{2} - \dfrac{F_0(b) - F_0(a)}{2} & \text{if } x \in (b, 1), \end{cases}$$

where

$$F_1(x) = \int \frac{K}{\gamma_1(x)}\, dx.$$

Hence

$$\begin{aligned} \delta W &= K\overline{\big(u_1(1) - u_0(1)) - (u_1(-1) - u_0(-1)\big)} = K\overline{\big(u_1(1) - u_0(1)\big)} \\ &= \frac{|K|^2}{2} \int_a^b \overline{\Big(\frac{1}{\gamma_1} - \frac{1}{\gamma_0}\Big)}\, dx, \\ \Re(\delta W) &= \frac{|K|^2}{2} \int_a^b \left(\frac{\sigma_0}{\sigma_0^2 + \epsilon_0^2} - \frac{\sigma_1}{\sigma_1^2 + \epsilon_1^2} \right) dx, \end{aligned}$$

and

$$\Im(\delta W) = \frac{|K|^2}{2}\int_a^b \left(-\frac{\epsilon_0}{\sigma_0^2+\epsilon_0^2} + \frac{\epsilon_1}{\sigma_1^2+\epsilon_1^2}\right) dx.$$

So, if one of the monotonicity conditions

$$\frac{\sigma_1}{\sigma_1^2+\epsilon_1^2} > (<) \frac{\sigma_0}{\sigma_0^2+\epsilon_0^2} \quad \text{in} \quad (-1,1)$$

or

$$\frac{\epsilon_1}{\sigma_1^2+\epsilon_1^2} > (<) \frac{\epsilon_0}{\sigma_0^2+\epsilon_0^2} \quad \text{in } (-1,1)$$

holds, then either $\Re(\delta W) \neq 0$ or $\Im(\delta W) \neq 0$ and δW recovers (a,b) uniquely.

In particular observe that if $\epsilon_0 = 0$ we find that $\Im(\delta W) \neq 0$ if ϵ_1 has constant sign in $(-1,1)$ and $\Re(\delta W) \neq 0$ if $\sigma_1 - \sigma_0 > 0$ in $(-1,1)$ which are exactly the condition (**H3ii**). If the above conditions fail uniqueness does not hold. Consider, for example, $\gamma_0 = (2+ix)^2$ for $x \in (-1,1)$ and $\gamma_1 = 17/4$. Then one easily sees that

$$\Re(\delta W) = |K|^2(b-a)\Big(\frac{4}{17} - \frac{4-ab}{(4+b^2)(4+a^2)}\Big),$$
$$\Im(\delta W) = |K|^2(b-a)\Big(-\frac{2(a+b)}{(4+b^2)(4+a^2)}\Big)$$

and clearly $\Re(\delta W) = \Im(\delta W) = 0$ for $a = 1/2$ and $b = -1/2$.

4. Main tools: quantitative estimates of unique continuation

We list now various forms of the quantitative estimates of unique continuation that we will need in the sequel. Throughout this section we will assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{0,1}$ with constants r_0 and M_0. and A is a symmetric $n \times n$ matrix with real entries defined in $\mathbb{R}^n$ satisfying:

(*Uniform ellipticity*) For a given λ_0, $0 < \lambda_0 \leq 1$,

$$\lambda_0|\xi|^2 \leq A(x)\xi\cdot\xi \leq {\lambda_0}^{-1}|\xi|^2, \quad \text{for every} \quad \xi \in \mathbb{R}^n,\ x \in \mathbb{R}^n. \tag{4.1}$$

(*Lipschitz regularity*) For a given $\lambda_1 > 0$,

$$|A(x) - A(y)| \leq \frac{\lambda_1}{r_0}\,|x-y|, \quad \text{for every} \quad x, y \in \mathbb{R}^n. \tag{4.2}$$

Theorem 4.1 (Three spheres inequality, [3]). *Let $u \in H^1(\Omega)$ be a solution to the equation*

$$\operatorname{div}(A(x)\nabla u(x)) = 0 \quad \text{in } \Omega.$$

For every $r_1, r_2, r_3, \bar{r}$, $0 < r_1 < r_2 < r_3 \leq \bar{r}$, and for every $x_0 \in \Omega_{\bar{r}}$,

$$\int_{B_{r_2}(x_0)} |\nabla u_0|^2 \leq C\Big(\int_{B_{r_1}(x_0)} |\nabla u|^2\Big)^{\theta}\Big(\int_{B_{r_3}(x_0)} |\nabla u|^2\Big)^{1-\theta},$$

where $C > 0$ and θ, $0 < \theta < 1$, only depend on λ_0, λ_1, r_1/r_3, and r_2/r_3.

Theorem 4.2 (Lipschitz propagation of smallness, [3]). *Let h satisfy* **(H4)** *and let $u \in H^1(\Omega)$ be the solution of the Neumann problem*

$$\begin{cases} \operatorname{div}(A(x)\nabla u(x)) = 0 & \text{in } \Omega, \\ A\nabla u \cdot \nu = h & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

For every $\rho > 0$ and for every $x \in \Omega_{2\rho}$, we have

$$\int_{B_\rho(x)} |\nabla u|^2 \geq C^{-1} \int_\Omega |\nabla u|^2,$$

where $C \geq 1$ only depends on λ_0, λ_1, M_0, M_1, $F(h)$, and ρ/r_0.

The *three spheres inequality* and the *Lipschitz propagation of smallness* in [3] are obtained for real valued functions u and h but with straightforward modifications they apply to complex valued functions.

Theorem 4.3 (Doubling inequality). *Let $u \in H^1(B_{r_0}(x_0))$ be the solution of*

$$\operatorname{div}(A(x)\nabla u(x)) = 0 \quad \text{in } B_{r_0}(x_0). \tag{4.4}$$

Then, there exist positive constants α and C, depending only on λ_0 and on λ_1, such that

$$\int_{B_{2r}(x_0)} |\nabla u|^2 \leq C \left(\frac{\int_{B_{r_0}(x_0)} |\nabla u|^2}{\int_{B_{r_0/2}(x_0)} |\nabla u|^2} \right)^\alpha \int_{B_r(x_0)} |\nabla u|^2, \tag{4.5}$$

for every r such that $0 < r \leq r_0/2$.

The *doubling inequality* was first derived by Garofalo and Lin in [15]. Later it was also derived by Kukavica in [20] using Rellich's identity. In the appendix, for the convenience of the reader, we will give the proof of the *doubling inequality* following the proof in [20], showing the modifications one must make in the case of complex-valued functions and estimating more carefully the constant occurring in the inequality.

Lemma 4.4. *Let Ω satisfy* **(H1)**, *let $\overline{r}$ and R be positive numbers such that $3\sqrt{n}R < \overline{r}$, and let $u \in H^1(\Omega)$ be a nontrivial solution of*

$$\operatorname{div}(A(x)\nabla u(x)) = 0 \quad \text{in } \Omega.$$

Assume that $\Omega_{\overline{r}} \neq \emptyset$. Then, for every $x_0 \in \Omega_{\overline{r}}$ and for every measurable set $E \subset Q_R(x_0)$, we have

$$\frac{|E|}{|Q_R(x_0)|} \leq \left(\frac{H \int_E |\nabla u|^2}{\int_{Q_R(x_0)} |\nabla u|^2} \right)^{1/p}, \tag{4.6}$$

where H and $p>1$ are given by

$$p = 1 + \frac{\log 4F_{\overline{r}}(u)}{\log(17/16)}, \quad \textit{and} \quad H = \big(27\, F_{\overline{r}}(u)\big)^{p(p-1)},$$

where

$$F_{\overline{r}}(u) = C\Big(\frac{\int_\Omega |\nabla u|^2}{\int_{\Omega_{\overline{r}/2}} |\nabla u|^2}\Big)^C \tag{4.7}$$

and C depends on λ_0, λ_1, M_0, M_1, and $\overline{r}/r_0$.

Sketch of the proof. Lemma 4.4 can be proved by adapting the proof of Lemma 2.4 in [21]. We describe in detail the necessary changes. The most important difference between Lemma 2.4 in [21] and our lemma is that $|\nabla u|^2$ appears in the bound (4.6) while in [21] $|u|^2$ is involved.

Observe that $|\nabla u|^2$ satisfies the following reverse Hölder inequality (RHI):

$$\begin{aligned}\Big(\frac{1}{|Q_R(x_0)|}\int_{Q_R(x_0)} \big(|\nabla u|^2\big)^{1+\delta}\Big)^{1/(1+\delta)} \\ \le \frac{C}{|Q_R(x_0)|}\Big(\frac{\int_{B_{\overline{r}}(x_0)} |\nabla u|^2}{\int_{B_{\overline{r}/2}(x_0)} |\nabla u|^2}\Big)^\alpha \int_{Q_R(x_0)} |\nabla u|^2,\end{aligned} \tag{4.8}$$

for any $x_0 \in \Omega_{\overline{r}}$ and R such that $0 < 2\sqrt{n}R \le \overline{r}$, where C and α depend only on λ_0 and λ_1 and $\delta > 0$ is arbitrary.

In fact, if we set

$$\tau = \frac{1}{|Q_R(x_0)|}\int_{Q_R(x_0)} u(x)dx,$$

from [16], the Poincaré inequality and (4.5) we get,

$$\begin{aligned}\sup_{Q_R(x_0)} |\nabla u|^2 &\le \frac{C}{R^2}\sup_{Q_{3/2R}(x_0)} |u-\tau|^2 \le \frac{C'}{R^{n+2}}\int_{Q_{2R}(x_0)} |u-\tau|^2 \\ &\le \frac{C''}{R^n}\int_{Q_{2R}(x_0)} |\nabla u|^2 \le \frac{C''}{R^n}\int_{B_{2\sqrt{n}R}(x_0)} |\nabla u|^2 \\ &\le \frac{C'''}{R^n}\Big(\frac{\int_{B_{\overline{r}}(x_0)} |\nabla u|^2}{\int_{B_{\overline{r}/2}(x_0)} |\nabla u|^2}\Big)^\alpha \int_{Q_R(x_0)} |\nabla u|^2,\end{aligned}$$

where C', C'' and C''' and α depend on λ_0 and λ_1 only. We derive (4.8) in a trivial manner.

Using iteratively the *three spheres inequality* we get the estimate (see [4])

$$\int_{\Omega_{\overline{r}/2}} |\nabla u|^2 \le C\Big(\int_{B_{\overline{r}/2}(x_0)} |\nabla u|^2\Big)^\theta \Big(\int_\Omega |\nabla u|^2\Big)^{1-\theta}, \tag{4.9}$$

where $0 < \theta < 1$ and θ and C depend on $\lambda_0, \lambda_1, M_0, M_1$ and $\overline{r}/r_0$. From (4.9) we have trivially

$$\frac{\int_{B_{\overline{r}}(x_0)} |\nabla u|^2}{\int_{B_{\overline{r}/2}(x_0)} |\nabla u|^2} \leq \frac{\int_\Omega |\nabla u|^2}{\int_{B_{\overline{r}/2}(x_0)} |\nabla u|^2} \leq \left(\frac{C\int_\Omega |\nabla u|^2}{\int_{\Omega_{\overline{r}/2}} |\nabla u|^2}\right)^{1/\theta}. \tag{4.10}$$

From (4.10) and (4.8) we get the following version of RHI:

$$\left(\frac{1}{|Q_R(x_0)|}\int_{|Q_R(x_0)} \left(|\nabla u|^2\right)^{1+\delta}\right)^{1/(1+\delta)} \leq \frac{F}{|Q_R(x_0)|}\int_{Q_R(x_0)} |\nabla u|^2, \tag{4.11}$$

for any $x_0 \in \Omega_{\overline{r}}$ and for any R such that $R \in (0, \frac{\overline{r}}{2\sqrt{n}}]$ and $\delta > 0$, where

$$F = \left(\frac{C\int_\Omega |\nabla u|^2}{\int_{\Omega_{\overline{r}/2}} |\nabla u|^2}\right)^{\alpha/\theta}, \tag{4.12}$$

with C, α and θ depending only on $\lambda_0, \lambda_1, M_0, M_1$, and $\overline{r}/r_0$. In order to prove (4.9) we used the Lipschitz regularity of $\partial\Omega$ in order to guarantee that Ω_ρ is a connected set for ρ sufficiently small. If (4.11) holds for $\overline{r}$ small then it clearly holds also for large $\overline{r}$. The most difficult part of the proof is to show that the lemma follows from (4.11) but this can be found in Theorem 2.11 in [14], while an explicit evaluation of the constants can be found in [21]. □

5. Proof of the main results

In this section we will use the quantitative unique continuation estimates stated in the previous section and regularity results for solutions of elliptic equations to get upper and lower bounds of the measure $|D|$ of the inclusion D, in terms of the energy related to the background potential u_0.

Throughout this section we will assume that A is a symmetric real $n \times n$ matrix defined in $\mathbb{R}^n$ satisfying (4.1) and (4.2)

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ satisfy* **(H1)** *with $\partial\Omega$ of class $C^{0,1}$ and let $D \subset \Omega$ satisfy* **(H2a)**. *Let h satisfy* **(H4a)** *and let $u \in H^1(\Omega)$ be a solution to the Neumann problem* (4.3) *such that*

$$\int_{\partial\Omega} u = 0. \tag{5.1}$$

Then

$$\left(\frac{|D|}{|\Omega|}\right)^p \leq C\left(\frac{\int_D |\nabla u|^2}{\int_\Omega |\nabla u|^2}\right), \tag{5.2}$$

where p, $p > 1$ and C depend only on $d_0/r_0, M_0, M_1, \lambda_0, \lambda_1$, and $F(h)$, where $F(h)$ is given by (2.1).

Proof. Let $\delta = d_0/(4\sqrt{n})$ and cover D with pairwise internally disjoint closed cubes, Q_j, $j = 1, \dots, N$, of side length δ. Assume that $Q_j \cap D \neq \emptyset$ for $j = 1, \dots, N$. We have

$$D \subset \cup_{j=1}^{N} Q_j \subset \Omega_{\frac{3}{4}d_0}. \tag{5.3}$$

The value of $p > 1$ will be chosen later. From (5.3) and Hölder inequality (in what follows $p' = (p-1)/p$) we get

$$\begin{aligned} |D| &= \sum_{j=1}^{N} |D \cap Q_j| = \sum_{j=1}^{N} \frac{|D \cap Q_j|}{|Q_j|} |Q_j| \\ &\leq \Big(\sum_{j=1}^{N} \Big(\frac{|D \cap Q_j|}{|Q_j|} \Big)^p \Big)^{1/p} \Big(\sum_{j=1}^{N} |Q_j|^{p'} \Big)^{1/p'} \leq |\Omega_{\frac{3}{4}d_0}| \Big(\sum_{j=1}^{N} \Big(\frac{|D \cap Q_j|}{|Q_j|} \Big)^p \Big)^{1/p}. \end{aligned}$$

Hence, for any $p > 1$ we have

$$\Big(\frac{|D|}{|\Omega|} \Big)^p \leq \sum_{j=1}^{N} \Big(\frac{|D \cap Q_j|}{|Q_j|} \Big)^p. \tag{5.4}$$

Now, in order to choose p and to bound the right-hand side of (5.4) we apply Lemma 4.4 with $\overline{r} = \frac{3}{4}d_0$ and we bound $F_{\overline{r}}(h)$ defined in (4.7) from above. We bound $\int_{\Omega_{\overline{r}/2}} |\nabla u|^2$ from below by observing that, for $\bar{x} \in \Omega_{\overline{r}}$, applying the *Lipschitz propagation of smallness* (LPS) with $\rho = \overline{r}/2$, we get

$$\int_{\Omega_{\overline{r}/2}} |\nabla u|^2 \geq \int_{B_{\overline{r}/2}(\bar{x})} |\nabla u|^2 \geq C_1^{-1} \int_{\Omega} |\nabla u|^2, \tag{5.5}$$

where $C_1 \geq 1$ depends on $d_0/r_0, M_0, M_1, \lambda_0, \lambda_1$, and $F(h)$, where $F(h)$ is given by (2.1). Hence, by (5.5), we obtain that

$$F_{\overline{r}}(h) \leq C_1.$$

Now let

$$p = 1 + \frac{\log 4C_1^2}{\log(17/16)}. \tag{5.6}$$

By Lemma 4.4 we have

$$\Big(\frac{|D \cap Q_j|}{|Q_j|} \Big)^p \leq (27\, C_1^2)^{p(p-1)} \frac{\int_{D \cap Q_j} |\nabla u|^2}{\int_{Q_j} |\nabla u|^2}, \quad j = 1, \dots, N. \tag{5.7}$$

We use the LPS property again to estimate the right-hand side of (5.7) from above. Denoting by x_j the center of the cube Q_j we have

$$\int_{Q_j} |\nabla u|^2 \geq \int_{B_{\delta/2}(x_j)} |\nabla u|^2 \geq C_2^{-1} \int_{\Omega} |\nabla u|^2, \tag{5.8}$$

where $C_2 \geq 1$ depends on $d_0/r_0, M_0, M_1, \lambda_0, \lambda_1$, and $F(h)$. By (5.8), (5.7), (5.6), and (5.4) we get the claim. □

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ satisfy* **(H1)** *with $\partial\Omega$ of class $C^{1,1}$, let $D \subset \Omega$ satisfy* **(H2b)***, let the function σ_0 be as in* **(H3)***, and let h satisfy* **(H4b)***. Let $u \in H^1(\Omega)$ be the solution to the Neumann problem*

$$\begin{cases} \operatorname{div}((\sigma_0(x)\nabla u(x)) = 0 & \text{in } \Omega, \\ \sigma_0 \nabla u \cdot \nu = h & \text{on } \partial\Omega, \end{cases}$$

satisfying the normalization condition (5.1)*. Then*

$$\Big(\frac{|D|}{|\Omega|}\Big)^p \leq C\Big(\frac{\int_D |\nabla u|^2}{\int_\Omega |\nabla u|^2}\Big),$$

where p, $p > 1$ and C depend only on $r_1/r_0, M_0, M_1, c_0, L,$ and $F(h)$, where $F(h)$ is given by (2.1)*.*

Proof. Define $\Gamma := \partial\Omega \cap \overline{B}_{r_1}(P)$. First we construct a suitable family of cylinders covering $\partial\Omega\backslash\Gamma$.

Let

$$r_2 = \min\Big\{\frac{r_1}{4\sqrt{n}}, \frac{r_0}{2\sqrt{n}M_0}\Big\}$$

and fix $r \in (0, r_2]$, to be chosen later. Let $\{Q_j\}_{j=1}^J$ a family of closed mutually internally disjoint cubes of side length $2r$ such that

$$(\partial\Omega\backslash\Gamma) \bigcap Q_j \neq \emptyset, \quad j = 1, \ldots, J, \quad \text{and} \quad \partial\Omega\backslash\Gamma \subset \bigcup_{j=1}^J Q_j.$$

Fix $j \in \{1, \ldots, J\}$ and let $x_j \in (\partial\Omega\backslash\Gamma) \cap Q_j$. Let ν_j be the exterior unit normal vector to $\partial\Omega$ at x_j on Let R_j the cylinder centered at x_j with axis parallel to ν_j and with base a ball of radius $2\sqrt{n}r$ and with height $2\sqrt{n}M_0 r$. Setting $\tilde{R}_j = 2(R_j - x_j) + x_j$ one sees easily that

$$\bigcup_{j=1}^J \tilde{R}_j \supset \Omega\backslash\Omega_{2\sqrt{n}r}$$

and hence

$$\operatorname{dist}\Big(\Omega\backslash \bigcup_{j=1}^J \tilde{R}_j, \partial\Omega\Big) \geq 2\sqrt{n}r. \tag{5.9}$$

Furthermore, since the interiors of the cubes Q_j, $j = 1, \ldots, J$, are pairwise disjoint and since, obviously,

$$\bigcup_{j=1}^J Q_j \subset \big\{x \in \mathbb{R}^n \,:\, \operatorname{dist}(x, \partial\Omega) < 2\sqrt{n}r\big\},$$

we obtain for J the estimate

$$J \leq (2r)^{-n}\Big|\bigcup_{j=1}^J Q_j\Big| \leq C\Big(\frac{r_0}{r}\Big)^{n-1}, \tag{5.10}$$

where C depends only on M_0 and M_1. Let

$$D' = D\bigcap\Big(\bigcup_{j=1}^{J}\tilde{R}_j\Big), \quad \text{and} \quad D'' = D\backslash D'.$$

From (5.9) we have

$$\mathrm{dist}(D'',\partial\Omega) \geq 2\sqrt{n}r.$$

From this last inequality and Proposition 5.1 we get

$$\Big(\frac{|D''|}{|\Omega|}\Big)^p \leq C_r\Big(\frac{\int_{D''}|\nabla u|^2}{\int_\Omega|\nabla u|^2}\Big), \tag{5.11}$$

where C_r depends only on $r/r_0, M_0, M_1, c_0, L$, and $F(h)$.

Let, for a fixed index $j\in\{1,\ldots,J\}$,

$$D_j := \tilde{R}_j\cap D' \quad \text{and} \quad \hat{R}_j := 2(\tilde{R}_j - x_j) + x_j.$$

It is easy to see that if

$$r \leq \min\Big\{\frac{r_1}{16\sqrt{n}\sqrt{1+M_0^2}}, \frac{r_2}{2}\Big\}$$

then

$$\mathrm{dist}(\hat{R}_j,\Gamma_0) \geq \frac{r_1}{4}, \tag{5.12}$$

where we recall that $\Gamma_0 = \Gamma\cap\overline{B_{r_1/2}(P)}$. Furthermore, up to a rigid transformation such that $x_j = 0$, we have

$$\hat{R}_j\cap\Omega = \big\{(x',x_n)\in\mathbb{R}^n : x_n > \psi(x'),\ |x'|\leq 8\sqrt{n}r,\ |x_n|\leq 8\sqrt{n}M_0r\big\},$$

where

$$\psi(0) = |\nabla\psi(0)| = 0$$

and

$$\|\psi\|_{L^\infty} + r_0\|\nabla\psi\|_{L^\infty} + r_0^2\|D^2\psi\|_{L^\infty} \leq M_0r_0.$$

Without loss of generality we may assume that $\sigma_0(0) = 1$. Following the arguments of [1] or [2] we can construct a function $\Psi\in C^{1,1}(\overline{B_{\rho_0}(0)},\mathbb{R}^n)$, where $\rho_0 = 16\sqrt{n}\sqrt{1+M_0^2}\,r$ such that

$$\Psi(x',\psi(x')) = (x',0), \quad \forall x'\in B'_{\rho_0}(0), \tag{5.13}$$

$$\Psi(\hat{R}_j\cap\Omega)\subset\big\{(x',x_n) : x_n > 0\big\}. \tag{5.14}$$

Moreover, there exist $C_1, C_2 \geq 1$ depending only on M_0 such that

$$C_1^{-1}|x-z| \leq |\Psi(x)-\Psi(z)| \leq C_1|x-z|, \quad \forall x,z\in B_{\rho_0}(0), \tag{5.15}$$

$$C_2^{-1} \leq |\mathrm{det}D\Psi(x)| \leq C_2, \quad \forall x\in B_{\rho_0}(0), \tag{5.16}$$

and, setting $A(y) = \{a_{ij}(y)\}_{ij=1}^n$, where

$$A(y) = |\det D\Psi^{-1}(x)|\,(D\Psi)\,(\Psi^{-1}(y))\,\sigma_0\,(\Psi^{-1}(y))\,(D\Psi)^T(\Psi^{-1}(y)), \tag{5.17}$$

$$v(y) = u(\Psi^{-1}(y)), \tag{5.18}$$

we have

$$A(y) = \mathrm{Id}, \tag{5.19}$$

$$a_{nk}(y', 0) = a_{kn}(y', 0) = 0, \quad k = 1, \dots, n, \tag{5.20}$$

$$C_3^{-1}\,|\xi|^2 \le A(y)\xi\cdot\xi \le C_3\,|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\ \forall y \in \Psi(\Omega \cap \hat{R}_j), \tag{5.21}$$

$$|A(y) - A(z)| \le \frac{C_4}{r}\,|y - z|, \quad \forall y, z \in \Psi(\Omega \cap \hat{R}_j), \tag{5.22}$$

where, in (5.19), Id denotes the identity matrix and $C_3, C_4 \ge 1$ depend only on M_0. Furthermore, recalling $\hat{R}_j \cap \Gamma_0 \ne \emptyset$ and (5.12) we have

$$\begin{cases} \mathrm{div}((A(y)\nabla_y v(y)) = 0 & \text{in } \Psi(\Omega \cap \hat{R}_j), \\ \dfrac{\partial v}{\partial y_n}(y', 0) = 0 & \text{on } \Psi(\partial\Omega \cap \hat{R}_j). \end{cases} \tag{5.23}$$

From the properties of the matrix A, in particular from (5.20), we have that the function $\tilde{v}$ defined by

$$\tilde{v}(y', y_n) := v(y', |y_n|) \tag{5.24}$$

solves an elliptic equation with Lipschitz coefficients in the principal part. More precisely, let $\tilde{A}(y) = \{\tilde{a}_{ij}(y)\}_{i,j=1}^n$ be the matrix with entries

$$\tilde{a}_{ij}\,(y', |y_n|) = a_{ij}\,(y', |y_n|), \quad \text{if} \quad i, j \in \{1, \dots, n-1\} \quad \text{or} \quad i = j = n,$$

$$\tilde{a}_{ij}\,(y', y_n) = \tilde{a}_{ij}\,(y', y_n) = \mathrm{sgn}(y_n)\,a_{nj}\,(y', |y_n|), \quad \text{if} \quad i, j \in \{1, \dots, n-1\} \quad \text{or} \quad i = j.$$

Then we have

$$\mathrm{div}\big((A(y)\nabla_y \tilde{v}(y)\big) = 0 \quad \text{in } \hat{\Lambda}_j,$$

where

$$\hat{\Lambda}_j = \big\{(y', y_n) \in \mathbb{R}^n : (y', |y_n|) \in \hat{\Lambda}_j^+\big\},$$

with

$$\hat{\Lambda}_j^+ = \Psi(\Omega \cap \hat{R}_j).$$

It is easy to see that the matrix $\tilde{A}$ satisfies uniform ellipticcity and Lipschitz continuity with the same constants as in (5.21) and (5.22).

In the sequel we will use the notation

$$\tilde{\Lambda}_j^+ := \Psi(\Omega \cap \tilde{R}_j),$$
$$\tilde{\Lambda}_j := \big\{(y', y_n) \in \mathbb{R}^n : (y', |y_n|) \in \tilde{\Lambda}_j^+\big\},$$
$$\tilde{D}_j := \Psi(D_j).$$

Since our aim is to bound $|D_j|$, we proceed initially as in the proof of Proposition 5.1.

First we note that from (5.15) we get

$$\operatorname{dist}(\tilde{D}_j, \partial\hat{\Lambda}_j) \geq \delta_0 := \frac{2\sqrt{n}r}{C_1},$$

where C_1 is the constant appearing in (5.15). Cover $\tilde{D}_j$ by pairwise internally disjoint closed cubes, $Q_{j,k}, k = 1, \dots, N_j$, of side length $\delta_1 := \delta_0/(4\sqrt{n})$. We have

$$\tilde{D}_j \subset \bigcup_{k=1}^{N_j} Q_{j,k} \subset \hat{\Lambda}_{\frac{3}{4}\delta_0}.$$

Since we are interested in applying Lemma 4.4 with $\Omega = \hat{\Lambda}_j$ and $\bar{r} = \delta_0/4$ we need to prove first the following claim

Claim 1. *There exists a constant C depending only on $c_0, L, M_0, M_1, r/r_0$, and $F(h)$ such that*

$$\tilde{F}_{j,\bar{r}}(\tilde{v}) := \frac{\int_{\hat{\Lambda}_j} |\nabla\tilde{v}|^2}{\int_{\hat{\Lambda}_{j,\bar{r}/2}} |\nabla\tilde{v}|^2} \leq C, \quad j = 1, \dots, J, \tag{5.25}$$

(*with C independent of j*).

Proof of the claim. Since for $\bar{r} = \delta_0/4$ we have that $\hat{\Lambda}_{j,\bar{r}/2} \supset \tilde{\Lambda}$, recalling that $\tilde{v}$ is the even reflection of $v = u \circ \Psi^{-1}$, by a change of variables we derive

$$\tilde{F}_{j,\bar{r}}(\tilde{v}) \leq C \frac{\int_{\tilde{R}_j \cap \Omega} |\nabla u|^2}{\int_{\tilde{R}_{j,\bar{r}/2} \cap \Omega} |\nabla u|^2}, \quad j = 1, \dots, J, \tag{5.26}$$

where C depends only on c_0, L, M_0, M_1, and r/r_0. Now, since

$$\tilde{R}_j \cap \Omega \supset B_{\sqrt{n}r}(x_j - 2\sqrt{n}r\nu) := B^{(j)},$$
$$\operatorname{dist}(B^{(j)}, \partial(\tilde{R}_j \cap \Omega)) \geq \sqrt{n}r,$$

estimating the right-hand side of (5.26) and applying the LPS property we get

$$\tilde{F}_{j,\bar{r}}(\tilde{v}) \leq C \frac{\int_{\Omega} |\nabla u|^2}{\int_{B^{(j)}} |\nabla u|^2} \leq C',$$

where C' depends only on $c_0, L, M_0, M_1, r/r_0$, and $F(h)$.

We choose $r = r_2$. Proceeding as in the proof of Proposition 5.1 and using (5.26) we obtain

$$|\tilde{D}_j| \leq |\hat{\Lambda}_j| \left(\frac{\int_{\tilde{D}_j} |\nabla\tilde{v}|^2}{\int_{\hat{\Lambda}_j} |\nabla\tilde{v}|^2} \right)^{1/p}, \quad j = 1, \dots, J, \tag{5.27}$$

where C and $p \in (1, \infty)$ depend on $c_0, L, M_0, M_1, r_1/r_0$, and $F(h)$.

From the definitions of $\tilde{v}$ and of $\hat{\Lambda}_j$, with some simple change of variables and using again the LPS property, we derive from (5.27)

$$|D_j| \le C\,|\Omega| \left(\frac{\int_{D_j} |\nabla u|^2}{\int_\Omega |\nabla u|^2} \right)^{1/p}, \quad j = 1, \dots, J, \tag{5.28}$$

where C and $p \in (1, +\infty)$ depend on $c_0, L, M_0, M_1, r_1/r_0$, and $F(h)$.

From (5.28) and from (5.10) we have

$$|D'| \le \sum_{j=1}^{J} |D_j| \le C\,|\Omega| \left(\frac{\int_{D'} |\nabla u|^2}{\int_\Omega |\nabla u|^2} \right)^{1/p}, \tag{5.29}$$

where C and $p \in (1, \infty)$ depend on $c_0, L, M_0, M_1, r_1/r_0$, and $F(h)$. From (5.29) and (5.11) the claim follows. □

Proposition 5.3. *Under the same hypotheses of Proposition 5.2 we have*

$$\frac{\int_D |\nabla u|^2}{\int_\Omega |\nabla u|^2} \le C\, r_1^{-n}\, |D|\,,$$

where C depends only on $r_1/r_0, M_0, M_1, c_0$ and L.

Proof. Trivially we have

$$\int_D |\nabla u|^2 \le |D|\, \|\nabla u\|^2_{L^\infty(D)} \le |D|\, \|\nabla u\|^2_{L^\infty(\Omega \setminus B_{r_1}(P))}. \tag{5.30}$$

Since $\sigma_0 \nabla u \cdot \nu = 0$ on $\partial\Omega \setminus B_{r_1/2}(P)$, from standard estimates for elliptic equations, [16], and from the Poincaré inequality, we have that, letting $\tau = \frac{1}{|\Omega|}\int_\Omega u$, there hold

$$\begin{aligned} \|\nabla u\|^2_{L^\infty(\Omega\setminus B_{r_1}(P))} &\le \frac{C_1}{r_1^2}\, \|u - \tau\|^2_{L^\infty(\Omega\setminus B_{\frac{3}{4}r_1}(P))} \le \frac{C_1\, C_2}{r_1^{n+2}}\, \|u-\tau\|^2_{L^2(\Omega\setminus B_{r_{1/2}}(P))} \\ &\le \frac{C_1\, C_2}{r_1^{n+2}}\, \|u - \tau\|^2_{L^2(\Omega)} \le \frac{C_1\, C_2\, C_3\, r_0^2}{r_1^{n+2}}\, \|\nabla u\|^2_{L^2(\Omega)}, \end{aligned} \tag{5.31}$$

where C_1 depends only on $r_1/r_0, M_0, M_1, c_0$ and L; C_2 depends on $r_1/r_0, M_0, M_1$ and c_0; and C_3 depends on M_0 and M_1. From (5.30) and (5.31) we get

$$\frac{\int_D |\nabla u|^2}{\int_\Omega |\nabla u|^2} \le C_4\, r_1^{-n}\, |D|,$$

where C_4 depends only on $r_1/r_0, M_0, M_1, c_0$ and L. □

We are now ready to prove our main results.

Proof of Theorem 2.3. By standard elliptic estimates, we have

$$\sup_D |\nabla u_0| \le C \sup_{\Omega_{d_0/2}} |u_0| \le C \|u_0\|_{L^2(\Omega)},$$

From the trivial estimate

$$\int_\Omega |\nabla u_0|^2 \leq c_0^{-1} \int_\Omega \nabla u_0 \cdot \nabla u_0 = c_0^{-1} W_0,$$

and from the Poincaré inequality, we have

$$\sup_D |\nabla u_0| \leq C\, W_0^{1/2}, \tag{5.32}$$

where C depends on c_0, L, d_0/r_0, and M_0. Hence from (5.32) we get for $|D|$ the lower bound

$$\int_D |\nabla u|^2 \leq |D|\, \|\nabla u\|^2_{L^\infty(D)} \leq C\,|D|\,W_0. \tag{5.33}$$

By Proposition 5.1 and (5.33) we obtain

$$\tilde{C}_1 \frac{\int_D |\nabla u_0|^2}{W_0} \leq \frac{|D|}{|\Omega|} \leq \tilde{C}_2 \Big(\frac{\int_D |\nabla u_0|^2}{W_0} \Big)^{1/p},$$

where $\tilde{C}_1$ depends only on d_0/r_0, M_0, M_1, c_0 and L and $\tilde{C}_2$ depends only on d_0/r_0, M_0, M_1, c_0, L and $F(h)$. Finally, applying Proposition 3.3 if γ_0, γ_1 are constant and satisfy **(H3i)** or applying Proposition 3.4 if γ_0 and γ_1 satisfy **(H3ii)**, we get

$$C_1 \Big| \frac{\delta W}{W_0} \Big| \leq \frac{|D|}{|\Omega|} \leq C_2 \Big| \frac{\delta W}{W_0} \Big|^{1/p},$$

where C_1 depends only on the a priori constants c_0, μ_0, M_0, M_1, d_0/r_0, L, and the number $p > 1$ and C_2 depends on the same parameters and $F(h)$. □

Proof of Theorem 2.4. By Propositions 5.2 and 5.3 applied to the background potential u_0 that solves (2.3) (in the constant case up to a rescaling by a constant) we get

$$C_1' \frac{\int_D |\nabla u_0|^2}{W_0} \leq \frac{|D|}{|\Omega|} \leq C_2' \Big(\frac{\int_D |\nabla u_0|^2}{W_0} \Big)^{1/p},$$

where C_1' depends only on r_1/r_0, M_0, M_1, c_0 and L and C_2' depends on r_1/r_0, M_0, M_1, c_0, L, and $F(h)$. Finally applying Proposition 3.3 if γ_0, γ_1 are constant and satisfy **(H3i)** or applying Proposition 3.4 if γ_0 and γ_1 satisfy **(H3ii)** we get

$$C_1 \Big| \frac{\delta W}{W_0} \Big| \leq \frac{|D|}{|\Omega|} \leq C_2 \Big| \frac{\delta W}{W_0} \Big|^{1/p},$$

where C_1 depends only on the a priori constants $c_0, \mu_0, M_0, M_1, r_1/r_0, L$, and the number $p > 1$ only and C_2 depends on the same parameters and $F(h)$. □

6. Appendix

Proof of Theorem 4.3. The *doubling inequality* proved in [20] and [4] can be extended with straightforward arguments to the case of complex valued solutions of

$$\mathrm{div}(A(x)\nabla u(x)) = 0 \quad \text{in } B_{r_0}(x_0).$$

We give an idea of the modifications that need to be done to the proof. We assume that

$$A(0) = \mathrm{Id} \tag{6.1}$$

and define, for $0 < r < R_0$,

$$H(r) = \int_{\partial B_r} \frac{A(x)x\cdot x}{|x|^2}|v(x)|^2, \quad I(r) = \int_{B_r} A(x)\nabla v\cdot\overline{\nabla v}, \quad N(r) = \frac{rI(r)}{H(r)}. \tag{6.2}$$

If, instead of Rellich's identity used in [20], we use the relation

$$2\Re\big[(\beta\cdot\nabla\bar v)\mathrm{div}(A\nabla v)\big] = \mathrm{div}\big[2\Re((\beta\cdot\nabla\bar v)A\nabla v) - \beta(A\nabla v\cdot\nabla\bar v)\big]$$
$$+ (\mathrm{div}\beta)A\nabla v\cdot\nabla\bar v - 2\Re\big[\partial_l\,\beta_j\,a_{lk}\,\partial_k v\,\partial_j\bar v\big] + \beta_j(\partial_j a_{lk})\,\partial_k v\partial_l\bar v,$$

with β sufficiently smooth vector field on $\mathbb{R}^n$, we get that there exist constants $C_1 > 1, C_2$, and c, with $C_1 > 1, C_2$, depending only on λ_0 and λ_1 and c an absolute constant, such that

$$\int_{B_r}|v|^2 \le \lambda_0^2 r\int_{\partial B_r}|v|^2, \qquad \text{for } r\le\frac{R_0}{C_1}, \tag{6.3}$$

$$\Big|H'(r) - \frac{n-1}{r}H(r) - 2I(r)\Big| \le \frac{c\lambda_1}{R_0}H(r), \tag{6.4}$$

$$N(r)e^{C_2\,r/R_0} \quad \text{increasing in } \ (0,R_0]. \tag{6.5}$$

From (6.4) we have

$$\frac{d}{dr}\Big(\log\frac{H(r)}{r^{n-1}}\Big) \le \frac{c\lambda_1}{R_0} + \frac{2N(r)}{r}, \tag{6.6}$$

$$\frac{2N(r)}{r} \le \frac{d}{dr}\Big(\log\frac{H(r)}{r^{n-1}}\Big) + \frac{c\lambda_1}{R_0}. \tag{6.7}$$

Let $R_1 := R_0/C_1$ and $\rho, R\in(0,R_1]$ be such that $3\rho\le R$. Integrating both sides of (6.6) in the interval $[\rho,3\rho]$ we get, using (6.5),

$$\log\frac{H(3\rho)}{3^{n-1}H(\rho)} \le \frac{2c\lambda_1\rho}{R_0} + \int_\rho^{3\rho}\frac{2N(r)}{r} \le \frac{2c\lambda_1\rho}{R_0} + \int_\rho^{3\rho}\frac{2N(r)}{r}\,e^{C_2 r/R_0}$$
$$\le \frac{2\,c\,\lambda_1\,\rho}{R_0} + 2N(3\rho)\,e^{3C_2\rho/R_0}\log 3 \le \frac{2c\lambda_1 R}{3R_0} + 2N(R)\,e^{3C_2R/R_0}\log 3.$$

Hence, for $\rho \in (0, R/3]$, and $R \in (0, R_1]$, one has

$$\frac{1}{R}\log\frac{H(3\rho)}{3^{n-1}H(\rho)} \leq \frac{2c\lambda_1}{3R_0} + 2e^{C_2 R/R_0}\frac{N(R)}{R}\log 3. \tag{6.8}$$

From (6.8) and (6.7) we get, for $\rho \in (0, R_1/3]$ and $R \in (0, R_1]$,

$$\frac{1}{R}\log\frac{H(3\rho)}{3^{n-1}H(\rho)} \leq \frac{C_3}{R_0} + e^{C_2}(\log 3)\frac{d}{dR}\Big(\log\frac{H(R)}{R^{n-1}}\Big), \tag{6.9}$$

where $C_3 = e^{C_2\log 3} + 2c\lambda_1/3$. This last inequality implies, in particular, that for any $\rho \in (0, R_1/9]$ and $R \in (R_1/2, 3R_1/4]$ one has (integrating both sides of (6.9) over $[R_1/3, R]$)

$$\begin{aligned}\log 6\log\frac{H(3\rho)}{3^{n-1}H(\rho)} &\leq \log\frac{R}{R_1/3}\int_{R_1/3}^{R}\frac{1}{t}\log\frac{H(3\rho)}{3^{n-1}H(\rho)}\,dt\\ &\leq C_3\,\lambda_1\frac{R-R}{R_0} + e^{C_2}(\log 3)\log\frac{H(R)}{(3R/R_1)^{n-1}H(R_1/3)}\\ &\leq C_3\,\lambda_1 + e^{C_2}(\log 3)\log\frac{H(R)}{(3/2)^{n-1}H(R_1/3)}.\end{aligned}$$

Hence, for $\rho \in (0, R_1/9]$ and$R \in (R_1/2, 3R_1/4]$, by the elementary properties of the logarithm, we have,

$$H(3\rho) \leq C_4\Big(\frac{H(R)}{H(R_1/3)}\Big)^{C_5}H(\rho), \tag{6.10}$$

where C_4 and C_5 depend only on λ_0 and λ_1. Integrating both sides of (6.10), we derive, for every $\rho \in (0, R_1/9]$ and $R \in (R_1/2, 3R_1/4]$,

$$\int_0^\rho H(3s)\,ds \leq C_4\Big(\frac{H(R)}{H(R_1/3)}\Big)^{C_5}\int_0^\rho H(s)\,ds.$$

From (6.2) we get

$$\int_0^\rho H(s)\,ds \leq \lambda_0^{-1}\int_{B_\rho}|v|^2 \quad\text{and}\quad \int_0^\rho H(3s)\,ds \geq \frac{\lambda_0}{3}\int_{B_{3\rho}}|v|^2.$$

From the last two inequalities and from (6.10) one has

$$\int_{B_{3\rho}}|v|^2 \leq 3\,\lambda_0^{-2}\,C_4\left(\frac{H(R)}{H(R_1/3)}\right)^{C_5}\int_{B_\rho}|v|^2, \tag{6.11}$$

for $\rho \in (0, R_1/9]$ and $R \in (R_1/2, 3R_1/4]$.

Now, (6.11) holds also if instead of v we insert $v - \tau_\rho$ where $\tau_\rho = \frac{1}{|B_\rho|}\int_{B_\rho} v$.

Denoting by $\tilde{H}(r)$ the function upon substituting $v-\tau_\rho$ for v in (6.2), we have, recalling local boundness of solutions to elliptic equations, [16],

$$\tilde{H}(R) \leq \lambda_0^{-2}\int_{\partial B_R}|v-\tau_\rho|^2 \leq 4\,\lambda_0^{-2}\,R^{n-1}\,\|v\|_{L^\infty(B_R)} \leq \frac{C}{R_1}\int_{B_{R_1}}|v|^2, \tag{6.12}$$

for every $R \in (R_1/2, 3R_1/4]$ where C depends only on λ_0. On the other hand, applying (6.3), we obtain

$$\tilde{H}(R_1/3) \geq \lambda_0 \int_{\partial B_{R_1/3}} |v-\tau_\rho|^2 \geq \int_{B_{R_1/3}} |v-\tau_\rho|^2 \geq \frac{R_1}{C} \int_{B_{R_1/6}} |\nabla v|^2, \tag{6.13}$$

where $C \geq 1$ depends only on λ_0. From (6.12), (6.13), and (6.11) we get

$$\int_{B_{3\rho}} |v-\tau_\rho|^2 \leq C_6 \left(\frac{\int_{B_{R_1}} |v|^2}{R_1^2 \int_{B_{R_1/6}} |v|^2} \right)^{C_5} \int_{B_\rho} |v-\tau_\rho|^2, \tag{6.14}$$

where $C_6 \geq 1$ depends on λ_0, λ_1. Using the Poincaré inequality and the Caccioppoli inequality to bound the right-hand side of (6.14) from above and the left-hand side of (6.14) from below, we obtain, for any $\rho \in (0, R_1/3]$,

$$\int_{B_{2\rho}} |\nabla v|^2 \leq C_7 \left(\frac{\int_{B_{R_1}} |v|^2}{R_1^2 \int_{B_{R_1/6}} |\nabla v|^2} \right)^{C_5} \int_{B_\rho} |\nabla v|^2, \tag{6.15}$$

where $C_6 \geq 1$ depends on λ_0 and λ_1.

Iterating (6.15), by simple calculations we get

$$\int_{B_{\alpha\rho}} |\nabla v|^2 \leq C_8\, N_v'\, \alpha^{\log N_v / \log 2} \int_{B_\rho} |\nabla v|^2, \tag{6.16}$$

for any $\alpha \geq 1$ and ρ such that $3\alpha\rho \leq R_1$. Here we have set

$$N_v' = \left(\frac{\int_{B_{R_1}} |v|^2}{R_1^2 \int_{B_{R_1/6}} |\nabla v|^2} \right)^{C_5}$$

and C_8 depends on λ_0 and λ_1 only. Now we remove condition (6.1). To this end, let $A(x)$ be a symmetric matrix satisfying (4.1) and (4.2) and let $v \in H^1(B_{R_0})$ a weak solution of (4.4). Let us introduce the change of variables $y = Jx$ where $J = \sqrt{A^{-1}(0)}$ and consider, for any $r > 0$, the ellipsoids

$$E_r := \left\{ x \in \mathbb{R}^n : A^{-1}(0)x \cdot x < r^2 \right\} = J^{-1}(B_r).$$

Setting $w(y) = v(J^{-1}y)$ and $\tilde{A}(y) = JA(J^{-1}y)J$ one has

$$\begin{aligned}
&\operatorname{div}\big(\tilde{A}(y)\nabla_y w(y)\big) = 0 \quad \text{in } B_{R_0\sqrt{\lambda_0}}, \\
&\lambda_0^2 |\xi|^2 \leq \tilde{A}(y)\,\xi\cdot\xi \leq \lambda_0^{-2}|\xi|^2, \quad \forall y \in \mathbb{R}^n,\ \forall \xi \in \mathbb{R}^n, \\
&|\tilde{A}(y_1) - \tilde{A}(y_2)| \leq \frac{\lambda_0^{-3/2}\lambda_1}{R_0}|y_1 - y_2|, \quad \forall y_1, y_2 \in \mathbb{R}^n \ \text{ and } \ \tilde{A}(0) = \mathrm{Id}.
\end{aligned}$$

Furthermore, since

$$B_{\sqrt{\lambda_0}r} \subset E_r \subset B_{r/\sqrt{\lambda_0}}, \quad \forall r > 0,$$

by simple changes of variables we have

$$\lambda_0^{n/2+1} \int_{B_{\sqrt{\lambda_0}r}} |\nabla v|^2\, dx \leq \int_{B_r} |\nabla w|^2\, dy \leq \lambda_0^{-(n/2+1)} \int_{B_{r/\sqrt{\lambda_0}}} |\nabla v|^2\, dx. \tag{6.17}$$

for all $r > 0$. We apply (6.16) to w and we obtain

$$\int_{B_{\alpha\rho}} |\nabla w|^2\, dy \leq C_8'\, N_w''\, \alpha^{\log N_w''/\log 2} \int_{B_\rho} |\nabla w|^2\, dy \tag{6.18}$$

for any $\alpha \geq 1$ and ρ such that $3\alpha\rho \leq R_1\sqrt{\lambda_0} := R_2$, where

$$N_w'' = \left(\frac{\int_{B_{R_2}} |w|^2 dy}{R_2^2 \int_{B_{R_2/6}} |\nabla w|^2} \right)^{C_5'}$$

and C_5' and C_8' depend on λ_0 and λ_1 only. From (6.17) and (6.18) we derive easily

$$\begin{aligned}
\int_{B_{\alpha\rho}} |\nabla v|^2\, dx &\leq \lambda_0^{-(n/2+1)} \int_{B_{\alpha\rho/\sqrt{\lambda_0}}} |\nabla w|^2\, dy \\
&\leq C_8'\, N_w''\, (\lambda_0^{-1}\alpha)^{\log N_w''/\log 2}\, \lambda_0^{-(n/2+1)} \int_{B_{\rho\sqrt{\lambda_0}}} |\nabla w|^2\, dy \\
&\leq C_8'\, \lambda_0^{-(n+2)}\, N_w''\, (\lambda_0^{-1}\alpha)^{\log N_w''/\log 2} \int_{B_\rho} |\nabla v|^2\, dx, \qquad (6.19)
\end{aligned}$$

for any $\alpha \geq 1$ and ρ such that $3\alpha\rho \leq R_2$. From (6.19) and using (6.17) to estimate N_w'' in terms of v we get

$$\int_{B_{2\rho}} |\nabla v|^2\, dx \leq C_{10} \left(\frac{\int_{B_{R_1}} |v|^2}{R_1^2 \int_{B_{R_1\lambda_0/6}} |\nabla v|^2} \right)^{C_9} \int_{B_\rho} |\nabla v|^2\, dx \tag{6.20}$$

for any $\rho \leq R_1\lambda_0/6$ and where C_9 and C_{10} depend on λ_0 and λ_1 only.

Applying (6.20) to $v - \frac{1}{|B_{R_1}|}\int_{B_{R_1}} v$ and using the Poincaré inequality we have

$$\int_{B_{2\rho}} |\nabla v|^2\, dx \leq C_{11} \left(\frac{\int_{B_{R_1}} |\nabla v|^2}{R_1^2 \int_{B_{R_1\lambda_0/6}} |\nabla v|^2} \right)^{C_9} \int_{B_\rho} |\nabla v|^2\, dx \tag{6.21}$$

for $\rho \leq R_1\lambda_0/6$. Finally we want to prove (4.5). Let $\rho \in [R_1\lambda_0/6, R_0/2]$. We trivially have

$$\begin{aligned}
\int_{B_{2\rho}} |\nabla v|^2\, dx \leq \int_{B_{R_0}} |\nabla v|^2\, dx &= \left(\frac{\int_{B_{R_0}} |\nabla v|^2\, dx}{\int_{B_{R_1\lambda_0/6}} |\nabla v|^2\, dx} \right) \int_{B_{R_1\lambda_0/6}} |\nabla v|^2\, dx \\
&\leq \left(\frac{\int_{B_{R_0}} |\nabla v|^2\, dx}{\int_{B_{R_1\lambda_0/6}} |\nabla v|^2\, dx} \right) \int_{B_\rho} \nabla v|^2\, dx.
\end{aligned}$$

From last inequality and from (6.21) we immediately get, for $\rho \in [0, R_0/2]$,

$$\int_{B_{2\rho}} |\nabla v|^2\, dx \leq C_{11} \left(\frac{\int_{B_{R_0}} |\nabla v|^2\, dx}{\int_{B_{R_1\lambda_0/6}} |\nabla v|^2\, dx} \right) \int_{B_\rho} |\nabla v|^2\, dx. \tag{6.22}$$

Now, we apply the three spheres inequality

$$\int_{B_{R_0/2}} |\nabla v|^2\,dx \le C_{12}\Big(\int_{B_{R_0}} |\nabla v|^2\,dx\Big)^{\theta}\Big(\int_{B_{R_1\lambda_0/6}} |\nabla v|^2\,dx\Big)^{1-\theta}, \tag{6.23}$$

where C_{12} and $\theta \in (0,1)$ depend only on λ_0 and λ_1. From (6.23) we have trivially

$$\frac{\int_{B_{R_0}} |\nabla v|^2\,dx}{\int_{B_{R_1\lambda_0/6}} |\nabla v|^2\,dx} \le C_{12}\bigg(\frac{\int_{B_{R_0}} |\nabla v|^2\,dx}{\int_{B_{R_0/2}} |\nabla v|^2\,dx}\bigg)^{1/(1-\theta)}.$$

From this last inequality and (6.22) we finally get (4.5). □

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