



On trilinear oscillatory integrals

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Abstract. We examine a certain class of trilinear integral operators which incorporate oscillatory factors e^{iP} , where P is a real-valued polynomial, and prove smallness of such integrals in the presence of rapid oscillations.

1. Introduction

This note continues the study of multilinear oscillatory integral expressions of the form

$$I(\lambda P; f_1, \dots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^n f_j \circ \pi_j(x) \eta(x) dx,$$

where $\lambda \in \mathbb{R}$ is a parameter, $P: \mathbb{R}^m \rightarrow \mathbb{R}$ is a real-valued polynomial, $\pi_j: \mathbb{R}^m \rightarrow V_j$ are orthogonal projections onto some subspaces V_j of \mathbb{R}^m , $f_j: V_j \rightarrow \mathbb{C}$ are locally integrable functions with respect to Lebesgue measure on V_j , and $\eta \in C_0^1(\mathbb{R}^m)$ is compactly supported. All the subspaces V_j are assumed to have the same dimension, which is denoted by κ .

Christ, Li, Tao, and Thiele [5] initiated this study, exploring conditions on the polynomial phase P and on the projections $\{\pi_j\}$ which ensure decay estimates of the form

$$(1.1) \quad |I(\lambda P; f_1, \dots, f_n)| \leq C \langle \lambda \rangle^{-\varepsilon} \prod_{j=1}^n \|f_j\|_{L^\infty(V_j)}.$$

Their results were restricted to the comparatively extreme cases $\kappa = 1$ and $\kappa = m - 1$, and the small codimension case $n \leq m/(m - \kappa)$, leaving most cases open.

In the present paper we consider the trilinear situation in $\mathbb{R}^m = \mathbb{R}^{2\kappa}$ for arbitrary $\kappa \geq 2$. A typical expression of this type is then

$$I(P; f_1, f_2, f_3) = \iint_{\mathbb{R}^{2\kappa}} e^{iP(x,y)} f_1(x) f_2(y) f_3(x+y) \eta(x,y) dx dy,$$

with coordinates $(x, y) \in \mathbb{R}^{\kappa+\kappa}$. Before stating our main theorem, we introduce some notation and recall relevant results from the literature.

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1.1. Review

Let $d \geq 1$ be a positive integer, and let $\{\pi_j\}_{j=1}^3$ be surjective linear mappings from $\mathbb{R}^{2\kappa}$ to \mathbb{R}^κ . A polynomial $P: \mathbb{R}^{2\kappa} \rightarrow \mathbb{R}$ is said to be degenerate (with respect to the projections $\{\pi_j\}$) if there exist polynomials $p_j: \mathbb{R}^\kappa \rightarrow \mathbb{R}$ such that $P = \sum_{j=1}^3 p_j \circ \pi_j$. The vector space of all degenerate polynomials $P: \mathbb{R}^{2\kappa} \rightarrow \mathbb{R}$ of degree $\leq d$ is a subspace $\mathcal{P}_{\text{degen}}$ of the vector space $\mathcal{P}(d)$ of all polynomials $P: \mathbb{R}^{2\kappa} \rightarrow \mathbb{R}$ of degree $\leq d$. Denote the quotient space by $\mathcal{P}(d)/\mathcal{P}_{\text{degen}}$, by $[P]$ the equivalence class of P in $\mathcal{P}(d)/\mathcal{P}_{\text{degen}}$, and by $\|\cdot\|_{nd}$ some fixed choice of norm for this quotient space. In a similar way, let $\|\cdot\|_{nc}$ denote some fixed choice of norm for the quotient space of polynomials $P: \mathbb{R}^{2\kappa} \rightarrow \mathbb{R}$ of degree $\leq d$ modulo constants.

It will be convenient to work with norms defined by inner products. If $P(x, y) = \sum_{\alpha, \beta} c_{\alpha\beta} x^\alpha y^\beta$, then set

$$\|P\|_{\mathcal{P}(d)} = \left(\sum_{\alpha, \beta} |c_{\alpha\beta}|^2 \right)^{1/2}, \quad \|P\|_{nc} = \left(\sum_{(\alpha, \beta) \neq (0,0)} |c_{\alpha\beta}|^2 \right)^{1/2}.$$

One likewise defines the norm $\|\cdot\|_{nd}$ by choosing some Hilbert space structure for $\mathcal{P}(d)/\mathcal{P}_{\text{degen}}$.

The norm $\|\cdot\|_{nc}$ controls oscillatory integrals of the first kind, in light of the following version of stationary phase.

Theorem 1.1. *Let $p(t) = \sum_{|\alpha| \leq d} c_\alpha t^\alpha$, $c_\alpha \in \mathbb{R}$, be a polynomial in m variables of degree $d \geq 1$. Then*

$$\left| \int_{[0,1]^m} e^{ip(t)} dt \right| \leq C_{d,m} \left(\sum_{0 < |\alpha| \leq d} |c_\alpha| \right)^{-1/d}.$$

Theorem 1.1 is a straightforward consequence of the well-known lemma of van der Corput [8].

On the other hand, the norm $\|\cdot\|_{nd}$ controls multilinear oscillatory integrals (in particular, oscillatory integrals of the second kind), as is shown in [5]. The following theorem is most relevant to our discussion.

Theorem 1.2. *Suppose that $n < 2m$ and $d < \infty$. Then, for any family $\{V_j\}_{j=1}^n$ of one-dimensional subspaces of \mathbb{R}^m which lie in general position, there exist constants $C < \infty$ and $\varepsilon > 0$ such that*

$$|I(P; f_1, \dots, f_n)| \leq C \langle \|P\|_{nd} \rangle^{-\varepsilon} \prod_{j=1}^n \|f_j\|_{L^2(V_j)}$$

for all polynomials $P: \mathbb{R}^m \rightarrow \mathbb{R}$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R})$. Moreover, ε can be taken to depend only on n, m , and d .

1.2. Result

Let $\{\pi_1, \pi_2, \pi_3\}$ be a collection of three surjective linear mappings from $\mathbb{R}^{2\kappa}$ to \mathbb{R}^κ . We say that these lie in general position if for any two indices $i \neq j \in \{1, 2, 3\}$, the nullspace of π_i is transverse to the nullspace of π_j .

In the present paper we prove the following:

Theorem 1.3. *Let $\kappa \geq 1$ and $d < \infty$. Let $\{\pi_1, \pi_2, \pi_3\}$ be a collection of three surjective linear mappings from $\mathbb{R}^{2\kappa}$ to \mathbb{R}^κ , which lie in general position. Then*

$$(1.2) \quad |I(P; f_1, f_2, f_3)| \leq C \langle \|P\|_{nd} \rangle^{-\varepsilon} \prod_{j=1}^3 \|f_j\|_{L^2(\mathbb{R}^\kappa)},$$

for all polynomials $P: \mathbb{R}^{2\kappa} \rightarrow \mathbb{R}$ of degree $\leq d$ and for all functions $f_j \in L^2(\mathbb{R}^\kappa)$, with constants $C, \varepsilon \in \mathbb{R}^+$ which depend only on κ, d, η , and $\{\pi_j\}$.

A more general result is established in the sequel [4]. Both of these works rely on dichotomies between structure and what may be termed pseudorandomness, but the notions of structure employed are quite different. A general function f_j is decomposed as the sum of pseudorandom and structured parts, the structured part being in turn an infinite sum of atomic functions with a particular structure. Fundamental questions are how such a dichotomy can usefully be defined for inequalities (1.2), what form the atomic structured components should take, and how a function can be efficiently decomposed into pseudorandom and structured summands so that good quantitative bounds can be obtained.

In the present paper, the relevant atomic functions take the form $e^{i\phi(x)}$, where $\phi(x_1, x_2, \dots, x_\kappa)$ is a real-valued polynomial of controlled degree with respect to x_κ , whose coefficients are unknown measurable real functions of $(x_1, x_2, \dots, x_{\kappa-1})$. Related reasoning is iterated to gain control of those coefficients. This argument is relatively direct.

In contrast, [4] uses a cruder reductive argument which works with a class of atomic functions which are less closely adapted to the oscillatory integral analysis, yet do suffice for a proof of a bound (1.2) with some exponent $\varepsilon > 0$. It is shown that there exist coordinates $x = (x', x'') \in \mathbb{R}^{\kappa'} \times \mathbb{R}^{\kappa''}$, with both κ' and κ'' strictly less than κ , with respect to which a structured component f_j can be expressed as a rapidly convergent sum of product functions $f'_j(x')f''_j(x'')$. No further structural information is initially known about these atomic functions. This reduction is applied iteratively to the resulting factors, until a multilinear expression is reached in which each factor f_j depends only on a single real variable. Since that situation was treated in [5], the proof is then complete. Fundamentally, after the iteration, one is implicitly working with atomic functions of the form $e^{i\phi}$ where $\phi = \sum_{s=1}^\kappa \phi_s$ with each ϕ_s being a polynomial of a single real variable. Most of the reasoning is concerned with the linear algebra needed to show the existence of suitable coordinates, rather than with any direct analysis of oscillatory integrals. The present paper does involve such direct analysis.

In both works, as in [5], a key element (partly implicit rather than fully explicit) is the representation of a function f_j as a sum of functions with specific

structural properties, but the particular structural properties that arise here are quite different from those in [4]. Since optimal decay exponents ε in the inequality (1.2) remain unknown, and the fundamental problem of devising a truly efficient way of decomposing functions into pseudorandom and structured components remains unresolved, both the approach of the present work, and that of [4] are of potential interest.

As in [5], the reasoning here is organized in terms of a priori inequalities in such a manner that the decomposition of a structured function as an infinite sum of atomic functions is not made explicit. It is implicit below in the reduction to (4.3), which is the inequality in question with one f_j replaced by an atomic function $e^{i\phi}$.

If x and y are real numbers, we will write $x \lesssim y$ if there exists a finite constant C such that $x \leq Cy$. The constant C may depend on some parameters which will be clear from the context. The notation $\langle x \rangle$ is shorthand for $(1 + |x|^2)^{1/2}$.

2. First reduction

It is no loss of generality to restrict attention to the case where $\mathbb{R}^{2\kappa}$ is identified with $\mathbb{R}_x^\kappa \times \mathbb{R}_y^\kappa$, and $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, and $\pi_3(x, y) = x + y$. Indeed, since the nullspaces of π_1 and π_2 are transverse, we may adopt coordinates $(x, y) \in \mathbb{R}^{\kappa+\kappa}$ such that the nullspace of π_1 is $\{(0, y)\}$, while the nullspace of π_2 is $\{(x, 0)\}$. Writing $\pi_3(x, y) = Ax + By$ where $A, B : \mathbb{R}^\kappa \rightarrow \mathbb{R}^\kappa$ are linear, the transversality hypothesis implies that both A and B are injective. Therefore it is possible to make invertible changes of coordinates in \mathbb{R}_x^κ and \mathbb{R}_y^κ so that A and B become the identity operator, and $\pi_3(x, y) = x + y$. Next, $\pi_1(x, y) = Dx$ for some invertible $D : \mathbb{R}^\kappa \rightarrow \mathbb{R}^\kappa$. By making a change of variables in the range of D , we may achieve $\pi_1(x, y) \equiv x$. Finally, a corresponding change of coordinates in the range of π_2 makes $\pi_2(x, y) \equiv y$.

In order to keep the notation simple, we will discuss in detail the case $\kappa = 2$, then will indicate in §7 how the analysis extends without additional difficulty to arbitrary dimensions.

3. Second reduction

We will restrict our attention to polynomial phases of the form λP , where $\lambda \in (0, \infty)$ and $\|P\|_{nd} = 1$, for, if $\|P\|_{nd} = 0$, then the conclusion of Theorem 1.3 is trivial. In particular, henceforth P will be assumed nondegenerate with respect to the projections $\{\pi_j\}_{j=1}^3$.

In the following lemma, $P_{(x_2, y_2)}(x_1, y_1) := P(x_1, y_1, x_2, y_2)$, and $\|\cdot\|_{nd}$ denotes a norm on the space of polynomials of degree $\leq d$ in x_1 and y_1 modulo degenerate polynomials with respect to the projections $(x_1, y_1) \mapsto x_1, y_1, x_1 + y_1$. We will sometimes write $I(P)$ as a shorthand for $I(P; f_1, f_2, f_3)$.

Let $K \subset \mathbb{R}_{x_2, y_2}^2$ be the projection of the support of η onto the (x_2, y_2) -plane.

Lemma 3.1. *Let $P: \mathbb{R}^4 \rightarrow \mathbb{R}$ be a real-valued polynomial of degree $\leq d$. If the polynomial $(x_1, y_1) \mapsto P(x, y)$ is nondegenerate with respect to the one-dimensional projections $(x_1, y_1) \mapsto x_1$, $(x_1, y_1) \mapsto y_1$, and $(x_1, y_1) \mapsto x_1 + y_1$ for some $(x_2, y_2) \in \mathbb{R}^2$, then there exists a constant $C < \infty$ such that*

$$|I(\lambda P; f_1, f_2, f_3)| \leq C \left(\lambda \cdot \sup_{(x_2, y_2) \in K} \|P_{(x_2, y_2)}\|_{nd} \right)^{-\sigma} \prod_{j=1}^3 \|f_j\|_2,$$

for all functions $f_j \in L^2(\mathbb{R}^2)$, where $\sigma > 0$ is a constant which depends only on d .

Proof. We can apply Theorem 1.2 with $m = 2$ and $n = 3$ to conclude that

$$J_{x_2, y_2}(\lambda P) := \iint_{\mathbb{R}^2} e^{i\lambda P(x_1, y_1, x_2, y_2)} f_1(x_1, x_2) f_2(y_1, y_2) f_3(x_1 + y_1, x_2 + y_2) \cdot \eta(x_1, y_1, x_2, y_2) dx_1 dy_1$$

satisfies

$$\begin{aligned} |J_{x_2, y_2}(\lambda P)| &\leq C(1 + \lambda^2 |Q(x_2, y_2)|)^{-\rho} \|f_1(\cdot, x_2)\|_2 \|f_2(\cdot, y_2)\|_2 \|f_3(\cdot, x_2 + y_2)\|_2 \\ &= C(1 + \lambda^2 |Q(x_2, y_2)|)^{-\rho} g_1(x_2) g_2(y_2) g_3(x_2 + y_2), \end{aligned}$$

for some $\rho > 0$ depending only on d , where $g_j(t) = \|f_j(\cdot, t)\|_2$ and $Q(x_2, y_2) = \|P_{(x_2, y_2)}(\cdot)\|_{nd}^2$ is a polynomial of degree $\leq 2d$.

For $\varepsilon > 0$, let

$$E_\varepsilon := \{(x, y) \in K : |Q(x, y)| < \varepsilon\}.$$

A basic sublevel set estimate [1] yields

$$|E_\varepsilon| \leq C \|Q\|_{L^\infty(K)}^{-\delta'} \varepsilon^{\delta'} \quad \text{for } \delta' = \frac{1}{\deg(Q)}$$

and some absolute constant $C < \infty$, if Q has positive degree. We now split the original integral $I(\lambda P; f_1, f_2, f_3)$ into two pieces and estimate each of them separately. On the one hand, Hölder’s and Young’s inequalities imply

$$\begin{aligned} \iint_{E_\varepsilon} |J_{x_2, y_2}(\lambda P)| dx_2 dy_2 &\leq C \iint_{E_\varepsilon} g_1(x_2) g_2(y_2) g_3(x_2 + y_2) dx_2 dy_2 \\ &\leq C |E_\varepsilon|^{1/4} \left(\iint_{\mathbb{R}^2} g_1^{4/3}(x_2) g_2^{4/3}(y_2) g_3^{4/3}(x_2 + y_2) dx_2 dy_2 \right)^{3/4} \\ &\leq C |E_\varepsilon|^{1/4} \prod_{j=1}^3 \|g_j^{4/3}\|_{3/2}^{3/4} = C |E_\varepsilon|^{1/4} \prod_{j=1}^3 \|g_j\|_2 \leq C \|Q\|_{L^\infty(K)}^{-\delta} \varepsilon^\delta \prod_{j=1}^3 \|f_j\|_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \iint_{\mathbb{R}^2 \setminus E_\varepsilon} |J_{x_2, y_2}(\lambda P)| dx_2 dy_2 &\leq C(1 + \lambda^2 \varepsilon)^{-\rho} \iint_{\mathbb{R}^2} g_1(x_2) g_2(y_2) g_3(x_2 + y_2) dx_2 dy_2 \\ &\leq C(1 + \lambda^2 \varepsilon)^{-\rho} \prod_{j=1}^3 \|g_j\|_{3/2} \leq C(1 + \lambda^2 \varepsilon)^{-\rho} \prod_{j=1}^3 \|f_j\|_2. \end{aligned}$$

If Q has degree zero then the same conclusion is reached more simply, with $\varepsilon^{\delta'}$ replaced by 1. Thus

$$|I(\lambda P; f_1, f_2, f_3)| \leq C \left[\|Q\|_{L^\infty(K)}^{-\delta} \varepsilon^\delta + (\lambda^2 \varepsilon)^{-\rho} \right] \prod_{j=1}^3 \|f_j\|_2.$$

Since $\|Q\|_{L^\infty(K)} = \sup_{(x_2, y_2) \in K} \|P_{(x_2, y_2)}\|_{nd}^2$, optimizing in ε yields an upper bound

$$|I(\lambda P; f_1, f_2, f_3)| \leq C \left(\lambda \sup_{(x_2, y_2) \in K} \|P_{(x_2, y_2)}\|_{nd} \right)^{-\frac{2\rho\delta}{\rho+\delta}} \prod_{j=1}^3 \|f_j\|_2. \quad \square$$

4. Third reduction

The goal of this step is to show that it is enough to consider functions of the form $f_j(u_1, u_2) = e^{i\phi_j(u_1, u_2)}$, where ϕ_j has polynomial dependence on u_1 of bounded degree and is real valued.

From the last section, we get the desired decay rate unless P is “almost degenerate” with respect to the projections $(x_1, y_1) \mapsto x_1, y_1, x_1 + y_1$ for almost every $(x_2, y_2) \in \mathbb{R}^2$, in the sense that

$$\sup_{(x_2, y_2) \in K} \|P_{(x_2, y_2)}\|_{nd} \lesssim \lambda^{-1+\tau} \quad \text{for some } \tau > 0.$$

We have the freedom to choose τ arbitrarily small later in the argument.

In this case, one can decompose

$$P(x, y) = q_1(x_1, x_2, y_2) + q_2(y_1, x_2, y_2) + q_3(x_1 + y_1, x_2, y_2) + R(x, y),$$

for some measurable functions q_j and R which are polynomials of degree $\leq d$ in x_1 and y_1 , and where the remainder R satisfies

$$|R(x, y)| \lesssim \lambda^{-1+\tau} \quad \text{if } (x, y) \in K'$$

for any fixed compact set $K' \subset \mathbb{R}^4$. To justify this, for each integer $k \geq 0$ choose some Hilbert space norm for the vector space of all degree k homogeneous polynomials in x_1 and y_1 . Write $P_{(x_2, y_2)}(x_1, y_1) = P(x, y)$. Write $P(x, y) = \sum_{k=0}^d P_{k, (x_2, y_2)}(x_1, y_1)$ where $P_{k, (x_2, y_2)}$ is a homogeneous polynomial of degree k in (x_1, y_1) , whose coefficients are polynomials in (x_2, y_2) . Now we use two facts shown implicitly in [5]. First, if $p = \sum_k p_k$ is a decomposition of a polynomial in (x_1, y_1) into its homogeneous summands of degree k , then $\sum_k \|p_k\|_{nd}$ is comparable to $\|p\|_{nd}$. Second, if $p(x_1, y_1)$ is homogeneous of degree k , then for any $d \geq k$, the norm of p in the space of all homogeneous polynomials of degree k modulo polynomials $cx_1^k + c'y_1^k + c''(x_1 + y_1)^k$ is comparable to the norm $\|p\|_{nd}$ of p in the space of all polynomials of degrees $\leq d$ modulo all degenerate polynomials of degrees $\leq d$, where degenerate polynomials are those which are sums of polynomials

in x_1 , polynomials in y_1 , and polynomials in $x_1 + y_1$. Qualitative versions of these two facts were established in [5]; the quantitative versions stated here follow from the equivalence of all norms in any finite-dimensional vector space.

For each k , the projection $Q_{k,(x_2,y_2)}$ of $P_{k,(x_2,y_2)}$ onto the span of x_1^k, y_1^k , and $(x_1 + y_1)^k$ has polynomial dependence on (x_2, y_2) . Moreover, all coefficients of $P_{k,(x_2,y_2)} - Q_{k,(x_2,y_2)}$ are $O(\lambda^{-1+\tau})$ for $(x_2, y_2) \in K$, and therefore for (x_2, y_2) in any fixed bounded set. For $k \geq 2$, $Q_{k,(x_2,y_2)}$ decomposes uniquely as $q_{1,k}(x_2, y_2)x_1^k + q_{2,k}(x_2, y_2)y_1^k + q_{3,k}(x_2, y_2)(x_1 + y_1)^k$; these coefficients $q_{i,k}$ continue to have polynomial dependence on (x_2, y_2) . For $k = 1$ there is likewise a unique such decomposition, with the additional condition $q_{3,k} \equiv 0$, and for $k = 0$, with two additional conditions $q_{2,k} \equiv q_{3,k} \equiv 0$. Recombining terms gives the claim.

We use this to compute J_{x_2,y_2} (a similar calculation occurs in [5], p. 340):

$$\begin{aligned} J_{x_2,y_2}(\lambda P) &= \iint_{\mathbb{R}^2} e^{i\lambda P(x,y)} f_1(x) f_2(y) f_3(x+y) \eta(x,y) dx_1 dy_1 \\ &= \iint_{\mathbb{R}^2} e^{i\lambda q_1(x_1,x_2,y_2)} f_1(x_1, x_2) e^{i\lambda q_2(y_1,x_2,y_2)} f_2(y_1, y_2) \\ &\quad \cdot e^{i\lambda q_3(x_1+y_1,x_2,y_2)} f_3(x_1 + y_1, x_2 + y_2) e^{i\lambda R} \eta dx_1 dy_1 \\ &= \iint_{\mathbb{R}^2} g_1(x_1) g_2(y_1) g_3(x_1 + y_1) \zeta(x_1, y_1) dx_1 dy_1 \\ &= C \iint \left(\int \widehat{g}_1(\xi_1) e^{ix_1 \xi_1} d\xi_1 \right) g_2(y_1) g_3(x_1 + y_1) \\ &\quad \cdot \left(\iint \widehat{\zeta}(\xi_2, \xi_3) e^{i(x_1,y_1) \cdot (\xi_2, \xi_3)} d\xi_2 d\xi_3 \right) dx_1 dy_1 \\ &= C \iiint \widehat{g}_1(\xi_1) \widehat{\zeta}(\xi_2, \xi_3) \\ &\quad \cdot \left[\int g_2(y_1) e^{iy_1 \xi_3} \left(\int g_3(x_1 + y_1) e^{ix_1(\xi_1 + \xi_2)} dx_1 \right) dy_1 \right] d\xi_1 d\xi_2 d\xi_3 \\ &= C \iiint \widehat{g}_1(\xi_1) \widehat{g}_2(\xi_1 + \xi_2 - \xi_3) \widehat{g}_3(-\xi_1 - \xi_2) \widehat{\zeta}(\xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3. \end{aligned}$$

Implicit in this notation is the dependence of the functions $g_j := e^{i\lambda q_j} f_j$ and $\zeta := e^{i\lambda R} \eta$ on x_2 and y_2 .

Since λR is a polynomial in x_1 and y_1 of bounded degree which is $O(\lambda^\tau)$ on $\text{supp}(\eta)$ and the same holds for all its derivatives, we have that

$$|\widehat{\zeta}(\xi)| \leq C_{N,\eta} \lambda^{N\tau} (1 + |\xi|)^{-N}, \quad \forall \xi \in \mathbb{R}^2, \quad \forall N \in \mathbb{N},$$

provided $\eta \in C^N$. In particular, if¹ $\eta \in C_0^3(\mathbb{R}^4)$, then

$$|\widehat{\zeta}(\xi)| \leq C \lambda^{3\tau} (1 + |\xi|)^{-3}, \quad \forall \xi \in \mathbb{R}^2.$$

¹We lose no generality in assuming this extra smoothness on η : by the usual decomposition of a compactly supported Hölder continuous function $\zeta = f + g$ into a smooth part f such that $\|f\|_{C^n} = O(\lambda^{C_n \delta})$ and a bounded remainder g such that $\|g\|_\infty = O(\lambda^{-\delta})$, it is easy to see that if the result holds for some $\eta \in C_0^n$ ($n \in \mathbb{N}$) with a constant $C = O(\|\eta\|_{C^n})$, then it will continue to hold for all η which are compactly supported and Hölder continuous of order α .

We use this (together with the Cauchy–Schwarz inequality and the Plancherel formula) to conclude that

$$\begin{aligned}
 |J_{x_2, y_2}(\lambda P)| &\leq C \lambda^{3\tau} \|\widehat{g}_1\|_\infty \iiint \frac{|\widehat{g}_2(\xi_1 + \xi_2 - \xi_3)| |\widehat{g}_3(-\xi_1 - \xi_2)|}{(1 + |(\xi_2, \xi_3)|)^3} d\xi_1 d\xi_2 d\xi_3 \\
 &\leq C \lambda^{3\tau} \|\widehat{g}_1\|_\infty \|g_2\|_2 \|g_3\|_2.
 \end{aligned}$$

Let $\delta > 3\tau$ and consider the set

$$F := \{(x_2, y_2) \in \mathbb{R}^2 : \|\widehat{g}_1\|_\infty \lesssim \lambda^{-\delta}\}.$$

There are two possibilities:

- (i) If $|F^c| \lesssim \lambda^{-\delta}$, then $|I(\lambda P)| \lesssim \lambda^{-(\delta-3\tau)}$, as is easily seen by splitting the integral

$$I(\lambda P) = \iint_{\mathbb{R}^2} J_{x_2, y_2}(\lambda P) dx_2 dy_2$$

over the regions F and F^c .

- (ii) If $|F^c| \gtrsim \lambda^{-\delta}$, we set $E := F^c$. Note that $\|\widehat{g}_1\|_\infty \gtrsim \lambda^{-\delta}$ for every $(x_2, y_2) \in E$.

Since condition (i) yields the desired decay, henceforth we restrict attention to the case in which condition (ii) holds. Then there exists a measurable subset $E \subset \mathbb{R}^2$ such that $|E| \gtrsim \lambda^{-\delta}$ and $\|\widehat{g}_1\|_\infty \gtrsim \lambda^{-\delta}$ for every $(x_2, y_2) \in E$. We still have the freedom to choose $\delta > 0$ as small as we wish later in the argument.

Why is this conclusion of interest? Since

$$\lambda^{-\delta} \lesssim \|\widehat{g}_1\|_\infty = \sup_{\xi} \left| \int_{\mathbb{R}} e^{i\lambda q_1(x_1, x_2, y_2)} f_1(x_1, x_2) e^{-ix_1 \xi} dx_1 \right|,$$

we can find measurable functions θ and $\tilde{\theta}$ such that, for $(x_2, y_2) \in E$,

$$\begin{aligned}
 \lambda^{-\delta} &\lesssim \left| \int e^{i\lambda q_1(x_1, x_2, y_2)} f_1(x_1, x_2) e^{-ix_1 \theta(x_2, y_2)} dx_1 \right| \\
 &= e^{-i\tilde{\theta}(x_2, y_2)} \int f_1(x_1, x_2) e^{i\lambda q_1(x_1, x_2, y_2) - ix_1 \theta(x_2, y_2)} dx_1.
 \end{aligned}$$

Because we are working in a fixed bounded region, there exist a measurable subset $E_1 \subset \mathbb{R}$ such that $|E_1| \gtrsim \lambda^{-\delta}$ and a single number $\bar{y}_2 \in \mathbb{R}$ such that for each $x_2 \in E_1$, $(x_2, \bar{y}_2) \in E$. Thus

$$\begin{aligned}
 \lambda^{-\delta} &\lesssim e^{-i\tilde{\theta}(x_2, \bar{y}_2)} \int f_1(x_1, x_2) e^{i\lambda q_1(x_1, x_2, \bar{y}_2) - ix_1 \theta(x_2, \bar{y}_2)} dx_1 \\
 &= \int f_1(x_1, x_2) e^{i\varphi_1(x_1, x_2)} dx_1, \quad \text{if } x_2 \in E_1,
 \end{aligned}$$

where

$$\varphi_1(x_1, x_2) = \lambda q_1(x_1, x_2, \bar{y}_2) - x_1 \theta(x_2, \bar{y}_2) - \tilde{\theta}(x_2, \bar{y}_2)$$

is a real-valued polynomial in x_1 of degree $\leq d$, whose coefficients are measurable functions of x_2 .

We would like to use this to conclude that the inner product of f_1 with $e^{-i\varphi_1}$ is reasonably large. While this is not necessarily true, the following *extension argument* will prove sufficient for our purposes: for every $x_2 \in \mathbb{R}$, choose $\theta^*(x_2)$ in a measurable way and such that

$$e^{i\theta^*(x_2)} \int_{\mathbb{R}} f_1(x_1, x_2) e^{i\varphi_1(x_1, x_2)} dx_1 \geq 0.$$

We can guarantee that $\theta^* \equiv 0$ on E_1 .

Define $\phi_1(x_1, x_2) := \theta^*(x_2) + \varphi_1(x_1, x_2)$. Then ϕ_1 is likewise a real-valued polynomial in x_1 of degree $\leq d$, whose coefficients are measurable functions of x_2 . Now

$$\int_{\mathbb{R}} f_1(x_1, x_2) e^{i\phi_1(x_1, x_2)} dx_1 \geq 0, \quad \text{for every } x_2 \in \mathbb{R},$$

while, for any $x_2 \in E_1$,

$$\int_{\mathbb{R}} f_1(x_1, x_2) e^{i\phi_1(x_1, x_2)} dx_1 \gtrsim \lambda^{-\delta}.$$

Therefore, since $|E_1| \gtrsim \lambda^{-\delta}$,

$$|\langle f_1, e^{-i\phi_1} \rangle| \gtrsim \lambda^{-2\delta}.$$

Since $\|f_1\|_{L^2} = 1$ and f_1 is supported in a fixed bounded set,

$$(4.1) \quad \|f_1 - \langle f_1, e^{-i\phi_1} \rangle e^{-i\phi_1}\|_2^2 \leq (1 - C\lambda^{-4\delta}).$$

Let $\mathbf{A}(\lambda)$ be the best constant in the inequality

$$|I(\lambda P; f_1, f_2, f_3)| \leq \mathbf{A}(\lambda) \|f_1\|_{L^2} \|f_2\|_{L^2} \|f_3\|_{L^2}.$$

That $\mathbf{A}(\lambda)$ is finite is an immediate consequence of the dual form of Young's convolution inequality and the fact that, in this context, $L^2 \subset L^{3/2}$. Now (4.1) implies

$$\begin{aligned} |I(\lambda P; f_1, f_2, f_3)| &= \\ &= |I(\lambda P; f_1 - \langle f_1, e^{-i\phi_1} \rangle e^{-i\phi_1}, f_2, f_3) + I(\lambda P; \langle f_1, e^{-i\phi_1} \rangle e^{-i\phi_1}, f_2, f_3)| \\ &\leq \mathbf{A}(\lambda) \|f_1 - \langle f_1, e^{-i\phi_1} \rangle e^{-i\phi_1}\|_2 \|f_2\|_2 \|f_3\|_2 + |\langle f_1, e^{-i\phi_1} \rangle| |I(\lambda P; e^{-i\phi_1}, f_2, f_3)| \\ &\leq \mathbf{A}(\lambda) (1 - C\lambda^{-4\delta})^{1/2} + C |I(\lambda P; e^{-i\phi_1}, f_2, f_3)|. \end{aligned}$$

Therefore

$$\mathbf{A}(\lambda) \leq \mathbf{A}(\lambda) (1 - C\lambda^{-4\delta})^{1/2} + C \sup_{\phi_1, f_2, f_3} |I(\lambda P; e^{-i\phi_1}, f_2, f_3)|$$

where the supremum is taken over all functions f_2 and f_3 supported in the specified regions satisfying $\|f_j\|_{L^2} = 1$, and over all real-valued functions $\phi_1(x_1, x_2)$ which

are polynomials of degree $\leq d$ with respect to x_1 , with coefficients depending measurably on x_2 . Since $\mathbf{A}(\lambda) < \infty$, it follows that

$$(4.2) \quad \mathbf{A}(\lambda) \leq C \lambda^{4\delta} \sup_{\phi_1, f_2, f_3} |I(\lambda P; e^{-i\phi_1}, f_2, f_3)|.$$

Therefore it suffices to prove that

$$(4.3) \quad |I(\lambda P; e^{-i\phi_1}, f_2, f_3)| \leq C \lambda^{-\varepsilon} \|f_2\|_{L^2} \|f_3\|_{L^2}$$

for a certain $\varepsilon > 0$; for δ may then be chosen to equal $\varepsilon/5$.

By repeating the above steps for g_2 and g_3 , we conclude that it suffices to prove that there exists $\varepsilon > 0$ such that

$$(4.4) \quad |I(\lambda P; e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})| \leq C \lambda^{-\varepsilon}$$

uniformly for all $\lambda \geq 1$, all polynomials P satisfying $\|P\|_{nd} = 1$, and all real-valued measurable functions $\phi_j(u_1, u_2)$ which are polynomials of degree $\leq d$ with respect to u_1 .

5. Handling remainders

In the last section we have reduced matters to the case where the f_j have the special form

$$\begin{cases} f_1(x_1, x_2) = e^{i\phi_1(x_1, x_2)} \\ f_2(y_1, y_2) = e^{i\phi_2(y_1, y_2)} \\ f_3(x_1 + y_1, x_2 + y_2) = e^{i\phi_3(x_1 + y_1, x_2 + y_2)}, \end{cases}$$

where ϕ_j are partial polynomials in the sense described following (4.4). Write

$$\begin{cases} \phi_1(x_1, x_2) = \sum_{j=0}^d \theta_{1,j}(x_2) x_1^j \\ \phi_2(y_1, y_2) = \sum_{k=0}^d \theta_{2,k}(y_2) y_1^k \\ \phi_3(x_1 + y_1, x_2 + y_2) = \sum_{l=0}^d \theta_{3,l}(x_2 + y_2) (x_1 + y_1)^l \end{cases}$$

where $\theta_{1,j}$, $\theta_{2,k}$, and $\theta_{3,l}$ are measurable and real-valued. Also express

$$P(x, y) = \sum_{j,k} p_{jk}(x_2, y_2) x_1^j y_1^k.$$

Then

$$\begin{aligned} I(\lambda P; f_1, f_2, f_3) &= \iint e^{i\lambda P} e^{i\phi_1} e^{i\phi_2} e^{i\phi_3} \eta \, dx \, dy \\ &= \iint \left(\iint e^{i \sum_{j,k} \psi_{jk}(x_2, y_2) x_1^j y_1^k} \eta \, dx_1 \, dy_1 \right) dx_2 \, dy_2, \end{aligned}$$

where

$$\begin{aligned} \psi_{jk}(x_2, y_2) = & \left\{ \begin{array}{ll} \theta_{1,j}(x_2) & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{array} \right\} + \left\{ \begin{array}{ll} \theta_{2,k}(y_2) & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{array} \right\} \\ & + \binom{j+k}{k} \theta_{3,j+k}(x_2 + y_2) + \lambda p_{jk}(x_2, y_2). \end{aligned}$$

The desired bound $|I(\lambda P; e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})| \leq C\lambda^{-\varepsilon}$ follows directly from Theorem 1.1, unless there exists a measurable subset $E \subset \mathbb{R}^2$ of measure $|E| \gtrsim \lambda^{-\delta}$ such that

$$(5.1) \quad \sum_{(j,k) \neq (0,0)} |\psi_{jk}(x_2, y_2)| \lesssim \lambda^r, \quad \forall (x_2, y_2) \in E.$$

We may choose $\delta, r > 0$ to be as small as may be desired for later purposes, at the expense of taking ε sufficiently small in (4.4).

The proof of the following lemma will be given later.

Lemma 5.1. *Let $P: \mathbb{R}^2 \rightarrow \mathbb{R}^D$ be a real vector-valued polynomial of degree d , and let $f, g: [0, 1] \rightarrow \mathbb{R}^D$ be measurable functions. Let $E \subseteq [0, 1]^2$ be a measurable subset of the unit square of Lebesgue measure $|E| = \varepsilon > 0$. Assume that*

$$(5.2) \quad |f(x) + g(y) + P(x, y)| \leq 1 \quad \text{for all } (x, y) \in E.$$

Then there exist \mathbb{R}^D -valued polynomials Q_1 and Q_2 of degrees $\leq d$ and measurable sets $E_1, E_2 \subseteq [0, 1]$ such that

$$\begin{cases} |f(x) - Q_1(x)| \lesssim \varepsilon^{-C} & \text{for } x \in E_1 \\ |g(y) - Q_2(y)| \lesssim \varepsilon^{-C} & \text{for } y \in E_2 \\ |E_1| \geq c\varepsilon, \quad |E_2| \geq c\varepsilon. \end{cases}$$

The constants $c, C \in \mathbb{R}^+$ depend only on d .

The phase estimates (5.1), together with Lemma 5.1, allow us to control most of the terms θ_i . Letting $k = 0$, we have that

$$|\psi_{j0}(x_2, y_2)| = \left| \theta_{1,j}(x_2) + \theta_{3,j}(x_2 + y_2) + \lambda p_{j0}(x_2, y_2) \right| \lesssim \lambda^r$$

if $1 \leq j \leq d$ and $(x_2, y_2) \in E$. Since $|E| \gtrsim \lambda^{-\delta}$, Lemma 5.1 implies that, for every $1 \leq j \leq d$, there exists a real-valued polynomial $\tilde{Q}_{1,j}$ of degree $\leq d$ such that

$$|\lambda^{-r} \theta_{1,j}(x_2) - \tilde{Q}_{1,j}(x_2)| \lesssim (\lambda^{-\delta})^{-C}$$

whenever $x_2 \in E_1$; here, $E_1 \subset \mathbb{R}$ is a measurable subset which does not depend on j and such that $|E_1| \gtrsim \lambda^{-\delta}$. A similar conclusion can be drawn for each of the terms $\theta_{3,l}$ with $1 \leq l \leq d$. Choosing $j = 0$ we control the terms $\theta_{2,k}$ for $1 \leq k \leq d$ in an analogous way.

We conclude that, for every $1 \leq j, k, l \leq d$,

$$\begin{cases} \theta_{1,j}(x_2) = Q_{1,j}(x_2) + \tilde{R}_{1,j}(x_2) \\ \theta_{2,k}(y_2) = Q_{2,k}(y_2) + \tilde{R}_{2,k}(y_2) \\ \theta_{3,l}(x_2 + y_2) = Q_{3,l}(x_2 + y_2) + \tilde{R}_{3,l}(x_2 + y_2) \end{cases}$$

where $Q_{1,j}$, $Q_{2,k}$, and $Q_{3,l}$ are polynomials of degree $\leq d$, and the remainders satisfy

$$\begin{cases} |\tilde{R}_{1,j}(x_2)| \lesssim \lambda^\beta & \text{if } x_2 \in E_1, \\ |\tilde{R}_{2,k}(y_2)| \lesssim \lambda^\beta & \text{if } y_2 \in E_2, \\ |\tilde{R}_{3,l}(x_2 + y_2)| \lesssim \lambda^\beta & \text{if } x_2 + y_2 \in E_3, \end{cases}$$

for certain measurable subsets $E_1, E_2, E_3 \subset \mathbb{R}$ which satisfy

$$|E_i| \gtrsim \lambda^{-\delta}.$$

The parameter $\beta := r + C\delta$ is a function of r, δ , and the constant $C = C(d)$ from Lemma 5.1.

We have estimates for the remainders R_i in rather small sets only, but it is possible to reduce to the case in which these estimates hold globally, via an extension argument similar to the one used in the previous section. Set $Q_1(x) = \sum_{j=1}^d Q_{1,j}(x_2)x_1^j$ and $\tilde{R}_1(x) = \sum_{j=1}^d \tilde{R}_{1,j}(x_2)x_1^j$. By modifying each $\tilde{R}_{1,j}$ suitably at each point of the complement of E_1 , we produce a function $\Phi_1 = \theta_{1,0} + Q_1 + R_1$ such that $\theta_{1,0}$ is a measurable and real-valued function of x_2 , $Q_1(x)$ is a polynomial function of $x \in \mathbb{R}^2$ of degree $\leq d$, $R_1(x_1, x_2)$ is a polynomial in x_1 of degree $\leq d$ whose coefficients are measurable functions of x_2 , $|R_1(x)| \lesssim \lambda^\beta$ for every $x \in \mathbb{R}^2$, all functions are real-valued, and

$$(5.3) \quad \langle e^{i\phi_1}, e^{i\Phi_1} \rangle \gtrsim \lambda^{-\delta}.$$

By the same argument used to reduce from general f_j to $e^{i\phi_j}$ in (4.2),

$$(5.4) \quad \mathbf{A}(\lambda) \lesssim \lambda^{C\delta} \sup_{\Phi_1, \phi_2, \phi_3} |I(\lambda P; e^{i\Phi_1}, e^{i\phi_2}, e^{i\phi_3})|,$$

where the supremum is taken over all Φ_1, ϕ_2 , and ϕ_3 of the above form. This argument can be repeated twice more to give

$$(5.5) \quad \mathbf{A}(\lambda) \lesssim \lambda^{C\delta} \sup_{\Phi_1, \Phi_2, \Phi_3} |I(\lambda P; e^{i\Phi_1}, e^{i\Phi_2}, e^{i\Phi_3})|,$$

where each of the functions Φ_i shares the properties indicated above for Φ_1 .

Now

$$\begin{aligned} I(\lambda P; e^{i\Phi_1}, e^{i\Phi_2}, e^{i\Phi_3}) &= \iint e^{i\theta_{1,0}(x_2)} e^{i\theta_{2,0}(y_2)} e^{i\theta_{3,0}(x_2+y_2)} \\ &\cdot \left(\iint e^{i\lambda P(x,y)} e^{iQ_1(x)} e^{iQ_2(y)} e^{iQ_3(x+y)} e^{i(R_1(x)+R_2(y)+R_3(x+y))} \eta \, dx_1 dy_1 \right) dx_2 dy_2. \end{aligned}$$

Let $\tilde{P} := P + \lambda^{-1}Q_1 + \lambda^{-1}Q_2 + \lambda^{-1}Q_3$. Since $[P] = [\tilde{P}]$, $\|P\|_{nd} = \|\tilde{P}\|_{nd}$. We are left with

$$I(\lambda P) = \iint e^{i\theta_{1,0}(x_2)} e^{i\theta_{2,0}(y_2)} e^{i\theta_{3,0}(x_2+y_2)} \cdot \left(\iint e^{i\lambda\tilde{P}(x,y)} e^{i(R_1(x)+R_2(y)+R_3(x+y))} \eta \, dx_1 dy_1 \right) dx_2 dy_2,$$

where all the functions in the exponents are real-valued, \tilde{P} is a polynomial of degree $\leq d$ such that $\|\tilde{P}\|_{nd} = \|P\|_{nd} = 1$, the $\theta_{j,0}$ and the R_j are measurable functions, and the remainders $R_j(u_1, u_2)$ are polynomial functions of u_1 of degrees $\leq d$ which satisfy $|R_j(u)| \lesssim \lambda^\beta$ for all $u \in \mathbb{R}^2$.

6. The end of the proof

Decompose

$$(6.1) \quad \tilde{P} = P_0 + P^* \quad \text{where} \quad P_0(x_2, y_2) = \tilde{P}(0, 0, x_2, y_2)$$

and $P^* = \tilde{P} - P_0$.

Since

$$\psi_{00}(x_2, y_2) = \lambda P_0(x_2, y_2) + \theta_{1,0}(x_2) + \theta_{2,0}(y_2) + \theta_{3,0}(x_2 + y_2),$$

we can write

$$(6.2) \quad I(\lambda P) = \iint e^{i\psi_{00}(x_2, y_2)} \left(\iint e^{i\lambda P^*(x,y)} e^{i(R_1(x)+R_2(y)+R_3(x+y))} \eta \, dx_1 dy_1 \right) dx_2 dy_2.$$

Our main assumption, namely that P is nondegenerate with respect to the projections $\{\pi_j\}_{j=1}^3$, has not yet come into play. To apply it, we need a lemma:

Lemma 6.1. *For any $d \in \mathbb{N}$ there exists $c > 0$ with the following property. Let $\tilde{P}: \mathbb{R}^4 \rightarrow \mathbb{R}$ be any real-valued polynomial of degree $\leq d$. Decompose $\tilde{P} = P_0 + P^*$ as in (6.1). Then*

$$\| \|P^*_{(x_2, y_2)} \|_{nc} \|_{\mathcal{P}(d)} + \|P_0\|_{nd} \geq c \|\tilde{P}\|_{nd}.$$

The expression $\|P_0\|_{nd}$ in this lemma has two natural interpretations, but these define the same quantity since

$$(6.3) \quad \min_{\deg(p_i) \leq d} \|P_0(x_2, y_2) + p_1(x_1, x_2) + p_2(y_1, y_2) + p_3(x_1 + y_1, x_2 + y_2)\|_{\mathcal{P}(d)} \\ = \min_{\deg(q_i) \leq d} \|P_0(x_2, y_2) + q_1(x_2) + q_2(y_2) + q_3(x_2 + y_2)\|_{\mathcal{P}(d)}.$$

The two inequalities implicit in this equality are obtained by setting $q_j(t) = p_j(0, t)$, and by setting $p_j(t_1, t_2) = q_j(t_2)$, respectively.

Proof. The left-hand side defines a seminorm on the finite-dimensional vector space of polynomials of degrees $\leq d$ modulo degenerate polynomials, so it suffices to show that if $\| \|P_{(x_2, y_2)}^*\|_{nc} \|_{\mathcal{P}(d)}$ vanishes then P^* is degenerate, and correspondingly for P_0 . For P_0 this is a tautology, in view of (6.3). On the other hand, $P^*(x, y) = \sum_{(j,k) \neq (0,0)} q_{j,k}(x_2, y_2) x_1^j y_1^k$ where the $q_{j,k}$ are uniquely determined polynomials, and

$$\| \|P_{(x_2, y_2)}^*\|_{nc} \|_{\mathcal{P}(d)} = 0 \quad \text{if and only if } q_{j,k} \equiv 0 \text{ for each } j \text{ and } k.$$

Thus $P^* \equiv 0$, so, in particular, P^* is degenerate. □

Therefore there exists a constant $c_d > 0$ such that $\| \|P_{(x_2, y_2)}^*\|_{nc} \|_{\mathcal{P}(d)} \geq c_d$ or $\|P_0\|_{nd} \geq c_d$.

6.1. Case 1: $\| \|P_{(x_2, y_2)}^*\|_{nc} \|_{\mathcal{P}(d)} \geq c_d$

We have that

$$I(\lambda P) = \iint e^{i(\lambda P_0(x_2, y_2) + \theta_{1,0}(x_2) + \theta_{2,0}(y_2) + \theta_{3,0}(x_2 + y_2))} \cdot \left(\iint e^{i\lambda P^*(x, y)} e^{i(R_1(x) + R_2(y) + R_3(x+y))} \eta \, dx_1 \, dy_1 \right) dx_2 \, dy_2,$$

where by Theorem 1.1, the absolute value of the inner integral is

$$\lesssim \left(\lambda \|P_{(x_2, y_2)}^*\|_{nc} + \lambda^{-1} R_{1,(x_2)} + \lambda^{-1} R_{2,(y_2)} + \lambda^{-1} R_{3,(x_2+y_2)} \right)_{nc}^{-1/d}.$$

Since $|R_j| \leq \lambda^\beta$ and $\beta < 1$, we have that, if λ is large enough, then

$$\left\| P_{(x_2, y_2)}^* + \lambda^{-1} R_{1,(x_2)} + \lambda^{-1} R_{2,(y_2)} + \lambda^{-1} R_{3,(x_2+y_2)} \right\|_{nc} \gtrsim \|P_{(x_2, y_2)}^*\|_{nc}$$

for every $(x_2, y_2) \in \mathbb{R}^2$. We conclude that the absolute value of the inner integral is

$$\lesssim (\lambda \|P_{(x_2, y_2)}^*\|_{nc})^{-1/d}.$$

If $(x_2, y_2) \in \mathbb{R}^2$ is such that

$$\|P_{(x_2, y_2)}^*\|_{nc} \gtrsim \lambda^{-1+\tau} \quad \text{for some } \tau > 0,$$

we get the desired decay. Otherwise, observe that (i) implies a sublevel set estimate of the form

$$|\{(x_2, y_2) \in \mathbb{R}^2 : \|P_{(x_2, y_2)}^*\|_{nc} < \lambda^{-1+\tau}\}| \lesssim (\lambda^{-1+\tau})^\delta.$$

Therefore the contribution of the set of such points (x_2, y_2) to the integral is small, and this concludes the analysis of case 1.

6.2. Case 2: $\|P_0\|_{nd} \geq c_d$

By Fubini’s theorem,

$$I(\lambda P) = \iint \left(\iint e^{i\lambda[P_0(x_2,y_2)+P^*(x,y)]} \cdot e^{i(\theta_{1,0}+R_{1,(x_1)})(x_2)} e^{i(\theta_{2,0}+R_{2,(y_1)})(y_2)} e^{i(\theta_{3,0}+R_{3,(x_1+y_1)})(x_2+y_2)} \eta \, dx_2 dy_2 \right) dx_1 dy_1.$$

By Theorem 1.2, the absolute value of the inner integral in the last expression is less than or equal to a constant multiple of $(1 + \lambda^2 \|P_0 + P^*_{(x_1,y_1)}\|_{nd}^2)^{-\rho}$, for some $\rho = \rho(d) > 0$. It follows that

$$|I(\lambda P)| \lesssim \iint (1 + \lambda^2 \|P_0 + P^*_{(x_1,y_1)}\|_{nd}^2)^{-\rho} dx_1 dy_1.$$

To handle this integral, note that $P^*(0, 0, x_2, y_2) = 0$ and hypothesis (ii) together imply that

$$\left| \{ (x_1, y_1) \in \mathbb{R}^2 : \|P_0 + P^*_{(x_1,y_1)}\|_{nd}^2 < \varepsilon \} \right| \lesssim \varepsilon^\delta$$

because $(x_1, y_1) \mapsto \|P_0 + P^*_{(x_1,y_1)}\|_{nd}^2$ is a polynomial of degree $\leq 2d$. An argument entirely analogous to the one used to prove Lemma 3.1 concludes the analysis.

7. Higher dimensions and generalization

So far we have only discussed the case where the domain of P is \mathbb{R}^4 , but the case of $\mathbb{R}^{2\kappa}$ for $\kappa > 2$ is treated in essentially the same way. Now write $x = (x', x_\kappa)$, $y = (y', y_\kappa)$ where $x', y' \in \mathbb{R}^{\kappa-1}$. The proof proceeds by induction on κ . The only significant change in the proof is that in case 2 of the final step of the proof, since $\mathbb{R}^{\kappa-1}$ is no longer \mathbb{R}^1 , Theorem 1.2 does not apply; instead, one simply invokes the induction hypothesis.

Our result may be generalized to include arbitrary smooth phases and not just polynomial ones. The details are a straightforward modification of those in [6] and therefore they are not included.

8. Proof of Lemma 5.1

There remains to prove Lemma 5.1. The proof will rely on the following related, but simpler, result. Let X be a normed linear space. We write $|x|$ for the norm of a vector in X .

Lemma 8.1. *Let Ω and Ω' be probability spaces with measures μ and μ' , and let f and f' be X -valued functions defined on these spaces. Let $0 < r < 1$ and $R \in (0, \infty)$. Let $E \subset \Omega \times \Omega'$ satisfy $(\mu \times \mu')(E) \geq r$. Suppose that*

$$|f(x) - f'(x')| \leq R \quad \text{for all } (x, x') \in E.$$

Then there exist $a \in X$, $G \subset \Omega$, and $G' \subset \Omega'$ such that

$$\begin{aligned} \mu(G) &\geq cr, \\ \mu'(G') &\geq cr, \\ |f(x) - a| &\leq CR \quad \text{for all } x \in G, \\ |f'(x') - a| &\leq CR \quad \text{for all } x' \in G'. \end{aligned}$$

Here c and C are certain absolute constants.

Proof. Let $|E|$ denote $(\mu \times \mu')(E)$. Let $\pi_1 : \Omega \times \Omega' \rightarrow \Omega$ and $\pi_2 : \Omega \times \Omega' \rightarrow \Omega'$ denote the canonical projections. For $x \in \pi_1(E)$ and $x' \in \pi_2(E)$, consider the slices

$$\begin{cases} E^x := \{x' \in \Omega' : (x, x') \in E\}, \\ E_{x'} := \{x \in \Omega : (x, x') \in E\}. \end{cases}$$

By Fubini's theorem, E^x is μ' -measurable for μ -a.e. x and $E_{x'}$ is μ -measurable for μ' -a.e. x' .

Claim. There exists $(x_0, x'_0) \in E$ such that

$$\begin{cases} G := E_{x'_0} \subset \Omega & \text{is } \mu\text{-measurable and } \mu(G) \geq cr, \\ G' := E^{x_0} \subset \Omega' & \text{is } \mu'\text{-measurable and } \mu'(G') \geq cr. \end{cases}$$

Assuming the claim, we have that

$$\begin{cases} |f(x) - f'(x'_0)| \leq R & \text{for every } x \in G, \\ |f(x_0) - f'(x'_0)| \leq R, \\ |f(x_0) - f'(x')| \leq R & \text{for every } x' \in G'. \end{cases}$$

Let $a := (f(x_0) + f'(x'_0))/2$. Then, for any $x \in G$,

$$|f(x) - a| \leq |f(x) - f'(x'_0)| + |f'(x'_0) - a| \leq \frac{3}{2}R,$$

and similarly for $x' \in G'$.

To prove the claim, start by assuming that E^x and $E_{x'}$ are measurable for every $(x, x') \in E$. Express E as a disjoint union $E = \mathcal{G} \cup \mathcal{B}$, where

$$\begin{cases} \mathcal{G} = \{(x, x') \in E : \mu'(E^x) \geq r/4 \text{ and } \mu(E_{x'}) \geq r/4\}, \\ \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 := \{(x, x') \in E : \mu'(E^x) < r/4\} \cup \{(x, x') \in E : \mu(E_{x'}) < r/4\}. \end{cases}$$

We prove the stronger statement $|\mathcal{G}| > 0$. Suppose on the contrary that $|E| = |\mathcal{B}|$. Then

$$r \leq |E| = |\mathcal{B}| = |\mathcal{B}_1 \cup \mathcal{B}_2| \leq |\mathcal{B}_1| + |\mathcal{B}_2|,$$

whence $|\mathcal{B}_1| \geq r/2$ or $|\mathcal{B}_2| \geq r/2$. Without loss of generality assume that the former holds. Then

$$\left| \left\{ (x, x') \in E : \mu'(E^x) \geq \frac{r}{4} \right\} \right| \leq |E| - \frac{r}{2}.$$

However, defining $S_1 := \{x \in \pi_1(E) : \mu'(E) \geq r/4\}$ and $S_2 := \pi_1(E) \setminus S_1$, we have that

$$|E| = \int_{\pi_1(E)} \mu'(E^x) d\mu(x) = \int_{S_1} \mu'(E^x) d\mu(x) + \int_{S_2} \mu'(E^x) d\mu(x) \leq \left(|E| - \frac{r}{2}\right) + \frac{r}{4},$$

a contradiction. □

Proof of Lemma 5.1. Let A be the norm of P in the quotient space of \mathbb{R}^D -valued polynomials of degree $\leq d$ modulo degenerate polynomials with respect to the pair of projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$. It is well known (see [8], p. 416) that a decay bound of the form (1.1) holds for these projections. That is, for any d there exists $\rho > 0$ such that for any compact sets $K, K' \subset \mathbb{R}$ there exists $C < \infty$ such that

$$\left| \int_{\mathbb{R}^2} e^{iQ(x,y)} f(x)g(y) dx dy \right| \leq C(1 + \|Q\|_{nd})^{-\rho} \|f\|_2 \|g\|_2$$

for all functions f and g supported on K and K' respectively, for all real-valued polynomials Q of degree $\leq d$. This in turn, applied to the individual components of P , implies a sublevel set inequality of the form

$$|E| \leq C_\gamma A^{-\gamma},$$

where $C_\gamma = C/(1 - \gamma)$, C is an absolute constant, and γ depends only on d ; see for instance the discussion in [3] for the simple derivation.

Hence $A \leq C_\gamma \varepsilon^{-1/\gamma}$. That is,

$$\inf_{\deg p, q \leq d} \sup_{(x,y) \in [0,1]^2} |P(x, y) - p(x) - q(y)| \leq C_\gamma \varepsilon^{-1/\gamma}.$$

The infimum is actually a minimum, so there exist polynomials p and q of degree $\leq d$ such that

$$\sup_{(x,y) \in [0,1]^2} |P(x, y) - p(x) - q(y)| \leq C_\gamma \varepsilon^{-1/\gamma}.$$

In particular, for $(x, y) \in E$, we have that

$$\begin{aligned} |(f(x) + p(x)) + (g(y) + q(y))| &\leq |f(x) + g(y) + P(x, y)| + |p(x) + q(y) - P(x, y)| \\ &\leq 1 + C_\gamma \varepsilon^{-1/\gamma} = C'_\gamma \varepsilon^{-1/\gamma}. \end{aligned}$$

Apply Lemma 8.1 to conclude the existence of $a \in \mathbb{C}$ and $E_1 \subset [0, 1]$ such that $|E_1| \geq \varepsilon/4$ and

$$|f(x) + p(x) - a| \leq C_\gamma \varepsilon^{-1/\gamma} = C_\gamma |E|^{-1/\gamma} \quad \text{for every } x \in E_1.$$

Proceeding similarly for $g + q$ completes the proof. □

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