Rev. Mat. Iberoam. **30** (2014), no. 2, 537–550
DOI 10.4171/RMI/792

Groups with restrictions on subgroups of infinite rank

Maria De Falco, Francesco de Giovanni, Carmela Musella and Nadir Trabelsi

Abstract. It is known that a (generalized) soluble group whose proper subgroups of infinite rank are abelian either is abelian or has finite rank. It is proved here that if G is a group of infinite rank such that all its proper subgroups of infinite rank have locally finite commutator subgroup, then the commutator subgroup G' of G is locally finite, provided that G satisfies a suitable generalized solubility condition. Moreover, a similar result is obtained for groups whose proper subgroups of infinite rank are quasihamiltonian.

1. Introduction

A group G is said to have *finite* (*Prüfer*) *rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property. A classical theorem of A. I. Mal'cev, proved in [14], states that a locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank. On the other hand, in [7], M. R. Dixon, M. J. Evans and H. Smith proved that a (generalized) soluble group, in which all proper subgroups of infinite rank are abelian, either is abelian or has finite rank. The investigation of groups whose proper subgroups of infinite rank have a given property has been continued in a series of papers (see, for instance, [8], [9], [10]).

It is not difficult to show that if G is a (generalized) soluble group of infinite rank and all its proper subgroups of infinite rank have finite commutator subgroup, then also the commutator subgroup G' of G is finite. This and some more consequences of the previous results will be proved in Section 3.

The aim of this paper is to provide a further contribution to this topic, and our first main result is the following.

Mathematics Subject Classification (2010): 20F19.
Keywords: Prüfer rank, strongly locally graded group, quasihamiltonian group.

Theorem A. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G have locally finite commutator subgroup, then the commutator subgroup G' of G is also locally finite.*

We will work within the universe of strongly locally graded groups, a class of generalized soluble groups that can be defined as follows.

Recall that a group G is *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index. Let $\mathfrak{D}$ be the class of periodic locally graded groups, and let $\bar{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ under the operators $\acute{\boldsymbol{P}}, \grave{\boldsymbol{P}}, \boldsymbol{R}$ and $\boldsymbol{L}$ (these and other relevant operators will be defined in Section 2, and we shall use the first chapter of the monograph [18] as a general reference for definitions and properties of closure operations on group classes). It is easy to prove that any $\bar{\mathfrak{D}}$-group is locally graded, and that the class $\bar{\mathfrak{D}}$ is closed with respect to forming subgroups. Moreover, in [2] N.S. Černikov proved that every $\bar{\mathfrak{D}}$-group with finite rank contains a locally soluble subgroup of finite index. Obviously, all residually finite groups belong to $\bar{\mathfrak{D}}$, and hence the consideration of any free nonabelian group shows that the class $\bar{\mathfrak{D}}$ is not closed with respect to homomorphic images. We shall say that a group G is *strongly locally graded* if every section of G is a $\bar{\mathfrak{D}}$-group. Thus strongly locally graded groups form a large $\boldsymbol{S}$ and $\boldsymbol{H}$-closed class of generalized soluble groups, containing in particular all locally (soluble-by-finite) groups.

The last section of the paper deals with groups whose proper subgroups of infinite rank are quasihamiltonian. Recall that a subgroup H of a group G is said to be *permutable* if $HX = XH$ for each subgroup X of G, and the group G is *quasihamiltonian* if all its subgroups are permutable, i.e., if $XY = YX$ for all subgroups X and Y of G. The structure of quasihamiltonian groups has been described by K. Iwasawa (see Chapter 2 of [22]). It has recently been proved by M.R. Dixon and Z.Y. Karatas in [9] that if G is a strongly locally graded group whose subgroups of infinite rank are permutable, then either G is quasihamiltonian or it has finite rank. Our second main theorem deals with groups whose subgroups of infinite rank are quasihamiltonian.

Theorem B. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are quasihamiltonian, then also G is quasihamiltonian.*

It is well known that a group G is quasihamiltonian if and only if it is locally nilpotent and the lattice $\mathfrak{L}(G)$ of all subgroups of G is modular. Theorem B will be obtained as a consequence of a similar result on groups for which the lattice of subgroups of any proper subgroup of infinite rank is modular. The proof of this latter theorem uses Theorem A, as locally graded groups with modular subgroup lattice have locally finite commutator subgroup.

We mention here that Theorem A is also a necessary step in the study of groups whose proper subgroups of infinite rank have finite conjugacy classes; such groups are investigated in [5].

Most of our notation is standard and can be found in [18].

2. Preliminary results

Theorem A will be obtained as a special case of a general theorem of the same type concerning abstract group classes with certain suitable properties. We recall here the definitions of the main closure operations on group classes that will be used in our arguments. If $\mathfrak{X}$ is any class of groups, then:

- $\boldsymbol{S}\mathfrak{X}$ is the class of all groups whose subgroups belong to $\mathfrak{X}$,
- $\boldsymbol{H}\mathfrak{X}$ is the class of all groups whose homomorphic images belong to $\mathfrak{X}$,
- $\boldsymbol{N}\mathfrak{X}$ is the class of all groups generated by normal $\mathfrak{X}$-subgroups,
- $\boldsymbol{L}\mathfrak{X}$ is the class of all groups such that every finite subset is contained in an $\mathfrak{X}$-subgroup; in particular, if $\boldsymbol{S}\mathfrak{X} = \mathfrak{X}$, then $\boldsymbol{L}\mathfrak{X}$ is the class of all groups whose finitely generated subgroups belong to $\mathfrak{X}$,
- $\boldsymbol{P}\mathfrak{X}$ is the class of all groups admitting a finite series with $\mathfrak{X}$-factors,
- $\acute{\boldsymbol{P}}\mathfrak{X}$ is the class of all groups admitting an ascending series with $\mathfrak{X}$-factors,
- $\grave{\boldsymbol{P}}\mathfrak{X}$ is the class of all groups admitting a descending series with $\mathfrak{X}$-factors,
- $\boldsymbol{R}\mathfrak{X}$ is the class of all groups which can be embedded in a cartesian product of $\mathfrak{X}$-subgroups,
- if $\mathfrak{Y}$ is any group class, $\mathfrak{X}\mathfrak{Y}$ is the class of all groups G containing a normal $\mathfrak{X}$-subgroup N such that G/N belongs to $\mathfrak{Y}$.

Lemma 2.1. *Let $\mathfrak{X}$ be an $\boldsymbol{S}$-closed class of groups such that $(\boldsymbol{L}\mathfrak{X})\mathfrak{X} = \boldsymbol{L}\mathfrak{X}$. Then the class $\boldsymbol{L}\mathfrak{X}$ is $\boldsymbol{P}$-closed, i.e., $(\boldsymbol{L}\mathfrak{X})(\boldsymbol{L}\mathfrak{X}) = \boldsymbol{L}\mathfrak{X}$.*

Proof. Let G be a group containing a normal $\boldsymbol{L}\mathfrak{X}$-subgroup N such that the factor group G/N is locally $\mathfrak{X}$, and let E be any finitely generated subgroup of G. Then EN/N belongs to $\mathfrak{X}$ and so E lies in $(\boldsymbol{L}\mathfrak{X})\mathfrak{X} = \boldsymbol{L}\mathfrak{X}$. Therefore E is an $\mathfrak{X}$-group and G is locally $\mathfrak{X}$. □

Corollary 2.2. *Let $\mathfrak{X}$ be a class of groups such that $\boldsymbol{S}\mathfrak{X} = \boldsymbol{H}\mathfrak{X} = \mathfrak{X}$ and $(\boldsymbol{L}\mathfrak{X})\mathfrak{X} = \boldsymbol{L}\mathfrak{X}$. Then the class $\boldsymbol{L}\mathfrak{X}$ is $\boldsymbol{N}$-closed.*

If $\mathfrak{X}$ is a class of groups, recall that the $\mathfrak{X}$-*radical* of a group G is the subgroup generated by all normal $\mathfrak{X}$-subgroups of G. Although the $\mathfrak{X}$-radical of a group need not belong to $\mathfrak{X}$ in general, this turns out to be true for some relevant group classes. In fact, if $\mathfrak{X}$ is any class of groups such that $\boldsymbol{S}\mathfrak{X} = \boldsymbol{H}\mathfrak{X} = \mathfrak{X}$ and $(\boldsymbol{L}\mathfrak{X})\mathfrak{X} = \boldsymbol{L}\mathfrak{X}$, it follows from Corollary 2.2 that in any group G the $\boldsymbol{L}\mathfrak{X}$-radical is locally $\mathfrak{X}$ and contains all ascendant $\boldsymbol{L}\mathfrak{X}$-subgroups of G (see Lemma 1.31 in part 1 of [18]).

Lemma 2.3. *Let G be a strongly locally graded group whose proper subgroups have finite rank. Then G has finite rank.*

Proof. Assume, aiming at a contradiction, that the group G has infinite rank. Then G has no proper subgroups of finite index, so that in particular it cannot be finitely generated. It follows that every finitely generated subgroup of G has finite rank, and so it is soluble-by-finite by Černikov's theorem [2]. Since G has infinite rank, it must contain a locally soluble subgroup H of infinite rank (see [6]), and hence $G = H$ is locally soluble. Therefore G has finite rank (see Lemma 1 of [7]), and this contradiction proves the lemma. □

The knowledge of the structure of groups whose proper normal subgroups have finite rank will be relevant in our considerations. This structure is described by the following result.

Lemma 2.4. *Let G be a strongly locally graded group of infinite rank whose proper normal subgroups have finite rank. Then either G is locally nilpotent or it has a simple homomorphic image of infinite rank.*

Proof. The factor group G/G' has finite rank, as it cannot contain proper subgroups of infinite rank; then G' has infinite rank and so $G = G'$ is perfect. Moreover, any proper normal subgroup of G is a strongly locally graded group of finite rank, so that it contains a locally soluble subgroup of finite index. It follows that the product of two arbitrary locally soluble normal subgroups of G is locally soluble, and hence the join L of all locally soluble normal subgroups of G is likewise locally soluble. It is also clear that all proper normal subgroups of G/L are finite. Assume first that G is not locally soluble, i.e., $L \neq G$. As G does not contain proper subgroups of finite index, G/L cannot be covered by its finite normal subgroups, and so it contains a largest proper normal subgroup K/L. Then K/L is finite and G/K is a simple group of infinite rank.

Suppose now that G is locally soluble. Then for each proper normal subgroup N of G there is a positive integer k, depending only on the rank of N, such that the subgroup $N^{(k)}$ is hypercentral (see Lemma 10.39 of part 2 of [18]). In particular, as G is not simple, it contains an abelian non-trivial normal subgroup. On the other hand, the hypotheses are obviously inherited by homomorphic images, and so G is a hyperabelian group. Assume, aiming a contradiction, that G is not locally nilpotent, so that G properly contains its Hirsch–Plotkin radical H. Then H has finite rank, and hence G contains a normal subgroup V such that the index $|G : V|$ is finite and the subgroup V' is hypercentral (see Theorem 8.16 of part 2 of [18]). It follows that $V = G$, so that $G = G'$ is hypercentral, and this contradiction completes the proof of the lemma. □

We consider now the case of simple groups whose proper subgroups of infinite rank have a certain property.

Lemma 2.5. *Let $\mathfrak{X}$ be a class of groups which is* **S** *and* **L***-closed, and let G be a simple strongly locally graded group of infinite rank. If all proper subgroups of G of infinite rank belong to $\mathfrak{X}$, then either G is an $\mathfrak{X}$-group or it is locally finite.*

Proof. Assume that G is not an $\mathfrak{X}$-group. As G is simple and locally graded, it cannot be finitely generated. Moreover, since $\mathfrak{X}$ is a local class, G contains a finitely generated subgroup G_1 which is not an $\mathfrak{X}$-group. If E is any finitely generated subgroup of G, the proper subgroup $\langle E, G_1\rangle$ does not belong to $\mathfrak{X}$, and hence it has finite rank. Therefore all finitely generated subgroups of G have finite rank, and Černikov's result yields that they are soluble-by-finite, i.e., G is locally (soluble-by-finite).

Since G has infinite rank, the ranks of its finitely generated subgroups are unbounded, and so there exists in G an ascending chain

$$G_1 < G_2 < \cdots < G_n < G_{n+1} < \cdots$$

of finitely generated subgroups such that $r(G_n) < r(G_{n+1})$ for each n. Clearly, the countable subgroup

$$\langle G_n \mid n \in \mathbb{N} \rangle$$

has infinite rank and does not belong to $\mathfrak{X}$, so that

$$G = \langle G_n \mid n \in \mathbb{N} \rangle.$$

For each positive integer n, let R_n be the soluble radical of G_n, and put

$$R = \langle R_n \mid n \in \mathbb{N} \rangle.$$

As the subgroup R_n is normalized by R_i for every $i \leq n$, it follows that

$$\langle R_1, \dots, R_n \rangle = R_1 \dots R_n$$

is soluble, and hence R is locally soluble. Moreover, G_1 is contained in the normalizer $N_G(R)$, so that the index $|G_1R : R|$ is finite and G_1R is (locally soluble)-by-finite. In particular, G_1R is a proper subgroup of G which does not belong to $\mathfrak{X}$, and so G_1R has finite rank. However the subgroups G_n have unbounded ranks, and hence the orders of the finite groups G_n/R_n are unbounded. It follows that also the socles of the finite groups G_n/R_n have unbounded ranks (see Propositions 2.1 and 2.4 of [6]).

As R is a locally soluble group of finite rank, there exists a positive integer k such that the k-th term $R^{(k)}$ of the derived series of R is periodic (see Lemma 10.39 in part 2 of [18]). It is well known that each subgroup G_n is minimax (see, for instance, [17]). Thus for every n we can choose a locally finite normal subgroup L_n of G_n in such a way that the subgroups L_n have unbounded ranks (see [6], Proposition 3.3). The subgroup L_n is normalized by L_i for all $i \leq n$, so that

$$L = \langle L_n \mid n \in \mathbb{N} \rangle$$

is locally finite. Moreover, L has infinite rank and $G_1 \leq N_G(L)$, so that $G_1L = G$ and L is normal in G. Therefore $G = L$ is locally finite. □

Corollary 2.6. *Let $\mathfrak{X}$ be a class of groups which is $\boldsymbol{S}$ and $\boldsymbol{L}$-closed and contains all finite groups, and let G be a simple strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G belong to $\mathfrak{X}$, then G is an $\mathfrak{X}$-group.*

Our next two lemmas are useful for reducing our arguments to certain special situations.

Lemma 2.7. *Let $\mathfrak{X}$ be an $\boldsymbol{S}$-closed class of groups, and let G be a group of infinite rank whose proper subgroups of infinite rank belong to $\mathfrak{X}$. If the commutator subgroup G' of G has finite rank, then all proper subgroups of G belong to $\mathfrak{X}$.*

Proof. Let X be any subgroup of finite rank of G. Then the product XG' likewise has finite rank, and so the factor group G/XG' has infinite rank. It follows that XG' is contained in a proper subgroup of G of infinite rank, and hence X belongs to $\mathfrak{X}$. □

Lemma 2.8. *Let $\mathfrak{X}$ be a class of groups which is* ***S*** *and* ***L****-closed, and let G be a group whose proper subgroups of infinite rank belong to $\mathfrak{X}$. If the commutator subgroup G' of G has infinite rank, then either G is an $\mathfrak{X}$-group or G/G' is finitely generated.*

Proof. Assume that G is not an $\mathfrak{X}$-group. As $\mathfrak{X}$ is a local class, there exists a finitely generated subgroup E of G which is not in $\mathfrak{X}$. Then the subgroup EG' has infinite rank and does not belong to $\mathfrak{X}$, so that $EG' = G$ and hence G/G' is finitely generated. □

The last lemma of this section deals with the case of locally nilpotent groups.

Lemma 2.9. *Let $\mathfrak{X}$ be a group class which is* ***L****-closed and contains all abelian groups. If G is a locally nilpotent group of infinite rank whose proper subgroups of infinite rank belong to $\mathfrak{X}$, then either G is an $\mathfrak{X}$-group or it contains a maximal subgroup which belongs to $\mathfrak{X}$.*

Proof. Suppose that G does not belong to $\mathfrak{X}$. As G is locally nilpotent, it contains an abelian subgroup A of infinite rank (see p. 38 in part 2 of [18]). On the other hand, the class $\mathfrak{X}$ is ***L***-closed, and hence by Zorn's Lemma there exists a maximal $\mathfrak{X}$-subgroup M of G containing A. If x is any element of $G\setminus M$, the subgroup $\langle x, M\rangle$ has infinite rank and does not belong to $\mathfrak{X}$, so that $\langle x, M\rangle = G$. Therefore M is a maximal subgroup of G, and the lemma is proved. □

3. Restrictions on commutator subgroups

We begin this section with some results concerning groups in which proper subgroups of infinite rank have finite commutator subgroup or certain slightly stronger properties. The structure of groups whose proper subgroups have finite commutator subgroup has been investigated by V. V. Belyaev and N. F. Sesekin in [1]; in particular, it turns out that any locally graded minimal non-(finite-by-abelian) group is a Černikov group, and so in particular it is locally finite and has finite rank. Recall here that, if $\mathfrak{X}$ is a group class, a group G is *minimal non-$\mathfrak{X}$* if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$.

Proposition 3.1. *Let G be a strongly locally graded group of infinite rank. If evert proper subgroup of infinite rank of G has a finite commutator subgroup, then also the commutator subgroup G' of G is finite.*

Proof. As every proper subgroup of G either is finite-by-abelian or has finite rank, it follows that G' has finite rank (see [8]). Then every proper subgroup of G has a finite commutator subgroup by Lemma 2.7, and hence G' is finite by the theorem of Belyaev and Sesekin. □

Corollary 3.2. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are central-by-finite, then the factor group $G/Z(G)$ is finite.*

Proof. A famous theorem of Schur proves that every central-by-finite group has a finite commutator subgroup. Then every proper subgroup of infinite rank of G has finite commutator subgroup (see, for instance, Theorem 4.12 of part 1 of [18]), and hence it follows from Proposition 3.1 that the commutator subgroup G' of G is likewise finite. In particular, every element of G has finitely many conjugates and so its centralizer has finite index in G. Of course, it can be assumed that G is not abelian, so that it must contain a proper subgroup H of finite index. Then $H/Z(H)$ is finite, so that G is abelian-by-finite and hence $G/Z(G)$ is finite, as G' is finite. □

A group is said to be *metahamiltonian* if all its nonabelian subgroups are normal. Metahamiltonian groups have been introduced and investigated by G. M. Romalis and N. F. Sesekin ([19], [20] and [21]), who proved in particular that the commutator subgroup of any locally graded metahamiltonian group is finite with prime-power order.

Corollary 3.3. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are metahamiltonian, then also G is metahamiltonian.*

Proof. As a locally graded metahamiltonian group has a finite commutator subgroup, it follows from Proposition 3.1 that the commutator subgroup G' of G is finite. Then all proper subgroups of G are metahamiltonian by Lemma 2.7, and hence G itself is metahamiltonian, because locally graded minimal nonmetahamiltonian groups are finite (see [3], Lemma 4.2). □

The rest of this section is devoted to the proof of a theorem concerning groups whose proper subgroups of infinite rank have commutator subgroups in a given class. Theorem A will be obtained as a special case of this result. In the following, we shall denote by $\mathfrak{A}$ the class of all abelian groups, so that, for any $\boldsymbol{S}$-closed group class $\mathfrak{X}$, $\mathfrak{X}\mathfrak{A}$ is the class of groups whose commutator subgroup belongs to $\mathfrak{X}$. Our next two results provide, in particular, information on the structure of minimal non-$(\boldsymbol{L}\mathfrak{F})\mathfrak{A}$ groups.

Lemma 3.4. *Let $\mathfrak{X}$ be a class of groups such that $\boldsymbol{S}\mathfrak{X}=\boldsymbol{H}\mathfrak{X}=\mathfrak{X}$ and $(\boldsymbol{L}\mathfrak{X})\mathfrak{X}=\boldsymbol{L}\mathfrak{X}$, and suppose that all strongly locally graded minimal non-$\mathfrak{X}\mathfrak{A}$ groups are locally $\mathfrak{X}$. If G is a minimal non-$(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ group, then G is finitely generated and has no proper subgroups of finite index.*

Proof. As the class $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ is clearly $\boldsymbol{S}$ and $\boldsymbol{L}$-closed, the group G must be finitely generated. Assume now that G contains a proper normal subgroup H of finite index. Then H is an $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$-group and so its commutator subgroup H' belongs to $\boldsymbol{L}\mathfrak{X}$. Moreover, the factor group G/H' is a finitely generated abelian-by-finite group, and in particular it satisfies the maximal condition on subgroups, so that all its proper subgroups belong to $\mathfrak{X}\mathfrak{A}$. As minimal non-$\mathfrak{X}\mathfrak{A}$ homomorphic images of G are locally $\mathfrak{X}$, it follows that G/H' belongs to $\mathfrak{X}\mathfrak{A}$. Therefore G'/H' is an $\mathfrak{X}$-group and hence G' belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{X}=\boldsymbol{L}\mathfrak{X}$. This contradiction proves that the group G contains no proper subgroups of finite index. □

Corollary 3.5. *Let $\mathfrak{X}$ be a class of groups such that $\boldsymbol{S}\mathfrak{X}=\boldsymbol{H}\mathfrak{X}=\mathfrak{X}$ and $(\boldsymbol{L}\mathfrak{X})\mathfrak{X}=\boldsymbol{L}\mathfrak{X}$, and suppose that all strongly locally graded minimal non-$\mathfrak{X}\mathfrak{A}$ groups are locally $\mathfrak{X}$. If G is a group whose proper subgroups are in the class $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ and either G is locally graded or $G' \neq G$, then G itself belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$.*

Recall that a group class $\mathfrak{X}$ is said to be a *Schur class* if for any group G such that the factor group $G/Z(G)$ belongs to $\mathfrak{X}$ also the commutator subgroup G' of G is an $\mathfrak{X}$-group. Thus the already quoted theorem of Schur on the finiteness of the commutator subgroup of a central-by-finite group just states that the class $\mathfrak{F}$ of finite groups is a Schur class. Moreover, if $\mathfrak{X}$ is any Schur class which is $\boldsymbol{S}$ and $\boldsymbol{H}$-closed, it is easy to show that the class $\boldsymbol{L}\mathfrak{X}$ consisting of all (locally $\mathfrak{X}$) groups also has the Schur property; in particular, the class $\boldsymbol{L}\mathfrak{F}$ of all locally finite groups is a Schur class. Many other Schur classes of groups can be introduced by imposing suitable finiteness conditions (see, for instance, [11]).

Lemma 3.6. *Let $\mathfrak{X}$ be a Schur class containing all finite groups and such that $\boldsymbol{S}\mathfrak{X} = \boldsymbol{H}\mathfrak{X} = \mathfrak{X}$ and $(\boldsymbol{L}\mathfrak{X})\mathfrak{X} = \boldsymbol{L}\mathfrak{X}$, and let G be a strongly locally graded group whose proper subgroups of infinite rank belong to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$. If G contains a normal subgroup K such that G/K is simple of infinite rank, then G belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$.*

Proof. The class $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ is clearly $\boldsymbol{S}$ and $\boldsymbol{L}$-closed, so that it follows from Corollary 2.6 that the simple group G/K belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ and hence it must be locally $\mathfrak{X}$. Moreover, Lemma 2.3 yields that G/K contains a proper subgroup H/K of infinite rank. Then H belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ and so its commutator subgroup H' is locally $\mathfrak{X}$. Let R be the $\boldsymbol{L}\mathfrak{X}$-radical of K. Then $H' \cap K$ is contained in R, so that K/R is abelian and $[H', K] \leq R$. Therefore $H'R/R$ is contained in $C_{G/R}(K/R)$, and so K/R is a proper subgroup of its centralizer (observe here that if H' is contained in K, then H/R is abelian and hence it centralizes K/R). It follows that K/R is contained in the centre of G/R, and so the group $G'R/R$ is locally $\mathfrak{X}$, as $\boldsymbol{L}\mathfrak{X}$ is a Schur class. On the other hand, the subgroup R belongs to the class $\boldsymbol{L}\mathfrak{X}$ by Corollary 2.2, so that G' is locally $\mathfrak{X}$ by Lemma 2.1 and G lies in $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$. □

Lemma 3.7. *Let $\mathfrak{X}$ be a group class containing all finite groups and such that $\boldsymbol{S}\mathfrak{X} = \boldsymbol{H}\mathfrak{X} = \mathfrak{X}$, $(\boldsymbol{L}\mathfrak{X})\mathfrak{X} = \boldsymbol{L}\mathfrak{X}$ and all strongly locally graded minimal non-$\mathfrak{X}\mathfrak{A}$ groups are locally $\mathfrak{X}$. If G is a group of infinite rank whose proper subgroups of infinite rank belong to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$, and $G' \neq G$, then G belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$.*

Proof. Assume first that the commutator subgroup G' of G has finite rank, so that all proper subgroups of G belong to the class $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ by Lemma 2.7. Then G itself lies in $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ by Corollary 3.5.

Suppose now that G' has infinite rank, so that G' belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$, and hence G'' is locally $\mathfrak{X}$. If G'' has infinite rank, then all proper subgroups of G/G'' lie in $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$ and another application of Corollary 3.5 yields that G'/G'' is locally $\mathfrak{X}$. As the class $\boldsymbol{L}\mathfrak{X}$ is $\boldsymbol{P}$-closed by Lemma 2.1, it follows that G' belongs to $\boldsymbol{L}\mathfrak{X}$, and so G is in $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$. Assume finally that G'' has finite rank. Then G/G'' has infinite rank, and it is enough to show that the group G/G'' belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$, so that replacing G by G/G'' we may suppose without loss of generality that G is metabelian. Assume, aiming at a contradiction, that G' is not locally $\mathfrak{X}$, and let R be the $\boldsymbol{L}\mathfrak{X}$-radical of G'. As R is locally $\mathfrak{X}$ by Corollary 2.2, it is properly contained in G' and so G/R is not abelian. On the other hand, if X is any proper subgroup of infinite rank of G, then X' is locally $\mathfrak{X}$, and hence it is contained in R. Therefore all proper subgroups of infinite rank of G/R are abelian and it follows that G/R must have finite rank (see [7]). Thus R has infinite rank, so that all proper subgroups of G/R are abelian, and G/R is a soluble minimal nonabelian group. Therefore G/R is finite and G is locally $\mathfrak{X}$, since all finite groups belong to $\mathfrak{X}$. This contradiction proves the lemma. □

We are now ready to prove the main result of this section.

Theorem 3.8. *Let $\mathfrak{X}$ be a Schur class containing all finite groups and such that $\boldsymbol{S}\mathfrak{X} = \boldsymbol{H}\mathfrak{X} = \mathfrak{X}$, $(\boldsymbol{L}\mathfrak{X})\mathfrak{X} = \boldsymbol{L}\mathfrak{X}$ and suppose that all strongly locally graded minimal non-$\mathfrak{X}\mathfrak{A}$ groups are locally $\mathfrak{X}$. If G is a strongly locally graded group of infinite rank whose proper subgroups of infinite rank belong to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$, then G belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$.*

Proof. By Lemma 3.7 we may suppose that the group G is perfect. Suppose first that G contains a proper normal subgroup N of infinite rank. Then all proper subgroups of G/N belong to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$, so that also G/N lies in this class by Corollary 3.5, and hence G/N is locally $\mathfrak{X}$ as $G' = G$. Let V be the $\boldsymbol{L}\mathfrak{X}$-radical of N, and let g be any element of G. Clearly $\langle g, N\rangle$ is a proper subgroup of infinite rank of G, so that it belongs to $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$, and in particular $[N, g]$ is locally $\mathfrak{X}$. Then $[N, g]$ is contained in V, and so N/V lies in the centre of G/V. Since $\boldsymbol{L}\mathfrak{X}$ is a Schur class and G is perfect, it follows that G/V belongs to $\boldsymbol{L}\mathfrak{X}$, so that G is locally $\mathfrak{X}$, as V is in $\boldsymbol{L}\mathfrak{X}$ and $\boldsymbol{L}\mathfrak{X}$ is $\boldsymbol{P}$-closed.

Assume now that all proper normal subgroups of G have finite rank. It follows from Lemma 3.6 that the statement is true if G has a simple homomorphic image of infinite rank, so that by Lemma 2.4 we may suppose that G is locally nilpotent. Assume, aiming at a contradiction, that G does not belong to the class $(\boldsymbol{L}\mathfrak{X})\mathfrak{A}$, so that it contains a finitely generated subgroup E with the same property. Clearly, G has a countable subgroup H of infinite rank, and we have $G = \langle E, H\rangle$, so that G must be countable and in particular all its cyclic subgroups are ascendant. Let W be the $\boldsymbol{L}\mathfrak{X}$-radical of G. Then W is locally $\mathfrak{X}$ and the $\boldsymbol{L}\mathfrak{X}$-radical of G/W is trivial, as $\boldsymbol{L}\mathfrak{X}$ is $\boldsymbol{P}$-closed. It follows that G/W has no cyclic $\mathfrak{X}$-subgroups, and hence W contains all locally $\mathfrak{X}$ subgroups of G. Therefore, the commutator subgroup of any proper subgroup of infinite rank of G is contained in W, and so all proper subgroups of infinite rank of G/W are abelian. On the other hand, G/W has infinite rank (as W is a proper subgroup of G), so that G/W is abelian (see [7]) and hence $G' \leq W$ is locally $\mathfrak{X}$. This contradiction completes the proof of the theorem. □

The result of Belyaev and Sesekin quoted at the beginning of this section, together with Schur's theorem, allows us to apply Theorem 3.8 when $\mathfrak{X}$ is the class $\mathfrak{F}$ of all finite groups, and shows that Theorem A is a special case of this latter result.

4. Quasihamiltonian groups

Recall that a lattice $\mathfrak{L}$ is *modular* if the identity

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

holds for all elements x, y and z of $\mathfrak{L}$ such that $x \leq z$. Obviously, if G is any abelian group, the lattice $\mathfrak{L}(G)$, consisting of all subgroups of G, is modular, and hence groups with modular subgroup lattice naturally arise in the study of isomorphisms between lattices of subgroups. On the other hand, there exist also infinite simple groups with modular subgroup lattices, such as, for instance, Tarski groups (i.e., infinite simple groups all of whose proper nontrivial subgroups have prime order). The main result of this section describes the behavior of groups whose proper subgroups of infinite rank have a modular subgroup lattice. The structure of groups with modular subgroup lattices has been completely described by K. Iwasawa and R. Schmidt. Our next statement collects some of their results (see Theorems 2.4.11, 2.4.16 and 2.4.21 of [22]); for other properties of groups with modular subgroup lattices we refer to chapter 2 of [22].

Lemma 4.1. *Let G be a locally graded group with modular subgroup lattice. Then G is metabelian, the elements of finite order of G form a locally finite subgroup T, and G/T is abelian. Moreover, if G is nonperiodic, then it is quasihamiltonian, and either G is abelian or G/T has rank 1.*

We will also need the following information on the subgroup lattices of the projective special linear group and the Suzuki group over a locally finite field.

Lemma 4.2. *Let K be an infinite locally finite field. Then the simple groups $PSL(2, K)$ and $Sz(K)$ contain proper subgroups of infinite rank with nonmodular subgroup lattice.*

Proof. Let G be one of the groups $PSL(2, K)$ and $Sz(K)$. In [16], J. Otal and J. M. Peña proved that G contains a subgroup H which is a semidirect product of a normal subgroup V by a subgroup U of infinite rank such that there exist subgroups of V which are not normalized by U. In this situation it is well known that the lattice $\mathfrak{L}(H)$ is not modular (see, for instance, Lemma 3.3 of [4]). □

Lemma 4.3. *Let G be a locally finite group containing a normal subgroup N of infinite rank such that each subgroup of N has finitely many conjugates in G. Then every finite subgroup of G is contained in a proper subgroup of infinite rank.*

Proof. Since N has finite conjugacy classes of subgroups, the factor group $N/Z(N)$ is finite (see [15]), and so $Z(N)$ contains a subgroup A which is the direct product of infinitely many groups of prime order. If H is any finite subgroup of G, there exists a subgroup B of A such that

$$A = (A \cap H) \times B,$$

and we may consider a proper subgroup of finite index V of B. Since V has only finitely many conjugates in G and the index $|A : V|$ is finite, also the core W of V in G has finite index in A. In particular, W has infinite rank and the product HW is a proper subgroup of G containing H. □

We are now ready to prove the following result.

Theorem 4.4. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G have a modular subgroup lattice, then also the lattice of subgroups of G is modular.*

Proof. The commutator subgroup of any locally graded group with modular subgroup lattice is locally finite by Lemma 4.1, so that it follows from Theorem A that the commutator subgroup G' of G is locally finite. In particular, the elements of finite order of G form a subgroup T and G/T is torsion-free abelian.

If G' has finite rank, we obviously have that G is not finitely generated; moreover, by Lemma 2.7, every proper subgroup of G has modular subgroup lattice, and hence G itself has modular subgroup lattice since the class of groups with modular subgroup lattices is $\boldsymbol{L}$-closed (see, for instance, Lemma 5.1 of [12]).

Suppose that G' has infinite rank, so that Lemma 2.8 allows us to suppose that G/G' is finitely generated and in particular G/T is a free abelian group of finite rank. Assume first that G is not periodic. If G/T has rank at least 2, then

$$G/T = \langle aT \rangle \times H/T,$$

where a is an element of infinite order and $H \neq T$. For each prime number p, the proper subgroup $\langle a^p, H \rangle$ of G has infinite rank and so it has modular subgroup lattice; it follows that $\langle a^p, H \rangle$ is abelian, since the rank of $\langle a^p, H \rangle/T$ is greater than 1. Then $[H, a] = \{1\}$ and $G = \langle a, H \rangle$ is likewise abelian. On the other hand, if G/T is cyclic, we have $G = \langle a \rangle \ltimes T$ for some a, and for each prime number p the lattice $\mathfrak{L}(\langle a^p, T \rangle)$ is modular; then T is abelian, a normalizes all subgroups of T and $[x, a] = 1$ for each element x of T whose order is either a prime or 4, and so it follows that also the subgroup lattice of G is modular (see Theorem 2.4.11 of [22]).

Suppose now that G is periodic (and so even locally finite), so that G/G' is finite. If every proper normal subgroup of G has finite rank, it follows from Lemma 2.4 that either G is locally nilpotent or it has a simple homomorphic image G/K of infinite rank. On the other hand, all proper subgroups of G are (locally soluble)-by-finite by Černikov's result and hence in the latter case the group G/K must be isomorphic either to $PSL(2, F)$ or to $Sz(F)$ for some infinite locally finite field (see [13]), and this is impossible by Lemma 4.2. Therefore G is locally nilpotent and has no maximal subgroups, so that G is quasihamiltonian by Lemma 2.9.

Assume finally that G contains a proper normal subgroup N of infinite rank. Then the lattice $\mathfrak{L}(N)$ is modular and every proper subgroup of G/N has modular subgroup lattice, so that either the lattice $\mathfrak{L}(G/N)$ is modular or G/N is a finite group whose proper subgroups are supersoluble (note here that a finite group with modular subgroup lattice is supersoluble). In both cases G/N is soluble, and hence G is a soluble group. Therefore G' is a proper subgroup of infinite rank, so that G' has modular subgroup lattice and hence G'' is abelian and all subgroups of G'' are normal in G'. In particular, every subgroup of G'' has finitely many conjugates in G. If G'' has infinite rank, it follows from Lemma 4.3 that each finite subgroup of G is contained in a proper subgroup of infinite rank, so that every finite subgroup has modular subgroup lattice and also the lattice $\mathfrak{L}(G)$ is modular. Finally, if G'' has finite rank, we have obviously that G'/G'' has infinite rank, and another application of Lemma 4.3 yields that every finite subgroup of G/G'' is contained in a proper subgroup of infinite rank, and hence it has modular subgroup lattice. Therefore the lattice $\mathfrak{L}(G/G'')$ is modular, and so all subgroups of G'/G'' are normal in G/G''. Let U/G'' be a subgroup of infinite rank of G'/G'' such that also G/U has infinite rank. For each finite subgroup X of G, the product XU is a proper subgroup of infinite rank, so that $\mathfrak{L}(XU)$ is modular and hence X has modular subgroup lattice. Therefore the lattice $\mathfrak{L}(G)$ is modular also in this case, and the proof is complete. □

We just mention here that if G is a strongly locally graded group whose proper subgroups of infinite rank have distributive subgroup lattices, then G has finite rank. In fact, it is well known that a group has a distributive subgroup lattice if and only if it is locally cyclic (see Theorem 1.2.3 of [22]), and hence the above condition is just the requirement that all proper subgroups of G have finite rank; in this situation, it follows from Lemma 2.3 that G must have finite rank.

Proof of Theorem B. Assume, aiming at a contradiction, that the group G is not quasihamiltonian. As the lattice $\mathfrak{L}(G)$ is modular by Theorem 4.4, it follows from Lemma 4.1 that G is locally finite, and so it contains a finite subgroup which is not quasihamiltonian. Then G' has infinite rank by Lemma 2.7. Moreover, G' is abelian and all its subgroups are normal in G, so that it follows from Lemma 4.3 that each finite subgroup of G is contained in a proper subgroup of infinite rank. This contradiction proves the theorem. □

References

[1] Belyaev, V. V. and Sesekin, N. F.: On infinite groups of Miller–Moreno type. *Acta Math. Acad. Sci. Hungar.* **26** (1975), 369–376.

[2] Černikov, N. S.: A theorem on groups of finite special rank. *Ukrainian Mat. J.* **42** (1990), no. 7, 855–861.

[3] De Falco, M., de Giovanni, F. and Musella, C.: Groups whose finite homomorphic images are metahamiltonian. *Comm. Algebra* **37** (2009), no. 7, 2468–2476.

[4] De Falco, M., de Giovanni, F., Musella, C. and Sysak, Y. P.: Periodic groups with nearly modular subgroup lattice. *Illinois J. Math.* **47** (2003), no. 1-2, 189–205.

[5] De Falco, M., de Giovanni, F., Musella, C. and Trabelsi, N.: Groups whose proper subgroups of infinite rank have finite conjugacy classes. *Bull. Aust. Math. Soc.* **89** (2014), no. 1, 41–48.

[6] Dixon, M. R., Evans, M. J. and Smith, H.: Locally (soluble-by-finite) groups of finite rank. *J. Algebra* **182** (1996), no. 3, 756–769.

[7] Dixon, M. R., Evans, M. J. and Smith, H.: Locally soluble-by-finite groups with all proper non-nilpotent subgroups of finite rank. *J. Pure Appl. Algebra* **135** (1999), no. 1, 33–43.

[8] Dixon, M. R., Evans, M. J. and Smith, H.: Groups with all proper subgroups (finite rank)-by-nilpotent. *Arch. Math. (Basel)* **72** (1999), no. 5, 321–327.

[9] Dixon, M. R. and Karatas, Z. Y.: Groups with all subgroups permutable or of finite rank. *Centr. Eur. J. Math.* **10** (2012), no. 3, 950–957.

[10] Evans, M. J. and Kim, Y.: On groups in which every subgroup of infinite rank is subnormal of bounded defect. *Comm. Algebra* **32** (2004), no. 7, 2547–2557.

[11] Franciosi, S., de Giovanni, F. and Kurdachenko, L. A.: The Schur property and groups with uniform conjugacy classes. *J. Algebra* **174** (1995), no. 3, 823–847.

[12] de Giovanni, F., Musella, C. and Sysak, Y. P.: Groups with almost modular subgroup lattice. *J. Algebra* **243** (2001), no. 2, 738–764.

[13] Kleidman, P. B. and Wilson, R. A.: A characterization of some locally finite simple groups of Lie type. *Arch. Math. (Basel)* **48** (1987), no. 1, 10–14.

[14] Mal'cev, A. I.: On certain classes of infinite solvable groups. *Mat. Sb.* **28** (1951), 567–588. *Amer. Math. Soc. Transl. (2)* **2** (1956), 1–21.

[15] Neumann, B. H.: Groups with finite classes of conjugate subgroups. *Math. Z.* **63** (1955), 76–96.

[16] Otal, J. and Peña, J. M.: Infinite locally finite groups of type $PSL(2, K)$ or $Sz(K)$ are not minimal under certain conditions. *Publ. Mat.* **32** (1988), no. 1, 43–47.

[17] Robinson, D. J. S.: A note on groups of finite rank. *Compositio Math.* **31** (1969), 240–246.

[18] Robinson, D. J. S.: *Finiteness conditions and generalized soluble groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete 62-63, Springer-Verlag, New York-Berlin, 1972.

[19] Romalis, G. M. and Sesekin, N. F.: Metahamiltonian groups. *Ural. Gos. Univ. Mat. Zap.* **5** (1966), 101–106.

[20] Romalis, G. M. and Sesekin, N. F.: Metahamiltonian groups II. *Ural. Gos. Univ. Mat. Zap.* **6** (1968), 50–52.

[21] Romalis, G. M. and Sesekin, N. F.: Metahamiltonian groups III. *Ural. Gos. Univ. Mat. Zap.* **7** (1969/70), 195–199.

[22] Schmidt, R.: *Subgroup lattices of groups.* De Gruyter Expositions in Mathematics 14, Walter de Guyter, Berlin, 1994.

This work was partially supported by MIUR-PRIN 2009 (Teoria dei Gruppi e Applicazioni). The first three authors are members of GNSAGA (INdAM); the last author is grateful for the hospitality and financial support of the University of Napoli Federico II during the preparation of this paper.

Received June 26, 2012.

Maria De Falco: Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Complesso Universitario Monte S. Angelo, Via Cintia, 80126 Napoli, Italy.
E-mail: mdefalco@unina.it

Francesco de Giovanni: Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Complesso Universitario Monte S. Angelo, Via Cintia, 80126 Napoli, Italy.
E-mail: degiovan@unina.it

Carmela Musella: Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Complesso Universitario Monte S. Angelo, Via Cintia, 80126 Napoli, Italy.
E-mail: cmusella@unina.it

Nadir Trabelsi: Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, University of Setif, Setif 19000, Algeria.
E-mail: nadir_trabelsi@yahoo.fr