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# On inhomogeneous Strichartz estimates for the Schrödinger equation

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**Abstract.** In this paper we consider inhomogeneous Strichartz estimates in the mixed norm spaces which are given by taking temporal integration before spatial integration. We obtain some new estimates, and discuss the necessary conditions.

## 1. Introduction

To begin, we consider the Cauchy problem

$$\begin{cases} iu_t + \Delta u = F(x,t), & (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x). \end{cases}$$

By Duhamel's principle we have the solution

$$u(x,t) = e^{it\Delta} f(x) - i \int_0^t e^{i(t-s)\Delta}\, F(s)\, ds.$$

Here $e^{it\Delta}$ is the free propagator given by

$$e^{it\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)}\, \widehat{f}(\xi)\, d\xi.$$

The estimates for the solution in terms of $f$ and $F$ play important roles in the study of nonlinear Schrödinger equations (see [5] and [23]). Estimating the solution $u$ consists in two parts, the homogeneous ($F = 0$) and the inhomogeneous ($f = 0$) part.

It is well known that the homogeneous Strichartz estimate

$$\big\| e^{it\Delta} f \big\|_{L^q_t L^r_x} \le C\, \|f\|_2 \tag{1.1}$$

holds if and only if $2/q = n(1/2 - 1/r)$, $q \ge 2$, and $(q,r,n) \ne (2,\infty,2)$ (see [11], [13] and the references therein).

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However, the determination of optimal range of $(q, r)$ and $(\widetilde{q}', \widetilde{r}')$ for which the inhomogeneous Strichartz estimate

$$\Big\| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \Big\|_{L_t^q L_x^r} \le C \, \|F\|_{L_t^{\widetilde{q}'} L_x^{\widetilde{r}'}} \tag{1.2}$$

holds is not yet completed. By duality the homogeneous estimates imply some inhomogeneous estimates but it was observed that the estimate (1.2) is valid on a wider range than that given by admissible pairs $(q, r)$ and $(\widetilde{q}', \widetilde{r}')$ for the homogeneous estimates (1.1) (see [6] and [12]). Foschi and Vilela in their independent works ([10] and [25]) obtained the currently best known range of $(q, r)$ and $(\widetilde{q}', \widetilde{r}')$ for which (1.2) holds. However, there still remain some gaps between their range and the known necessary conditions. See also [19] for a new necessary condition and some weak endpoint estimates.

### 1.1. Time-space estimates

We now consider estimates in different mixed norms which are given by taking time integration before spatial integration. We call (1.1) and (1.2) *space-time estimates*, and by *a time-space estimate* we mean an estimate given in $L_x^r L_t^q$ norms; e.g. (1.3) or (1.4). Besides the estimate (1.3) the homogeneous time-space estimates

$$\|e^{it\Delta} f\|_{L_x^r L_t^q} \le C \|f\|_{\dot{H}^s}, \quad s = n/2 - 2/q - n/r, \tag{1.3}$$

have been of interest. Here $\dot{H}^s$ denotes the homogeneous Sobolev space of order $s$. Even though (1.1) and (1.3) have the same scaling, they are of different natures. In particular, for the time-space estimate Galilean invariance is no longer valid. The condition $1/q + (n+1)/r \le n/2$ is necessary for (1.3) even with a frequency localized initial datum $f$ as is easily seen by using Knapp's example. It is currently conjectured that (1.3) holds whenever $1/q + (n+1)/r \le n/2$, $2 \le q < \infty$. When $n = 1$, this is known to be true [14]. In higher dimensions (1.3) is known for $q$ and $r$ satisfying $1/q + (n+1)/r \le n/2$, and additionally $r > 16/5$ when $n = 2$ and for $r > 2(n+3)/(n+1)$ when $n \ge 3$ ([17]). The estimate (1.3) is closely related to the maximal Schrödinger estimate which has been studied to obtain almost everywhere convergence to the initial data. See [4], [8], [21], [24], [14], [17], [20] and references therein for further discussions and related issues. Also see [15], [1] for recent results.

In this paper we seek the optimal range of $(\widetilde{r}', r)$ for which the time-space inhomogeneous Strichartz estimate

$$\Big\| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \Big\|_{L_x^r L_t^q} \le C \|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}} \tag{1.4}$$

holds for some $q$ and $\widetilde{q}'$. Obviously, this is weaker than (1.2) if $q \le r$ and $\widetilde{q}' \ge \widetilde{r}'$ since one can get (1.4) from (1.2) via Minkowski's inequality. However, it turns out that the range for (1.4) is quite different from that of (1.2). The currently known range of $(1/\widetilde{r}', 1/r)$ for which (1.2) is valid for some $q$ and $\widetilde{q}'$ is contained in

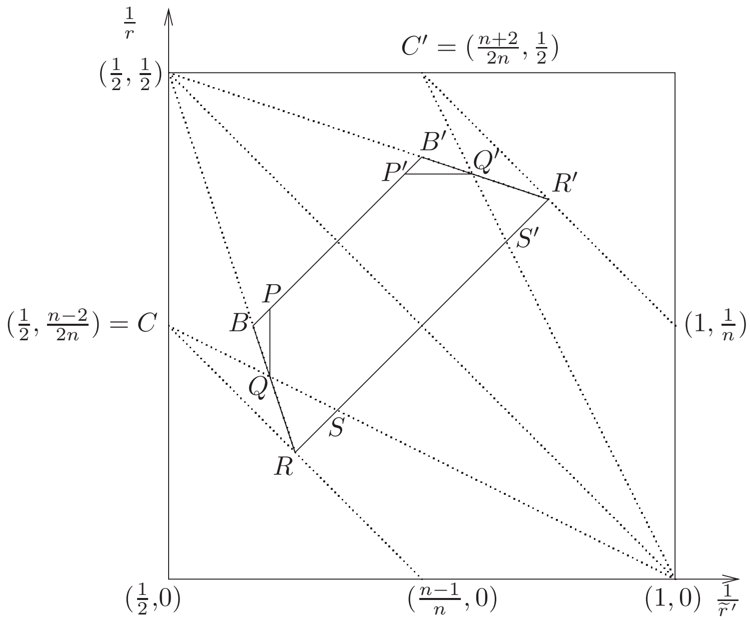


FIGURE 1. The points $B$, $C$, $P$, $Q$, $R$ and $S$, and the dual points $B'$, $C'$, $P'$, $Q'$, $R'$ and $S'$, when $n \geq 3$.

the closed pentagon with vertices $(1/2, 1/2)$, $C'$, $S'$, $S$, and $C$ (see Figure 1) and it is known that (1.2) fails unless $(1/\widetilde{r}', 1/r)$ is contained in the closed pentagon with vertices $(1/2, 1/2)$, $C'$, $R'$, $R$, and $C$. We will show that (1.4) is possible only if $(1/\widetilde{r}', 1/r)$ is contained in the closed trapezoid $B$, $R$, $R'$, $B'$ from which the points $R$ and $R'$ are removed. In [9] it was shown that if $1 \leq \widetilde{r}' \leq 2 \leq r \leq \infty$ and $|1/r + 1/\widetilde{r}' - 1| < 1/n$, there are $q$ and $\widetilde{q}'$ which allow the time delayed estimates in time-space norm. But in contrast to the space-time estimate (1.2) the above discussion shows that mere existence of such $q$ and $\widetilde{q}'$ for time delayed estimate is not enough to obtain (1.4) and accurate information on possible range of $q$ and $\widetilde{q}'$ is important.

To show (1.4) we work on the Fourier transform side by making use of the fact that the Duhamel part is similar to a multiplier of negative order (see [7], [18]). This allows us to take advantage of localization on the Fourier transform side. This plays important roles in our argument. We believe that this method is more flexible than the conventional argument which relies heavily on the dispersive estimate.

**Necessary conditions.** We now discuss the conditions on $(q, r)$ and $(\widetilde{q}', \widetilde{r}')$ which are necessary for (1.4). By scaling, the condition

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} + \frac{n}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{r}\right) = 1 \tag{1.5}$$

should be satisfied.

Using the examples in [10], [25], we see that the conditions which are needed for (1.2) are also necessary for (1.4):

$$\widetilde{r}' < 2 < r, \qquad \frac{1}{\widetilde{r}'} - \frac{1}{r} \le \frac{2}{n}, \qquad 1 - \frac{1}{n} \le \frac{1}{\widetilde{r}'} + \frac{1}{r} \le 1 + \frac{1}{n}, \tag{1.6}$$

$$\widetilde{q}' \le q, \qquad \frac{1}{q} < n\Big(\frac{1}{2} - \frac{1}{r}\Big), \qquad \frac{1}{\widetilde{q}'} > 1 - n\Big(\frac{1}{\widetilde{r}'} - \frac{1}{2}\Big). \tag{1.7}$$

By considering additional test functions, we get the following conditions which will be shown later (see Section 4):

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} + (n+1)\Big(\frac{1}{\widetilde{r}'} - \frac{1}{r}\Big) \ge 2, \tag{1.8}$$

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} \ge \frac{2n}{r} - n + 1, \quad \frac{1}{\widetilde{q}'} - \frac{1}{q} \ge n + 1 - \frac{2n}{\widetilde{r}'}. \tag{1.9}$$

To facilitate the statement of our results, for $n \ge 3$, we define points $B$, $C$, $P$, $Q$, $R$, and $S$ which are contained in $[1/2, 1] \times [0, 1/2]$ by setting

$$B = \Big(\frac{n+3}{2(n+2)}, \frac{n-1}{2(n+2)}\Big), \ C = \Big(\frac{1}{2}, \frac{n-2}{2n}\Big), \ P = \Big(\frac{n+2}{2(n+1)}, \frac{n^2}{2(n+1)(n+2)}\Big),$$
$$Q = \Big(\frac{n+2}{2(n+1)}, \frac{n-2}{2(n+1)}\Big), \ R = \Big(\frac{n+1}{2n}, \frac{n-3}{2n}\Big), \ S = \Big(\frac{n}{2(n-1)}, \frac{(n-2)^2}{2n(n-1)}\Big),$$

and we also define the dual points $B'$, $C'$, $P'$, $Q'$, $R'$, and $S'$ by setting $X' = (1-b, 1-a)$ when $X = (a, b)$. (See Figure 1.) Let $\mathcal{N}(n)$ be the closed trapezoid with vertices $B$, $B'$, $R$, and $R'$ from which the points $R$ and $R'$ are removed. Combined with (1.5), (1.8) gives

$$\frac{1}{\widetilde{r}'} - \frac{1}{r} \ge \frac{2}{n+2}, \tag{1.10}$$

and the first and second conditions in (1.9) give

$$n\Big(1 - \frac{1}{2\widetilde{r}'}\Big) \ge \frac{3n}{2r}, \qquad \frac{3n}{2\widetilde{r}'} \ge n\Big(1 - \frac{1}{2r}\Big), \tag{1.11}$$

respectively. Also, by (1.5) and (1.7), we see that $(1/\widetilde{r}', 1/r) \ne R$ and $(1/\widetilde{r}', 1/r) \ne R'$. Hence, from this, (1.6), (1.10) and (1.11), it follows that (1.4) holds only if $(1/\widetilde{r}', 1/r) \in \mathcal{N}(n)$.

**Sufficiency part.** We will show the stronger estimate

$$\Big\| \int_{-\infty}^{t} e^{i(t-s)\Delta} F(s)\, ds \Big\|_{L^r_x L^q_t} \le C \|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}, \tag{1.12}$$

which implies (1.4) and $\| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} F(s)\, ds \|_{L^r_x L^q_t} \le C \|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}$. As mentioned above, if $q \le r$ and $\widetilde{q}' \ge \widetilde{r}'$, from the known range of the space-time estimate ([10], [24]), one can get (1.12) for $(1/\widetilde{r}', 1/r)$ contained in the closed hexagon $\mathcal{H}$ with vertices $P$, $Q$, $S$, $P'$, $Q'$ and $S'$, from which the line segments $[P, Q]$ and $[P', Q']$

and the points $S, S'$ are removed[1]. We extend the range further to include the triangular region $\Delta QRS$ and $\Delta Q'R'S'$. It should be noted that no inhomogeneous space-time estimate (1.2) is known for $(1/\widetilde{r}', 1/r)$ contained in the interior of $\Delta QRS$ and $\Delta Q'R'S'$.

**Theorem 1.1.** *Let $n \geq 3$ and $\mathcal{S}(n)$ be the open hexagon with vertices $P$, $Q$, $R$, $P'$, $Q'$, and $R'$ to which the line segments $(P, P')$ and $(R, R')$ are added. If $(1/\widetilde{r}', 1/r) \in \mathcal{S}(n)$, then (1.12) holds for some $q$ and $\widetilde{q}'$.*

For $(1/\widetilde{r}', 1/r)$ contained in the region $\Delta QRS \setminus [Q, R]$, the estimate (1.12) is available if $(\widetilde{q}', q)$ satisfies (1.5) and (1.7) and additionally $1/q < n(1/\widetilde{r}' - 1/2)$, $1/\widetilde{q}' > 1 - n(1/2 - 1/r)$. With (1.5), these additional conditions are due to the third inequality of (3.2) and its dual. By duality the same holds for $(1/\widetilde{r}', 1/r)$ contained in the region $\Delta Q'R'S' \setminus [Q', R']$. Making use of the currently known time-space homogeneous estimates (1.3) (see [16] and [17]) together with the argument of this paper, it is possible to obtain further estimates on a larger range of $q$ and $\widetilde{q}'$ but these estimates are not enough to extend the range of $(\widetilde{r}', r)$.

When $n = 2$, (1.12) holds if $(1/\widetilde{r}', 1/r)$ is contained in the open pentagon with vertices $P$, $Q$, $(1, 0)$, $Q'$ and $P'$, to which the line segment $(P, P')$ is added, but this is not new; it follows from the known range of the space-time estimate ([10], [25]). When $n = 1$, it is possible to obtain the full range except for some endpoint estimates. In fact, from the necessary conditions, (1.4) is possible only if $(1/\widetilde{r}', 1/r)$ is contained in the closed triangle $\Delta$ with vertices $(2/3, 0)$, $(1, 0)$, and $(1, 1/3)$.

**Theorem 1.2.** *Let $n = 1$. Then (1.12) holds for some $q$ and $\widetilde{q}'$ provided that $(1/\widetilde{r}', 1/r)$ is contained in $\Delta \setminus \big([(2/3, 0), (1, 0)] \cup [(1, 1/3), (1, 0)]\big)$. In fact, (1.12) holds if $q$ and $\widetilde{q}'$ satisfy $1 < \widetilde{q}' < 2 < q < \infty$ and $1/\widetilde{r}' - 1/r + 1/2\widetilde{q}' - 1/2q \geq 1$.*

The rest of this paper is organized as follows. In Section 2 we obtain some frequency localized estimates which will be used in later sections. Then, using these estimates and a summation method, we prove Theorem 1.1 and Theorem 1.2 in Section 3. Nextly, we show the necessary conditions (1.8) and (1.9) in Section 4.

Throughout the paper, the letter C stands for a constant which is possibly different at each occurrence. In addition to the symbol $\widehat{\ }$, we use $\mathcal{F}(\cdot)$ to denote the Fourier transform, and $\mathcal{F}^{-1}(\cdot)$ to denote the inverse Fourier transform. Finally, we denote by $\chi_E$ the characteristic function of a set $E$.

## 2. Preliminaries

In this section we prove several preliminary estimates which will be used for the proof of Theorem 1.1, which is to be shown in Section 3.

We define the operator $T_\delta$ for dyadic numbers $\delta \in 2^{\mathbb{Z}} := \{2^z : z \in \mathbb{Z}\}$ by

$$T_\delta F = \int \delta\, \phi(\delta(t-s))\, e^{i(t-s)\Delta} F(s)\, ds\,, \tag{2.1}$$

[1] In fact, when $q \leq r$ and $\widetilde{q}' \geq \widetilde{r}'$, (1.5) and (1.7) are satisfied for $(1/\widetilde{r}', 1/r)$ contained in $\mathcal{H}$. Hence, one can use the known space-time estimate.

where $\phi$ is a smooth function supported in $(1/2, 2)$ such that $\sum_{k=-\infty}^{\infty}\phi(2^k t) = 1$ for $t > 0$. Then we can write

$$TF := \int_{-\infty}^{t} e^{i(t-s)\Delta}F(s)\,ds = \sum_{\delta\in 2^{\mathbb{Z}}} \delta^{-1}\,T_\delta F. \tag{2.2}$$

By direct computation it is easy to see that

$$\widehat{T_\delta F}(\xi,\tau) = \widehat{\phi}\Big(\frac{\tau+|\xi|^2}{\delta}\Big)\,\widehat{F}(\xi,\tau). \tag{2.3}$$

By this dyadic decomposition in time, the boundedness problem for $T$ is essentially reduced to obtaining suitable bounds for $T_\delta$ in terms of $\delta$. From this one may view the operator $F \to \int_{-\infty}^{t} e^{i(t-s)\Delta}F(s)ds$ as the multiplier operator of negative order 1 which is associated to the paraboloid.

**Proposition 2.1.** *Let $n \ge 2$. Suppose that Fourier transform of $F$ is supported in $\{(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R} : 1/2 \le |\xi| \le 2\}$. Then we have*

$$\|T_\delta F\|_{L^r_x L^2_t} \le C\,\delta^{-(n-1)/2+n/\widetilde{r}'}\,\|F\|_{L^{\widetilde{r}'}_x L^2_t}$$

*for $r$ and $\widetilde{r}'$ satisfying $1 \le \widetilde{r}' \le 2$ and $(n+1)/r \le (n-1)(1-1/\widetilde{r}')$.*

*Proof.* By interpolation, it is enough to consider the cases $(\widetilde{r}', r) = (2, \frac{2(n+1)}{n-1})$ and $(1,\infty)$. This actually gives the estimates along the line $(n+1)/r = (n-1)(1-1/\widetilde{r}')$. The other estimates follow from Bernstein's inequality because the spatial Fourier transform of $F$ is compactly supported.

*The case $(\widetilde{r}', r) = (2, 2(n+1)/(n-1))$.* By duality it is enough to show that

$$\|T_\delta F\|_{L^2_x L^2_t} \le C\,\delta^{1/2}\,\|F\|_{L^{(2n+2)/(n+3)}_x L^2_t}.$$

Since $\widehat{F}(\cdot,\tau)$ is supported in $\{|\xi|\sim 1\}$, by (2.3) and Plancherel's theorem we have

$$\|T_\delta F\|^2_{L^2_x L^2_t} \le C \iint_{1/2\le|\xi|\le 2} \Big|\,\widehat{\phi}\Big(\frac{\tau+|\xi|^2}{\delta}\Big)\,\widehat{F}(\xi,\tau)\Big|^2\,d\xi\,d\tau.$$

Thus, we are reduced to showing that

$$\iint_{1/2\le|\xi|\le 2} \Big|\,\widehat{\phi}\Big(\frac{\tau+|\xi|^2}{\delta}\Big)\,\widehat{F}(\xi,\tau)\Big|^2\,d\xi\,d\tau \le C\,\delta\,\|F\|^2_{L^{(2n+2)/(n+3)}_x L^2_t}. \tag{2.4}$$

The left-hand side equals to

$$\iint_{1/2}^{2} \Big|\,\widehat{\phi}\Big(\frac{\tau+r^2}{\delta}\Big)\Big|^2 \int_{S^{n-1}} |\mathcal{F}_x\mathcal{F}_t F(r\theta,\tau)|^2 d\theta\, r^{n-1}\,dr\,d\tau.$$

Using the Tomas–Stein theorem [22] (the $L^{(2n+2)/(n+3)}$-$L^2$-restriction estimate to the sphere $rS^{n-1}$, $r\sim 1$), we see that

$$\int_{S^{n-1}} |\mathcal{F}_x\mathcal{F}_t F(r\theta,\tau)|^2 d\theta \le C\,\|\mathcal{F}_t F(\cdot,\tau)\|^2_{L^{(2n+2)/(n+3)}_x}.$$

Integrating in $r$, it follows that

$$\iint_{1/2\le|\xi|\le 2} \Big|\,\widehat{\phi}\Big(\frac{\tau+|\xi|^2}{\delta}\Big)\,\widehat{F}(\xi,\tau)\Big|^2\,d\xi\,d\tau \le C\,\delta\,\|\mathcal{F}_tF(\tau)\|^2_{L^2_\tau L_x^{(2n+2)/(n+3)}}.$$

By Minkowski's inequality and Plancherel's theorem, we get (2.4).

*The case* $(\widetilde{r}', r) = (1,\infty)$. Note that $T_\delta F$ can be written as

$$T_\delta F(x,t) = \iint K_\delta(x-y,t-s)F(y,s)\,dy\,ds,$$

where

$$\begin{aligned}K_\delta(y,s) &= \delta\phi(\delta s)\iint e^{i(-sr^2+ry\cdot\theta)}\,\psi(r)\,d\theta\,dr \\ &= \iiint \widehat{\phi}\Big(\frac{\tau+r^2}{\delta}\Big)\,e^{i(s\tau+ry\cdot\theta)}\,\psi(r)\,d\theta\,dr\,d\tau\end{aligned} \tag{2.5}$$

and $\psi\in C_0^\infty(1/2,2)$. Since $|K_\delta(y,s)|\le C|\delta\phi(\delta s)|$, by Young's inequality we have $\|T_\delta F\|_{L^\infty_x L^2_t}\le C\|F\|_{L^1_xL^2_t}$. We may assume that $\delta\lesssim 1$.

By the choice of $\phi$, $K_\delta\neq 0$ for $s\sim\delta^{-1}$. Hence, by integration by parts we see that, for any large $M$ and $N$, $|K_\delta(y,s)|\le C\delta^M$ if $|y|\le\delta^{-1}/100$ and $|K_\delta(y,s)|\le C(1+|y|)^{-N}$ if $|y|\ge 100\delta^{-1}$. Set $\chi_\delta(y) = \chi_{\{\delta^{-1}/100\le|y|\le 100\delta^{-1}\}}$, $\widetilde{K}_\delta(y,s) = K_\delta(y,s)\chi_\delta(y)$, and

$$\widetilde{T}_\delta F(x,t) = \iint \widetilde{K}_\delta(x-y,t-s)F(y,s)\,dy\,ds.$$

Then it is enough to show that

$$\|\widetilde{T}_\delta F\|_{L^\infty_x L^2_t}\le C\,\delta^{(n+1)/2}\,\|F\|_{L^1_xL^2_t}. \tag{2.6}$$

From (2.5), it follows that

$$\begin{aligned}&\mathcal{F}_t(\widetilde{T}_\delta F)(x,\tau)\\ &\quad= \int \mathcal{F}_tF(y,\tau)\int\widehat{\phi}\left(\frac{\tau+r^2}{\delta}\right)\chi_\delta(x-y)\psi(r)\Big(\int_{S^{n-1}} e^{ir(x-y)\cdot\theta}d\theta\Big)\,dr\,dy.\end{aligned}$$

Hence by Plancherel's theorem we see that $\|\widetilde{T}_\delta F(x,\cdot)\|^2_{L^2_t}$ is bounded by

$$\int\Big[\int|\mathcal{F}_tF(y,\tau)|\int\Big|\,\widehat{\phi}\Big(\frac{\tau+r^2}{\delta}\Big)\chi_\delta(x-y)\psi(r)\int_{S^{n-1}}e^{ir(x-y)\cdot\theta}d\theta\Big|\,dr\,dy\Big]^2d\tau.$$

Using the fact that $\int e^{irx\cdot\theta}d\theta = O(|x|^{-(n-1)/2})$ for large $|x|$, and integrating in $r$,

$$\|\widetilde{T}_\delta F(x,\cdot)\|^2_{L^2_t}\le C\delta^{n+1}\int\Big(\int|\mathcal{F}_tF(y,\tau)|\,dy\Big)^2d\tau.$$

By Minkowski's inequality and Plancherel's theorem,

$$\|\mathcal{F}_tF\|_{L^2_\tau L^1_x}\le\|\mathcal{F}_tF\|_{L^1_xL^2_\tau} = \|F\|_{L^1_xL^2_t}.$$

Hence we get (2.6). □

Throughout this paper we use several times the following summation lemma which is due to Bourgain [2] (see also [3] for a generalization). The lemma is a version of Lemma 2.3 in [18] for Banach-valued functions. (For a proof we refer the reader to [18].)

**Lemma 2.2.** *Let $\varepsilon_1, \varepsilon_2 > 0$. Let $1 \le q \le \infty$ and $1 \le r_1, r_2 < \infty$. Suppose that $f_1(y,z), f_2(y,z), \dots$ are a sequence of functions defined on $\mathbb{R}^l \times \mathbb{R}^m$ for which $\|f_j\|_{L^{r_1}_y L^q_z} \le M_1 2^{\varepsilon_1 j}$ and $\|f_j\|_{L^{r_2}_y L^q_z} \le M_2 2^{-\varepsilon_2 j}$ holds. Then*

$$\Big\| \sum f_j \Big\|_{L^{r,\infty}_y L^q_z} \le C\, M_1^{\theta}\, M_2^{1-\theta},$$

*where $\theta = \varepsilon_2/(\varepsilon_1+\varepsilon_2)$ and $1/r = \theta/r_1 + (1-\theta)/r_2$. Here we denote by $L^{r,\infty}_y$ the weak $L^r$ space.*

Using this lemma, we remove the assumption that the spatial Fourier transform of $F$ is supported in $\{|\xi| \sim 1\}$.

**Proposition 2.3.** *Let $n \ge 3$. Suppose that the spatial Fourier transform of $F$ is supported in $\{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R} : |\xi| \le 2\}$. Then we have*

$$\|T_\delta F\|_{L^r_x L^2_t} \le C\, \delta^{-(n-1)/2 + n/\widetilde{r}'}\ \|F\|_{L^{\widetilde{r}'}_x L^2_t}$$

*for $r$ and $\widetilde{r}'$ satisfying $n/(n-1) < \widetilde{r}' \le 2$ and $1/r + 1/\widetilde{r}' \le (n-1)/n$.*

*Proof.* Since we are assuming that $F$ is supported in $\{(\xi,\tau)\colon |\xi| \le 2\}$, we may decompose $T_\delta$ so that

$$T_\delta = \sum_{j \ge -1} T^j_\delta,$$

where $T^j_\delta$ is given by

$$\widehat{T^j_\delta F}(\xi,\tau) = \widehat{\phi}\Big(\frac{\tau + |\xi|^2}{\delta}\Big)\, \phi(2^j|\xi|)\, \widehat{F}(\xi,\tau).$$

By rescaling we have

$$T^j_\delta\, F(x,t) = T_{2^{2j}\delta}\, F_j(2^{-j}x, 2^{-2j}t),$$

where $F_j = \phi(|D|)F(2^j\cdot, 2^{2j}\cdot)$. Thus, by Proposition 2.1 we see that

$$\|T^j_\delta F\|_{L^r_x L^2_t} \le C\, \delta^{-(n-1)/2 + n/\widetilde{r}'}\ 2^{jn(1/r + 1/\widetilde{r}' - (n-1)/n)}\ \|F\|_{L^{\widetilde{r}'}_x L^2_t} \tag{2.7}$$

for $r$ and $\widetilde{r}'$ satisfying $1 \le \widetilde{r}' \le 2$ and $(n+1)/r \le (n-1)(1-1/\widetilde{r}')$. If $1/r + 1/\widetilde{r}' < (n-1)/n$, we can sum to get the desired estimate. To obtain the estimates for the endpoint cases $1/r + 1/\widetilde{r}' = (n-1)/n$, we use Lemma 2.2.

Fix $\widetilde{r}'$ and $r$ such that $1/r + 1/\widetilde{r}' = (n-1)/n$ and $n/(n-1) < \widetilde{r}' < 2$. We now choose $r_1$ and $r_2$, so that $(n+1)/r_i \le (n-1)(1-1/\widetilde{r}')$, $i = 1, 2$, and

$$\frac{1}{r_2} + \frac{1}{\widetilde{r}'} < \frac{n-1}{n} < \frac{1}{r_1} + \frac{1}{\widetilde{r}'}.$$

Note that $(1/\widetilde{r}', 1/r)$ is on the open segment joining $(1/\widetilde{r}', 1/r_1)$ and $(1/\widetilde{r}', 1/r_2)$. From (2.7) we see

$$\|T_\delta^j F\|_{L_x^{r_i} L_t^2} \le C\, \delta^{-(n-1)/2+n/\widetilde{r}'}\, 2^{jn(1/r_i+1/\widetilde{r}'-(n-1)/n)}\, \|F\|_{L_x^{\widetilde{r}'} L_t^2}$$

for $i = 1, 2$. Now we can apply Lemma 2.2 with $\varepsilon_i = n|1/r_i + 1/\widetilde{r}' - (n-1)/n|$. We get

$$\|T_\delta^j F\|_{L_x^{r,\infty} L_t^2} \le C\, \delta^{-(n-1)/2+n/\widetilde{r}'}\, \|F\|_{L_x^{\widetilde{r}'} L_t^2}.$$

This weak type estimate for $1/r + 1/\widetilde{r}' = (n-1)/n$ and $n/(n-1) < \widetilde{r}' < 2$ can be strengthened to strong type by real interpolation. Finally, the estimate for $(1/\widetilde{r}', 1/r) = (1/2, (n-2)/(2n))$ can be obtained directly from

$$\|T_\delta F\|_{L_t^2 L_x^{2n/(n-2)}} \le C\, \delta^{1/2}\, \|F\|_{L_t^2 L_x^2} \tag{2.8}$$

via Minkowski's inequality. This also follows from the endpoint space-time homogeneous estimate. Indeed, by Hölder's inequality we see

$$|T_\delta F(x,t)| \le C\, \delta^{1/2}\, \big\| e^{i(t-s)\Delta} F(s) \big\|_{L_s^2},$$

and so

$$\|T_\delta F\|_{L_t^2 L_x^{2n/(n-2)}} \le C\, \delta^{1/2}\, \big\| e^{i(t-s)\Delta} F(s) \big\|_{L_s^2 L_t^2 L_x^{2n/(n-2)}}$$

by Minkowski's inequality. Applying (1.1) with $(q, r) = (2, 2n/(n-2))$, we get (2.8). □

## 3. Sufficiency part: proofs of Theorems 1.1 and 1.2

In this section we will prove Theorems 1.1 and 1.2. We may assume that the space time Fourier transform of $F$ is supported in the set $\{(\xi, \tau) : |\xi| \le 2,\ |\tau| \le 2\}$ since this additional assumption can be simply removed by rescaling together with the condition (1.5).

*Proof of Theorem* 1.1. Since we already have the estimates in the hexagon $\mathcal{H}$, to show (1.12) it suffices to show the estimates when $(1/\widetilde{r}', 1/r) \in \big(\Delta QRS \cup \Delta Q'R'S'\big) \setminus \big([Q, R] \cup [Q', R']\big)$. By duality and complex interpolation, it is enough to show the case where $(1/\widetilde{r}', 1/r) \in \Delta QRS \setminus ([Q, R] \cup [Q, S])$.

Let $\Omega = \Omega(n)$ denote the closed triangle with vertices $C$, $((n-1)/n, 0)$, and $(1, 0)$ from which the point $((n-1)/n, 0)$ is removed. The proof is then based on the following estimate. For $(1/\widetilde{r}', 1/r) \in \Omega$,

$$\|T_\delta F\|_{L_x^r L_t^q} \le C\, \delta^{1/\widetilde{q}'-1/q+\frac{n}{2}(1/\widetilde{r}'-1/r)}\, \|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}} \tag{3.1}$$

holds provided that

$$-\frac{n}{2}\Big(\frac{1}{r} + \frac{1}{\widetilde{r}'} - 1\Big) \le \frac{1}{q} \le \frac{1}{\widetilde{q}'} \le 1 + \frac{n}{2}\Big(\frac{1}{r} + \frac{1}{\widetilde{r}'} - 1\Big).$$

This can be shown by interpolating the case $(1/\widetilde{r}', 1/r) = (1, 0)$ and the case in which $(1/\widetilde{r}', 1/r)$ is on the line segment joining $C$ and $((n-1)/n, 0)$. Since Proposition 2.3 already gives the estimates on the line segment, we only need to show that

$$\|T_\delta F\|_{L^\infty_x L^q_t} \le C\, \delta^{1/\widetilde{q}' - 1/q + n/2}\, \|F\|_{L^1_x L^{\widetilde{q}'}_t}$$

for $1 \le \widetilde{q}' \le q \le \infty$. By Minkowski's inequality, it is enough to show that

$$\|T_\delta F\|_{L^q_t L^\infty_x} \le \delta^{1/\widetilde{q}' - 1/q + n/2}\, \|F\|_{L^{\widetilde{q}'}_t L^1_x}.$$

Using the fact that $\big\|e^{i(t-s)\Delta} g\,\big\|_{L^\infty_x} \le C|t-s|^{-n/2}\|g\|_{L^1_x}$ (dispersive estimate), this follows from (2.1) and Young's inequality.

Now we fix $\widetilde{r}'$ and $r$ such that $(1/\widetilde{r}', 1/r) \in \Delta QRS \setminus ([Q,S] \cup [Q,R])$. We claim that there is $(1/\widetilde{q}', 1/q) \in [0,1] \times [0,1]$ which satisfies (1.5) and

$$-\frac{n}{2}\Big(\frac{1}{r} + \frac{1}{\widetilde{r}'} - 1\Big) < \frac{1}{q} \le \frac{1}{\widetilde{q}'} < 1 + \frac{n}{2}\Big(\frac{1}{r} + \frac{1}{\widetilde{r}'} - 1\Big). \tag{3.2}$$

Indeed, since $(1/\widetilde{r}', 1/r) \in \Delta QRS \setminus ([Q,S] \cup [Q,R])$, it follows that

$$0 \le 1 - \frac{n}{2}\Big(\frac{1}{\widetilde{r}'} - \frac{1}{r}\Big) < 1 - n\Big(\frac{1}{\widetilde{r}'} - \frac{1}{2}\Big) < 1 + \frac{n}{2}\Big(\frac{1}{r} + \frac{1}{\widetilde{r}'} - 1\Big) < 1. \tag{3.3}$$

The third inequality in (3.3) says that $(1/\widetilde{r}', 1/r)$ lies above the line joining $Q$ and $R$. Hence, there exists $1 < \widetilde{q}' < \infty$ such that

$$1 - n\Big(\frac{1}{\widetilde{r}'} - \frac{1}{2}\Big) < \frac{1}{\widetilde{q}'} < 1 + \frac{n}{2}\Big(\frac{1}{r} + \frac{1}{\widetilde{r}'} - 1\Big). \tag{3.4}$$

(Note that the first inequality is also one of the necessary conditions in (1.7).) Now just set $1/q = 1/\widetilde{q}' + n/2\,(1/\widetilde{r}' - 1/r) - 1$ and (1.5) is obviously satisfied . Then the first inequality in (3.3) gives the second in (3.2), and the first in (3.4) implies the first in (3.2). From (3.2), we can find a small neighborhood $V$ of $(1/\widetilde{r}', 1/r)$, contained in $\Omega$, such that, for $(1/a, 1/b) \in V$,

$$-\frac{n}{2}\Big(\frac{1}{b} + \frac{1}{a} - 1\Big) < \frac{1}{q} \le \frac{1}{\widetilde{q}'} < 1 + \frac{n}{2}\Big(\frac{1}{b} + \frac{1}{a} - 1\Big).$$

Therefore, by (3.1) we have, for $(1/a, 1/b) \in V$,

$$\|\delta^{-1}\, T_\delta F\|_{L^b_x L^q_t} \le C\, \delta^{1/\widetilde{q}' - 1/q + \frac{n}{2}(1/a - 1/b) - 1}\, \|F\|_{L^a_x L^{\widetilde{q}'}_t}. \tag{3.5}$$

Once this is obtained, we can prove the desired estimates by repeating the argument in the proof of Proposition 2.3. In fact, we consider a point $(1/a_0, 1/b_0) \in V$ on the line $1/a - 1/b = 1/\widetilde{r}' - 1/r$ and choose two points $(1/a_0, 1/b_i) \in V$, $i = 1, 2$, such that

$$\frac{1}{\widetilde{q}'} - \frac{1}{q} + \frac{n}{2}\Big(\frac{1}{a_0} - \frac{1}{b_1}\Big) - 1 < 0 < \frac{1}{\widetilde{q}'} - \frac{1}{q} + \frac{n}{2}\Big(\frac{1}{a_0} - \frac{1}{b_2}\Big) - 1.$$

Replacing $(a,b)$ with $(a_0,b_1)$ and $(a_0,b_2)$ in the inequality (3.5), we have two estimates to which we can apply Lemma 2.2 with $\varepsilon_i = |1/\widetilde{q}' - 1/q + \frac{n}{2}(1/a_0 - 1/b_i) - 1|$. Hence, we get

$$\Big\| \sum_{\delta\in 2^{\mathbb{Z}}} \delta^{-1}\, T_\delta F \Big\|_{L_x^{b_0,\infty} L_t^q} \le C\|F\|_{L_x^{a_0} L_t^{\widetilde{q}'}}$$

for all $(1/a_0, 1/b_0) \in V$ if $1/a - 1/b = 1/\widetilde{r}' - 1/r$. We now interpolate these estimates to get the strong type estimate, in particular, at $(1/\widetilde{r}', 1/r)$. This completes the proof. □

*Proof of Theorem* 1.2. First we claim that, for $1 \le \widetilde{r}', \widetilde{q}' \le 2 \le r, q \le \infty$, and $0 < \delta \ll 1$,

$$\|T_\delta F\|_{L_x^r L_t^q} \le C\, \delta^{1/\widetilde{r}'-1/r+\frac{1}{2}(1/\widetilde{q}'-1/q)}\, \|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}} \tag{3.6}$$

whenever $\widehat{F}$ is supported in $\{(\xi,\tau) \in \mathbb{R}\times\mathbb{R}\colon |\xi| \le 1, |\tau| \sim 1\}$. From (2.3) we see that the Fourier transform of $T_\delta$ is essentially supported in the $\delta$-neighborhood of $\{(\xi,\tau)\colon \tau = -|\xi|^2, |\tau| \sim 1\}$. Hence it is sufficient to show (3.6) under the assumption that the Fourier support of $F$ is contained in $\{(\xi,\tau) \in \mathbb{R}\times\mathbb{R}\colon |\xi| \sim 1, |\tau| \lesssim 1\}$. The contribution from the other region is negligible.

Under this assumption, by (2.3), Plancherel's theorem in $t$, and Hölder's inequality it follows that

$$\begin{aligned}\|T_\delta F(x,\cdot)\|_{L_t^2} &\le C\Big( \int \Big| \int_{1/2\le|\xi|\le 2} e^{ix\xi}\, \widehat{\phi}\Big(\frac{\tau+|\xi|^2}{\delta}\Big)\, \widehat{F}(\xi,\tau)\, d\xi \Big|^2 d\tau \Big)^{1/2} \\ &\le C\, \delta^{1/2} \Big( \int \int_{1/2\le|\xi|\le 2} \big|\widehat{F}(\xi,\tau)\big|^2 d\xi\, d\tau \Big)^{1/2}.\end{aligned}$$

Plancherel's theorem gives $\|T_\delta F\|_{L_x^\infty L_t^2} \le C\, \delta^{1/2}\, \|F\|_{L_x^2 L_t^2}$. By this and duality we have $\|T_\delta F\|_{L_x^\infty L_t^2} \le C\delta \|F\|_{L_x^1 L_t^2}$, and from the dispersive estimate $\|T_\delta F\|_{L_x^\infty L_t^\infty} \le C\, \delta^{3/2}\, \|F\|_{L_x^1 L_t^1}$. Interpolation between these two estimates gives, for $1 \le \widetilde{q}' \le 2 \le q \le \infty$,

$$\|T_\delta F\|_{L_x^\infty L_t^q} \le C\, \delta\, \delta^{\frac{1}{2}(1/\widetilde{q}'-1/q)}\, \|F\|_{L_x^1 L_t^{\widetilde{q}'}}.$$

Let $Q \subset \mathbb{R}^{1+1}$ be a cube of side length $\delta^{-1}$ and let $\widetilde{Q}$ be the cube of side length $C\,\delta^{-1}$ which has the same center as $Q$. Here $C > 0$ is a sufficiently large constant. By Hölder's inequality we have for $1 \le \widetilde{r}', \widetilde{q}' \le 2 \le r, q \le \infty$, and $0 < \delta \ll 1$,

$$\|T_\delta(\chi_{\widetilde{Q}} F)\|_{L_x^r L_t^q(Q)} \le C\, \delta^{1/\widetilde{r}'-1/r+\frac{1}{2}(1/\widetilde{q}'-1/q)}\, \|F\|_{L_x^{\widetilde{r}'} L_t^{\widetilde{q}'}}. \tag{3.7}$$

Now we deduce (3.6) from this. First, from the assumption that the Fourier transform of $F$ is contained in $\{(\xi,\tau) \in \mathbb{R}\times\mathbb{R} : |\xi| \sim 1\}$, we observe that $T_\delta$ is localized at scale $\delta^{-1}$ in $x$. More precisely, the kernel $K_\delta$ of $T_\delta$ satisfies that

$$|K_\delta(x,t)| \le C\, \delta^M\, \chi_{[1/2\delta,\, 2/\delta]}(|t|)\, (1+|x|)^{-M}$$

for any $M$ if $|x| \ge C\delta^{-1}$. (See (2.5) and the paragraph following it).

Hence it follows that if $(x,t)\in Q$, then

$$|T_\delta F(x,t)| \le C|\, T_\delta \chi_{\widetilde{Q}} F(x,t)| + C\delta^M (\mathcal{E}_\delta * |F|)(x,t) \tag{3.8}$$

for some large $M>0$ where $\mathcal{E}_\delta = \chi_{[1/2\delta,\,2/\delta]}(t)(1+|x|)^{-M}$. Let $\{Q\}$ be a collection of (essentially disjoint) cubes of side length $\delta^{-1}$ which cover $\mathbb{R}^{1+1}$. Then by (3.8) we have

$$\|T_\delta F\|_{L^r_x L^q_t} \le C \Big\| \sum_Q \chi_Q \big| T_\delta(\chi_{\widetilde{Q}} F)\big| \Big\|_{L^r_x L^q_t} + C\,\delta^M \|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}$$

because $r \ge \widetilde{r}'$ and $q \ge \widetilde{q}'$. Hence, by Minkowski's inequality and (3.7) we have

$$\begin{aligned}\|T_\delta F\|_{L^r_x L^q_t} &\le C \Big( \sum_Q \|T_\delta(\chi_{\widetilde{Q}} F)\|^p_{L^r_x L^q_t(Q)} \Big)^{1/p} + C\,\delta^M \, \|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t} \\ &\le C\,\delta^{1/\widetilde{r}'-1/r+\frac12(1/\widetilde{q}'-1/q)} \Big( \sum_Q \|\chi_{\widetilde{Q}} F\|^p_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t} \Big)^{1/p} + C\,\delta^M \, \|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t},\end{aligned}$$

where $p=\min(q,r)$. Since $r\ge \widetilde{r}'$ and $q \ge \widetilde{q}'$, using Minkowski's inequality again, we get the desired inequality (3.6).

For $j\in\mathbb{Z}$, define the multiplier operators $\mathcal{P}_j F$ by

$$\widehat{\mathcal{P}_j F}(\xi,\tau) = \phi(2^j\tau)\widehat{F}(\xi,\tau).$$

Using (2.2), (3.6), Lemma 2.2, and repeating the previous argument, one can show that

$$\Big\| \mathcal{P}_0\Big( \int_{-\infty}^t e^{i(t-s)\Delta} F(s)\,ds \Big) \Big\|_{L^r_x L^q_t} = \Big\| \sum_{\delta\in\mathbb{Z}} \delta^{-1}\, T_\delta(\mathcal{P}_0 F) \Big\|_{L^r_x L^q_t} \le C\|\mathcal{P}_0 F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t} \tag{3.9}$$

provided that $1<\widetilde{r}'<2<r<\infty$, $1\le \widetilde{q}'<2<q\le\infty$, and $1/\widetilde{r}'-1/r+\frac12(1/\widetilde{q}'-1/q)\ge 1$.

In fact, the case $1/\widetilde{r}'-1/r+\frac12(1/\widetilde{q}'-1/q) > 1$ can be obtained by direct summation because $\|T_\delta F\|_{L^r_x L^q_t} \le C\|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}$ for $\delta \ge 1$. Now by rescaling it follows that

$$\Big\| \mathcal{P}_j\Big( \int_{-\infty}^t e^{i(t-s)\Delta} F(s)\,ds \Big) \Big\|_{L^r_x L^q_t} \le C\, 2^{j(1-\frac12(1/\widetilde{r}'-1/r)-(1/\widetilde{q}'-1/q))}\, \|\mathcal{P}_j F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}.$$

Hence we have uniform bounds if $\frac12(1/\widetilde{r}'-1/r)+1/\widetilde{q}'-1/q=1$ and the condition for (3.9) is satisfied. Now note that if $1/\widetilde{r}'-1/r\ge 2/3$, there are $\widetilde{q}'$ and $q$ satisfying $\frac12(1/\widetilde{r}'-1/r)+1/\widetilde{q}'-1/q=1$ and $1/\widetilde{r}'-1/r+\frac12(1/\widetilde{q}'-1/q)\ge 1$. Therefore, if $1/\widetilde{r}'-1/r\ge 2/3$ and $1<\widetilde{r}'<2<r<\infty$, we have, for some $1<\widetilde{q}'<2<q<\infty$,

$$\Big\| \mathcal{P}_j\Big( \int_{-\infty}^t e^{i(t-s)\Delta} F(s)\,ds \Big) \Big\|_{L^r_x L^q_t} \le C\,\|\mathcal{P}_j F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}.$$

We put these estimates together using the Littlewood–Paley theorem in $t$. Since $1 < \widetilde{r}' < 2 < r < \infty$ and $1 < \widetilde{q}' \le 2 \le q < \infty$, by the Littlewood–Paley theorem and Minkowski's inequality

$$\Big\| \sum_j T(\mathcal{P}_j F) \Big\|_{L^r_x L^q_t} \lesssim \Big\| \Big( \sum_j \|T(\mathcal{P}_j F)\|^2_{L^q_t} \Big)^{1/2} \Big\|_{L^r_x} \lesssim \Big( \sum_j \|T(\mathcal{P}_j F)\|^2_{L^r_x L^q_t} \Big)^{1/2}$$
$$\lesssim \Big( \sum_j \|\mathcal{P}_j F\|^2_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t} \Big)^{1/2} \lesssim \Big\| \Big( \sum_j \|\mathcal{P}_j F\|^2_{L^{\widetilde{q}'}_t} \Big)^{1/2} \Big\|_{L^{\widetilde{r}'}_x} \lesssim \|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t}.$$

This completes the proof. □

## 4. Necessary conditions

By constructing some counterexamples, we show the conditions (1.8) and (1.9).

*Proof of* (1.9). Let $M > 0$ be a sufficiently large number and set

$$\widehat{F}(\xi, \tau) = \varphi(|\xi|)\, \psi(M^{1/2}(\tau + 1)),$$

where $\psi \in \mathcal{S}(\mathbb{R})$ with supp $\mathcal{F}^{-1}(\psi) \in [0, 1]$ and $\varphi$ is a smooth function supported in $(1/2, 2)$ with $\varphi(1) = 1$. Note that if $|t| \sim M$, then we write

$$\int_0^t e^{i(t-s)\Delta} F(s)\, ds = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, e^{-it|\xi|^2}\, \widehat{F}(\xi, -|\xi|^2)\, d\xi$$

because the support of $F(y, \cdot)$ is contained in $[0, M^{1/2}]$ for all $y$. Since we have $\int_{S^{n-1}} e^{ix\cdot\xi}\, d\sigma(\xi) = C|x|^{-(n-2)/2} J_{(n-2)/2}(|x|)$, by the asymptotic behavior of the Bessel function [22], we see that

$$\Big| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \Big| \sim |x|^{-(n-1)/2}\, |I(x, t)|$$

for sufficiently large $|x|$, where

$$I(x, t) = \int_0^\infty r^{-(n-1)/2}\, \varphi(r)\, \psi\big(M^{1/2}(r^2 - 1)\big)\, e^{-itr^2} \cos\big(r|x| - \pi(n-1)/4\big)\, dr.$$

We set $\widetilde{\varphi} = r^{-(n-1)/2}\, \varphi(r)$. Then we have

$$I(x, t) = \int \widetilde{\varphi}(1)\, \psi\big(M^{1/2}(r^2 - 1)\big)\, e^{-itr^2} \cos\big(r|x| - \pi(n-1)/4\big)\, dr$$
$$+ \int \big(\widetilde{\varphi}(r) - \widetilde{\varphi}(1)\big)\, \psi\big(M^{1/2}(r^2 - 1)\big)\, e^{-itr^2} \cos\big(r|x| - \pi(n-1)/4\big)\, dr.$$

By the rapid decay of $\psi$, the support of $\widetilde{\varphi}$, and the mean value theorem it is easy to see that the second term on the right-hand side is $O(M^{-1})$. Similarly, for a large constant $B > 0$,

$$I(x,t) = \int_{|r-1|\le BM^{-1/2}} \psi\big(M^{1/2}(r^2-1)\big)\, e^{-itr^2} \cos\big(r|x| - \pi(n-1)/4\big)\, dr \\ + O(M^{-1/2}/B^{100}) + O(M^{-1}).$$

Hence, we get

$$I(x,t) = \frac{1}{2}\Big(e^{-i\pi(n-1)/4}\, I_-(x,t) + e^{i\pi(n-1)/4}\, I_+(x,t)\Big) + O(M^{-1/2}/B^{100}),$$

where

$$I_\pm(x,t) = \int_{|r-1|\le BM^{-1/2}} e^{-i(tr^2 \pm r|x|)}\, \psi\big(M^{1/2}(r^2-1)\big)\, dr.$$

By changing variables $r \to r+1$ and $r \to M^{-1/2}r$, it follows that

$$I_-(x,t) = e^{-i(t-|x|)} M^{-1/2} \int_{|r|\le B} e^{-i\left(tM^{-1}r^2 + (2t-|x|)M^{-1/2}r\right)} \psi(2r + M^{-1/2}r^2)dr.$$

Thus, if $|t| \sim M/B^2$ and $|2t - |x|| \lesssim M^{1/2}/B$, we get $|I_-(x,t)| \gtrsim M^{-1/2}$. On the other hand, we see that $|I_+(x,t)| \lesssim B^2 M^{-1}$ if $|t| \sim M$ and $|x| \sim M$. Consequently, if $|t| \sim M$, $|x| \sim M$, and $|2t - |x|| \lesssim M^{1/2}$, then $|I(x,t)| \gtrsim M^{-1/2}$. Therefore, we see that

$$\Big\| \int_0^t e^{i(t-s)\Delta} F(s)\, ds \Big\|_{L_x^r L_t^q} \gtrsim M^{-n/2} M^{1/2q} M^{n/r}.$$

Also, it is easy to see that $\|F\|_{L_x^{\tilde r'} L_t^{\tilde q'}} \lesssim M^{-1/2} M^{1/2\tilde q'}$. Hence, the estimate (1.4) implies that

$$M^{-n/2} M^{1/2q} M^{n/r} \lesssim M^{-1/2} M^{1/2\tilde q'}.$$

By letting $M \to \infty$, we get the first inequality in (1.9), and the second one follows from duality. □

*Proof of* (1.8). Write

$$U(F)(x,t) = \int_0^t e^{i(t-s)\Delta} F(s)\, ds.$$

Then, using the kernel of $e^{it\Delta}$, we have

$$U(F)(x,t) = \int_0^t \int_{\mathbb{R}^n} (t-s)^{-n/2} \exp\Big(\frac{i|x-y|^2}{4(t-s)}\Big) F(y,s)\, dy\, ds.$$

For $0 < \delta \ll 1$, we set

$$F(y,s) = \Phi\big(\delta^{1/2}(y_1 + 2s), \delta^{1/2}\,\overline{y}, \delta s\big)\, e^{-i(y_1+s)},$$

where $\Phi(y_1, \overline{y}, s) = \chi(y_1)\chi(y_2)\cdots\chi(y_n)\chi(s)$ and $\chi = \chi_{[0,1]}$.

By the change of variables $y_1 \to y_1 - 2s$, we see that

$$e^{i(x_1+t)} U(F)(x,t) = \int_0^t \int_{\mathbb{R}^n} (t-s)^{-n/2}\, e^{iP(x,y,t,s)}\, \Phi\big(\delta^{1/2} y_1, \delta^{1/2}\, \overline{y}, \delta s\big)\, dy\, ds,$$

where

$$P(x,y,t,s) = \frac{|\overline{x}-\overline{y}|^2 + (x_1 - y_1 + 2t)^2}{4(t-s)}.$$

Note that $|P(x,y,t,s)| \lesssim 1$ if $(x_1+2t)^2 \le \delta^{-1}$, $|\overline{x}| \le \delta^{-1/2}$, and $100\delta^{-1} \le t \le 200\delta^{-1}$. We see

$$|U(F)(x,t)| \gtrsim \delta^{n/2} \Big| \int_0^t \int_{\mathbb{R}^n} \Phi\, dy\, ds \Big| \gtrsim \delta^{-1},$$

provided that $(x_1+2t)^2 \le \delta^{-1}$, $|\overline{x}| \le \delta^{-1/2}$ and $100\delta^{-1} \le t \le 200\delta^{-1}$. Hence

$$\|U(F)\|_{L^r_x L^q_t} \gtrsim \delta^{-1}\, \delta^{-1/2q}\, \delta^{-(n+1)/2r}.$$

On the other hand, $\|F\|_{L^{\widetilde{r}'}_x L^{\widetilde{q}'}_t} \le C \delta^{-1/2\widetilde{q}'} \delta^{-(n+1)/2\widetilde{r}'}$. From (1.4) we get

$$\delta^{-1}\, \delta^{-1/2q}\, \delta^{-(n+1)/2r} \lesssim \delta^{-1/2\widetilde{q}'}\, \delta^{-(n+1)/2\widetilde{r}'}.$$

By letting $\delta \to 0$, we get (1.8). □

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