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Uniqueness of area minimizing surfaces for extreme curves

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Abstract. Let M be a compact, orientable, mean convex 3-manifold with boundary ∂M. We show that the set of all simple closed curves in ∂M which bound unique area minimizing disks in M is dense in the space of simple closed curves in ∂M which are nullhomotopic in M. We also show that the set of all simple closed curves in ∂M which bound unique absolutely area minimizing surfaces in M is dense in the space of simple closed curves in ∂M which are nullhomologous in M.

1. Introduction

The Plateau problem investigates the existence of an area minimizing disk (or surface) with a given boundary curve in a given manifold M. Besides the solution of this problem, there have been many important results on the regularity and embeddedness of solutions, and on the number of solutions. In this paper, we focus on the number of solutions and give new uniqueness results.

The main question along this line is if, for a given curve, there is a unique area minimizing disk or surface in the ambient manifold M. The first result about this question was obtained by Radó in the early 1930s. He showed that if a curve can be projected bijectively to a convex plane curve, then it bounds a unique minimal disk [13]. In 1973, in [12], Nitsche proved uniqueness of minimal disks for boundary curves with total curvature less than 4π. Then, Tromba [14] showed that a generic curve in $\mathbb{R}^3$ bounds a unique area minimizing disk. Morgan [11] proved a similar result concerning absolutely area minimizing surfaces. Later, White proved a very strong generic uniqueness result for fixed topological type in any dimension and codimension [15]. In particular, he showed that a generic k-dimensional $C^{j,\alpha}$-submanifold of a Riemannian manifold cannot bound two smooth, minimal $(k+1)$-manifolds of equal area.

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In [2], the first author proved generic uniqueness results for both versions of the Plateau problem under the condition that $H_2(M;\mathbb{Z}) = 0$. In this paper, we generalize these results by removing the assumption on homology. Our techniques are simple and topological. The first main result is the following:

Theorem 3.1. *Suppose that M is a compact, orientable, mean convex 3-manifold. Let $\mathcal{E}$ be the set of simple closed curves on the boundary of M which are nullhomotopic in M, and let $\mathcal{U} \subset \mathcal{E}$ comprise those curves that bound unique area minimizing disks in M. Then $\mathcal{U}$ is not only dense but also a countable intersection of open dense subsets of $\mathcal{E}$ with respect to the C^0-topology.*

The second main result is a similar theorem for absolutely area minimizing surfaces:

Theorem 4.1. *Suppose that M is a compact, orientable, mean convex 3-manifold. Let $\mathcal{F}$ be the set of simple closed curves on the boundary of M which are nullhomologous in M, and let $\mathcal{V} \subset \mathcal{F}$ comprise those curves that bound unique absolutely area minimizing surfaces in M. Then, $\mathcal{V}$ is not only dense but also a countable intersection of open dense subsets of $\mathcal{F}$ with respect to the C^0-topology.*

For natural generalizations of these results to the smooth category see the last section of this paper.

The "lens" technique introduced in [2] to prove generic uniqueness results does not generalize to manifolds with nontrivial homology, mainly because, in general, the disks or surfaces need not be separating in M, hence one cannot construct a canonical neighborhood (*lens*) $N_\Gamma = [\Sigma_\Gamma^-, \Sigma_\Gamma^+]$ which contains all area minimizing disks (or surfaces) for a given nullhomotopic (or nullhomologous) $\Gamma \subset \partial M$. Provided that these neighborhoods exist and are disjoint for disjoint curves on the boundary, a summation argument which involves the thickness (or volume) of N_Γ would give the desired uniqueness results. But these lenses are the key element in the proof, and without them, the whole argument collapses.

In the disk case, in general, we still have the disjointness of the area minimizing disks for disjoint boundaries by [9]. Even though we can not construct disjoint lenses N_Γ for a given curve $\Gamma \subset \partial M$ as in [2] because of nontrivial homology, when we consider the behavior of area minimizing disks near the boundary, we still get disjoint canonical neighborhoods (*lens with a big hole*) near ∂M for disjoint curves, and the summation argument in [2] works. Hence the proof of Theorem 3.1 can be achieved with a modification of the original argument in [2].

On the other hand, in the surface case, we do not have the disjointness of the absolutely area minimizing surfaces for disjoint boundaries when the ambient manifold has nontrivial second homology [3]. Hence, the arguments we used in the disk case do not work here either. In order to prove the result in the surface case, we use a completely new approach. The main idea of the proof is as follows. First, we isometrically embed the original manifold M into a larger manifold $\widehat{M}$. Then, we utilize the fact that for any separating curve γ in an absolutely area minimizing surface Σ in M, γ bounds a unique absolutely area minimizing surface $S \subset \Sigma$ in M in the following way. For any simple closed curve $\Gamma \subset \partial M$, consider a nearby simple closed curve $\widehat{\Gamma}$ in $\widehat{M} - M$. Then, if $\widehat{\Sigma}$ is an absolutely area minimizing surface in $\widehat{M}$

with $\partial\widehat{\Sigma} = \widehat{\Gamma}$, then the curve $\Gamma' = \widehat{\Sigma} \cap \partial M$ will be a uniqueness curve in ∂M near Γ. This shows density in the surface case. Having this density result, we can adapt the summation argument in [2] to finish the proof of Theorem 4.1.

Note that this argument using an imbedding in a larger manifold can easily be adapted to the disk case to reprove all the results in Section 3, hence the results in [2], too. Note also that the mean convexity of M is crucial in employing this approach (See Remark 4.5).

The organization of the paper is as follows: In the next section we cover some basic results which will be used later. Section 3 contains the proof of Theorem 3.1. In section 4, we prove the analogous result regarding absolutely area minimizing surfaces. Section 5 is devoted to further remarks.

2. Preliminaries

In this section, we review the basic results which will be used in subsequent sections.

Definition 2.1. Let M be a compact Riemannian 3-manifold with boundary. Then M is called *mean convex* (or *sufficiently convex*) if the following conditions hold:

- ∂M is piecewise smooth.
- Each smooth subsurface of ∂M has nonnegative curvature with respect to an inward normal.
- There exists a Riemannian manifold N such that M is isometric to a submanifold of N and each smooth subsurface S of ∂M extends to a smooth embedded surface S' in N such that $S' \cap M = S$.

We call a simple closed curve *extreme* if it is on the boundary of its convex hull. Our results apply to the extreme curves as the convex hull naturally satisfies the conditions above. Note that a simple closed curve in the boundary of a mean convex manifold M is called a *weak extreme* or an *H-extreme curve*.

Definition 2.2. An *area minimizing disk* is a disk which has the smallest area among disks having a given boundary. An *absolutely area minimizing surface* is a surface which has the smallest area among all orientable surfaces (with no topological restriction) having a given boundary.

Now we state the main facts which we use in the following sections.

Lemma 2.3 ([9], [10]). *Let M be a compact, mean convex 3-manifold, and let $\Gamma \subset \partial M$ be a simple closed curve nullhomotopic in M. Then there exists an area minimizing disk $D \subset M$ with $\partial D = \Gamma$. All such disks are properly embedded in M, i.e., their boundaries are in ∂M, and they are pairwise disjoint. Moreover, area minimizing disks spanning disjoint simple closed curves in ∂M are also disjoint.*

Note that the claim in the last sentence in Lemma 2.3 is known as the Meeks–Yau exchange roundoff trick. The main idea is as follows. If two area minimizing disks D_1 and D_2 with disjoint boundaries intersect, the intersection will contain a closed curve β. Let $D_i^\beta \subset D_i$ be the smaller disk bounded by β. Then, by swapping D_1^β and D_2^β, we get a new area minimizing disk $D_1' = (D_1 - D_1^\beta) \cup D_2^\beta$ with a folding curve β. Pushing D_1' along the folding curve β to the convex side decreases area which contradicts with D_1' being area minimizing.

An analogous statement for absolutely area minimizing surfaces is obtained by combining the following results.

Theorem 2.4 ([5], [1], [6]). *Let M be a compact, strictly mean convex 3-manifold and let $\Gamma \subset \partial M$ be a nullhomologous simple closed curve. Then there exists an absolutely area minimizing surface $\Sigma \subset M$ with $\partial\Sigma = \Gamma$, and each such Σ is smooth away from its boundary and is smooth around points of its boundary where Γ is smooth.*

Hass proved the following statement for closed 3-manifolds. It can be generalized with a slight modification of his argument. This lemma can be regarded as the adaptation of Meeks–Yau exchange roundoff trick to the surface case.

Lemma 2.5 ([7]). *Let M be an orientable, mean convex 3-manifold, and let Σ_1 and Σ_2 be two homologous, properly embedded, absolutely area minimizing surfaces in M. If $\partial\Sigma_1$ and $\partial\Sigma_2$ are disjoint or the same, then Σ_1 and Σ_2 are disjoint.*

Proof. Since Σ_1 and Σ_2 are in the same homology class, they separate a codimension-0 submanifold M' from M, and $\Sigma_1 \cup \Sigma_2 \subset \partial M'$. Then, Σ_1 and Σ_2 separate each other [7]. Let $\Sigma_1 \setminus \Sigma_2 = S_1^+ \cup S_1^-$, and $\Sigma_2 \setminus \Sigma_1 = S_2^+ \cup S_2^-$. Assuming $\partial S_1^- = \partial S_2^- = \Sigma_1 \cap \Sigma_2$ (S_1^+ and S_2^+ are the components containing $\partial\Sigma_1$ and $\partial\Sigma_2$ respectively), $\Sigma_1' = (\Sigma_1 \setminus S_1^-) \cup S_2^-$ would be another absolutely area minimizing surface in M with boundary $\partial\Sigma_1$. This is because Σ_1 and Σ_2 are absolutely area minimizing surfaces, and $\partial S_1^- = \partial S_2^-$ implies $|S_1^-| = |S_2^-|$. However, Σ_1' is singular along $\Sigma_1 \cap \Sigma_2$ which contradicts the regularity theorem for absolutely area minimizing surfaces [4]. □

Now, we state a lemma about the limit of area minimizing disks in a mean convex manifold.

Lemma 2.6 ([8]). *Let M be a compact, mean convex 3-manifold and let $\{D_i\}$ be a sequence of properly embedded area minimizing disks in M. Then there is a subsequence of $\{D_i\}$ which converges to a countable collection of properly embedded area minimizing disks in M.*

3. Uniqueness of area minimizing disks

This section is devoted to the proof of the following theorem.

Theorem 3.1. *Suppose that M is a compact, orientable, mean convex 3-manifold. Let $\mathcal{E}$ be the set of simple closed curves on the boundary of M which are nullhomotopic in M, and let $\mathcal{U} \subset \mathcal{E}$ comprise those curves that which bound unique area minimizing disks in M. Then $\mathcal{U}$ is not only dense but also a countable intersection of open dense subsets of $\mathcal{E}$ with respect to the C^0-topology.*

Remark 3.2. In the proof of this theorem we ignore the curves in $\mathcal{E}$ that bound area minimizing disks in ∂M. This is justified by the fact that a curve γ in the interior of an area minimizing disk $D \in \partial M$ cannot bound a properly embedded area minimizing disk D' since swapping the disk in D bounded by γ with D' and rounding off (the exchange-roundoff trick) would give a disk with the same boundary as D but with strictly smaller area than D. In particular, such a curve γ is clearly an interior point of $\mathcal{U}$.

Proof. For each $\Gamma \in \mathcal{E}$ fix an annular neighborhood $A_\Gamma \subset \partial M$ and a properly embedded annulus $A'_\Gamma \subset M$ with $\partial A_\Gamma = \partial A'_\Gamma$ as in Lemma 3.3, i.e.,

- $A_\Gamma \cup A'_\Gamma$ bounds a solid torus in M, and
- if the boundary of a properly embedded area minimizing disk $D \subset M$ is essential in A_Γ, then D intersect A'_Γ in a unique essential simple closed curve (see Figure 1).

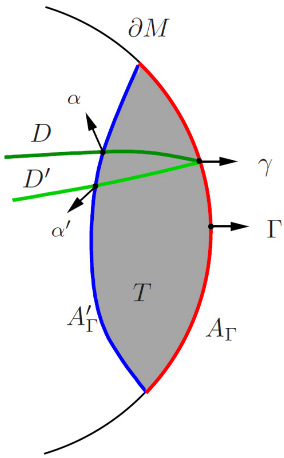

FIGURE 1. For any $\gamma \subset A_\Gamma \subset \partial M$, any area minimizing disk D with $\partial D = \gamma$ intersects A'_Γ in a unique essential curve α. The grey region represents the solid torus T in M with $\partial T = A_\Gamma \cup A'_\Gamma$.

For an essential simple closed curve γ in A_Γ, let R^Γ_γ denote (as in Lemma 3.5) the smallest annulus in A'_Γ which contains the intersection of A'_Γ with all the area minimizing disks spanning γ. Note that $\gamma \in \mathcal{U}$ if and only if $|R^\Gamma_\gamma| = 0$, where $|\cdot|$ denotes the area.

First we will prove that $\mathcal{U}$ is dense in $\mathcal{E}$. Let $\Gamma \in \mathcal{E}$, and foliate A_Γ by essential simple closed curves $\{\Gamma_t : t \in [-\epsilon, \epsilon]\}$ such that $\Gamma_0 = \Gamma$. By Lemma 3.4, the regions $R^\Gamma_{\Gamma_t}$ and $R^\Gamma_{\Gamma_s}$ in A'_Γ are disjoint for $s \neq t$. Therefore,

$$\sum_{t \in [-\epsilon, \epsilon]} |R^\Gamma_{\Gamma_t}| < |A'_\Gamma| < \infty \ .$$

Hence $|R^{\Gamma}_{\Gamma_t}| > 0$ only for countably many $t \in [-\epsilon, \epsilon]$, i.e., Γ_t bounds a unique area minimizing disk for uncountably many $t \in [-\epsilon, \epsilon]$. Since we began with an arbitrary $\Gamma \in \mathcal{E}$, this proves that $\mathcal{U}$ is dense in $\mathcal{E}$.

To prove that $\mathcal{U}$ is the intersection of countably many open dense subsets of $\mathcal{E}$ let

$$U_n = \{\gamma \in \mathcal{E} | \text{ there exists } \Gamma \in \mathcal{E} \text{ such that } \gamma \text{ is essential in } A_\Gamma \text{ and } |R^{\Gamma}_{\gamma}| < 1/n\}$$

for every $n \in \mathbb{Z}_+$. Observe that $\mathcal{U} = \cap_{n\in\mathbb{N}} U_n$, and in particular, each U_n is dense. It remains to show that every U_n is open. Let $\gamma \in U_n$, choose $\Gamma \in \mathcal{E}$ such that γ is essential in A_Γ with $|R^{\Gamma}_{\gamma}| < 1/n$, and choose an annular region R in A'_Γ with $|R| < 1/n$ whose interior contains R^{Γ}_{γ}. Since $\mathcal{U}$ is dense in $\mathcal{E}$, there is a sequence $\{\gamma_n\}$ of pairwise disjoint, essential curves in A_Γ converging to γ such that each γ_n bounds a unique area minimizing disk D_n in M. We can arrange that all these curves are in a prescribed component of $A_\Gamma \setminus \gamma$. By Lemma 2.6, the sequence $\{D_n\}$ has a subsequence converging to a countable collection of area minimizing disks spanning γ. This implies the existence of essential curves γ^+ and γ^- in A_Γ such that

- the curves γ^+ and γ^- are contained in different components of $A_\Gamma \setminus \gamma$,
- each of $\gamma^\pm$ bounds a unique area minimizing disk $D^\pm$, and
- $D^\pm \cap A'_\Gamma \subset R$.

Let A_γ be the open annulus in A_Γ bounded by $\gamma^\pm$, and let V_γ be the set of all simple closed curves essential in A_γ. Note that V_γ is an open neighborhood of γ in $\mathcal{E}$. Moreover, $V_\gamma \subset U_n$ because $D^+ \cup D^-$ separates the solid torus bounded by $A_\Gamma \cup A'_\Gamma$, and an area minimizing disk spanning any $\alpha \in V_\gamma$ has to be disjoint from $D^+ \cup D^-$, forcing R^{Γ}_{α} to remain inside R. This proves that U_n is open in $\mathcal{E}$ and finishes the proof. □

In the rest of this section we will prove the lemmas used in the proof of Theorem 3.1.

Lemma 3.3. *For every $\Gamma \in \mathcal{E}$, there exist annuli A_Γ and A'_Γ with common boundary, the former a neighborhood of Γ in ∂M and the latter properly embedded in M, such that $A_\Gamma \cup A'_\Gamma$ bounds a solid torus in M and any properly embedded area minimizing disk in M spanning an essential curve in A_Γ intersects A'_Γ in a unique essential curve.*

Proof. Given $\Gamma \in \mathcal{E}$, we choose an annular neighborhood A_Γ and a solid torus neighborhood $N_\Gamma \supset A_\Gamma$ of Γ in ∂M and M, respectively. Although we shrink the annular neighborhood as we proceed, we abuse notation by continuing to denote it by A_Γ. Note that, by [9], we can choose A_Γ sufficiently small that there is an area minimizing annulus A in M with boundary ∂A_Γ. If there is such an area minimizing annulus A that is *properly embedded*, then let A'_Γ be A. Otherwise A_Γ is the unique area minimizing annulus with boundary ∂A_Γ, and we will now explain how to construct A'_Γ in this case.

Let Γ^+ and Γ^- denote the boundary components of A_Γ, let $D^\pm$ be area minimizing disks spanning $\Gamma^\pm$, and let $\{\gamma^\pm_n\}$ be sequences of disjoint simple closed curves in the interior of $D^{\pm}$ converging to $\Gamma^{\pm}$. Let $\widehat{M}$ be the component of $\overline{M \setminus (D^+ \cup D^-)}$ that contains Γ. Note that $\widehat{M}$ is mean convex as the $D^{\pm}$ are minimal, and the $\gamma_n^{\pm}$ can be regarded as simple closed curves in $\partial\widehat{M}$. Therefore, by choosing A_Γ sufficiently small and n sufficiently large, we can guarantee that there is an area minimizing annulus A_n in $\widehat{M}$ spanning $\gamma_n^+ \cup \gamma_n^-$. Let A'_Γ be the union of A_n and the obvious (area minimizing) annuli in $D_\pm$ between $\gamma_n^{\pm}$ and $\Gamma^{\pm}$.

Before we proceed, we will prove that for sufficiently small A_Γ and sufficiently large n, A_n is an area minimizing surface not only in $\widehat{M}$ but also in M. Assume that there is an annulus $A'_n \subset M$ such that $\partial A'_n = \partial A_n = \gamma_n^+ \cup \gamma_n^-$ and $|A'_n| < |A_n|$. Since A_n is area minimizing in $\widehat{M}$, A'_n cannot be embedded in $\widehat{M}$. Without loss of generality, assume that $A'_n \cap (D^+ \setminus \gamma_n^+) \neq \emptyset$. Any component α of $A'_n \cap D^+$ has to be essential in A'_n, since otherwise we could swap the disks bounded by α in A'_n and in D^+ to get a contradiction, using the exchange-roundoff trick. If a component α of $A'_n \cap D^+$ and γ_n^+ are concentric in D^+, then we get a contradiction (again by the exchange roundoff trick) by swapping the annular regions between γ_n^+ and α in D^+ and in A'_n. Therefore any component α of $A'_n \cap D^+$ has to be essential in A'_n and nullhomotopic in $D^+ \setminus D_n$, where D_n denotes the disk in D^+ bounded by γ_n^+. Consider the annulus A''_n in A'_n with $\partial A''_n = \alpha \cup \gamma_n^+$ and the disk D_α in D^+ bounded by α. Note that the disk $D = A''_n \cup D_\alpha$ bounds γ_n^+ hence $|D_n| \leq |D| \leq |A''_n| + |D_\alpha|$. The facts that D_α is a subset of $D^+ \setminus D_n$ and the sequence $\{\gamma_n^+ = \partial D_n\}_n$ converges to $\Gamma^+ = \partial D^+$ imply that $|D_\alpha|$ can be made arbitrarily small. Hence to get a contradiction, all we need to do is make A_Γ sufficiently small and n sufficiently large, forcing $|A''_n| + |D_\alpha| < |D_n|$.

Now we have defined A'_Γ regardless of whether A_Γ bounds a properly embedded area minimizing annulus in M or not. Note that A'_Γ is properly embedded in M, $\partial A'_\Gamma = \partial A_\Gamma$, and $A'_\Gamma \cup A_\Gamma$ bounds a solid torus T in M (at least when we choose A_Γ small enough to ensure that A'_Γ remains in the solid torus neighborhood N_Γ of Γ). Also note that A'_Γ is either area minimizing or it is the union of three area minimizing annuli glued along $\gamma_n^{\pm}$.

In the rest of the proof, we will show that for any properly embedded area minimizing disk D_γ spanning an essential curve γ in A_Γ, $D_\gamma \cap A'_\Gamma$ is the unique essential curve in A'_Γ: First, since γ is essential in A_Γ, it is also essential in the solid torus T and cannot bound any surface in T. Therefore D_γ has to intersect A'_Γ. Moreover, any component α of $D_\gamma \cap A'_\Gamma$ has to be an essential curve in A'_Γ since otherwise we could swap the disks bounded by α in D_γ and in A'_Γ to get a contradiction using the exchange-roundoff trick.

Now, assume that $D_\gamma \cap A'_\Gamma$ has two components α_1 and α_2. These curves cannot be concentric in D_γ since, otherwise, again by using the exchange-roundoff trick, we would get a contradiction with the area minimizing property of D_γ after swapping the annular regions between the α_i in D_γ and in A'_Γ. We eliminate the remaining possibility of nonconcentric α_i by choosing A_Γ with sufficiently small area compared to that of an area minimizing disk D_Γ spanning Γ. Let α be any component of $D_\gamma \cap A'_\Gamma$ and D_α be the disk in D_γ bounded by α. We have the following inequalities by area minimizing properties of D_γ, D_Γ, D^+, and that of A'_Γ (or, depending on the construction of A'_Γ, A_Γ and A_n, and the convergence of $\{\gamma_n^+\}$ to Γ^+):

$$|D_\gamma| + |A_\Gamma| > |D^+| \ , \ |D^+| + |A_\Gamma| > |D_\gamma| \ ,$$
$$|D_\Gamma| + |A_\Gamma| > |D^+| \ , \ |D^+| + |A_\Gamma| > |D_\Gamma| \ ,$$
$$|D_\alpha| + |A'_\Gamma| > |D^+| \ , \ |A_\Gamma| \geq |A'_\Gamma| \ .$$

It follows that

$$|D_\gamma \setminus D_\alpha| = |D_\gamma| - |D_\alpha| < |D^+| + |A_\Gamma| - |D_\alpha| < |A'_\Gamma| + |A_\Gamma| \leq 2|A_\Gamma| \ .$$

Assuming that the components α_1 and α_2 of $D_\gamma \cap A'_\Gamma$ are not concentric in D_γ, we get

$$|D_\gamma| = |(D_\gamma \setminus D_{\alpha_1}) \cup (D_\gamma \setminus D_{\alpha_2})| < |(D_\gamma \setminus D_{\alpha_1})| + |(D_\gamma \setminus D_{\alpha_2})| < 4|A_\Gamma| \ .$$

Hence

$$|D_\gamma| > |D^+| - |A_\Gamma| > |D_\Gamma| - 2|A_\Gamma|$$

leads to

$$|D_\Gamma| < 6\,|A_\Gamma|$$

which is impossible once we choose $|A_\Gamma|$ sufficiently small since $|D_\Gamma|$ is independent of this choice. □

In the Lemma 3.4 and Lemma 3.5, we fix an arbitrary $\Gamma \in \mathcal{E}$ and annuli A_Γ and A'_Γ as in Lemma 3.3.

Lemma 3.4. *Let γ and γ' be disjoint, essential simple closed curves in A_Γ, let D_1 and D_2 be distinct properly embedded area minimizing disks in M bounding γ, let $\alpha_i = D_i \cap A'_\Gamma$, and let $R \subset A'_\Gamma$ be the annulus bounded by α_1 and α_2. Then any area minimizing disk in M spanning γ' is disjoint from R.*

Proof. Observe that each of the disks D_1 and D_2 separates the solid torus T with $\partial T = A_\Gamma \cup_{\Gamma^\pm} A'_\Gamma$ into two pieces. Since $D_1 \cap D_2 = \gamma \subset \partial T$, $D_1 \cup_\gamma D_2$ separates T into three pieces, and R is "half" (the annulus $(D_1 \cup_\gamma D_2) \cap T$ being the other "half") of the boundary of the "middle" piece T_0. Note that $T_0 \cap A_\Gamma = \gamma$, therefore γ' does not intersect T_0. If an area minimizing disk spanning γ' were to intersect R, this would force it to intersect either D_1 or D_2, but this is impossible since properly embedded area minimizing disks with disjoint boundaries do not intersect by Lemma 2.3. □

Lemma 3.5. *For every simple closed curve γ which is essential in A_Γ and bounds a properly embedded area minimizing disk in M there is a subset R^Γ_γ of A'_Γ such that*

1. *the intersection of A'_Γ and any area minimizing disk spanning γ belongs to R^Γ_γ,*
2. *R^Γ_γ is an annulus if $\gamma \notin \mathcal{U}$,*
3. *R^Γ_γ is a simple closed curve if $\gamma \in \mathcal{U}$, and*
4. *if γ and γ' are disjoint, so are R^Γ_γ and $R^\Gamma_{\gamma'}$.*

Proof. If $\gamma \in \mathcal{U}$, then the definition of R^Γ_γ is obvious and (3) is a consequence of Lemma 3.3. Assume that $\gamma \notin \mathcal{U}$, and consider all the curves obtained as the intersection of A'_Γ with an area minimizing disk spanning γ. Let R^Γ_γ be the union of all the annuli bounded by any pair of such curves. Claims (1) and (2) hold by definition and the connectedness of R^Γ_γ. Claim (4) is a consequence of Lemma 3.4. □

4. Uniqueness of absolutely area minimizing surfaces

This section is devoted to the proof of the following theorem.

Theorem 4.1. *Suppose that M is a compact, orientable, mean convex 3-manifold. Let $\mathcal{F}$ be the set of simple closed curves on the boundary of M which are nullhomologous in M, and let $\mathcal{V} \subset \mathcal{F}$ comprise those that bound a unique absolutely area minimizing surface in M. Then, $\mathcal{V}$ is not only dense but also countable intersection of open dense subsets of $\mathcal{F}$ with respect to the C^0-topology.*

Remark 4.2. In order to prove this theorem, one might want to use a method similar to that used in the disk case. However, the crucial step in this method is Lemma 2.3, i.e., the area minimizing disks D_1 and D_2 in M bounding the simple closed curves Γ_1 and Γ_2 in ∂M are disjoint provided that the curves Γ_1 and Γ_2 are disjoint, and this is not true in the absolutely area minimizing surfaces case. There exist disjoint H-extreme curves which bound intersecting absolutely area minimizing surfaces [3].

Remark 4.3. As in the disk case, we will ignore the curves in $\mathcal{F}$ that bound absolutely area minimizing surfaces in ∂M (See Remark 3.2).

Proposition 4.4. *$\mathcal{V}$ is dense in $\mathcal{F}$ with respect to the C^0-topology.*

Proof. Assume otherwise. Then there is a simple closed curve Γ with a neighborhood $N_\epsilon(\Gamma)$ in ∂M such that any simple closed curve $\Gamma' \subset N_\epsilon(\Gamma)$, i.e., $d(\Gamma, \Gamma') < \epsilon$ in the C^0-metric, bounds at least two absolutely area minimizing surfaces Σ'_1 and Σ'_2 in M.

This implies that an absolutely area minimizing surface Σ in M with $\partial\Sigma = \Gamma$ cannot lie in ∂M. Indeed, since M is mean convex, by the maximum principle, $\Sigma \cap \partial M = \Gamma$. This is because if $\Sigma \subset \partial M$, then for any simple closed curve α near Γ in $\Sigma \subset \partial M$, α must bound a unique absolutely area minimizing surface. Otherwise, if α bounds $\Sigma_1 \subset \Sigma$ and another absolutely area minimizing surface Σ_2 in M, then $\Sigma' = (\Sigma \setminus \Sigma_1) \cup \Sigma_2$ would be yet another absolutely area minimizing surface with boundary Γ since $|\Sigma| = |\Sigma'|$. However, there is a singularity along α in Σ'. This contradicts the regularity theorem for absolutely area minimizing surfaces [4].

Now, embed M into a larger 3-manifold N isometrically as in Definition 2.1, i.e., M is isometric to a codimension-0 submanifold of N. We abuse the notation and denote this submanifold by M. For every $\delta > 0$, let M_δ denote the δ-neighborhood of M in N.

For each $j \in \mathbb{Z}_+$, consider a sequence of curves $\{\widehat{\Gamma}_i^j\}_{i=1}^\infty$ in $M_{1/j} \setminus M$ which converges to Γ as i tends to ∞. For every $i, j \in \mathbb{Z}_+$, let $\widehat{\Sigma}_i^j$ be an absolutely area minimizing surface in $M_{1/j}$ with $\partial\widehat{\Sigma}_i^j = \widehat{\Gamma}_i^j$. For each j, by Federer's compactness theorem [4], a subsequence of $\{\widehat{\Sigma}_i^j\}_i$ converges to an absolutely area minimizing surface Σ^j in $M_{1/j}$ with $\partial\Sigma^j = \Gamma$. As a further consequence of compactness, the sequence $\{\Sigma^j\}_{j=1}^\infty$ has a subsequence converging to an absolutely area minimizing surface Σ in M with $\partial\Sigma = \Gamma$.

Claim: There exists $j \in \mathbb{Z}_+$ such that $\Sigma^j \subset M$, and hence Σ^j is an absolutely area minimizing surface in $M_{1/k}$ for every $k \geq j$.

Proof of the Claim: Assume that $\Sigma^j \setminus M \neq \emptyset$ for all j. Now, replace the sequence Σ^j with the sequence $\Sigma_M^j = \overline{\Sigma^j \cap \mathrm{int}(M)}$ which also converges to Σ. Since $\mathrm{int}(\Sigma) \cap \partial M = \emptyset$ by assumption, we can assume that Σ_M^j is connected by ignoring the smaller pieces if necessary. Now consider $\Gamma^j = \partial\Sigma_M^j$ in ∂M. If $\Gamma^j = \Gamma$ for infinitely many j, then a sequence of interior points of Σ^j's would converge to a point in ∂M, contradicting the assumption that $\mathrm{int}(\Sigma) \cap \partial M = \emptyset$. Therefore Γ^j is distinct from Γ (but can intersect it) for all but finitely many j. On the other hand, Γ^j converges to Γ since Σ_M^j converges to Σ. Hence for sufficiently large j, $\Gamma^j \subset N_\epsilon(\Gamma)$ and by assumption, Γ^j bounds in M at least one absolutely area minimizing surface S_2 other than $S_1 = \Sigma_M^j$ (see Figure 2). By swapping S_1 and S_2 in Σ^j, we get a new surface $\tilde{\Sigma}^j = (\Sigma^j \setminus S_1) \cup S_2$ which has the same area as Σ^j. Hence, $\tilde{\Sigma}^j$ is an absolutely area minimizing surface in $M_{1/j}$ with boundary Γ. However, $\tilde{\Sigma}^j$ is singular along Γ^j which contradicts the regularity theorem for absolutely area minimizing surfaces [4]. This finishes the proof of the claim.

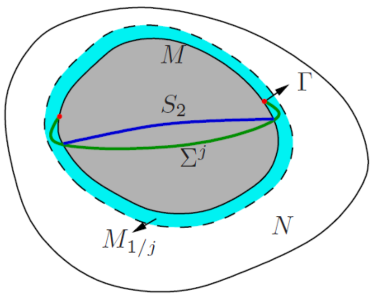

FIGURE 2. Σ^j is the absolutely area minimizing surface in $M_{1/j}$ with $\partial\Sigma^j = \Gamma$. S_2 is another absolutely area minimizing surface with $\partial S_2 = \partial(\Sigma^j \cap M)$.

Now, to finish the proof of the proposition, we get a contradiction as follows. By using the claim above, fix a positive integer j such that $\Sigma^j \subset M$. Then Σ^j is an absolutely area minimizing surface in M with $\partial\Sigma^j = \Gamma$. Let $\Sigma_i = \widehat{\Sigma}_i^j \cap M$, and $\Gamma_i = \partial\Sigma_i = \widehat{\Sigma}_i^j \cap \partial M$. Then Σ_i converges to Σ^j and Γ_i converges to Γ since $\widehat{\Sigma}_i^j$ converges to Σ^j (as i approaches to ∞). Therefore for sufficiently large i_o, $\Gamma_{i_o} \subset N_\epsilon(\Gamma)$, and consequently, Γ_{i_o} bounds at least one other absolutely area minimizing surface Σ'_{i_o} in M beside Σ_{i_o} by the assumption.

Let $\tilde{\Sigma}^j_{i_o} = (\widehat{\Sigma}^j_{i_o} \setminus \Sigma_{i_o}) \cup \Sigma'_{i_o}$. Since $\tilde{\Sigma}^j_{i_o}$ has the same area and boundary as $\widehat{\Sigma}^j_{i_o}$, it is also an absolutely area minimizing surface in $M_{1/j}$. However, it is singular along Γ_{i_o} contradicting the regularity theorem for absolutely area minimizing surfaces [4]. □

Remark 4.5. The mean convexity of M is crucial in the proof above. If M was not mean convex, then it is easy to construct examples where for any $j \in \mathbb{Z}_+$, the absolutely area minimizing surface $\Sigma^j \subset M_{\frac{1}{j}}$ satisfies $\partial\Sigma_j = \Gamma$ and $\Sigma_j \subset M_{\frac{1}{j}} - M$. One can simply take a 3-manifold M which is not mean convex, and the absolutely area minimizing surface Σ with boundary $\Gamma \subset \partial M$ completely lies in ∂M, i.e., $\Sigma \subset \partial M$. Then, for such a manifold, $\widehat{\Sigma}^j_i \cap M$ might be empty for any i, and the whole argument collapses (See also Remark 4.3).

Proposition 4.6. *$\mathcal{V}$ is a countable intersection of open dense subsets of $\mathcal{F}$ with respect to the C^0-topology.*

Proof. Let $\Gamma \in \mathcal{V}$ be a uniqueness curve, i.e., $\Gamma \subset \partial M$ bounds a unique absolutely area minimizing surface Σ in M. Let $\{\Gamma_i^+\}$ be a sequence of pairwise disjoint simple closed curves in $\mathcal{V}$ which converges to Γ. We also assume that every Γ_i^+ is on the same (say positive) side of Γ, i.e., $A_i^+ \subset A_j^+$ when $i > j$, where $A_i^+ = [\Gamma, \Gamma_i^+]$ is the annular component of $\partial M \setminus (\Gamma \cup \Gamma_i^+)$ for any i.

For each i, there exists a unique absolutely area minimizing surface Σ_i^+ in M with $\partial\Sigma_i^+ = \Gamma_i^+$. By the compactness theorem, a subsequence of $\{\Sigma_i^+\}$, which, by abuse of notation, will also be denoted $\{\Sigma_i^+\}$, converges to Σ which is the unique absolutely area minimizing surface in M with boundary Γ.

Take a tubular neighborhood $N(\Sigma) \simeq \Sigma \times (-1, 1)$ of Σ in M. Since Σ_i^+ converges to Σ, there exists an N_0 such that for any $i \geq N_0$, $\Sigma_i^+ \subset N(\Sigma)$ and Γ_i^+ is isotopic to Γ in $\partial N(\Sigma)$. Unlike the disk case, a priori we do not know that$\Sigma_i^+ \cap \Sigma = \emptyset$ even when $\Gamma_i^+ \cap \Gamma = \emptyset$ (see Remark 4.2). However, since Γ_i^+ separates the annulus $\partial N(\Sigma)$ for $i \geq N_0$, Σ_i^+ separates the product neighborhood $N(\Sigma)$. Therefore, for $i \geq N_0$, Σ_i^+ is in the same homology class as Σ, and consequently, by Lemma 2.5, Σ_i^+ and Σ are disjoint (See Figure 3 left). We denote the component of $M \setminus (\Sigma \cup \Sigma_i^+)$ whose boundary contains A_i^+ by $M_i^+ = [\Sigma, \Sigma_i^+]$.

Claim: There exists $N_1 \geq N_0$ such that for $i > N_1$, any absolutely area minimizing surface S whose boundary is C^0-close and isotopic to Γ in A_i^+ is contained in M_i^+. Consequently, S is in the same homology class with Σ, by the arguments above (See Figure 3 right).

Proof of the Claim: Assume otherwise, i.e., for any $i > N_0$, we can find a sequence of absolutely area minimizing surfaces S_i in M with $\partial S_i \subset A_i^+$ and $S_i \not\subseteq M_i^+$. If S_i and $\Sigma_{N_0}^+$ are disjoint, then Σ separates S_i since $\Sigma_{N_0}^+ \cup \Sigma$ separates M, but by using the swapping argument above, we get a new absolutely area minimizing surface S_i' with singularity along $S_i \cap \Sigma$ contradicting regularity. The assumption that S_i is disjoint from Σ leads to a similar contradiction. Therefore we have a sequence of absolutely area minimizing surfaces S_i in M such that for every $i \geq N_0$, S_i intersects both Σ and Σ_i^+.

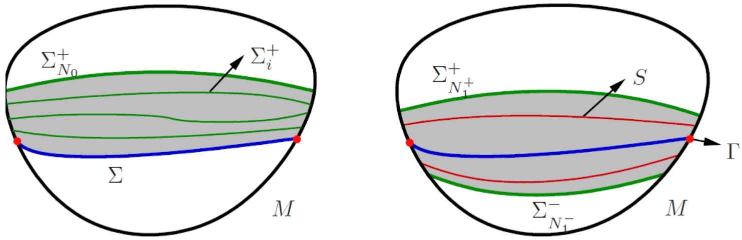

FIGURE 3. On the left: for $i \geq N_0$, Σ_i^+ is in the same homology class as Σ, and consequently, Σ_i^+ and Σ are disjoint. On the right: any absolutely area minimizing surface S with $\partial S \subset [\Gamma_{N_1^-}, \Gamma_{N_1^+}]$ is in the same homology class as Σ, and hence any such S and S' are disjoint whenever $\partial S \cap \partial S' = \emptyset$.

Since ∂S_i converges to Γ, and Γ is a uniqueness curve, by the compactness theorem, after passing to a subsequence if necessary, S_i converges to Σ. However, since $S_i \cap \Sigma^+_{N_0} \neq \emptyset$ for any $i > N_0$, and $\Sigma^+_{N_0}$ is compact, the limit of the sequence $\{S_i\}$ must have a limit point on $\Sigma^+_{N_0}$. Since $\Sigma^+_{N_0} \cap \Sigma = \emptyset$, this is a contradiction. The claim follows.

Obviously, a similar statement holds for the "negative side" of Γ. Therefore, every uniqueness curve Γ in $\mathcal{V}$, has a tubular neighborhood A_Γ in ∂M such that all absolutely area minimizing surfaces in M with boundary isotopic to Γ in A_Γ are in the same homology class. In particular, any two distinct absolutely area minimizing surfaces with the same boundary in A_Γ are disjoint by Lemma 2.5. Similarly, any two absolutely area minimizing surfaces with disjoint boundaries in A_Γ are also disjoint.

Now, we will show that $\mathcal{V}$ is a countable intersection of open dense subsets. We will follow the arguments proving the main theorem of [2]. Above, we showed that for any simple closed curve Γ in $\mathcal{V}$, there is a neighborhood N_Γ (corresponding to the curves isotopic to Γ in A_Γ above) in the C^0 topology such that for any $\Gamma' \in N_\Gamma$, an absolutely area minimizing surface S with $\partial S = \Gamma'$ is in the same homology class as Σ, where Σ is the unique absolutely area minimizing surface in M with $\partial \Sigma = \Gamma$. This implies that any two absolutely area minimizing surfaces with disjoint or matching boundaries in N_Γ must be disjoint. Now, let $\mathcal{G} = \bigcup_{\Gamma \in \mathcal{V}} N_\Gamma$. As $\mathcal{V}$ is dense in $\mathcal{F}$ by Proposition 4.4, $\mathcal{G}$ is open dense in $\mathcal{F}$.

The rest of the proof is along the same lines as the proof of Theorem 3.2 in [2], more precisely the part regarding Claim 2. Here we give an outline and refer the reader to [2] for further details. For each $\alpha \in \mathcal{G}$, we can construct a canonical neighborhood $\Omega_\alpha = [\Sigma^-_\alpha, \Sigma^+_\alpha]$, (the region between "extremal" absolutely area minimizing surfaces Σ^-_α and Σ^+_α with $\partial \Sigma^\pm_\alpha = \alpha$) which contains every absolutely area minimizing surface in M with boundary α. By construction, Ω_α is independent of N_Γ and depends only on α. By the disjointness of absolutely area minimizing surfaces with boundary in $\mathcal{G}$, if $\alpha \cap \beta = \emptyset$, then $\Omega_\alpha \cap \Omega_\beta = \emptyset$. Also, if α is a uniqueness curve, then $\Sigma^+_\alpha = \Sigma^-_\alpha$ and $\Omega_\alpha = \Sigma^\pm_\alpha$ should be regarded as a *degenerate* region (with no thickness).

Let s_α be the volume of Ω_α and define $U_i = \{\, \alpha \in \mathcal{G} \mid s_\alpha < \frac{1}{i} \,\}$ for each $i \in \mathbb{Z}_+$. Note that $\mathcal{V}$ is contained in every U_i since $s_\alpha = 0$ for every $\alpha \in \mathcal{V}$, by definition. In particular, U_i is dense in $\mathcal{F}$. Moreover, $\mathcal{V} = \bigcap_{i=1}^{\infty} U_i$, by construction. Finally, by using the arguments similar to those in the proof of Theorem 3.2 in [2], one can prove that U_i is open in $\mathcal{G}$, hence in $\mathcal{F}$. □

Remark 4.7. Notice that in the proof of Proposition 4.6, we show that for any simple closed curve $\Gamma \in \mathcal{V}$, there exists an annular neighborhood A_Γ of Γ in ∂M, such that any absolutely area minimizing surface with boundary in A_Γ must be in the same homology class as the unique absolutely area minimizing surface with boundary Γ (see Figure 3 right). This is interesting in its own right, and shows local constancy of the homology classes of absolutely area minimizing surfaces in some sense.

5. Further remarks

The density and genericity results in Sections 3 and 4 are about C^0 simple closed curves in ∂M with C^0-topology. Note that the arguments in these results easily generalize to the smooth case. In particular, let $\mathcal{E}^k$ be the set of C^k simple closed curves in ∂M which are nullhomotopic in M. Then Theorem 3.1 generalizes to $\mathcal{U}^k = \mathcal{U} \cap \mathcal{E}^k$ in the C^0-topology. Moreover, this implies that if ∂M smooth, then $\mathcal{U}^\infty$ is dense in $\mathcal{E}$ in the C^0-topology. In other words, when ∂M is smooth, then for any C^0 nullhomotopic simple closed curve Γ in ∂M, there exists a C^∞ simple closed curve Γ^∞ which is close to Γ in the C^0-topology such that Γ^∞ bounds a unique area minimizing disk in M. Similar results holds for the absolutely area minimizing surface case, too. It might be interesting to study these questions in C^k-topology.

We should note that the generic uniqueness results in [15] are not directly related with our results. In [15], for a fixed $(m-1)$-manifold X, White shows that a generic element in $C^{j,\alpha}$ embeddings of X into $\mathbb{R}^n$ bounds a unique absolutely area minimizing m-manifold in $\mathbb{R}^n$ ([15], Theorem 7). In particular, this result implies that a generic $C^{j,\alpha}$ simple closed curve in $\mathbb{R}^3$ bounds a unique absolutely area minimizing surface [11]. White's result also generalizes to closed manifolds of any dimension (see Section 8 in [15]). However, it does not generalize to manifolds with boundary (see the remarks in Section 8 in [15]). Hence, although it implies generic uniqueness for the curves in the interior of the manifold, it does not imply even the existence of a uniqueness curve in ∂M. In this sense, White's results are not directly related with the results in this paper. On the other hand, it might be interesting to generalize White's techniques to manifolds with boundary, and hence to solve the generic uniqueness question in the smooth category mentioned above.

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