Rev. Mat. Iberoam. **30** (2014), no. 4, 1149–1190
DOI 10.4171/RMI/811

H^∞ functional calculus and square function estimates for Ritt operators

Christian Le Merdy

Abstract. A Ritt operator $T\colon X \to X$ on a Banach space is a power bounded operator satisfying an estimate $n\|T^n - T^{n-1}\| \leq C$. When $X = L^p(\Omega)$ for some $1 < p < \infty$, we study the validity of square functions estimates $\big\|\big(\sum_k k|T^k(x) - T^{k-1}(x)|^2\big)^{1/2}\big\|_{L^p} \lesssim \|x\|_{L^p}$ for such operators. We show that T and T^* both satisfy such estimates if and only if T admits a bounded functional calculus with respect to a Stolz domain. This is a single operator analogue of the famous Cowling–Doust–McIntosh–Yagi characterization of bounded H^∞-calculus on L^p-spaces by the boundedness of certain Littlewood–Paley–Stein square functions. We also prove a similar result for Hilbert spaces. Then we extend the above to more general Banach spaces, where square functions have to be defined in terms of certain Rademacher averages. We focus on noncommutative L^p-spaces, where square functions are quite explicit, and we give applications, examples, and illustrations on such spaces, as well as on classical L^p.

1. Introduction

Let X be a Banach space and let $T\colon X \to X$ be a bounded operator. If $F \subset \mathbb{C}$ is any compact set containing the spectrum of T, a natural question is whether there is an estimate

$$\|\varphi(T)\| \leq K \sup\big\{|\varphi(\lambda))| : \lambda \in F\big\} \tag{1.1}$$

satisfied by all rational functions φ. The mapping $\varphi \mapsto \varphi(T)$ on rational functions is the most elementary form of a "holomorphic functional calculus" associated to T and (1.1) means that this functional calculus is bounded in an appropriate sense.

The most famous such functional calculus estimate is von Neumann's inequality, which says that if $F = \overline{\mathbb{D}}$ is the closed unit disc centered at 0, then (1.1) holds with $K = 1$ for any contraction T on Hilbert space. Von Neumann's inequality was a source of inspiration for the development of various topics around functional cal-

Mathematics Subject Classification (2010): Primary 47A60; Secondary 47A99.
Keywords: Functional calculus, Ritt operators, R-boundedness, square functions.

culus estimates on a Hilbert spaces, including polynomial boundedness, K-spectral sets and related similarity problems. We refer the reader to [5], [48], [49], [52] and the references therein for comprehensive information. See also [13], [11] for striking results in the case when F is equal to the numerical range of T.

When X is a non-Hilbertian Banach space, our knowledge on operators $T\colon X \to X$ and compact sets F satisfying (1.1) for some $K \geq 1$ is quite limited. Positive examples are provided by scalar type operators (see [15]). A more significant observation is that this issue is closely related to the H^∞-functional calculus associated to sectorial operators and indeed, that topic plays a key role in this paper. H^∞-functional calculus was introduced by McIntosh and his co-authors in [10] and [45] and was then developed and applied successfully to various areas, in particular to the study of maximal regularity for certain PDEs, to harmonic analysis of semigroups, and to multiplier theory. We refer the reader to [32] for more information.

In this paper we deal with a holomorphic functional calculus for Ritt operators. Recall that, by definition, $T\colon X \to X$ is a Ritt operator provided that T is power bounded and there exists a constant $C > 0$ such that $n\|T^n - T^{n-1}\| \leq C$ for any integer $n \geq 1$. In this case, the spectrum of T is included in the closure $\overline{B_\gamma}$ of a Stolz domain of the unit disc; see Section 2 and Figure 1 below for details. In accordance with the preceding discussion, this leads to the question whether T satisfies an estimate (1.1) for $F = \overline{B_\gamma}$. We will say that T has a bounded $H^\infty(B_\gamma)$ functional calculus in this case (this terminology will be justified in Section 2). The general problem motivating the present work is to characterize Ritt operators having a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \big(0, \pi/2\big)$ and to exhibit explicit classes of operators satisfying this property.

If $X = H$ is a Hilbert space and $T\colon H \to H$ is a bounded operator, we define the "square function"

$$\|x\|_T = \Big(\sum_{k=1}^{\infty} k\big\|T^k(x) - T^{k-1}(x)\big\|_H^2\Big)^{1/2}, \quad x \in H. \tag{1.2}$$

Likewise for any measure space (Ω, μ), for any $1 \leq p < \infty$ and for any $T\colon L^p(\Omega) \to L^p(\Omega)$, we consider

$$\|x\|_T = \Big\|\Big(\sum_{k=1}^{\infty} k\big|T^k(x) - T^{k-1}(x)\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}, \quad x \in L^p(\Omega). \tag{1.3}$$

Let $T\colon X \to X$ be a Ritt operator on either $X = H$ or $X = L^p(\Omega)$. It was implicitly proved in [39] that if T has a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \big(0, \pi/2\big)$, then it satisfies a uniform estimate $\|x\|_T \lesssim \|x\|$.

This paper has two main purposes. First we establish a converse to this result and prove the following. (Here $p' = p/(p-1)$ is the conjugate of p.)

Theorem 1.1. *Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a Ritt operator, with $1 < p < \infty$. The following assertions are equivalent.*

(i) *The operator T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in (0, \pi/2)$.*

(ii) *The operator T and its adjoint $T^*\colon L^{p'}(\Omega) \to L^{p'}(\Omega)$ both satisfy uniform estimates*

$$\|x\|_T \lesssim \|x\|_{L^p} \quad \textit{and} \quad \|y\|_{T^*} \lesssim \|y\|_{L^{p'}}$$

for $x \in L^p(\Omega)$ and $y \in L^{p'}(\Omega)$.

We also prove a similar result for Ritt operators on Hilbert space.

Second, we investigate relationships between the existence of a bounded $H^\infty(B_\gamma)$ functional calculus and adapted square function estimates on general Banach spaces. We pay a special attention to noncommutative L^p-spaces and prove square function estimates for large classes of Schur multipliers and self-adjoint Markov operators on those spaces.

Ritt operators can be considered as discrete analogues of sectorial operators of type $< \pi/2$, as explained, e.g., in [7], [8] or Section 2 of [39]. According to this analogy, Theorem 1.1 and its Hilbertian counterpart should be regarded as discrete analogues of the main results of [10], [45] showing the equivalence between the boundedness of H^∞-functional calculus and some square function estimates for sectorial operators. Likewise, in the noncommutative setting, our results are both an analogue and an extension of the main results of the memoir [25].

The definitions of the discrete square functions (1.2) and (1.3) go back at least to [56], where they were used to study self-adjoint Markov operators and diffusion semigroups on classical (= commutative) L^p-spaces. They appeared, in the context of Ritt operators, in [28] and [39], [40].

We now turn to a brief description of the paper. In Sections 2 and 3, we introduce the $H^\infty(B_\gamma)$ functional calculus and square functions for Ritt operators, and we prove basic preliminary results. Our definition of square functions on general Banach spaces relies on Rademacher averages. Regarding such averages as abstract square functions is a well-known principle; see e.g. [53], [30], [25] for illustrations. If T is a Ritt operator, then $A = I_X - T$ is a sectorial operator and we show in Section 4 that T has a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \pi/2$ if and only if A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \pi/2$. This observation, stated as Proposition 4.1, provides a tool for transferring bounded H^∞-calculus results from the sectorial setting to Ritt operators. There is apparently no similar way to compare square functions associated to T to square functions associated to A. This is at the root of most of the difficulties in our analysis of Ritt operators. Proposition 4.1 will be applied in Section 8, where we give applications and illustrations on Hilbert spaces, classical L^p-spaces, and noncommutative L^p-spaces.

We will make use of R-boundedness and the notion of R-Ritt operators. This class was introduced by Blunck [7], [8] as a discrete counterpart of R-sectorial operators. Our first main result, proved in Section 5, says that if a Ritt operator $T\colon L^p(\Omega) \to L^p(\Omega)$ satisfies condition (ii) in Theorem 1.1 above, then it is actually an R-Ritt operator. In Section 6, we show that on a Banach space X with finite cotype, any Ritt operator $T\colon X \to X$ with a bounded $H^\infty(B_\gamma)$ functional calculus satisfies square function estimates. This is based on the study of a strong form of the H^∞-functional calculus called "quadratic H^∞-functional calculus", where scalar valued holomorphic functions are replaced by ℓ^2-valued functions. Section 7 is devoted to the converse problem of whether square function estimates for T and T^* imply a bounded $H^\infty(B_\gamma)$ functional calculus. We show that this holds true whenever T is R-Ritt, and complete the proofs of Theorem 1.1 and similar equivalence results.

To close the introduction, we record some notation used throughout this paper. We let $B(X)$ denote the algebra of bounded operators on X and we let I_X (or simply I if there is no ambiguity on X) denote the identity operator on X. We let $\sigma(T)$ denote the spectrum of the operator T (bounded or not) and we let $R(\lambda, T) = (\lambda I_X - T)^{-1}$ denote the resolvent operator when λ belongs to the resolvent set $\mathbb{C} \setminus \sigma(T)$. Next, we let $\mathrm{Ran}(T)$ and $\mathrm{Ker}(T)$ denote the range and the kernel of T, respectively.

For any $a \in \mathbb{C}$ and $r > 0$, we let $D(a, r)$ denote the open disc of radius r centered at a. Also, we let $\mathbb{D} = D(0, 1)$ denote the open unit disc. For any nonempty open set $\mathcal{O} \subset \mathbb{C}$ and any Banach space Z, we let $H^\infty(\mathcal{O}; Z)$ denote the space of bounded holomorphic functions $\varphi \colon \mathcal{O} \to Z$. This is a Banach space for the supremum norm

$$\|\varphi\|_{H^\infty(\mathcal{O};Z)} = \sup\{\|\varphi(\lambda)\|_Z \,:\, \lambda \in \mathcal{O}\}.$$

In the scalar case, we write $H^\infty(\mathcal{O})$ instead of $H^\infty(\mathcal{O}; \mathbb{C})$ and $\|\varphi\|_{\infty,\mathcal{O}}$ instead of $\|\varphi\|_{H^\infty(\mathcal{O})}$. Finally we let $\mathcal{P}$ denote the algebra of complex polynomials.

In Theorem 1.1 and subsequently in the paper we use the notation $\lesssim$ to indicate an inequality valid up to a constant which does not depend on the particular element to which it applies. Then $A(x) \approx B(x)$ means that we have both $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$.

2. Ritt operators and their functional calculus

We start this section with some classical background on the H^∞-functional calculus associated with sectorial operators. The construction and basic properties below go back to [10], [45], see also [29], [35] for complementary information.

For any $\omega \in (0, \pi)$, we let

$$\Sigma_\omega = \{z \in \mathbb{C}^* \,:\, |\mathrm{Arg}(z)| < \omega\} \tag{2.1}$$

be the open sector of angle 2ω around the positive real axis $(0, \infty)$.

Let X be a Banach space. We say that a closed linear operator $A \colon D(A) \to X$ with dense domain $D(A) \subset X$ is sectorial of type ω if $\sigma(A) \subset \overline{\Sigma_\omega}$ and for any $\nu \in (\omega, \pi)$, the set

$$\{zR(z, A) \,:\, z \in \mathbb{C} \setminus \overline{\Sigma_\nu}\} \tag{2.2}$$

is bounded.

For any $\theta \in (0, \pi)$, let $H_0^\infty(\Sigma_\theta)$ denote the algebra of bounded holomorphic functions $f \colon \Sigma_\theta \to \mathbb{C}$ for which there exist positive real numbers $s, c > 0$ such that

$$|f(z)| \leq c\,\frac{|z|^s}{1 + |z|^{2s}}\,, \qquad z \in \Sigma_\theta.$$

Let $0 < \omega < \theta < \pi$ and let $f \in H_0^\infty(\Sigma_\theta)$. Then we set

$$f(A) = \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(z) R(z, A)\, dz\,, \tag{2.3}$$

where $\nu \in (\omega, \theta)$ and the boundary $\partial\Sigma_\nu$ is oriented counterclockwise. The sectoriality condition ensures that this integral is absolutely convergent and defines an element of $B(X)$. Moreover by Cauchy's theorem, this definition does not depend on the choice of ν. Further the resulting mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_0^\infty(\Sigma_\theta)$ into $B(X)$ which is consistent with the usual functional calculus for rational functions.

We say that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus if this last homomorphism is bounded, that is, there exists a constant $K > 0$ such that

$$\|f(A)\| \leq K\, \|f\|_{\infty,\Sigma_\theta}, \qquad f \in H_0^\infty(\Sigma_\theta).$$

If A has dense range and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, then the above homomorphism extends naturally to a bounded homomorphism $f \mapsto f(A)$ from the whole space $H^\infty(\Sigma_\theta)$ into $B(X)$.

It is well known that the above construction can be adapted to various contexts; see e.g. [23] and [16]. We shall briefly explain below such a functional calculus construction for Ritt operators. We first recall some background on this class.

We say that an operator $T\colon X \to X$ is a Ritt operator provided that the two sets

$$\big\{T^n \,:\, n \geq 0\big\} \quad \text{and} \quad \big\{n(T^n - T^{n-1}) \,:\, n \geq 1\big\} \tag{2.4}$$

are bounded. The following spectral characterization is crucial: T is a Ritt operator if and only if

$$\sigma(T) \subset \overline{\mathbb{D}} \quad \text{and} \quad \big\{(\lambda - 1)R(\lambda, T) \,:\, |\lambda| > 1\big\} \quad \text{is bounded.}$$

Indeed this condition is often taken as the definition of Ritt operators. We refer to [44], [46] for this characterization and also to [47], which contains the key argument, and to [7], [8] and Section 2 of [39] for complementary information. Let

$$A = I - T.$$

It follows from the above referenced papers that T is a Ritt operator if and only if

$$\sigma(T) \subset \mathbb{D} \cup \{1\} \quad \text{and} \quad A \text{ is a sectorial operator of type } < \pi/2\,. \tag{2.5}$$

We will need quantitative versions of the above equivalence property. For this purpose we introduce the Stolz domains B_γ as in Figure 1. Namely for any angle $\gamma \in \big(0, \pi/2\big)$, we let B_γ be the interior of the convex hull of 1 and the disc $D(0, \sin\gamma)$.

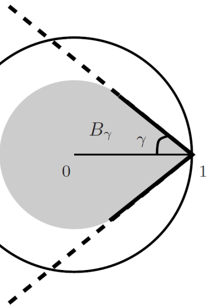

FIGURE 1.

Lemma 2.1. *An operator $T\colon X \to X$ is a Ritt operator if and only if there exists an angle $\alpha \in \big(0, \pi/2\big)$ such that*

$$\sigma(T) \subset \overline{B_\alpha}$$

and, for any $\beta \in \big(\alpha, \pi/2\big)$, the set

$$\big\{(\lambda - 1)R(\lambda, T)\ :\ \lambda \in \mathbb{C} \setminus \overline{B_\beta}\big\}$$

is bounded.

In this case, $A = I - T$ is a sectorial operator of type α.

Proof. Assume that T is a Ritt operator and apply (2.5). Let $\omega \in \big(0, \pi/2\big)$ be a sectorial type of A. Then $\sigma(T)$ is both included in $\mathbb{D} \cup \{1\}$ and in the cone $1 - \overline{\Sigma_\omega}$. Hence there exists $\omega \leq \alpha < \pi/2$ such that $\sigma(T) \subset \overline{B_\alpha}$.

Consider the function h on $\mathbb{C} \setminus \sigma(T)$ defined by $h(\lambda) = (\lambda - 1)R(\lambda, T)$. This function is bounded on $\mathbb{C} \setminus \overline{D}(0, 2)$. Indeed if we let $C_0 = \sup_{n \geq 0} \|T^n\|$, then writing

$$R(\lambda, T) = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

when $|\lambda| > 1$, we have $\|R(\lambda, T)\| \leq C_0/(|\lambda| - 1)$, and hence

$$|\lambda - 1|\, \|R(\lambda, T)\| \leq C_0\, \frac{|\lambda| + 1}{|\lambda| - 1}, \quad \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Let $\beta \in \big(\alpha, \pi/2\big)$. The compact set

$$\Lambda_\beta = \big\{\lambda \in 1 - \overline{\Sigma_\beta}\ :\ \mathrm{Re}(\lambda) \leq \sin^2\beta \quad \text{and} \quad \sin\beta \leq |\lambda| \leq 2\big\} \tag{2.6}$$

is contained in the resolvent set of T. Hence h is bounded on Λ_β. Furthermore

$$h(\lambda) = (1 - \lambda)R\big((1 - \lambda), A\big)$$

and A is sectorial of type α. Consequently h is bounded outside $1 - \overline{\Sigma_\beta}$. Altogether, this shows that h is bounded outside $\overline{B_\beta}$. The rest of the statement is obvious. □

The above lemma leads to the following.

Definition 2.2. We say that $T\colon X \to X$ is a Ritt operator of type $\alpha \in \big(0, \pi/2\big)$ if it satisfies the conclusions of Lemma 2.1.

Then we construct an H^∞-functional calculus as follows. For any $\gamma \in \big(0, \pi/2\big)$, we let $H_0^\infty(B_\gamma) \subset H^\infty(B_\gamma)$ be the space of bounded holomorphic functions $\varphi\colon B_\gamma \to \mathbb{C}$ for which there exist positive real numbers $s, c > 0$ such that

$$|\varphi(\lambda)| \leq c\,|1-\lambda|^s, \quad \lambda \in B_\gamma. \tag{2.7}$$

Assume that T has type α and $\gamma \in \big(\alpha, \pi/2\big)$. Then for any $\varphi \in H_0^\infty(B_\gamma)$, we define

$$\varphi(T) \,=\, \frac{1}{2\pi i}\int_{\partial B_\beta} \varphi(\lambda)\, R(\lambda, T)\, d\lambda\,, \tag{2.8}$$

where $\beta \in (\alpha, \gamma)$ and the boundary ∂B_β is oriented counterclockwise. The boundedness of $\big\{(\lambda-1)R(\lambda,T) \,:\, \lambda \in \partial B_\beta \setminus \{1\}\big\}$ and the assumption (2.7) imply that this integral is absolutely convergent and defines an element of $B(X)$. It does not depend on β and the mapping

$$H_0^\infty(B_\gamma) \longrightarrow B(X), \quad \varphi \mapsto \varphi(T),$$

is an algebra homomorphism. Proofs of these facts are similar to those in the sectorial case.

We state a technical observation for further use.

Lemma 2.3. *Let T be a Ritt operator of type α. Then rT is a Ritt operator for any $r \in (0,1)$ and:*

(1) *For any $\beta \in \big(\alpha, \pi/2\big)$, the set*

$$\big\{(\lambda-1)R(\lambda, rT) \,:\, r \in (0,1),\ \lambda \in \mathbb{C} \setminus B_\beta\big\}$$

is bounded;

(2) *For any $\gamma \in \big(\alpha, \pi/2\big)$ and any $\varphi \in H_0^\infty(B_\gamma)$, $\varphi(T) = \lim_{r\to 1^-} \varphi(rT)$.*

Proof. Consider $\beta \in \big(\alpha, \pi/2\big)$. It is clear that for any $\lambda \in \mathbb{C}\setminus B_\beta$ and any $r \in (0,1)$, that we have $\lambda/r \in \mathbb{C} \setminus \overline{B_\beta}$, $\lambda \notin \sigma(rT)$, and we have

$$(\lambda-1)R(\lambda, rT) = \frac{\lambda-1}{\lambda-r}\left(\frac{\lambda}{r}-1\right) R\Big(\frac{\lambda}{r}, T\Big).$$

Since the sets

$$\big\{(\lambda-1)(\lambda-r)^{-1} \,:\, r \in (0,1),\ \lambda \in \mathbb{C} \setminus B_\beta, \big\}$$

and

$$\big\{(\mu-1)R(\mu, T) \,:\, \mu \in \mathbb{C} \setminus \overline{B_\beta}\big\}$$

are bounded, we obtain (1).

Applying Lebesgue's theorem to (2.8), the assertion (2) follows at once. □

Let $H^\infty_{0,1}(B_\gamma) \subset H^\infty(B_\gamma)$ be the linear span of $H^\infty_0(B_\gamma)$ and constant functions. For any $\varphi = c + \psi$, with $c \in \mathbb{C}$ and $\psi \in H^\infty_0(B_\gamma)$, set $\varphi(T) = cI_X + \psi(T)$. Then $H^\infty_{0,1}(B_\gamma) \subset H^\infty(B_\gamma)$ is a unital algebra and $\varphi \mapsto \varphi(T)$ is a unital homomorphism from $H^\infty_{0,1}(B_\gamma)$ into $B(X)$. Note that $H^\infty_{0,1}(B_\gamma)$ contains rational functions with poles off $\overline{B_\gamma}$, and hence contains all polynomials.

For any T as above and any $r \in (0,1)$, $\sigma(rT) = r\sigma(T) \subset B_\beta$. Hence the definition of $\varphi(rT)$ provided by (2.8) is given by the usual Dunford–Riesz functional calculus of rT. It therefore follows from classical properties of that functional calculus and the approximation Lemma 2.3 that for any rational function φ with poles off $\overline{B_\gamma}$, the above definition of $\varphi(T)$ coincides with the one obtained by substituting T for the complex variable. In particular this applies to any $\varphi \in \mathcal{P}$.

Likewise, recall that since $I - T$ is sectorial one can define its fractional powers $(I-T)^\delta$ for any $\delta > 0$. Then this bounded operator coincides with $\varphi_\delta(T)$, where φ_δ is the element of $H^\infty_0(B_\gamma)$ given by $\varphi_\delta(\lambda) = (1-\lambda)^\delta$. See Section 6 of [45] and Chapter 3 of [23] for similar results.

Definition 2.4. Let T be a Ritt operator of type α and let $\gamma \in \big(\alpha, \pi/2\big)$. We say that T admits a bounded $H^\infty(B_\gamma)$ functional calculus if there exists a constant $K > 0$ such that

$$\|\varphi(T)\| \leq K\, \|\varphi\|_{\infty, B_\gamma}, \quad \varphi \in H^\infty_0(B_\gamma).$$

In this case $\varphi \mapsto \varphi(T)$ is a bounded homomorphism on $H^\infty_{0,1}(B_\gamma)$. The next statement shows that the above functional calculus property can be tested on polynomials only.

Proposition 2.5. *A Ritt operator T has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if there exists a constant $K \geq 1$ such that, for any $\varphi \in \mathcal{P}$,*

$$\|\varphi(T)\| \leq K\, \|\varphi\|_{\infty, B_\gamma}\,. \tag{2.9}$$

Proof. The 'only if' part is clear from the preceding discussion. To prove the 'if' part, assume (2.9) on $\mathcal{P}$ and consider $\varphi \in H^\infty_0(B_\gamma)$. Let $r \in (0,1)$, let $r' \in (r,1)$ be an auxiliary real number and let Γ be the boundary of $r'B_\gamma$ oriented counterclockwise.

By Runge's theorem (see, e.g., Theorem 13.9 in [55]), there exists a sequence $(\varphi_m)_{m\geq 1}$ of polynomials such that $\varphi_m \to \varphi$ uniformly on the compact set $r'\overline{B_\gamma}$. Since $\sigma(rT) \subset r'B_\gamma$, we deduce that

$$\varphi_m(rT) = \frac{1}{2\pi i}\int_\Gamma \varphi_m(\lambda) R(\lambda, rT)\, d\lambda \;\longrightarrow\; \frac{1}{2\pi i}\int_\Gamma \varphi(\lambda) R(\lambda, rT)\, d\lambda \,= \varphi(rT),$$

when $m \to \infty$. By (2.9),

$$\|\varphi_m(rT)\| \leq K\, \|\varphi_m\|_{\infty, rB_\gamma} \leq K\, \|\varphi_m\|_{\infty, r'B_\gamma}.$$

Passing to the limit yields

$$\|\varphi(rT)\| \leq K\, \|\varphi\|_{\infty, r'B_\gamma}.$$

Finally letting $r \to 1$ and applying Lemma 2.3 (2) we deduce $\|\varphi(T)\| \leq K\, \|\varphi\|_{\infty, B_\gamma}$. ∎

The above result is closely related to the following classical notion.

Definition 2.6. We say that a bounded operator $T\colon X \to X$ is polynomially bounded if there is a constant $K \geq 1$ such that

$$\|\varphi(T)\| \leq K\, \|\varphi\|_{\infty,\mathbb{D}}, \quad \varphi \in \mathcal{P}.$$

Obviously any Ritt operator with a bounded $H^\infty(B_\gamma)$ functional calculus is polynomially bounded. See Proposition 7.7 below for a partial converse.

According to Proposition 5.2 in [36], there exist Ritt operators on Hilbert space which are not polynomially bounded. Thus there exist Ritt operators without any bounded $H^\infty(B_\gamma)$ functional calculus. Note that various such (counter-)examples can be derived from Proposition 4.1 below or from Section 8a.

Remark 2.7. Let T be a Ritt operator of type α, let $\gamma \in (\alpha, \pi/2)$, and assume that $I - T$ has dense range. Then $I - T$ is one-to-one by Theorem 3.8 in [10], and arguing as in [10], [45], one can extend the definition of $\varphi(T)$ to any $\varphi \in H^\infty(B_\gamma)$. Namely let $\psi(z) = 1 - z$ and for any $\varphi \in H^\infty(B_\gamma)$, set $\varphi(T) = (I - T)^{-1}(\varphi\psi)(T)$, where $(\varphi\psi)(T)$ is defined by (2.8) and $\varphi(T)$ is defined on $D\big(\varphi(T)\big) = \{x \in X : (\varphi\psi)(T)x \in \mathrm{Ran}(I - T)\}$. It is easy to check that the domain of $\varphi(T)$ contains $\mathrm{Ran}(I-T)$, so that $\varphi(T)$ in densely defined, and that $\varphi(T)$ is closed. Consequently, $\varphi(T)$ is bounded if and only if $D\big(\varphi(T)\big) = X$.

Assume that T has a bounded $H^\infty(B_\gamma)$ functional calculus. Then $\varphi(T)$ is bounded for any $\varphi \in H^\infty(B_\gamma)$. Indeed let $\psi_n(z) = (1 - z)((1 - z) + n^{-1})^{-1}$ for any integer $n \geq 1$. The sectoriality of $(I - T)$ ensures that $(\psi_n(T))_{n\geq 1}$ is bounded. Hence there is a constant $K > 0$ such that $\|(\varphi\psi_n)(T)\| \leq K\, \|\varphi\|_{\infty,B_\gamma}$ for any $n \geq 1$. It is easy to check that $(\varphi\psi_n)(T)x \to \varphi(T)x$ for any $x \in \mathrm{Ran}(I - T)$. This shows the boundedness of $\varphi(T)$, with the estimate $\|\varphi(T)\| \leq K\, \|\varphi\|_{\infty,B_\gamma}$.

Remark 2.8. It is clear that the adjoint $T^*\colon X^* \to X^*$ of a Ritt operator $T \in B(X)$ (of type α) is a Ritt operator (of type α) as well. In this case, $\varphi(T)^* = \varphi(T^*)$ for any $\varphi \in H^\infty_0(B_\gamma)$ with $\gamma > \alpha$. Hence T^* has a bounded $H^\infty(B_\gamma)$ functional calculus if and only if T has one.

3. Square functions

On general Banach spaces, square functions of the forms (1.2) or (1.3) need to be replaced by suitable Rademacher averages. This short section is devoted to precise definitions of these abstract square functions, as well as to relevant properties of Rademacher norms on certain Banach spaces.

We let $(\varepsilon_k)_{k\geq 1}$ be a sequence of independent Rademacher variables on some probability space $(\mathcal{M}, d\mathbb{P})$. Given any Banach space X, we let $\mathrm{Rad}(X)$ denote the closed subspace of the Bochner space $L^2(\mathcal{M}; X)$ spanned by the set $\{\varepsilon_k \otimes x : k \geq 1, x \in X\}$. Thus for any finite family $(x_k)_{k\geq 1}$ of elements of X,

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} = \Big(\int_{\mathcal{M}} \Big\|\sum_k \varepsilon_k(u) x_k\Big\|_X^2 \, d\mathbb{P}(u)\Big)^{1/2}. \tag{3.1}$$

Moreover, elements of $\mathrm{Rad}(X)$ are sums of convergent series of the form

$$\sum_{k=1}^{\infty} \varepsilon_k \otimes x_k\,.$$

For any bounded operator $T\colon X \to X$, for any integer $m \geq 1$ and for any $x \in X$, we set

$$\|x\|_{T,m} \,=\, \Bigl\| \sum_{k=1}^{\infty} k^{m-1/2}\, \varepsilon_k \otimes T^{k-1}(I-T)^m(x) \Bigr\|_{\mathrm{Rad}(X)}.$$

More precisely for any $x \in X$ and any integer $k \geq 1$, set

$$x_k = k^{m-1/2}\, T^{k-1}(I-T)^m(x).$$

Then $\|x\|_{T,m}$ is equal to the $\mathrm{Rad}(X)$-norm of $\sum_{k=1}^{\infty} \varepsilon_k \otimes x_k$ if this series converges in $L^2(\mathcal{M};X)$, and $\|x\|_{T,m} = \infty$ otherwise.

If $X = L^p(\Omega)$ for some $1 \leq p < \infty$, then we have an equivalence

$$\Bigl\| \sum_k \varepsilon_k \otimes x_k \Bigr\|_{\mathrm{Rad}(L^p(\Omega))} \,\approx\, \Bigl\| \Bigl(\sum_k |x_k|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\Omega)} \tag{3.2}$$

for finite families $(x_k)_k$ of X (see, e.g., Theorem 1.d.6 in [41]). Hence for any $T\colon L^p(\Omega) \to L^p(\Omega)$ and any $m \geq 1$, we have

$$\|x\|_{T,m} \,\approx\, \Bigl\| \Bigl(\sum_{k=1}^{\infty} k^{2m-1} \bigl| T^{k-1}(I-T)^m(x) \bigr|^2 \Bigr)^{1/2} \Bigr\|_{L^p(\Omega)}, \qquad x \in L^p(\Omega). \tag{3.3}$$

In particular, the square function $\|\cdot\|_T$ defined by (1.3) is equivalent to $\|\cdot\|_{T,1}$.

Likewise, the Rademacher average (3.1) of a finite sequence $(x_k)_k$ on a Hilbert space H is equal to $(\sum_k \|x_k\|_H^2)^{1/2}$, hence for any $T \in B(H)$, we have

$$\|x\|_{T,m} \,=\, \Bigl(\sum_{k=1}^{\infty} k^{2m-1} \bigl\| T^{k-1}(I-T)^m(x) \bigr\|^2 \Bigr)^{1/2}, \qquad x \in H.$$

The square functions appearing in (3.3) are analogues of well-known square functions associated to sectorial operators on L^p-spaces. Namely let A be a sectorial operator of type $< \pi/2$ on $L^p(\Omega)$. Then $-A$ generates a bounded analytic semigroup $(e^{-tA})_{t\geq 0}$ on $L^p(\Omega)$ and for any integer $m \geq 1$, one can consider

$$\|x\|_{A,m} \,=\, \Bigl\| \Bigl(\int_0^{\infty} t^{2m-1} \bigl| A^m e^{-tA}(x) \bigr|^2\, dt \Bigr)^{1/2} \Bigr\|_{L^p(\Omega)}, \qquad x \in L^p(\Omega).$$

For any $t > 0$, $\partial^m/\partial t^m \bigl(e^{-tA}\bigr) = (-1)^m A^m e^{-tA}$. Hence, if we regard $\bigl(T^{k-1}(I-T)^m\bigr)_{k\geq 1}$ as the m-th discrete derivative of the sequence $(T^{k-1})_{k\geq 1}$, then $\|\cdot\|_{T,m}$ is the discrete analogue of the continuous square function $\|\cdot\|_{A,m}$. Thus Theorem 1.1 is a discrete analogue of the main result of [10] showing the equivalence between the boundedness of H^∞-functional calculus and square function estimates for sectorial operators.

Similar comments apply to the Hilbert space case.

In the sequel, the square functions $\|\cdot\|_{T,m}$ will be used for Ritt operators (although their definitions make sense for any operator).

Let X be a Banach space. The space $\mathrm{Rad}(\mathrm{Rad}(X))$ is the closure of finite sums

$$\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}$$

in $L^2(\mathcal{M}\times\mathcal{M}; X)$, where $x_{ij} \in X$ for any $i,j \geq 1$. We say that X has property (α) if the above decomposition is unconditional, that is, there exists a constant $C > 0$ such that for any finite family $(x_{ij})_{i,j\geq 1}$ of X and any family $(t_{ij})_{i,j\geq 1}$ of complex numbers,

$$\Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes t_{ij}\, x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \leq C \sup_{i,j} |t_{ij}| \Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.$$

Classical L^p-spaces (for $p < \infty$) have property (α). Indeed we have an equivalence

$$\Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(L^p(\Omega)))} \approx \Big\|\Big(\sum_{i,j} |x_{ij}|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \tag{3.4}$$

for finite families $(x_{ij})_{i,j}$ of $L^p(\Omega)$ which extends (3.2). This holds true as well for any Banach lattice with finite cotype in place of $L^p(\Omega)$.

On the contrary, infinite-dimensional noncommutative L^p-spaces (for $p \neq 2$) do not have property (α). This goes back to [50], where property (α) was introduced.

We shall now present more precise information, namely the so-called noncommutative Khintchine inequalities in one or two variables. In the one-variable case, these inequalities, stated as (3.5) and (3.6) below are due to Lust-Piquard for $1 < p < \infty$ ([42]) and to Lust-Piquard and Pisier for $p = 1$ ([43]). The two-variable inequalities (3.7) and (3.8) are taken from [51], pp. 111–112.

In the sequel we let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace and for any $1 \leq p \leq \infty$, we let $L^p(M)$ denote the associated noncommutative L^p-space. We refer the reader to [53] for background and general information on these spaces. Any element of $L^p(M)$ is a (possibly unbounded) operator and, for any such x, the modulus of x used in the following formulas is

$$|x| = (x^*x)^{1/2}.$$

The following equivalences, valid for finite families in $L^p(M)$, are the noncommutative counterparts of (3.2). If $2 \leq p < \infty$, then

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^p(M))} \approx \max\Big\{\Big\|\Big(\sum_k |x_k|^2\Big)^{1/2}\Big\|_{L^p(M)}, \Big\|\Big(\sum_k |x_k^*|^2\Big)^{1/2}\Big\|_{L^p(M)}\Big\}. \tag{3.5}$$

If $1 \leq p \leq 2$, then

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(L^p(M))} \approx \inf\Big\{\Big\|\Big(\sum_k |u_k|^2\Big)^{1/2}\Big\|_{L^p(M)} + \Big\|\Big(\sum_k |v_k^*|^2\Big)^{1/2}\Big\|_{L^p(M)}\Big\}, \tag{3.6}$$

where the infimum runs over all possible decompositions $x_k = u_k + v_k$ in $L^p(M)$.

Let $n \geq 1$ be an integer. The space $L^p(M_n(M))$ associated with the von Neumann algebra $M_n(M)$ can be canonically identified with the vector space of all $n \times n$ matrices with entries in $L^p(M)$. The following equivalences are the noncommutative counterparts of (3.4). If $2 \leq p < \infty$, then

$$\Big\|\sum_{i,j=1}^{n} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(L^p(M)))} \approx \max\Big\{\Big\|\Big(\sum_{i,j=1}^{n} |x_{ij}|^2\Big)^{1/2}\Big\|_{L^p(M)}, \Big\|\Big(\sum_{i,j=1}^{n} |x_{ij}^*|^2\Big)^{1/2}\Big\|_{L^p(M)}, \big\|[x_{ij}]\big\|_{L^p(M_n(M))}, \big\|[x_{ji}]\big\|_{L^p(M_n(M))}\Big\}. \tag{3.7}$$

If $1 \leq p \leq 2$, then

$$\Big\|\sum_{i,j=1}^{n} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(L^p(M)))} \approx \inf\Big\{\Big\|\Big(\sum_{i,j=1}^{n} |u_{ij}|^2\Big)^{1/2}\Big\|_{L^p(M)} + \Big\|\Big(\sum_{i,j=1}^{n} |v_{ij}^*|^2\Big)^{1/2}\Big\|_{L^p(M)} + \big\|[w_{ij}]\big\|_{L^p(M_n(M))} + \big\|[z_{ji}]\big\|_{L^p(M_n(M))}\Big\}, \tag{3.8}$$

where the infimum runs over all possible decompositions $x_{ij} = u_{ij} + v_{ij} + w_{ij} + z_{ij}$ in $L^p(M)$.

4. A transfer principle from sectorial operators to Ritt operators

Let $T\colon X \to X$ be a Ritt operator on an arbitrary Banach space. We remarked in Section 2 that

$$A = I - T$$

is a sectorial operator of type $< \pi/2$. The following transfer result will be extremely important for applications. Indeed it allows known results from the theory of the H^∞-calculus for sectorial operators to be used in our context. This principle will be illustrated in Section 8. The proof is a variant of that of Theorem 8.3 in [22], adapted to our situation (see also Proposition 3.2 in [39]).

Proposition 4.1. *The following are equivalent.*

(i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \big(0, \pi/2\big)$.*

(ii) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in \big(0, \pi/2\big)$.*

Proof. It will be convenient to set

$$\Delta_\gamma = 1 - B_\gamma.$$

for any $\gamma \in \big(0, \pi/2\big)$. This is a subset of the cone Σ_γ.

Assume (i). To any $f \in H_0^\infty(\Sigma_\gamma)$, associate φ given by $\varphi(\lambda) = f(1-\lambda)$. Then φ is defined on B_γ, its restriction to that set belongs to $H_0^\infty(B_\gamma)$, and $\|\varphi\|_{\infty,B_\gamma} =$

$\|f\|_{\infty,\Delta_\gamma} \leq \|f\|_{\infty,\Sigma_\gamma}$. Comparing (2.3) and (2.8) and applying Cauchy's theorem, we see that

$$f(A) = \varphi(T).$$

These observations imply that A has a bounded $H^\infty(\Sigma_\gamma)$ functional calculus.

Assume conversely that A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some θ in $\big(0, \pi/2\big)$. It follows from Lemma 2.1 that

$$\sigma(A) \subset \overline{\Delta_\alpha}$$

for some $\alpha \in \big(0, \pi/2\big)$. Taking θ close enough to $\pi/2$, we can assume that $\alpha < \theta$.

We fix $\gamma \in (\theta, \pi/2)$ and choose an arbitrary $\beta \in (\theta, \gamma)$. Let Γ_1 be the juxtaposition of the segments $[\cos(\beta)e^{i\beta}, 0]$ and $[0, \cos(\beta)e^{-i\beta}]$. Then let Γ_2 be the curve going from $\cos(\beta)e^{-i\beta}$ to $\cos(\beta)e^{i\beta}$ counterclockwise along the circle of center 1 and radius $\sin(\beta)$. Thus

$$\partial\Delta_\beta = \big\{\Gamma_1, \Gamma_2\big\}, \tag{4.1}$$

the juxtaposition of Γ_1 and Γ_2 (see Figure 2).

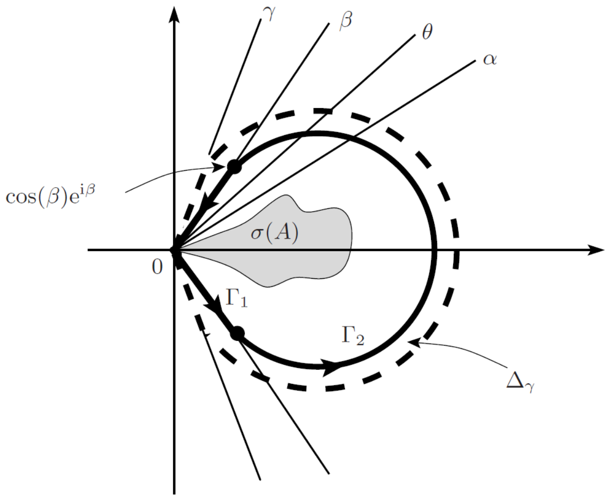

FIGURE 2.

Let $\varphi \in H_0^\infty(B_\gamma)$ and let $f\colon \Delta_\gamma \to \mathbb{C}$ be the holomorphic function defined by

$$f(z) = \varphi(1-z), \quad z \in \Delta_\gamma. \tag{4.2}$$

Then again we have $\|f\|_{\infty,\Delta_\gamma} = \|\varphi\|_{\infty,B_\gamma}$, moreover there exist two positive constants $c, s > 0$ such that

$$|f(z)| \leq c\,|z|^s, \quad z \in \Delta_\gamma. \tag{4.3}$$

We can define $f_1\colon \mathbb{C}\setminus\Gamma_1 \to \mathbb{C}$ and $f_2\colon \mathbb{C}\setminus\Gamma_2 \to \mathbb{C}$ by

$$f_1(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\lambda)}{\lambda - z}\, d\lambda \quad \text{and} \quad f_2(z) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\lambda)}{\lambda - z}\, d\lambda. \tag{4.4}$$

Clearly these functions are holomorphic on their domains. According to (4.1) and Cauchy's theorem, we have

$$\forall\, z \in \Delta_\beta, \quad f(z) = f_1(z) + f_2(z). \tag{4.5}$$

Since the distance between Γ_1 and $\Sigma_\theta \setminus \Delta_\theta$ is strictly positive and $\Gamma_1 \subset \Delta_\gamma$, there is a constant $C_1 \geq 0$ (not depending on f) such that

$$\forall\, z \in \Sigma_\theta \setminus \Delta_\theta, \quad |f_1(z)| \leq C_1 \,\|f\|_{\infty,\Delta_\gamma}. \tag{4.6}$$

Likewise there is a constant $C_2 \geq 0$ (not depending on f) such that

$$\forall\, z \in \Delta_\theta, \quad |f_2(z)| \leq C_2 \,\|f\|_{\infty,\Delta_\gamma}.$$

Combined with (4.5), this yields

$$\forall\, z \in \Delta_\theta, \quad |f_1(z)| \leq (1 + C_2)\,\|f\|_{\infty,\Delta_\gamma}.$$

Together with (4.6) this shows that $f_1 \in H^\infty(\Sigma_\theta)$ and that with $C_3 = \max\{C_1, 1 + C_2\}$, we have

$$\|f_1\|_{\infty,\Sigma_\theta} \leq C_3 \,\|f\|_{\infty,\Delta_\gamma}. \tag{4.7}$$

Now let $g\colon \Sigma_\theta \to \mathbb{C}$ be defined by

$$g(z) = f_1(z) + \frac{f_2(0)}{1+z}.$$

According to the definition of f_1 given by (4.4), $zf_1(z)$ is bounded when $|z| \to \infty$. Hence $zg(z)$ is bounded on Σ_θ. Further, f_2 is defined near 0, hence $|f_2(z) - f_2(0)| \lesssim |z|$ on Δ_θ. By (4.5), we have

$$g(z) = f(z) + \Bigl(\frac{f_2(0)}{1+z} - f_2(z)\Bigr) = f(z) + \bigl(f_2(0) - f_2(z)\bigr) \;-\; f_2(0)\,\frac{z}{1+z} \quad \text{on } \Delta_\theta.$$

Applying the above estimate and (4.3), we deduce that $|g(z)| \lesssim \max\{|z|^s, |z|\}$ on Δ_θ. These estimates show that g belongs to $H_0^\infty(\Sigma_\theta)$. We may therefore compute $g(A)$ by means of (2.3), and hence $f_1(A)$ by

$$f_1(A) = g(A) \;-\; f_2(0)(I + A)^{-1}.$$

From the assumption (ii), we get a constant $C_4 \geq 0$ (not depending on f) such that

$$\|f_1(A)\| \leq C_4 \,\|f_1\|_{\infty,\Sigma_\theta}.$$

Combining this with (4.7), we deduce

$$\|f_1(A)\| \leq C_3\, C_4\, \|f\|_{\infty,\Delta_\gamma}.$$

The holomorphic function f_2 is defined on an open neighborhood of the spectrum $\sigma(A)$. Hence $f_2(A)$ may be defined by the classical Riesz–Dunford functional calculus. Then by Fubini's theorem and (4.4), we have

$$f_2(A) \,=\, \frac{1}{2\pi i}\int_{\Gamma_2} f(\lambda)\, R(\lambda, A)\, d\lambda\,.$$

Consequently,

$$\|f_2(A)\| \,\leq\, \frac{1}{2\pi}\int_{\Gamma_2} |f(\lambda)|\, \|R(\lambda, A)\|\, |\, d\lambda|\,.$$

We deduce that there is a constant $C_5 \geq 0$ (not depending on f) such that

$$\|f_2(A)\| \,\leq C_5\, \|f\|_{\infty,\Delta_\gamma}.$$

Using (4.2) and (4.5) it is easy to check that

$$\varphi(T) \,=\, f_1(A) + f_2(A).$$

We deduce (with $C = C_3C_4 + C_5$) the estimate

$$\|\varphi(T)\| \leq C\, \|\varphi\|_{\infty,B_\gamma},$$

which shows the boundedness of the $H^\infty(B_\gamma)$ functional calculus. □

Remark 4.2. We mention another (easier) transfer principle. Let $(T_t)_{t\geq 0}$ be a bounded analytic semigroup on X, and let $-A$ denote its infinitesimal generator. For any fixed $t \geq 0$, T_t is a Ritt operator. This is easy to check; see Section 3 of [58] for more on this. Writing $\varphi(T_t) = f(A)$ with $f(z) = \varphi(e^{-tz})$, one shows that if A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in \big(0, \pi/2\big)$, then T_t admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \big(0, \pi/2\big)$.

5. *R*-boundedness and *R*-Ritt operators

This section starts with some background on R-boundedness, a notion which – by now – plays a prominent role in many questions concerning functional calculi; see in particular [29], [30], [59]. The notion of R-boundedness was introduced in [6] and developed in [9]. The resulting notion of R-Ritt operator (see below) was first studied by Blunck [7], [8].

Let X be a Banach space and let $E \subset B(X)$ be a set of bounded operators on X. We say that E is R-bounded if there exists a constant $C \geq 0$ such that for any finite family $(T_k)_k$ in E and any finite family $(x_k)_k$ in X,

$$\Big\|\sum_k \varepsilon_k \otimes T_k(x_k)\Big\|_{\mathrm{Rad}(X)} \,\leq\, C\,\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}.$$

In this case, we let $\mathcal{R}(E)$ denote the smallest possible C. Any R-bounded set E is bounded, with $\|T\| \leq \mathcal{R}(E)$ for any $T \in E$. If $X = H$ is a Hilbert space, the converse holds true, because of the isometric isomorphism $\mathrm{Rad}(H) = \ell^2(H)$. But if X is not isomorphic to a Hilbert space, then the unit ball of $B(X)$ is not R-bounded (see [1]).

We will use the following convexity result. This is a well-known consequence of Lemma 3.2 in [9], see also Lemma 4.2 in [25].

Lemma 5.1. *Let $J \subset \mathbb{R}$ be an interval, let $E \subset B(X)$ be an R-bounded set and let $K > 0$ be a constant. Then the set*

$$E_K = \Big\{ \int_J h(t)F(t)\,dt \,\Big|\, F\colon J \to E \ \textit{continuous},\ h \in L^1(J;dt)\ \int_J |h(t)|\,dt \leq K \Big\}$$

is R-bounded, with $\mathcal{R}(E_K) \leq 2K\mathcal{R}(E)$.

A sectorial operator A on X is called R-sectorial of R-type ω provided that $\sigma(A) \subset \overline{\Sigma_\omega}$ and for any $\nu \in (\omega, \pi)$, the set (2.2) is R-bounded.

Likewise, a Ritt operator T on X is called R-Ritt provided that the two sets in (2.4) are R-bounded. The following is an R-bounded version of (2.5) and Lemma 2.1. We refer to [7] for closely related results.

Lemma 5.2. *Let $T\colon X \to X$ be a Ritt operator and let $A = I - T$. The following are equivalent.*

(i) *T is R-Ritt.*

(ii) *A is R-sectorial of R-type $< \pi/2$.*

(iii) *There exists an angle $\alpha \in \big(0, \pi/2\big)$ such that $\sigma(T) \subset \overline{B_\alpha}$ and, for any $\beta \in \big(\alpha, \pi/2\big)$, the set*

$$\big\{(\lambda - 1)R(\lambda, T) \,:\, \lambda \in \mathbb{C} \setminus \overline{B_\beta}\big\}$$

is R-bounded.

Proof. The implications '(i)⇒(ii)' and '(iii)⇒(i)' follow from [7]. The proof of '(ii)⇒(iii)' is parallel to that of Lemma 2.1, using two elementary but important results on R-boundedness due to L. Weis. The first says that for any open set $\mathcal{O} \subset \mathbb{C}$ and for any compact set $F \subset \mathcal{O}$, any analytic function $\mathcal{O} \to B(X)$ maps F into an R-bounded subset of $B(X)$ (Proposition 2.6 in [59]). With the notation of the proof of Lemma 2.1, this implies that the two sets

$$E_1 = h(\Lambda_\beta) \quad \text{and} \quad E_2 = \{h(\lambda) \,:\, |\lambda| = 2\}$$

are R-bounded. The second is the maximum principle for R-boundedness (Proposition 2.8 in [59]). Together with the R-boundedness of E_2, it implies that $\{h(\lambda) : |\lambda| \geq 2\}$ is R-bounded. With these elements in hand, the adaptation of the proof of Lemma 2.1 is straightforward. □

We say that T is an R-Ritt operator of R-type α if it satisfies condition (iii) of Lemma 5.2. It is clear that in this case, $A = I - T$ is R-sectorial of R-type α.

In the rest of this section, we focus on *commutative* L^p-spaces; see however Remark 5.5. Our objective is the following theorem, which is a key step in the proof of Theorem 1.1.

Theorem 5.3. *Let (Ω,μ) be a measure space, let $1<p<\infty$, and let $T\colon L^p(\Omega)\to L^p(\Omega)$ be a power bounded operator. Assume that T satisfies uniform estimates*

$$\|x\|_{T,1}\lesssim\|x\|_{L^p}\quad\text{and}\quad\|y\|_{T^*,1}\lesssim\|y\|_{L^{p'}}\tag{5.1}$$

for $x\in L^p(\Omega)$ and $y\in L^{p'}(\Omega)$. Then T is R-Ritt.

Until the end of the proof of this theorem, we fix a bounded operator $T\colon L^p(\Omega)\to L^p(\Omega)$, with $1<p<\infty$. The following lemma is inspired by the proof of Theorem 4.7 in [28].

Lemma 5.4. *If T satisfies a uniform estimate*

$$\|x\|_{T,1}\lesssim\|x\|,\quad x\in L^p(\Omega),\tag{5.2}$$

then it automatically satisfies a uniform estimate

$$\|x\|_{T,2}\lesssim\|x\|,\quad x\in L^p(\Omega).\tag{5.3}$$

Proof. We will use the following elementary identity that the reader can easily check. For any integer $k\geq 1$,

$$\sum_{j=1}^{k} j(k+1-j)\,=\,\frac{1}{6}\,k(k+1)(k+2).\tag{5.4}$$

Let $x\in L^p(\Omega)$ and let $N\geq 1$ be an integer. According to the above identity we have an inequality

$$\sum_{k=1}^{N} k^3\,\bigl|T^{k-1}(I-T)^2x\bigr|^2\,\leq\,6\sum_{k=1}^{N}\sum_{j=1}^{k} j(k+1-j)\bigl|T^{k-1}(I-T)^2x\bigr|^2.$$

By a change of indices (letting $r=k+1-j$ for any fixed j), we have

$$\begin{aligned}\sum_{k=1}^{N}\sum_{j=1}^{k} j(k+1-j)\bigl|T^{k-1}(I-T)^2x\bigr|^2\,&=\,\sum_{j=1}^{N} j\sum_{k=j}^{N}(k+1-j)\bigl|T^{k-1}(I-T)^2x\bigr|^2\\&=\sum_{j=1}^{N} j\sum_{r=1}^{N+1-j} r\bigl|T^{r+j-2}(I-T)^2x\bigr|^2\leq\sum_{j=1}^{N} j\sum_{r=1}^{N} r\bigl|T^{r+j-2}(I-T)^2x\bigr|^2.\end{aligned}$$

According to (3.4), we have an estimate

$$\begin{aligned}&\Bigl\|\Bigl(\sum_{j,r=1}^{N} jr\bigl|T^{r+j-2}(I-T)^2x\bigr|^2\Bigr)^{1/2}\Bigr\|_{L^p(\Omega)}\\&\qquad\lesssim\Bigl\|\Bigl(\sum_{j,r=1}^{N} j^{1/2}\,r^{1/2}\,\varepsilon_j\otimes\varepsilon_r\otimes T^{r+j-2}(I-T)^2x\Bigr\|_{\mathrm{Rad}(\mathrm{Rad}(L^p(\Omega)))}.\end{aligned}$$

Furthermore, writing

$$T^{r+j-2}(I-T)^2x \,=\, T^{j-1}(I-T)\big[T^{r-1}(I-T)x\big],$$

and applying the assumption (5.2) twice, we see that

$$\Big\|\sum_{j,r=1}^{N} j^{1/2}\, r^{1/2}\varepsilon_j\otimes\varepsilon_r\otimes T^{r+j-2}(I-T)^2x\Big\|_{\mathrm{Rad}(\mathrm{Rad}(L^p(\Omega)))}$$

$$\lesssim \Big\|\sum_{r=1}^{N} r^{\frac{1}{2}}\varepsilon_r\otimes T^{r-1}(I-T)x\Big\|_{\mathrm{Rad}(L^p(\Omega))} \lesssim \|x\|.$$

Altogether, we obtain the estimate

$$\Big\|\Big(\sum_{k=1}^{N} k^3\,\big|T^{k-1}(I-T)^2x\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \lesssim \|x\|,$$

which proves (5.3). □

Proof of Theorem 5.3. Since T is power bounded and $X = L^p(\Omega)$ is reflexive, the mean ergodic theorem ensures that

$$X \,=\, \mathrm{Ker}(I-T)\oplus\overline{\mathrm{Ran}(I-T)}. \tag{5.5}$$

Furthermore the two square function estimates (5.1) imply that

$$\|x\|\approx\|x\|_{T,1},\qquad x\in\overline{\mathrm{Ran}(I-T)}.$$

Indeed this is implicit in Corollary 3.4 of [39], to which we refer for details. Let $(x_n)_{n\geq 1}$ be a finite family in $\overline{\mathrm{Ran}(I-T)}$, and let $(\eta_n)_{n\geq 1}$ be a sequence of ± 1. The above equivalence yields

$$\Big\|\sum_{n\geq 1}\eta_n\, x_n\Big\|_{L^p(\Omega)} \approx \Big\|\sum_{k\geq 1}\sum_{n\geq 1} k^{1/2}\,\eta_n\,\varepsilon_k\otimes T^{k-1}(I-T)x_n\Big\|_{\mathrm{Rad}(L^p(\Omega))}.$$

Averaging over the $\eta_n=\pm 1$ and applying (3.4), we obtain that

$$\Big\|\sum_{n\geq 1}\varepsilon_n\otimes x_n\Big\|_{\mathrm{Rad}(L^p(\Omega))} \approx \Big\|\Big(\sum_{k,n\geq 1} k\,\big|T^{k-1}(I-T)x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \tag{5.6}$$

for x_n in $\overline{\mathrm{Ran}(I-T)}$.

Applying Lemma 5.4 and similarly averaging the resulting estimates

$$\Big\|\sum_n \eta_n\, x_n\Big\|_{T,2} \lesssim \Big\|\sum_n \eta_n\, x_n\Big\|$$

over all $\eta_n=\pm 1$, we obtain that

$$\Big\|\Big(\sum_{k,n\geq 1} k^3\big|T^{k-1}(I-T)^2x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)} \lesssim \Big\|\sum_{n\geq 1}\varepsilon_n\otimes x_n\Big\|_{\mathrm{Rad}(L^p(\Omega))} \tag{5.7}$$

for x_n in $L^p(\Omega)$.

Our aim is to show that the two sets in (2.4) are R-bounded. Their restrictions to the kernel $\text{Ker}(I-T)$ clearly have this property. By (5.5) it therefore suffices to consider their restrictions to $\overline{\text{Ran}(I-T)}$.

Let $(x_n)_{n\geq 1}$ be a finite family in $\overline{\text{Ran}(I-T)}$. Each $T^n x_n$ belongs to that space. Hence, by (5.6), we have

$$\Big\|\sum_{n\geq 1}\varepsilon_n\otimes T^n x_n\Big\|_{\text{Rad}(L^p(\Omega))}\lesssim\Big\|\Big(\sum_{k,n\geq 1}k\,\big|T^{k+n-1}(I-T)x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}.$$

Moreover,

$$\begin{aligned}\Big\|\Big(\sum_{k,n\geq 1}k\big|T^{k+n-1}(I-T)x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}&\\ &\leq\Big\|\Big(\sum_{k,n\geq 1}(k+n)\big|T^{k+n-1}(I-T)x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}\\ &\leq\Big\|\Big(\sum_{n\geq 1}\sum_{k\geq n+1}k\big|T^{k-1}(I-T)x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}\\ &\leq\Big\|\Big(\sum_{k,n\geq 1}k\big|T^{k-1}(I-T)x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}.\end{aligned}$$

Using (5.6) we deduce that

$$\Big\|\sum_{n\geq 1}\varepsilon_n\otimes T^n x_n\Big\|_{\text{Rad}(L^p(\Omega))}\lesssim\Big\|\sum_{n\geq 1}\varepsilon_n\otimes x_n\Big\|_{\text{Rad}(L^p(\Omega))}.$$

This shows the R-boundedness of $\{T^n\,:\,n\geq 1\}$.

Likewise, using (5.6) and (5.7), we have

$$\begin{aligned}\Big\|\sum_{n\geq 1}\varepsilon_n\otimes nT^{n-1}(I-T)x_n\Big\|_{\text{Rad}(L^p(\Omega))}&\\ &\lesssim\Big\|\Big(\sum_{k,n\geq 1}k\,\big|n\,T^{k+n-2}(I-T)^2x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}\\ &\lesssim\Big\|\Big(\sum_{k,n\geq 1}(k+n)^3\big|T^{k+n-2}(I-T)^2x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}\\ &\lesssim\Big\|\Big(\sum_{n\geq 1}\sum_{k\geq n}(k+1)^3\big|T^{k-1}(I-T)^2x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}\\ &\lesssim\Big\|\Big(\sum_{k,n\geq 1}k^3\big|T^{k-1}(I-T)^2x_n\big|^2\Big)^{1/2}\Big\|_{L^p(\Omega)}\lesssim\Big\|\sum_{n\geq 1}\varepsilon_n\otimes x_n\Big\|_{\text{Rad}(L^p(\Omega))}.\end{aligned}$$

Thus the set $\{nT^{n-1}(I-T)\,:\,n\geq 1\}$ is R-bounded as well, which completes the proof. $\square$

Remark 5.5. It is easy to check that the above proof and hence Theorem 5.3 extend to the case when $L^p(\Omega)$ is replaced by a reflexive Banach space with property (α). In particular this holds true on any reflexive Banach lattice with finite cotype. However we do not know whether Theorem 5.3 holds true on *noncommutative* L^p*-spaces*.

The above proof can be adapted to the sectorial case, which yields a slight improvement of the main result of [10]. We explain this point in the separate note [38].

6. From H^∞ functional calculus to square functions

The main aim of this section is to determine when a Ritt operator T with a bounded $H^\infty(B_\gamma)$ functional calculus must satisfy square function estimates $\|x\|_{T,m} \lesssim \|x\|$. We will show that this holds true on Banach spaces with finite cotype. We refer the reader e.g. to [14] for information on cotype.

To this end, we investigate a strong form of bounded holomorphic functional calculus which is somehow natural for making connections with square functions. We consider both the sectorial case and the Ritt case.

Let $f_1, \ldots, f_n$ be a finite family in $H^\infty(\mathcal{O})$, for some nonempty open set $\mathcal{O} \subset \mathbb{C}$. In the sequel we let

$$\Big\| \Big(\sum_{l=1}^n |f_l|^2 \Big)^{1/2} \Big\|_{\infty,\mathcal{O}} = \sup \Big\{ \Big(\sum_{l=1}^n |f_l(z)|^2 \Big)^{1/2} \, : \, z \in \mathcal{O} \Big\}.$$

Equivalently, let $(e_1, \ldots, e_n)$ be the standard basis of the Hermitian space ℓ^2_n, then

$$\Big\| \Big(\sum_{l=1}^n |f_l|^2 \Big)^{1/2} \Big\|_{\infty,\mathcal{O}} = \Big\| \sum_{l=1}^n f_l \otimes e_l \Big\|_{H^\infty(\mathcal{O};\ell^2_n)}. \tag{6.1}$$

In the following definitions, X is an arbitrary Banach space.

Definition 6.1. (1) Let A be a sectorial operator of type $\omega \in (0, \pi)$ on X and let $\theta \in (\omega, \pi)$. We say that A admits a quadratic $H^\infty(\Sigma_\theta)$ functional calculus if there exists a constant $K > 0$ such that for any $n \geq 1$, for any $f_1, \ldots, f_n$ in $H^\infty_0(\Sigma_\theta)$, and for any $x \in X$,

$$\Big\| \sum_{l=1}^n \varepsilon_l \otimes f_l(A)x \Big\|_{\mathrm{Rad}(X)} \leq K \, \|x\| \, \Big\| \Big(\sum_{l=1}^n |f_l|^2 \Big)^{1/2} \Big\|_{\infty,\Sigma_\theta}. \tag{6.2}$$

(2) Let T be a Ritt operator of type $\alpha \in \big(0, \pi/2\big)$ on X and let $\gamma \in \big(\alpha, \pi/2\big)$. We say that T admits a quadratic $H^\infty(B_\gamma)$ functional calculus if there exists a constant $K > 0$ such that for any $n \geq 1$, for any $\varphi_1, \ldots, \varphi_n$ in $H^\infty_0(B_\gamma)$, and for any $x \in X$,

$$\Big\| \sum_{l=1}^n \varepsilon_l \otimes \varphi_l(T)x \Big\|_{\mathrm{Rad}(X)} \leq K \, \|x\| \, \Big\| \Big(\sum_{l=1}^n |\varphi_l|^2 \Big)^{1/2} \Big\|_{\infty,B_\gamma}.$$

Arguing as in Proposition 2.5, one can restrict to polynomials in part (2).

It is clear that any sectorial operator with a quadratic $H^\infty(\Sigma_\theta)$ functional calculus has a bounded $H^\infty(\Sigma_\theta)$ functional calculus. We will see in Proposition 6.7 that the converse does not hold. However we show that, up to a change of angle, the converse holds on a large class of Banach spaces. We will need the following remarkable estimate of Kaiser–Weis (Corollary 3.4 in [27]).

Lemma 6.2. ([27]) *Let X be a Banach space with finite cotype. Then there exists a constant $C>0$ such that*

$$\Big\|\sum_{k,l\geq 1}\alpha_{kl}\,\varepsilon_k\otimes\varepsilon_l\otimes x_k\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\leq C\sup_k\Big(\sum_l|\alpha_{kl}|^2\Big)^{1/2}\Big\|\sum_k\varepsilon_k\otimes x_k\Big\|_{\mathrm{Rad}(X)} \tag{6.3}$$

for any finite family $(\alpha_{kl})_{k,l\geq 1}$ of complex numbers and any finite family $(x_k)_{\geq 1}$ of X.

Theorem 6.3. *Assume that X has finite cotype and let A be a sectorial operator on X with a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Then A admits a quadratic $H^\infty(\Sigma_\nu)$ functional calculus for any $\nu\in(\theta,\pi)$.*

Proof. The proof relies on a decomposition principle for holomorphic functions due to E. Franks and A. McIntosh. Let $0<\theta<\nu<\pi$ be two angles. The decomposition principle says that there exist a constant $C>0$ and two sequences $(F_k)_{k\geq 1}$ and $(G_k)_{k\geq 1}$ in $H_0^\infty(\Sigma_\theta)$ such that:

(a) For any $z\in\Sigma_\theta$, we have $\sum_{k\geq 1}|F_k(z)|\leq C$.

(b) For any $z\in\Sigma_\theta$, we have $\sum_{k\geq 1}|G_k(z)|\leq C$.

(c) For any Banach space Z and for any function $F\in H^\infty(\Sigma_\nu;Z)$, there exists a bounded sequence $(b_k)_{k\geq 1}$ in Z such that

$$\|b_k\|\leq C\,\|F\|_{H^\infty(\Sigma_\nu;Z)},\qquad k\geq 1,$$

and

$$F(z)\,=\sum_{k=1}^\infty b_k\,F_k(z)\,G_k(z),\qquad z\in\Sigma_\theta.$$

Indeed, Proposition 3.1 in [16] and the last paragraph of Section 3 in [16] show this property for $Z=\mathbb{C}$. However it is easy to check that the proof works as well for Z-valued holomorphic functions.

Since A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus, we have a uniform estimate

$$\Big\|\sum_k\eta_kF_k(A)\Big\|\lesssim\sup_{z\in\Sigma_\theta}\Big|\sum_k\eta_k\,F_k(z)\Big|\leq\sup_k|\eta_k|\,\sup_{z\in\Sigma_\theta}\sum_k|F_k(z)|$$

for finite families $(\eta_k)_{k\geq 1}$ of complex numbers. Hence by (a), we have

$$\sup_{m\geq 1}\,\sup_{\eta_k=\pm 1}\Big\|\sum_{k=1}^m\eta_k\,F_k(A)\Big\|<\infty\,. \tag{6.4}$$

Likewise, (b) implies that

$$\sup_{m\geq 1}\ \sup_{\eta_k=\pm 1}\Bigl\|\sum_{k=1}^{m}\eta_k\, G_k(A)\Bigr\| < \infty\,. \tag{6.5}$$

We will apply property (c) with $Z=\ell^2_n$ for arbitrary $n\geq 1$. Let $f_1,\ldots,f_n$ be elements of $H^\infty_0(\Sigma_\nu)$ and consider

$$F=\sum_{l=1}^{n} f_l\otimes e_l\ \in H^\infty\bigl(\Sigma_\nu;\ell^2_n\bigr).$$

Let $(b_k)_{k\geq 1}$ be the bounded sequence of ℓ^2_n provided by (c), and write $b_k=(\alpha_{k1},\alpha_{k2},\ldots,\alpha_{kn})$ for any $k\geq 1$. Then

$$f_l(z)=\sum_{k=1}^{\infty}\alpha_{kl}\,F_k(z)\,G_k(z),\qquad z\in\Sigma_\theta, \tag{6.6}$$

for any $l=1,\ldots,n$, and

$$\sup_k\Bigl(\sum_l|\alpha_{kl}|^2\Bigr)^{1/2}\leq C\,\Bigl\|\Bigl(\sum_{l=1}^{n}|f_l|^2\Bigr)^{1/2}\Bigr\|_{\infty,\Sigma_\nu} \tag{6.7}$$

by (c) and (6.1).

For any $l=1,\ldots,n$ and any integer $m\geq 1$, we consider the function

$$h_{m,l}=\sum_{k=1}^{m}\alpha_{kl}\,F_k\,G_k,$$

which belongs to $H^\infty_0(\Sigma_\nu)$ and approximates f_l by (6.6).

Let $x\in X$. By the Khintchine–Kahane inequality (see, e.g., Theorem 1.e.13 in [41]), we have

$$\begin{aligned}\Bigl\|\sum_l\varepsilon_l\otimes h_{m,l}(A)x\Bigr\|_{\mathrm{Rad}(X)} &= \Bigl(\int_{\mathcal M}\Bigl\|\sum_{k,l}\varepsilon_l(u)\,\alpha_{kl}\,F_k(A)\,G_k(A)x\Bigr\|^2 d\mathbb{P}(u)\Bigr)^{1/2}\\ &\lesssim \int_{\mathcal M}\Bigl\|\sum_k F_k(A)\Bigl(\sum_l\varepsilon_l(u)\,\alpha_{kl}\,G_k(A)x\Bigr)\Bigr\|\,d\mathbb{P}(u)\,.\end{aligned}$$

For any $x_1,\ldots,x_m$ in X, we have

$$\sum_k F_k(A)x_k = \int_{\mathcal M}\Bigl(\sum_k\varepsilon_k(v)F_k(A)\Bigr)\Bigl(\sum_k\varepsilon_k(v)x_k\Bigr)\,d\mathbb{P}(v)\,,$$

hence, by (6.4),

$$\begin{aligned}\Bigl\|\sum_k F_k(A)x_k\Bigr\| &\leq \int_{\mathcal M}\Bigl\|\sum_k\varepsilon_k(v)F_k(A)\Bigr\|\,\Bigl\|\sum_k\varepsilon_k(v)x_k\Bigr\|\,d\mathbb{P}(v)\\ &\lesssim \int_{\mathcal M}\Bigl\|\sum_k\varepsilon_k(v)x_k\Bigr\|\,d\mathbb{P}(v).\end{aligned}$$

Applying this estimate with $x_k = \sum_l \varepsilon_l(u)\alpha_{kl} G_k(A)x$ and integrating over $(u,v) \in \mathcal{M}\times\mathcal{M}$, we deduce that

$$\Big\|\sum_l \varepsilon_l \otimes h_{m,l}(A)x\Big\|_{\mathrm{Rad}(X)} \lesssim \Big\|\sum_{k,l} \alpha_{kl}\, \varepsilon_k \otimes \varepsilon_l \otimes G_k(A)x\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.$$

By assumption, X has finite cotype. Hence it follows from Lemma 6.2 and (6.7) that

$$\Big\|\sum_l \varepsilon_l \otimes h_{m,l}(A)x\Big\|_{\mathrm{Rad}(X)} \lesssim \Big\|\Big(\sum_l |f_l|^2\Big)^{1/2}\Big\|_{\infty,\Sigma_\nu} \Big\|\sum_l \varepsilon_k \otimes G_k(A)x\Big\|_{\mathrm{Rad}(X)}.$$

Moreover, according to (6.5), we have $\|\sum_k \varepsilon_k \otimes G_k(A)x\|_{\mathrm{Rad}(X)} \lesssim \|x\|$. Thus we finally obtain

$$\Big\|\sum_l \varepsilon_l \otimes h_{m,l}(A)x\Big\|_{\mathrm{Rad}(X)} \lesssim \|x\| \,\Big\|\Big(\sum_l |f_l|^2\Big)^{1/2}\Big\|_{\infty,\Sigma_\nu}.$$

We deduce the expected result by an entirely classical approximation process, that we explain for the convenience of the reader. For any $\varepsilon \in (0,1)$, set $A_\varepsilon = (\varepsilon I + A)(I+\varepsilon A)^{-1}$. Then A_ε is bounded and invertible, its spectrum is a compact subset of Σ_θ, and it follows from Cauchy's theorem that for some contour Γ_ε of finite length included in the open set Σ_θ, we have

$$h(A_\varepsilon) = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} h(z)R(z,A_\varepsilon)\,dz$$

for any $h \in H_0^\infty(\Sigma_\theta)$. Since $h_{m,l} \to f_l$ pointwise and $\sup_{m,z} |h_{m,l}(z)| < \infty$, the above integral representation ensures that

$$\lim_{m\to\infty} h_{m,l}(A_\varepsilon) = f_l(A_\varepsilon), \quad l = 1,\dots,n.$$

Furthermore,

$$\lim_{\varepsilon\to 0} f_l(A_\varepsilon) = f_l(A)$$

for any $l = 1,\dots,n$, by Lemma 2.4 in [35].

Now observe that the A_ε uniformly admit a bounded $H^\infty(\Sigma_\theta)$ functional calculus. That is, there exists a constant $K > 0$ such that $\|h(A_\varepsilon)\| \le K\,\|h\|_{\infty,\Sigma_\theta}$ for any $h \in H_0^\infty(\Sigma_\theta)$ and any $\varepsilon \in (0,1)$. It therefore follows from the above proof that there is a constant $K' > 0$ such that

$$\Big\|\sum_l \varepsilon_l \otimes h_{m,l}(A_\varepsilon)x\Big\|_{\mathrm{Rad}(X)} \le K'\,\|x\|\,\Big\|\Big(\sum_l |f_l|^2\Big)^{1/2}\Big\|_{\infty,\Sigma_\nu} \tag{6.8}$$

for any $m \ge 1$ and any $\varepsilon \in (0,1)$. Then (6.2) follows from (6.8). □

We now state a similar result for Ritt operators and their functional calculus.

Theorem 6.4. *Assume that X has finite cotype and let T be a Ritt operator on X with a bounded $H^\infty(B_\gamma)$ functional calculus. Then T admits a quadratic $H^\infty(B_\nu)$ functional calculus for any $\nu \in (\gamma,\pi/2)$.*

Proof. There are two ways to get to this result. The first is to mimic the proof of Theorem 6.3, using a Franks–McIntosh decomposition adapted to Stolz domains. The existence of such decompositions follows from Section 5 of [16].

The second is to observe that the transfer principle stated as Proposition 4.1 holds (with essentially the same proof) for the quadratic functional calculus. Namely, with $A = I - T$, the following are equivalent:

(i) T admits a quadratic $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \big(0, \pi/2\big)$.

(ii) A admits a quadratic $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in \big(0, \pi/2\big)$.

Hence the result follows from Theorem 6.3, the implication '(i)⇒(ii)' of Proposition 4.1 and the implication '(ii)⇒(i)' above. □

Banach spaces with property (α) have finite cotype, hence Theorems 6.3 and 6.4 apply to such spaces. It turns out that a much stronger H^∞ calculus property holds for such spaces, as follows.

Proposition 6.5. *Assume that X has property (α).*

(1) *Let A be a sectorial operator on X with a bounded $H^\infty(\Sigma_\theta)$ functional calculus. Then for any $\nu \in (\theta, \pi)$, there exists a constant $K > 0$ such that*

$$\Big\|\sum_{l,j=1}^{n} \varepsilon_l \otimes f_{lj}(A)x_j\Big\|_{\text{Rad}(X)} \leq K \sup_{z\in\Sigma_\nu} \big\|[f_{lj}(z)]\big\|_{M_n} \Big\|\sum_{j=1}^{n} \varepsilon_j \otimes x_j\Big\|_{\text{Rad}(X)} \tag{6.9}$$

for any $n \geq 1$, for any matrix $[f_{lj}]$ of elements of $H_0^\infty(\Sigma_\nu)$ and for any $x_1, \ldots, x_n$ in X.

(2) *Let T be a Ritt operator on X with a bounded $H^\infty(B_\gamma)$ functional calculus. Then for any $\nu \in (\gamma, \pi/2)$, there exists a constant $K > 0$ such that*

$$\Big\|\sum_{l,j=1}^{n} \varepsilon_l \otimes \varphi_{lj}(T)x_j\Big\|_{\text{Rad}(X)} \leq K \sup_{z\in B_\nu} \big\|[\varphi_{lj}(z)]\big\|_{M_n} \Big\|\sum_{j=1}^{n} \varepsilon_j \otimes x_j\Big\|_{\text{Rad}(X)} \tag{6.10}$$

for any $n \geq 1$, for any matrix $[\varphi_{lj}]$ of elements of $H_0^\infty(B_\nu)$ and for any $x_1, \ldots, x_n$ in X.

Proof. A Banach space X with property (α) satisfies the following property: there exists a constant $C > 0$ such that for any $n \geq 1$, for any finite family $(b_k)_{k\geq 1}$ of elements of M_n that we denote by $b_k = [b_k(l,j)]_{1\leq l,j\leq n}$ and for any n-tuple $(x_{k1})_{k\geq 1}, \ldots, (x_{kn})_{k\geq 1}$ of families in X,

$$\begin{aligned}\Big\|\sum_{k\geq 1}\sum_{l,j=1}^{n} \varepsilon_k \otimes \varepsilon_l \otimes b_k(l,j)\, x_{kj}\Big\|_{\text{Rad}(\text{Rad}(X))} \\ \leq C \sup_k \|b_k\|_{M_n} \Big\|\sum_{k,j} \varepsilon_k \otimes \varepsilon_j \otimes x_{kj}\Big\|_{\text{Rad}(\text{Rad}(X))}.\end{aligned} \tag{6.11}$$

This strengthening of (6.3) for these spaces is due to Haak and Kunstmann, see Lemma 5.2 in [21] (see also [19]).

We explain (1). We consider an $n \times n$ matrix $[f_{lj}]$ of elements of $H_0^\infty(\Sigma_\nu)$ and we associate $F \in H^\infty(\Sigma_\nu; M_n)$ defined by

$$F(z) = \big[f_{lj}(z)\big], \quad z \in \Sigma_\nu.$$

Then, arguing as in the proof of Theorem 6.3 and applying the Franks–McIntosh decomposition principle with $Z = M_n$, we find a sequence $(b_k)_{k\geq 1}$ of $n\times n$ matrices $b_k = [b_k(l,j)]_{1\leq l,j\leq n}$ such that

$$f_{lj}(z) = \sum_{k=1}^{\infty} b_k(l,j)\, F_k(z)\, G_k(z), \quad z \in \Sigma_\theta,$$

for any $l, j = 1, \ldots, n$, and

$$\sup_k \|b_k\|_{M_n} \leq C \sup\big\{\big\|\big[f_{lj}(z)\big]\big\|_{M_n} \,:\, z \in \Sigma_\nu\big\}.$$

Using the above results in the place of (6.6) and (6.7), the estimate (6.11) in place of (6.3), and arguing as in the proof of Theorem 6.3, we obtain (6.9). The details are left to the reader.

Part (2) can be deduced from part (1) in the same manner that Theorem 6.4 was deduced from Theorem 6.3. □

Part (2) of the above proposition generalizes Theorem 3.3 in [39], where this property is proved for (commutative) L^p-spaces.

Remark 6.6. Property (6.9) means that the homomorphism $H_0^\infty(\Sigma_\nu) \to B(X)$ induced by the functional calculus is matricially R-bounded in the sense of [31], Section 4. Restricting this property to column matrices, we obtain the property proved in Theorem 6.3. On the other hand, restricting (6.9) to diagonal matrices, we recover the following result of Kalton–Weis [30] (see also Theorem 12.8 in [32]): if A has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on X with property (α), then for any $\nu \in (\theta, \pi)$, the functional calculus homomorphism $H_0^\infty(\Sigma_\nu) \to B(X)$ maps the unit ball of $H_0^\infty(\Sigma_\nu)$ into an R-bounded subset of $B(X)$.

We now show that Theorem 6.3 does not hold true for all Banach spaces. Namely, the next proposition shows that it fails on c_0. A similar construction shows that Theorem 6.4 also fails on c_0.

Proposition 6.7. *Let $A\colon c_0 \to c_0$ be defined by*

$$A(w) = \big(2^{-j} w_j\big)_{j\geq 1}, \quad w = (w_j)_{j\geq 1} \in c_0.$$

Then A is a sectorial operator and, for any $\theta \in (0, \pi)$,

(1) *A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus;*

(2) *A does not have a quadratic $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. The facts that A is sectorial and that property (1) holds are easy. Indeed, for any $\theta \in (0,\pi)$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$[f(A)](w) = (f(2^{-j})w_j)_{j\geq 1}$$

for any $w = (w_j)_{j\geq 1}$ in c_0, and hence

$$\|f(A)\| \leq \|f\|_{L^\infty(0,\infty)}.$$

To prove (2), let us assume that A admits a quadratic $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (0,\pi)$. Let $(e_j)_{j\geq 1}$ denote the standard basis of c_0. For any integers $n, m \geq 1$, for any $w_1,\ldots,w_m$ in $\mathbb{C}$, and any $f_1,\ldots,f_n$ in $H_0^\infty(\Sigma_\theta)$,

$$\sum_l \varepsilon_l \otimes f_l(A)\Bigl(\sum_j w_j e_j\Bigr) \,=\, \sum_{l,j} f_l(2^{-j})\, w_j\, \varepsilon_l \otimes e_j\,.$$

Hence there is a constant $C \geq 0$ (not depending on n, m or w_j) such that

$$\Bigl\|\sum_{l=1}^n\sum_{j=1}^m f_l(2^{-j})\, w_j\, \varepsilon_l \otimes e_j\Bigr\|_{\mathrm{Rad}(c_0)} \leq C\sup_j |w_j|\, \Bigl\|\Bigl(\sum_{l=1}^n |f_l|^2\Bigr)^{1/2}\Bigr\|_{\infty,\Sigma_\theta}$$

for any $f_1,\ldots,f_n$ in $H_0^\infty(\Sigma_\theta)$, By an entirely classical approximation argument, the above estimate holds also when the f_l belong to $H^\infty(\Sigma_\theta)$. Applying this with $w_j = 1$ for all j, one obtains, for $f_1,\ldots,f_n \in H^\infty(\Sigma_\theta)$,

$$\Bigl\|\sum_{l=1}^n\sum_{j=1}^m f_l(2^{-j})\, \varepsilon_l \otimes e_j\Bigr\|_{\mathrm{Rad}(c_0)} \leq C\,\Bigl\|\Bigl(\sum_{l=1}^n |f_l|^2\Bigr)^{1/2}\Bigr\|_{\infty,\Sigma_\theta}. \tag{6.12}$$

Let $Q_{n,m}\colon H^\infty(\Sigma_\theta;\ell_n^2) \to \ell_m^\infty(\ell_n^2)$ be defined by $Q_{n,m}(F) = \bigl(F(2^{-j})\bigr)_{1\leq j\leq m}$. Then Q_n is onto and the vectorial form of Carleson's interpolation theorem (see [17], VII. 2) ensures that its lifting constant is bounded by a universal constant not depending on m or n. Thus there is a constant $K \geq 1$ such that for any family $(\alpha_{lj})_{1\leq j\leq m,\, 1\leq l\leq n}$ of complex numbers there exist $f_1,\ldots,f_n$ in $H^\infty(\Sigma_\theta)$ such that

$$\Bigl\|\Bigl(\sum_{l=1}^n |f_l|^2\Bigr)^{1/2}\Bigr\|_{\infty,\Sigma_\theta} \leq K \sup_{1\leq j\leq m}\Bigl(\sum_{l=1}^n |\alpha_{lj}|^2\Bigr)^{1/2} \quad \text{and} \quad \alpha_{lj} = f_l(2^{-j})$$

for any $1 \leq j \leq m$ and $1 \leq l \leq n$. It therefore follows from (6.12) that

$$\Bigl\|\sum_{l=1}^n\sum_{j=1}^m \alpha_{lj}\, \varepsilon_l \otimes e_j\Bigr\|_{\mathrm{Rad}(c_0)} \leq CK \sup_{1\leq j\leq m}\Bigl(\sum_{l=1}^n |\alpha_{lj}|^2\Bigr)^{1/2}, \quad \alpha_{lj} \in \mathbb{C}. \tag{6.13}$$

Let $n \geq 1$ be an integer. Since the unit ball of ℓ_n^2 is compact, there exists a finite family $(y_1,\ldots,y_m)$ in that unit ball such that

$$\|y\|_{\ell_n^2} \leq 2\sup\bigl\{|\langle y, y_j\rangle| \,:\, j = 1,\ldots,m\bigr\} \tag{6.14}$$

for any $y \in \ell^2_n$. Let $(h_1,\dots,h_n)$ be an orthonormal basis of ℓ^2_n, and let

$$\alpha_{lj} = \langle h_l, y_j\rangle, \quad 1\le j\le m,\ 1\le l\le n.$$

Then the supremum in the right-hand side of (6.13) is equal to $\sup_j \|y_j\|$, hence is less than or equal to 1. Consequently,

$$\Big\|\sum_{l=1}^{n}\sum_{j=1}^{m}\langle h_l, y_j\rangle\, \varepsilon_l\otimes e_j\Big\|_{\mathrm{Rad}(c_0)} \le CK.$$

Now observe that, for any $u\in\mathcal{M}$,

$$\Big\|\sum_{l=1}^{n}\sum_{j=1}^{m}\langle h_l, y_j\rangle\, \varepsilon_l(u) e_j\Big\|_{c_0} = \Big\|\sum_{j=1}^{m}\Big\langle\sum_{l=1}^{n}\varepsilon_l(u)\, h_l, y_j\Big\rangle\, e_j\Big\|_{c_0} = \sup_j\Big|\Big\langle\sum_{l=1}^{n}\varepsilon_l(u)\, h_l, y_j\Big\rangle\Big|.$$

Since $(h_1,\dots,h_n)$ is an orthonormal basis, the norm of $\sum_{l=1}^{n}\varepsilon_l(u)\, h_l$ in ℓ^2_n is equal to $n^{1/2}$. Applying (6.14), we deduce that

$$n^{1/2}\le 2\,\Big\|\sum_{l=1}^{n}\sum_{j=1}^{m}\langle h_l, y_j\rangle\, \varepsilon_l(u)\, e_j\Big\|_{c_0}.$$

Integrating over $\mathcal{M}$, this yields $n^{1/2}\le 2CK$ for any $n\ge 1$, a contradiction. □

Now we return to the question addressed at the beginning of this section. The following classical result will be used in the next proof: If a Banach space X does not contain c_0 (as an isomorphic subspace), then a series $\sum_k \varepsilon_k\otimes x_k$ converges in $L^2(\mathcal{M};X)$ if and only if its partial sums are uniformly bounded (see [33]).

Proposition 6.8. *Assume that X does not contain c_0. Let $T\colon X\to X$ be a Ritt operator and assume that T has a quadratic $H^\infty(B_\gamma)$ functional calculus for some $\gamma\in\big(0,\pi/2\big)$. Then for any $m\ge 1$, T satisfies a uniform estimate*

$$\|x\|_{T,m}\lesssim \|x\|, \quad x\in X. \tag{6.15}$$

Proof. According to the property discussed before the statement of the proposition, it suffices to show the existence of a constant $K>0$ such that for any $n\ge 1$ and any $x\in X$,

$$\Big\|\sum_{l=1}^{n} l^{m-1/2}\,\varepsilon_l\otimes T^{l-1}\,(I-T)^m x\Big\|_{\mathrm{Rad}(X)}\le K\,\|x\|.$$

This is obtained by applying Definition 6.1 (2), with

$$\varphi_l(z) = l^{m-1/2}\, z^{l-1}(1-z)^l.$$

See the proof of Theorem 3.3 in [39] for the details. □

Finally we summarize what we obtain by combining Theorem 6.4 and Proposition 6.8. Recall that a Banach space with finite cotype cannot contain c_0.

Corollary 6.9. *Assume that X has finite cotype. Let $T\colon X\to X$ be a Ritt operator with a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma\in\bigl(0,\pi/2\bigr)$. Then it satisfies a square function estimate (6.15) for any $m\geq 1$.*

Remark 6.10. It follows from the proof of Proposition 6.7 that the spaces ℓ_n^∞ do not satisfy (6.3) uniformly, that is, there is no common constant $C>0$ such that (6.3) holds with $X=\ell_n^\infty$ for any $n\geq 1$. Moreover a Banach space with no finite cotype contains the ℓ_n^∞ uniformly as isomorphic subspaces (see, e.g., Theorem 14.1 in [14]) and hence cannot satisfy (6.3). Together with Lemma 6.2, this observation shows that a Banach space satisfies an estimate (6.3) if and only if it has finite cotype.

(Added in October 2012.) In an early version of this paper, Theorem 6.3 was stated under the assumption that X satisfies an estimate (6.3). I had overlooked Corollary 3.4 in [27] and realized only recently that (6.3) is the same as 'finite cotype'. This led to the present neater presentation of Section 6.

Bernhard Haak and Markus Haase have informed me that they obtained a variant of Theorem 6.3 in a work in progress (see [20]). This work is independent of mine, and was undertaken several months ago.

7. From square functions to H^∞ functional calculus

This section is devoted to the issue of showing that a Ritt operator has a bounded H^∞-functional calculus with respect to a Stolz domain B_γ, provided that it satisfies suitable square function estimates. We consider an arbitrary Banach space X and first establish a general result, namely Theorem 7.3 below. Then we consider special cases in the last part of the section.

Lemma 7.1. *Let $0<\alpha<\gamma<\pi/2$ and let $T\colon X\to X$ be a Ritt operator of type α (resp., an R-Ritt operator of R-type α). There exists a constant $C>0$ such that for any $\varphi\in H_0^\infty(B_\gamma)$, we have*

$$k\bigl\|\varphi(T)\bigl(T^k-T^{k-1}\bigr)\bigr\|\,\leq\, C\|\varphi\|_{\infty,B_\gamma},\quad k\geq 1$$

(resp. the set $\{k\varphi(T)(T^k-T^{k-1})\,:\,k\geq 1\}$ is R-bounded and

$$\mathcal{R}\bigl(\bigl\{k\varphi(T)\bigl(T^k-T^{k-1}\bigr)\,:\,k\geq 1\bigr\}\bigr)\,\leq\, C\|\varphi\|_{\infty,B_\gamma}\,).$$

Proof. We will prove this result in the 'R-Ritt case' only, the 'Ritt case' being similar and simpler. We fix a real number $\beta\in(\alpha,\gamma)$. Recall Lemma 5.2 and let

$$C_1=\mathcal{R}\bigl(\bigl\{(\lambda-1)R(\lambda,T)\,:\,\lambda\in\partial B_\beta\setminus\{1\}\bigr\}\bigr).$$

For any function $\varphi\in H_0^\infty(B_\gamma)$ and any integer $k\geq 1$, we have

$$k\,\varphi(T)\bigl(T^k-T^{k-1}\bigr)\,=\,\frac{1}{2\pi i}\int_{\partial B_\beta}k\,\varphi(\lambda)\lambda^{k-1}\bigl((\lambda-1)R(\lambda,T)\bigr)\,d\lambda\,.$$

Hence by Lemma 5.1, we have

$$\mathcal{R}\Big(\big\{k\varphi(T)\big(T^k - T^{k-1}\big) \, : \, k\geq 1\big\}\Big) \,\leq\, \frac{C_1}{\pi}\,\sup_{k\geq 1}\Big\{k\int_{\partial B_\beta} |\varphi(\lambda)|\,|\lambda|^{k-1}\,|d\lambda|\Big\}$$
$$\leq\, \frac{C_1}{\pi}\,\|\varphi\|_{\infty,B_\gamma}\,\sup_{k\geq 1}\Big\{k\int_{\partial B_\beta} |\lambda|^{k-1}\,|d\lambda|\Big\}.$$

The finiteness of the latter supremum is well known; see, e.g., Lemma 2.1 in [57] and its proof. The result follows at once. □

Lemma 7.2. *Let $T\colon X\to X$ be a Ritt operator. For any $x\in\overline{\mathrm{Ran}(I-T)}$, we have*

$$\sum_{k=1}^{\infty} k(k+1)\,T^{k-1}(I-T)^3 x \,=\, 2x.$$

Proof. Let $N\geq 1$ be an integer. First, we have

$$\sum_{k=1}^{N} k(k+1)\,T^{k-1}(I-T) \,=\, \sum_{k=1}^{N} k(k+1)\,T^{k-1} \,-\, \sum_{k=2}^{N+1} (k-1)k\,T^{k-1}$$
$$=\, 2\sum_{k=1}^{N} k\,T^{k-1} \,-\, N(N+1)T^N.$$

Then we compute

$$\sum_{k=1}^{N} k\,T^{k-1}(I-T) \,=\, \sum_{k=1}^{N} k\,T^{k-1} \,-\, \sum_{k=2}^{N+1} (k-1)\,T^{k-1} \,=\, \sum_{k=1}^{N} T^{k-1} \,-\, N\,T^N,$$

and we note that

$$\sum_{k=1}^{N} T^{k-1}(I-T) \,= I - T^N.$$

Combining these identities, we obtain that

$$\sum_{k=1}^{N} k(k+1)T^{k-1}(I-T)^3 \,=\, 2I-2T^N-2NT^N(I-T)-N(N+1)T^N(I-T)^2. \tag{7.1}$$

Since T is a Ritt operator, the four sequences

$$\mathcal{S}_0 = (T^N)_{N\geq 1}, \quad \mathcal{S}_1 = \big(N\,T^N(I-T)\big)_{N\geq 1}, \quad \mathcal{S}_2 = \big(N^2\,T^N(I-T)^2\big)_{N\geq 1}$$

and

$$\mathcal{S}_3 = \big(N^3\,T^N(I-T)^3\big)_{N\geq 1}$$

are bounded (see Lemma 2.1 in [57]).

If $x=(I-T)z$ is an element of $\mathrm{Ran}(I-T)$, the boundedness of $\mathcal{S}_3$ implies that

$$N(N+1)\,T^N(I-T)^2x\,=\,\frac{N+1}{N^2}\,N^3\,T^N(I-T)^3\,z\,\longrightarrow\,0\quad\text{when } N\to\infty.$$

Then the boundedness of the sequence $\mathcal{S}_2$ implies that we actually have

$$\lim_N N(N+1)\,T^N(I-T)^2x\,=\,0$$

for any x in the closure $\overline{\mathrm{Ran}(I-T)}$. Likewise, using $\mathcal{S}_2,\mathcal{S}_1$ and $\mathcal{S}_0$, we have

$$\lim_N NT^N(I-T)x\,=\,0\quad\text{and}\quad\lim_N T^Nx\,=\,0$$

for any $x\in\overline{\mathrm{Ran}(I-T)}$. Thus applying (7.1) yields the result. □

Theorem 7.3. *Let $T\colon X\to X$ be an R-Ritt operator of R-type $\alpha\in\big(0,\pi/2\big)$. If T and T^* both satisfy uniform estimates*

$$\|x\|_{T,1}\,\lesssim\,\|x\|\quad\textit{and}\quad\|y\|_{T^*,1}\,\lesssim\,\|y\|$$

for $x\in X$ and $y\in X^*$, then T admits a bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma\in\big(\alpha,\pi/2\big)$.

Proof. We fix γ in $\big(\alpha,\pi/2\big)$. Let $\omega=e^{2i\pi/3}$. Then the operators $\omega I-T$ and $\bar\omega I-T$ are invertible and $I-T^3=(I-T)(\omega I-T)(\bar\omega I-T)$. Hence

$$\mathrm{Ran}(I-T)\,=\,\mathrm{Ran}(I-T^3).$$

Note moreover that T^3 is a Ritt operator.

Let $\varphi\in\mathcal{P}$ such that $\varphi(1)=0$ and consider $x\in X$. Then $\varphi(T)x\in\mathrm{Ran}(I-T)$, so applying Lemma 7.2 to T^3 and using the above observations, we obtain

$$\sum_{k=1}^\infty k(k+1)\,T^{3(k-1)}(I-T^3)^3\varphi(T)x\,=\,2\,\varphi(T)\,x.$$

For convenience we set $\psi(T)=(I+T+T^2)^3/2$, so that $2\psi(T)(I-T)^3=(I-T^3)^3$. Then for any $y\in X^*$, we obtain

$$\big\langle\varphi(T)x,y\big\rangle\,=\,\sum_{k=1}^\infty\big\langle k(k+1)\,\psi(T)\,\varphi(T)\,T^{3(k-1)}(I-T)^3x,y\big\rangle$$

$$=\,\sum_{k=1}^\infty\Big\langle\big[(k+1)\varphi(T)T^{k-1}(I-T)\big]k^{1/2}T^{k-1}(I-T)x,k^{1/2}T^{*(k-1)}(I-T^*)\psi(T^*)y\Big\rangle.$$

Note that for any finite families $(x_k)_{k\geq1}$ in X and $(y_k)_{k\geq1}$ in X^*, we have

$$\sum_k\langle x_k,y_k\rangle\,=\,\int_{\mathcal{M}}\Big\langle\sum_k\varepsilon_k(u)\,x_k,\sum_k\varepsilon_k(u)\,y_k\Big\rangle\,d\mathbb{P}(u),$$

and hence

$$\Big|\sum_k \langle x_k, y_k\rangle\Big| \leq \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \Big\|\sum_k \varepsilon_k \otimes y_k\Big\|_{\mathrm{Rad}(X^*)}$$

by the Cauchy–Schwarz inequality.

Thus for any integer $N \geq 1$, we have

$$\begin{aligned}
&\Big|\sum_{k=1}^N \Big\langle \big[(k+1)\varphi(T)T^{k-1}(I-T)\big]k^{1/2}\,T^{k-1}(I-T)x, k^{1/2}\,T^{*(k-1)}(I-T^*)\psi(T^*)y\Big\rangle\Big|\\
&\quad\leq \Big\|\sum_{k=1}^N \varepsilon_k \otimes \big[(k+1)\,\varphi(T)\,T^{k-1}(I-T)\big]k^{1/2}\,T^{k-1}(I-T)x\Big\|_{\mathrm{Rad}(X)}\\
&\qquad\qquad \times \Big\|\sum_{k=1}^N \varepsilon_k \otimes k^{1/2}\,T^{*(k-1)}(I-T^*)\psi(T^*)y\Big\|_{\mathrm{Rad}(X^*)}\\
&\quad\leq \mathcal{R}\big(\big\{(k+1)\,\varphi(T)\,T^{k-1}(I-T)\,:\,k\geq 1\big\}\big)\,\|\psi(T)\|\,\|x\|_{T,1}\,\|y\|_{T^*,1}\\
&\quad\lesssim \|\varphi\|_{\infty,B_\gamma}\,\|x\|_{T,1}\,\|\psi(T^*)y\|_{T^*,1}
\end{aligned}$$

by Lemma 7.1. Applying our assumptions, we deduce that

$$\big|\langle\varphi(T)x, y\rangle\big| \lesssim \|\varphi\|_{\infty,B_\gamma}\,\|x\|\,\|y\|.$$

Since x and y are arbitrary, this implies an estimate $\|\varphi(T)\| \lesssim \|\varphi\|_{\infty,B_\gamma}$ for polynomials vanishing at 1. Writing any polynomial as $\varphi = \varphi(1) + (\varphi - \varphi(1))$, we immediately obtain a similar estimate for all polynomials. This yields the result by Proposition 2.5. □

Theorem 7.3 fails if we remove one of the two square function estimates in the assumption. This will follow from Proposition 8.2.

Finally we consider a special case and combinations with results from the previous sections. Following [29], we say that a Banach space X has property (Δ) if the triangular projection is bounded on $\mathrm{Rad}(\mathrm{Rad}(X))$, that is, there exists a constant $C > 0$ such that for finite doubly indexed families $(x_{kl})_{k,l\geq 1}$ in X,

$$\Big\|\sum_{k\geq 1}\sum_{l\geq k} \varepsilon_k \otimes \varepsilon_l \otimes x_{kl}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \leq C\,\Big\|\sum_{k\geq 1}\sum_{l\geq 1} \varepsilon_k \otimes \varepsilon_l \otimes x_{kl}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.$$

This condition is clearly weaker than (α). Furthermore any UMD Banach space has property (Δ), by Proposition 3.2 in [29]. Thus any noncommutative L^p-space with $1 < p < \infty$ has property (Δ). On the other hand, property (Δ) does not hold uniformly on the spaces ℓ^∞_n, hence any Banach space with property (Δ) has finite cotype.

It follows from Theorem 5.3 in [29] that if A is a sectorial operator on X with property (Δ) and A admits a bounded $H^\infty(\Sigma_\theta)$ for some $\theta < \pi/2$, then A is R-sectorial of R-type $< \pi/2$. Combining this with (the easy implication of) Proposition 4.1 and Lemma 5.2, we deduce the following.

Proposition 7.4. *Let T be a Ritt operator on X with property (Δ). If T admits a bounded $H^\infty(B_\gamma)$ for some $\gamma < \pi/2$, then T is R-Ritt.*

Combining Proposition 7.4 with Corollary 6.9, we obtain the following equivalence result.

Corollary 7.5.

(1) *Assume that X has property (Δ) and let $T\colon X \to X$ be a Ritt operator. The following assertions are equivalent.*

  (i) *T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \big(0, \pi/2\big)$.*

  (ii) *T is R-Ritt and T and T^* both satisfy uniform estimates*

$$\|x\|_{T,1} \lesssim \|x\| \quad \textit{and} \quad \|y\|_{T^*,1} \lesssim \|y\|$$

  for $x \in X$ and $y \in X^*$

(2) *Part (1) applies to Banach spaces with property (α) and to noncommutative L^p-spaces for $1 < p < \infty$.*

If $X = L^p(\Omega)$ is a commutative L^p-space with $1 < p < \infty$, then demanding that T be R-Ritt in condition (ii) is superfluous, by Theorem 5.3. In this case, the above statement yields Theorem 1.1. According to this discussion and Remark 5.5, Theorem 1.1 holds also on any reflexive space with property (α).

Remark 7.6. Let $m \geq 1$ be an integer. Theorem 7.3 remains valid if the uniform estimates $\|x\|_{T,1} \lesssim \|x\|$ and $\|y\|_{T^*,1} \lesssim \|y\|$ are replaced by

$$\|x\|_{T,m} \lesssim \|x\| \quad \text{and} \quad \|y\|_{T^*,m} \lesssim \|y\|.$$

Indeed the proof is essentially the same up to simple modifications left to the reader. Consequently, Corollary 7.5 is also valid with $\|\cdot\|_{T,1}$ and $\|\cdot\|_{T^*,1}$ replaced by $\|\cdot\|_{T,m}$ and $\|\cdot\|_{T^*,m}$, respectively.

We conclude this section with an observation of independent interest on the role of the R-Ritt condition in the study of $H^\infty(B_\gamma)$ functional calculus. Recall Definition 2.6.

Proposition 7.7. *Let $T\colon X \to X$ be an R-Ritt operator of R-type α. If T is polynomially bounded, then it admits a bounded $H^\infty(B_\gamma)$ functional calculus for any $\gamma \in \big(\alpha, \pi/2\big)$.*

Proof. As was observed in Section 5, the operator $A = I - T$ is R-sectorial of R-type α. Moreover the proof of the easy implication '(i)⇒(ii)' of Proposition 4.1 shows that A admits a bounded $H^\infty(\Sigma_{\pi/2})$ functional calculus. According to Proposition 5.1 in [29], this implies that for any $\theta \in (\alpha, \pi)$, A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus. The result therefore follows from Proposition 4.1. □

The R-boundedness assumption is essential in the above result. Indeed with F. Lancien we show in [34] the existence of Ritt operators that are polynomially bounded without admitting any bounded $H^\infty(B_\gamma)$ functional calculus.

8. Examples and illustrations

In this final section, we give additional results for the following 3 classes of Banach spaces: Hilbert spaces, commutative L^p-spaces, and noncommutative L^p-spaces. We give either characterizations of Ritt operators satisfying the equivalent conditions of Corollary 7.5, or exhibit classes of examples satisfying these conditions.

8.a. Hilbert spaces. Let H be a Hilbert space. Two bounded operators $S, T\colon H \to H$ are called similar provided that there is an invertible operator $V \in B(H)$ such that $S = V^{-1}TV$. In particular we say that T is similar to a contraction if there is an invertible operator $V \in B(H)$ such that $\|V^{-1}TV\| \leq 1$. This is equivalent to the existence of an equivalent Hilbert norm on H with respect to which T is contractive. Any T similar to a contraction is polynomially bounded (by von Neumann's inequality). Pisier's negative solution to the Halmos problem asserts that the converse is wrong; see [52] for details and related results on similarity problems. It is known, however, that any polynomially bounded Ritt operator is necessarily similar to a contraction; see [36], [12]. The next statement (which may be known to some similarity specialists) is a refinement of that result, also generalizing the Hilbert space version of Theorem 1.1.

Note that the class of Ritt operators is stable under similarity.

Theorem 8.1. *For any power bounded operator $T \in B(H)$, the following assertions are equivalent.*

(i) *T is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma \in \big(0, \pi/2\big)$.*

(ii) *T and T^* both satisfy uniform estimates*

$$\|x\|_{T,1} \lesssim \|x\| \quad \textit{and} \quad \|y\|_{T^*,1} \lesssim \|y\|$$

for $x, y \in H$.

(iii) *T is a Ritt operator and T is similar to a contraction.*

Proof. It follows from Theorem 4.7 in [28] that T is a Ritt operator if it satisfies (ii). With this result in hand, the equivalence between (i) and (ii) reduces to Corollary 7.5.

If T satisfies (iii), then it is polynomially bounded (see the discussion above). Hence it satisfies (i) by Proposition 7.7.

Assume (ii). Recall (5.5) (with $X = H$) and let $P\colon H \to H$ be the projection onto $\mathrm{Ker}(I - T)$ whose kernel equals $\overline{\mathrm{Ran}(I - T)}$. Then we have an equivalence

$$\|x\| \approx \big(\|P(x)\|^2 + \|x\|_{T,1}^2\big)^{1/2}, \quad x \in H. \tag{8.1}$$

Indeed this follows from the proof of Theorem 4.7 in [28], see also Corollary 3.4 in [39]. Let $|||x|||$ denote the right-hand side of (8.1). Then $|||\cdot|||$ is an equivalent Hilbert norm on H. Further for any $x \in H$,

$$\|T(x)\|_{T,1}^2 = \sum_{k=1}^{\infty} k\,\big\|T^{k+1}(x) - T^k(x)\big\|^2 \leq \sum_{k=2}^{\infty} k\,\big\|T^k(x) - T^{k-1}(x)\big\|^2 \leq \|x\|_{T,1}^2.$$

This implies that T is a contraction on $\big(H, |||\cdot|||\big)$. Thus T is similar to a contraction, which shows (iii). □

A natural question (also making sense on general Banach spaces) is whether one can get rid of one of the two square function estimates of (ii) in the above equivalence result. It turns out that the answer is negative.

Proposition 8.2. *There exists a Ritt operator T on a Hilbert space H that is not similar to a contraction, although it satisfies an estimate*

$$\|x\|_{T,1} \lesssim \|x\|, \quad x \in H.$$

Proof. This is a simple adaptation of Theorem 5.2 in [37], so the explanation will be brief. Let H be a separable infinite dimensional Hilbert space and let $(e_m)_{m\geq 1}$ be a normalized Schauder basis of H which satisfies an estimate

$$\Big(\sum_m |t_m|^2\Big)^{1/2} \lesssim \Big\|\sum_m t_m\, e_m\Big\| \tag{8.2}$$

for finite sequences $(t_m)_{m\geq 1}$ of complex numbers but for which there is no reverse estimate, that is,

$$\sup\Big\{\Big\|\sum_m t_m\, e_m\Big\| \, : \, \sum_m |t_m|^2 \leq 1\Big\} \,=\, \infty. \tag{8.3}$$

Let $T\colon H \to H$ be defined by

$$T\Big(\sum_m t_m\, e_m\Big) \,=\, \sum_m (1 - 2^{-m})\, t_m\, e_m.$$

According to e.g. Theorem 4.1 in [35], this operator is well-defined and $A = I - T$ is sectorial of any positive type. Moreover $\sigma(T) \subset [0, 1]$, hence T is a Ritt operator.

Arguing as in the proof of Theorem 5.2 in [37], one obtains an equivalence

$$\Big\|\sum_m t_m\, e_m\Big\|_{T,1} \,\approx\, \Big(\sum_m |t_m|^2\Big)^{1/2}$$

for finite sequences $(t_m)_{m\geq 1}$ of complex numbers.

In view of (8.2), this implies the square function estimate $\|x\|_{T,1} \lesssim \|x\|$. If T were similar to a contraction, it would satisfy an estimate $\|y\|_{T^*,1} \lesssim \|y\|$, by Theorem 8.1. It would therefore satisfy a reverse estimate $\|x\| \lesssim \|x\|_{T,1}$ by (8.1). This contradicts (8.3). □

8.b. Commutative L^p-spaces. Let (Ω, μ) be a measure space and let $1 < p < \infty$. The following is the main result of [39]. We provide a proof using the techniques of the present paper.

Theorem 8.3. [39] *Let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a positive contraction and assume that T is a Ritt operator. Then it satisfies the equivalent conditions of Theorem 1.1.*

Proof. Let $(T_t)_{t\geq 0}$ be the uniformly continuous semigroup on $L^p(\Omega)$ defined by

$$T_t = e^{-t}\, e^{tT}, \quad t \geq 0.$$

Then for any $t \geq 0$, T_t is positive and $\|T_t\| \leq e^{-t} e^{t\|T\|} \leq 1$. The generator of $(T_t)_{t\geq 0}$ is $T - I = -A$ and since T is a Ritt operator, A is sectorial of type $< \pi/2$. Hence A admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \pi/2$, by Proposition 2.2 in [39]. According to Proposition 4.1, this implies condition (i) of Theorem 1.1. □

For applications of this result to ergodic theory see [40].

Ritt operators on $L^p(\Omega)$ satisfying Theorem 1.1 do not have any description comparable to the one given by Theorem 8.1 for a Hilbert space. However in a separate joint work with C. Arhancet [4], we show that for an R-Ritt operator $T\colon L^p(\Omega) \to L^p(\Omega)$, T satisfies the conditions of Theorem 1.1 if and only if there exist a second measure space (Ω', μ'), two bounded maps $J\colon L^p(\Omega) \to L^p(\Omega')$ and $Q\colon L^p(\Omega') \to L^p(\Omega)$, and an isomorphism $U\colon L^p(\Omega') \to L^p(\Omega')$ such that $\{U^n : n \in \mathbb{Z}\}$ is bounded and

$$T^n = Q\, U^n J, \quad n \geq 0.$$

8.c. Noncommutative L^p-spaces. In this subsection, we let M be a semifinite von Neumann algebra equipped with a semifinite faithful trace τ. Thanks to the noncommutative Khintchine inequalities (3.5) and (3.6), Corollary 7.5 has a specific form on $L^p(M)$. We state it in the case $2 \leq p < \infty$. The dual case $(1 < p \leq 2)$ can be obtained by changing T into T^*. This is the noncommutative analogue of Theorem 1.1, the square functions (1.3) being replaced by their natural noncommutative versions.

Corollary 8.4. *Let $2 \leq p < \infty$ and let $T\colon L^p(M) \to L^p(M)$ be a Ritt operator. Then T admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \pi/2$ if and only if T is R-Ritt and there exists a constant $C > 0$ such that the following three estimates hold:*

(1) *For any $x \in L^p(M)$,*

$$\Big\| \Big(\sum_{k=1}^{\infty} k \left| T^k(x) - T^{k-1}(x) \right|^2 \Big)^{1/2} \Big\|_{L^p(M)} \leq C\, \|x\|_{L^p(M)}.$$

(2) *For any $x \in L^p(M)$,*

$$\Big\| \Big(\sum_{k=1}^{\infty} k \left| \big(T^k(x) - T^{k-1}(x)\big)^* \right|^2 \Big)^{1/2} \Big\|_{L^p(M)} \leq C\, \|x\|_{L^p(M)}.$$

(3) *For any $y \in L^{p'}(M)$, there exist two sequences $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ in $L^{p'}(M)$ such that*

$$k^{1/2}\left(T^{*k}(y) - T^{*(k-1)}(y)\right) = u_k + v_k$$

for any $k \geq 1$, and we both have

$$\Big\| \Big(\sum_{k=1}^{\infty} |u_k|^2 \Big)^{1/2} \Big\|_{L^{p'}(M)} \leq C\, \|y\|_{L^{p'}(M)},$$
$$\Big\| \Big(\sum_{k=1}^{\infty} |v_k^*|^2 \Big)^{1/2} \Big\|_{L^{p'}(M)} \leq C\, \|y\|_{L^{p'}(M)}.$$

We will now exhibit two classes of examples satisfying the conditions of the above corollary. We start with Schur multipliers. Here our von Neumann algebra is $B(\ell^2)$, the trace τ is the usual trace and the associated noncommutative L^p-spaces are the Schatten classes that we denote by S^p. We represent any element of $B(\ell^2)$ by a bi-infinite matrix in the usual way. We recall that a bounded Schur multiplier on $B(\ell^2)$ is a bounded map $T\colon B(\ell^2) \to B(\ell^2)$ of the form

$$[c_{ij}]_{i,j\geq 1} \overset{T}{\longmapsto} [t_{ij}c_{ij}]_{i,j\geq 1} \tag{8.4}$$

for some matrix $[t_{ij}]_{i,j\geq 1}$ of complex numbers. See, e.g., Theorem 5.1 in [52] for a description of those maps. It is well known (using duality and interpolation) that any bounded Schur multiplier $T\colon B(\ell^2) \to B(\ell^2)$ extends to a bounded map $T\colon S^p \to S^p$ for any $1 \leq p < \infty$, with

$$\|T\colon S^p \longrightarrow S^p\| \leq \|T\colon B(\ell^2) \longrightarrow B(\ell^2)\|.$$

In particular, any contractive Schur multiplier $T\colon B(\ell^2) \to B(\ell^2)$ extends to a contraction on S^p for any p. In this case, the complex numbers t_{ij} given by (8.4) have modulus ≤ 1. Moreover, $T\colon S^2 \to S^2$ is self-adjoint (in the usual Hilbertian sense) if and only if the associated matrix $[t_{ij}]_{i,j\geq 1}$ is real valued.

We say that a semigroup $(T_t)_{t\geq 0}$ of contractive Schur multipliers on $B(\ell^2)$ is w^*-continuous if $w^*\text{-}\lim_{t\to 0} T_t(x) = x$ for any $x \in B(\ell^2)$. In this case, $(T_t)_{t\geq 0}$ extends to a strongly continuous semigroup of S^p for any $1 \leq p < \infty$. Further we say that $(T_t)_{t\geq 0}$ is self-adjoint provided that $T_t\colon S^2 \to S^2$ is self-adjoint for any $t \geq 0$. See Chapter 5 of [25] for the more general notion of noncommutative diffusion semigroup.

In the sequel we let $\omega_p = \pi|1/p - 1/2|$. The following extends 8.C in [25].

Proposition 8.5. *Let $(T_t)_{t\geq 0}$ be a self-adjoint w^*-continuous semigroup of contractive Schur multipliers on $B(\ell^2)$. For any $1 < p < \infty$, let $-A_p$ be the infinitesimal generator of $(T_t)_{t\geq 0}$ on S^p. Then for any $\theta \in (\omega_p, \pi)$, A_p admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus.*

Proof. For any $1 < p < \infty$, let $(U_{t,p})_{t\in\mathbb{R}}$ be the translation semigroup on the Bochner space $L^p(\mathbb{R}; S^p)$. Then it follows from Corollary 4.3 and Theorem 5.3 in [3] that for any $b \in L^1(0,\infty)$,

$$\Big\| \int_0^\infty b(t)T_t\, dt \,:\, S^p \longrightarrow S^p \Big\| \leq \Big\| \int_0^\infty b(t)U_{t,p}\, dt \,:\, L^p(\mathbb{R}; S^p) \longrightarrow L^p(\mathbb{R}; S^p) \Big\|.$$

Let C_p be the negative generator of $(U_{t,p})_{t\in\mathbb{R}}$. By Lemma 2.12 in [35], the above inequality implies that for any $\theta > \pi/2$ and any $f \in H_0^\infty(\Sigma_\theta)$,

$$\|f(A_p)\| \leq \|f(C_p)\|.$$

Since S^p is a UMD Banach space, C_p has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \pi/2$ (see e.g. [24]). Hence the above estimate implies that in turn, A_p has a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \pi/2$.

We assumed that $(T_t)_{t\geq 0}$ is self-adjoint. Hence by Proposition 5.8 in [25], the above property holds true for any $\theta > \omega_p$. □

Recall Definition 2.6 for polynomial boundedness.

Corollary 8.6. *Let $T\colon B(\ell^2) \to B(\ell^2)$ be a contractive Schur multiplier associated with a real-valued matrix $[t_{ij}]_{i,j\geq 1}$ and let $1 < p < \infty$.*

1. *The induced operator $T\colon S^p \to S^p$ is polynomially bounded.*
2. *If there exists $\delta > 0$ such that $t_{ij} \geq -1 + \delta$ for any $i, j \geq 1$, then the induced operator $T\colon S^p \to S^p$ is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \pi/2$. When $p \geq 2$, it satisfies the conditions (1)–(3) of Corollary 8.4.*

Proof. We first prove (2). Assume that $t_{ij} \geq -1 + \delta$ for any $i, j \geq 1$. Then the spectrum of the self-adjoint map $T\colon S^2 \to S^2$ is contained in $[-1 + \delta, 1]$. Applying the Spectral Theorem, this readily implies that $T\colon S^2 \to S^2$ is a Ritt operator. According to Lemma 5.1 in [39], this implies that for any $1 < p < \infty$, $T\colon S^p \to S^p$ is a Ritt operator.

For any $t \geq 0$, $T_t = e^{-t}e^{tT}$ is a contractive self-adjoint Schur multiplier. Hence for any $1 < p < \infty$, $A = I - T$ has a bounded $H^\infty(\Sigma_\theta)$ functional calculus on S^p for any $\theta > \omega_p$, by Proposition 8.5. Note that $\omega_p < \pi/2$. Thus the result now follows from Proposition 4.1 and Corollary 8.4.

We now prove (1). Under our assumption, the square operator $T^2\colon B(\ell^2) \to B(\ell^2)$ is a contractive Schur multiplier, and its associated matrix is $[t_{ij}^2]_{i,j\geq 1}$. Hence T^2 satisfies part (2) of the present corollary. Let $1 < p < \infty$. Since polynomial boundedness is implied by the existence of a bounded $H^\infty(B_\gamma)$ functional calculus, we deduce from (2) that there exists a constant $K_p \geq 1$ such that

$$\|\varphi(T^2)\|_{B(S^p)} \leq K_p\, \|\varphi\|_{\infty,\mathbb{D}}, \quad \varphi \in \mathcal{P}.$$

Any polynomial φ admits a (necessarily unique) decomposition

$$\varphi(z) = \varphi_1(z^2) + z\varphi_2(z^2)$$

and it is easy to check that

$$\|\varphi_1\|_{\infty,\mathbb{D}} \leq \|\varphi\|_{\infty,\mathbb{D}} \quad \text{and} \quad \|\varphi_2\|_{\infty,\mathbb{D}} \leq \|\varphi\|_{\infty,\mathbb{D}}.$$

Writing $\varphi(T) = \varphi_1(T^2) + T\varphi_2(T^2)$, we deduce that

$$\begin{aligned}\|\varphi(T)\|_{B(S^p)} &\leq \|\varphi_1(T^2)\|_{B(S^p)} + \|\varphi_2(T^2)\|_{B(S^p)}\\ &\leq K_p\big(\|\varphi_1\|_{\infty,\mathbb{D}} + \|\varphi_2\|_{\infty,\mathbb{D}}\big) \leq 2K_p\|\varphi\|_{\infty,\mathbb{D}}.\end{aligned}$$

□

We now turn to our second class of examples. Here we assume that τ is finite and normalized, that is, $\tau(1) = 1$. In this case, $M \subset L^p(M)$ for any $1 \leq p < \infty$. Following [18], [54], we say that a linear map $T\colon M \to M$ is a Markov map if T is unital, completely positive and trace preserving. As is well-known, such a map is necessarily normal and for any $1 \leq p < \infty$, it extends to a contraction $T_p\colon L^p(M) \to L^p(M)$. We say that T is self-adjoint if its L^2-realization T_2 is self-adjoint in the usual Hilbertian sense.

Applying the techniques developed so far, the following analogue of Corollary 8.6 is a rather direct consequence of some recent work of M. Junge, É. Ricard and D. Shlyakhtenko.

Proposition 8.7. *Let $T\colon M \to M$ be a self-adjoint Markov map.*

(1) *For any $1 < p < \infty$, the operator $T_p\colon L^p(M) \to L^p(M)$ is polynomially bounded.*

(2) *If $-1 \notin \sigma(T_2)$, then for any $1 < p < \infty$, $T_p\colon L^p(M) \to L^p(M)$ is a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < 1/2$. When $p \geq 2$, it satisfies the conditions* (1)–(3) *of Corollary* 8.4.

Proof. Let $A_p = I_{L^p(M)} - T_p$ for any $1 < p < \infty$. Repeating the method applied to deduce Corollary 8.6 from Proposition 8.5, we see that it suffices to show that for any $1 < p < \infty$, A_p is sectorial and admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \pi/2$.

To this end, consider

$$T_t = e^{-t(I-T)}, \quad t \geq 0.$$

Then $(T_t)_{t\geq 0}$ is a 'noncommutative diffusion semigroup' in the sense of Chapter 5 of [25], and, for any $1 < p < \infty$, $-A_p$ is the generator of its L^p-realization. Hence A_p is sectorial by Proposition 5.4 in [25].

According to [26], each T_t is 'factorizable' in the sense of Definition 6.2 in [2] or Definition 1.3 in [18]. Writing $T_t = T_{t/2}^2$ and using Theorem 5.3 in [18], we deduce that each T_t satisfies the 'Rota dilation property' introduced in Definition 10.2 of [25] (see also Definition 5.1 in [18]).

We deduce the result by applying the reasoning in [25], 10.D. Indeed it is implicitly shown there that whenever $(T_t)_{t\geq 0}$ is a diffusion semigroup on a finite von Neumann algebra such that each T_t satisfies the Rota dilation property, then the negative generator of its L^p-realization admits a bounded $H^\infty(\Sigma_\theta)$ functional calculus for any $\theta > \omega_p$. □

Remark 8.8. (1) In regards to the necessity of having two parts in Proposition 8.7, we note that L^p-realizations of self-adjoint Markov maps are not necessarily Ritt operators. For instance, the mapping $T\colon \ell_2^\infty \to \ell_2^\infty$ defined by $T(t,s) = (s,t)$ is a Markov map but $-1 \in \sigma(T)$.

(2) If $T\colon M \to M$ satisfies the Rota dilation property, then it is a Markov map and its L^2-realization is positive in the Hilbertian sense. Hence it satisfies Proposition 8.7. In this case, the previous statement strengthens Corollary 10.9 in [25], where weaker square function estimates were established for operators with the Rota dilation property.

(3) For any self-adjoint Schur multiplier (resp. Markov map) T, the square operator T^2 satisfies the second part of Corollary 8.6 (resp. Proposition 8.7). Hence it satisfies an estimate

$$\Big\|\sum_{k=1}^{\infty} k^{1/2}\, \varepsilon_k \otimes \big(T^{k-1}(x) - T^{k+1}(x)\big)\Big\|_{\mathrm{Rad}(L^p(M))} \lesssim \|x\|_{L^p}.$$

References

[1] Arendt, W. and Bu, S.: The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.* **240** (2002), no. 2, 311–343.

[2] Anantharaman-Delaroche, C.: On ergodic theorems for free group actions on noncommutative spaces. *Probab. Theory Related Fields* **135** (2006), no. 4, 520–546.

[3] Arhancet, C.: On Matsaev's conjecture for contractions on noncommutative L_p-spaces. *J. Operator Theory* **69** (2013), no. 2, 387–421.

[4] Arhancet, C. and Le Merdy, C.: Dilation of Ritt operators on L^p-spaces. *Israel J. Math.* **201** (2014), no. 1, 373–414.

[5] Badéa, C., Beckermann, B. and Crouzeix, M.: Intersections of several disks of the Riemann sphere as K-spectral sets. *Comm. Pure Appl. Anal.* **8** (2009), no. 1, 37–54.

[6] Berkson, E. and Gillespie, T. A.: Spectral decompositions and harmonic analysis on UMD Banach spaces. *Studia Math.* **112** (1994), no. 1, 13–49.

[7] Blunck, S.: Maximal regularity of discrete and continuous time evolution equations. *Studia Math.* **146** (2001), no. 2, 157–176.

[8] Blunck, S.: Analyticity and discrete maximal regularity on L_p-spaces. *J. Funct. Anal.* **183** (2001), no. 1, 211–230.

[9] Clément, P., de Pagter, B., Sukochev, F. A. and Witvliet, H.: Schauder decompositions and multiplier theorems. *Studia Math.* **138** (2000), no. 2, 135–163.

[10] Cowling, M., Doust, I., McIntosh, A. and Yagi, A.: Banach space operators with a bounded H^∞ functional calculus. *J. Austral. Math. Soc. Ser. A* **60** (1996), no. 1, 51–89.

[11] Crouzeix, M.: Numerical range and functional calculus in Hilbert space. *J. Funct. Anal.* **244** (2007), no. 2, 668–690.

[12] De Laubenfels, R.: Similarity to a contraction, for power-bounded operators with finite peripheral spectrum. *Trans. Amer. Math. Soc.* **350** (1998), no. 8, 3169–3191.

[13] Delyon, B. and Delyon, F.: Generalization of von Neumann's spectral sets and integral representation of operators. *Bull. Soc. Math. France* **127** (1999), no. 1, 25–42.

[14] Diestel, J., Jarchow, H. and Tonge, A.: *Absolutely summing operators.* Cambridge Studies in Advanced Mathematics 43, Cambridge University Press, Cambridge, 1995.

[15] Dowson, H. R.: *Spectral theory of linear operators.* London Mathematical Society Monographs 12, Academic Press, London-New York, 1978.

[16] Franks, E. and McIntosh, A.: Discrete quadratic estimates and holomorphic functional calculi in Banach spaces. *Bull. Austral. Math. Soc.* **58** (1998), no. 2, 271–290.

[17] Garnett, J. B.: *Bounded analytic functions.* Pure and Applied Mathematics 96, Academic Press, New York-London, 1981.

[18] Haagerup, U. and Musat, M.: Factorization and dilation problems for completely positive maps on von Neumann algebras. *Commun. Math. Phys.* **303** (2011), no. 2, 555–594.

[19] Haak, B.: *Kontrolltheorie in Banachräumen und quadratische abschätzungen.* PhD Thesis, Karlsruhe, 2004.

[20] Haak, B. and Haase, M.: Square function estimates and functional calculi. AarXiv: 1311.0453, 2013.

[21] Haak, B. and Kunstmann, P. C.: Admissibility of unbounded operators and well-posedness of linear systems in Banach spaces. *Integral Equations Operator Theory* **55** (2006), no. 4, 497–533.

[22] Haase, M.: Spectral mapping theorems for holomorphic functional calculi. *J. London Math. Soc. (2)* **71** (2005), no. 3, 723–739.

[23] Haase, M.: *The functional calculus for sectorial operators.* Operator Theory: Advances and Applications 169, Birkhäuser Verlag, Basel, 2006.

[24] Hieber, M. and Prüss, J.: Functional calculi for linear operators in vector-valued L^p-spaces via the transference principle. *Adv. Differential Equations* **3** (1998), no. 6, 847–872.

[25] Junge, M., Le Merdy, C. and Xu, Q.: *H^∞-functional calculus and square functions on noncommutative L^p-spaces.* Astérisque **305**, 2006.

[26] Junge, M., Ricard, É. and Shlyakhtenko, D.: Noncommutative diffusion semigroups and free probability. In preparation.

[27] Kaiser, C. and Weis, L.: Wavelet transform for functions with values in UMD spaces. *Studia Math.* **186** (2008), no. 2, 101–126.

[28] Kalton, N. J. and Portal, P.: Remarks on ℓ_1 and ℓ_∞-maximal regularity for power bounded operators. *J. Aust. Math. Soc.* **84** (2008), no. 3, 345–365.

[29] Kalton, N. J. and Weis, L.: The H^∞-calculus and sums of closed operators. *Math. Ann.* **321** (2001), no. 2, 319–345.

[30] Kalton, N. J. and Weis, L.: The H^∞-functional calculus and square function estimates. Unpublished manusript, 2004.

[31] Kriegler, C. and Le Merdy, C.: Tensor extension properties of $C(K)$-representations and applications to unconditionality. *J. Aust. Math. Soc.* **88** (2010), no. 2, 205–230.

[32] Kunstmann, P.C. and Weis, L.: Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, 65–311. Lecture Notes in Math. 1855, Springer, Berlin, 2004.

[33] Kwapien, S.: On Banach spaces containing c_0. *Studia Math.* **52** (1974), 187–188.

[34] Lancien, F. and Le Merdy, C.: On functional calculus properties of Ritt operators. To appear in *Proc. Roy. Soc. Edinburgh*.

[35] Le Merdy, C.: H^∞-functional calculus and applications to maximal regularity. In *Semi-groupes d'opérateurs et calcul fonctionnel (Besançon, 1998),* 41–77. Publ. Math. UFR Sci. Tech. Besançon 16, Univ. Franche-Comté, Besançon, 1998.

[36] Le Merdy, C.: The similarity problem for bounded analytic semigroups on Hilbert spaces. *Semigroup Forum* **56** (1998), no. 2, 205–224.

[37] Le Merdy, C.: The Weiss conjecture for bounded analytic semigroups. *J. London Math. Soc. (2)* **67** (2003), no. 3, 715–738.

[38] Le Merdy, C.: A sharp equivalence between H^∞ functional calculus and square function estimates. *J. Evol. Equ.* **12** (2012), no. 4, 789–800.

[39] Le Merdy, C. and Xu, Q.: Maximal theorems and square functions for analytic operators on L^p-spaces. *J. Lond. Math. Soc. (2)* **86** (2012), no. 2, 343–365.

[40] Le Merdy, C. and Xu, Q.: Strong q-variation inequalities for analytic semigroups. *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 6, 2069–2097.

[41] Lindenstrauss, J. and Tzafriri, L.: *Classical Banach spaces. II. Function spaces.* Results in Mathematics and Related Areaas 97, Springer-Verlag, Berlin-New York, 1979.

[42] Lust-Piquard, F.: Inégalités de Khintchine dans C^p $(1 < p < \infty)$. *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), no. 7, 289–292.

[43] Lust-Piquard, F. and Pisier, G.: Noncommutative Khintchine and Paley inequalities. *Ark. Mat.* **29** (1991), no. 2, 241–260.

[44] Lyubich, Y.: Spectral localization, power boundedness and invariant subspaces under Ritt's type condition. *Studia Math.* **134** (1999), no. 2, 153–1677.

[45] McIntosh, A.: Operators which have an H^∞ functional calculus. In *Miniconference on operator theory and partial defferential equations (North Ryde, 1986),* 210–231. Proc. Centre Math. Anal. Austral. Nat. Univ. 14, Austral. Nat. Univ., Canberra, 1986.

[46] Nagy, B. and Zemanek, J.: A resolvent condition implying power boundedness. *Studia Math.* **134** (1999), no. 2, 143–151.

[47] Nevanlinna, O.: *Convergence of iterations for linear equations.* Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993.

[48] Paulsen, V. I.: Toward a theory of K-spectral sets. In *Surveys of some recent results in operator theory, Vol. I,* 221–240. Pitman Res. Notes Math. Ser. 171, Longman Sci. Tech., Harlow, 1988.

[49] Paulsen, V. I.: *Completely bounded maps and operator algebras.*, Cambridge Studies in Advanced Mathematics 78, Cambridge University Press, Cambridge, 2002.

[50] Pisier, G.: Some results on Banach spaces without local unconditional structure. *Compositio Math.* **37** (1978), no. 1, 3–19.

[51] Pisier, G.: *Non-commutative vector valued L_p-spaces and completely p-summing maps.* Astérisque 247, 1998.

[52] Pisier, G.: *Similarity problems and completely bounded maps.* Lecture Notes in Mathematics 1618. Springer-Verlag, Berlin, 2001.

[53] Pisier, G. and Xu, Q.: Non-commutative L^p-spaces. In *Handbook of the Geometry of Banach Spaces, Vol. II,* 1459–1517. North-Holland, Amsterdam, 2003.

[54] Ricard, É: A Markov dilation for self-adjoint Schur multipliers. *Proc. Amer. Math. Soc.* **136** (2008), no. 12, 4365–4372.

[55] Rudin, W.: *Real and complex analysis.* McGraw-Hill, New York, 1987.

[56] Stein, E. M.: *Topics in harmonic analysis related to the Littlewood–Paley theory.* Ann. Math. Studies 63, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1970.

[57] Vitse, P.: A band limited and Besov class functional calculus for Tadmor–Ritt operators. *Arch. Math. (Basel)* **85** (2005), no. 4, 374–385.

[58] Vitse, P.: A Besov class functional calculus for bounded holomorphic semigroups. *J. Funct. Anal.* **228** (2005), no. 2, 245–269.

[59] Weis, L.: Operator valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.* **319** (2001), no. 4, 735–758.

Received October 10, 2012.

Christian Le Merdy: Laboratoire de Mathématiques, Université de Franche-Comté, 25030 Besançon Cedex, France.
E-mail: clemerdy@univ-fcomte.fr

The author is supported by the research program ANR 2011 BS01 008 01.