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# A sharp multiplier theorem for Grushin operators in arbitrary dimensions

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**Abstract.** In a recent work by A. Martini and A. Sikora, sharp $L^p$ spectral multiplier theorems for the Grushin operators acting on $\mathbb{R}^{d_1}_{x'} \times \mathbb{R}^{d_2}_{x''}$ and defined by the formula

$$L = -\sum_{j=1}^{d_1} \partial_{x'_j}^2 - \Big(\sum_{j=1}^{d_1} |x'_j|^2\Big) \sum_{k=1}^{d_2} \partial_{x''_k}^2$$

are obtained in the case $d_1 \geq d_2$. Here we complete the picture by proving sharp results in the case $d_1 < d_2$. Our approach exploits $L^2$ weighted estimates with "extra weights" depending essentially on the second factor of $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ (in contrast to the mentioned work, where the "extra weights" depend only on the first factor) and gives a new unified proof of the sharp results without restrictions on the dimensions.

## 1. Introduction

Let X be $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, endowed with Lebesgue measure, and let $L$ be the Grushin operator on X, that is,

$$L = -\Delta_{x'} - |x'|^2 \Delta_{x''},$$

where $x'$ and $x''$ denote the two components of a point $x \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $\Delta_{x'}$ and $\Delta_{x''}$ are the corresponding partial Laplacians, and $|x'|$ is the Euclidean norm of $x'$. Since $L$ is an essentially self-adjoint operator on $L^2(\mathrm{X})$, a functional calculus for $L$ can be defined via spectral integration and, for all Borel functions $F : \mathbb{R} \to \mathbb{C}$, the operator $F(L)$ is bounded on $L^2(\mathrm{X})$ if and only if the function $F$, which is called a *spectral multiplier*, is essentially bounded with respect to the spectral measure.

The aim of this work is to give sufficient conditions for the $L^p$-boundedness (for $p \neq 2$) of an operator of the form $F(L)$, in terms of smoothness properties of the

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multiplier $F$. Namely, let $W_2^s(\mathbb{R})$ denote the $L^2$ Sobolev space on $\mathbb{R}$ of (fractional) order $s$, and define a scale-invariant local Sobolev norm by the formula

$$\|F\|_{MW_2^s} = \sup_{t>0} \|\eta\, F_{(t)}\|_{W_2^s},$$

where $F_{(t)}(\lambda) = F(t\lambda)$ and $\eta \in C_c^\infty(]0,\infty[)$ is a nontrivial auxiliary function (different choices of $\eta$ give rise to equivalent local norms). Our main results then read as follows.

**Theorem 1.** *Suppose that a function $F : \mathbb{R} \to \mathbb{C}$ satisfies*

$$\|F\|_{MW_2^s} < \infty$$

*for some $s > (d_1 + d_2)/2$. Then the operator $F(L)$ is of weak type $(1,1)$ and is bounded on $L^p(\mathrm{X})$ for all $p \in ]1,\infty[$. In addition,*

$$\|F(L)\|_{L^1 \to L^{1,\infty}} \leq C_s \|F\|_{MW_2^s}$$

*and, for all $p \in ]1,\infty[$,*

$$\|F(L)\|_{L^p \to L^p} \leq C_{p,s} \|F\|_{MW_2^s}.$$

**Theorem 2.** *Suppose that $\kappa > (d_1 + d_2 - 1)/2$. Then the Bochner–Riesz means $(1 - tL)_+^\kappa$ are bounded on $L^p(\mathrm{X})$ for all $p \in [1,\infty]$ uniformly in $t \in [0,\infty[$.*

These results are sharp, in the sense that the thresholds $(d_1 + d_2)/2$ on the order of differentiability $s$ in Theorem 1 and $(d_1 + d_2 - 1)/2$ on the order $\kappa$ of the Bochner–Riesz means in Theorem 2 cannot be decreased.

In the case $d_1 \geq d_2$, the results above are contained in joint work of the first named author and Adam Sikora [10], to which we refer for a discussion of the related literature (see also [5], [11], [1], [6], [13], [7], [14], [2], [4], [12], [15], [8]), and for a proof of the mentioned sharpness (based on [9]). In fact, [10] contains some results for the case $d_1 < d_2$ too, which however are not sharp. The new approach presented here differs from that of [10] even in the case $d_1 \geq d_2$, and gives a unified treatment of the sharp results without any restrictions on the pair $(d_1, d_2)$.

## 2. Structure of the proof

Let $\varrho$ be the control distance on X associated to the Grushin operator $L$, and denote by $B(x,r)$ the open $\varrho$-ball of center $x$ and radius $r$, and by $|B(x,r)|$ its Lebesgue measure. Moreover denote by $\mathcal{K}_{F(L)}$ the integral kernel of the operator $F(L)$. As shown in [10], Theorems 1 and 2 are consequences of the following $L^1$ weighted estimate (corresponding to Corollary 14 of [10] in the case $d_1 \leq d_2$).

**Proposition 3.** *For all $R > 0$, $\alpha \geq 0$, and $\beta > \alpha + (d_1 + d_2)/2$, and for all functions $F : \mathbb{R} \to \mathbb{C}$ such that $\operatorname{supp} F \subseteq \left[R^2, 4R^2\right]$,*

$$\operatorname*{ess\,sup}_{y \in \mathrm{X}} \left\|(1 + R\varrho(\cdot,y))^\alpha\, \mathcal{K}_{F(L)}(\cdot,y)\right\|_1 \leq C_{\alpha,\beta} \left\|F_{(R^2)}\right\|_{W_2^\beta}. \tag{2.1}$$

This estimate in turn follows via Hölder's inequality from an weighted $L^2$ estimate of the form

$$\operatorname*{ess\,sup}_{y \in \mathrm{X}} \left|B(y,1/R)\right|^{1/2} \left\| w_R(x,y)^\gamma \, (1+R\varrho(\cdot,y))^\alpha \, \mathcal{K}_{F(L)}(\cdot,y) \right\|_2 \leq C_{\alpha,\beta,\gamma} \|F_{(R^2)}\|_{W_2^\beta} \tag{2.2}$$

for suitable weight functions $w_R \colon \mathrm{X} \times \mathrm{X} \to [0,\infty[$ and constraints on $\alpha, \beta, \gamma \in [0,\infty[$.

In [10] the weights $w_R(x,y)$ depend only on the first components $x'$ and $y'$ of $x$ and $y$, and the proof of (2.2) is based on a subelliptic estimate satisfied by $L$. Such an approach corresponds to the one adopted in [6] for the sublaplacian on a Heisenberg(-type) group $G$, where a weight function is used that depends only on (the projection of the variable on) the first layer of $G$.

On the other hand, other work in the setting of Heisenberg groups [13], [14] exploits weight functions depending on both layers.

The approach presented below differs from all previous ones, since we use weight functions $w_R$ depending only on the second components $x''$ and $y''$ of the variables $x$ and $y$ (except for a rescaling factor due to the dependence of the volume of a ball of fixed radius on the center). In place of the subelliptic estimate used in [10], here we make a careful analysis based on properties of Hermite functions; in this sense, we are closer to the spirit of [13], [14], where instead identities for Laguerre functions are exploited.

We remark that the $L^2$ estimate (2.2) without the weights $w_R$ (that is, when $\gamma = 0$) holds true if $\beta > \alpha$, and this implies the $L^1$ estimate (2.1) when $\beta > \alpha + Q/2$, where $Q$ is the *homogeneous dimension* $d_1 + 2d_2$ of the doubling metric-measure space $\mathrm{X}$ with distance $\varrho$ and Lebesgue measure [3], [15]. The purpose of the "extra weights" $w_R$ is to pass from the homogeneous dimension $Q$ to the topological dimension $d_1 + d_2$. Since these two quantities differ by the dimension $d_2$ of the second factor of $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, it appears necessary, when $d_2$ is larger than $d_1$, to employ weights $w_R(x,y)$ that do not depend only on the first components $x', y'$. In fact the technique presented here, in contrast to the one in [10], does not put any constraint on the dimensions.

## 3. Weighted estimates and discrete differentiation

Given a point $x = (x', x'') \in \mathrm{X}$, we denote by $x'_j$ and $x''_k$ the $j$th component of $x'$ and the $k$th component of $x''$. For all $j \in \{1, \dots, d_1\}$ and $k \in \{1, \dots, d_2\}$, let $L_j$ and $T_k$ be the differential operators on $\mathrm{X}$ given by

$$L_j = (-i\partial_{x'_j})^2 + (x'_j)^2 \sum_{l=1}^{d_2} (-i\partial_{x''_l})^2, \quad T_k = -i\partial_{x''_k}.$$

If $(D_r)_{r>0}$ is the family of dilations on $\mathrm{X}$ defined by

$$D_r(x', x'') = (rx', r^2 x''),$$

then

$$L_j(f \circ D_r) = r^2 (L_j f) \circ D_r, \qquad T_k(f \circ D_r) = r^2 (T_k f) \circ D_r.$$

The Grushin operator $L$ on X is the sum $L_1 + \cdots + L_{d_1}$.

As shown in [10], the operators $L_1, \dots, L_{d_1}, T_1, \dots, T_{d_2}$ have a joint functional calculus; moreover, if $\mathbf{L}$ and $\mathbf{T}$ denote the vectors of operators $(L_1, \dots, L_{d_1})$ and $(T_1, \dots, T_{d_2})$, one can obtain a quite explicit formula for the integral kernel $\mathcal{K}_{G(\mathbf{L},\mathbf{T})}$ of an operator $G(\mathbf{L}, \mathbf{T})$ in the functional calculus in terms of Hermite functions. Namely, for all $\ell \in \mathbb{N}$, let $h_\ell$ denote the $\ell$th Hermite function, that is,

$$h_\ell(t) = (-1)^\ell \, (\ell! \, 2^\ell \sqrt{\pi})^{-1/2} \, e^{t^2/2} \left(\frac{d}{dt}\right)^\ell e^{-t^2},$$

and set, for all $n \in \mathbb{N}^{d_1}$, $u \in \mathbb{R}^{d_1}$, and $\xi \in \mathbb{R}^{d_2}$,

$$\tilde{h}_n(u, \xi) = |\xi|^{d_1/4} h_{n_1}(|\xi|^{1/2} u_1) \cdots h_{n_{d_1}}(|\xi|^{1/2} u_{d_1}).$$

Finally, denote by $e_1, \dots, e_{d_1}$ the standard basis vectors of $\mathbb{R}^{d_1}$, and by $\tilde{1}$ the element $(1, \dots, 1) = e_1 + \cdots + e_{d_1}$ of $\mathbb{N}^{d_1}$.

**Proposition 4.** *For all bounded Borel functions $G : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{C}$ compactly supported in $\mathbb{R}^{d_1} \times (\mathbb{R}^{d_2} \setminus \{0\})$, if*

$$m(n, \xi) = \begin{cases} G(|\xi|(2n + \tilde{1}), \xi) & \text{when } n \in \mathbb{N}^{d_1}, \\ 0 & \text{when } n \in \mathbb{Z}^{d_1} \setminus \mathbb{N}^{d_1}, \end{cases} \tag{3.1}$$

*then*

$$\mathcal{K}_{G(\mathbf{L},\mathbf{T})}(x, y) = (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} \sum_{n \in \mathbb{N}^{d_1}} m(n, \xi) \, \tilde{h}_n(y', \xi) \, \tilde{h}_n(x', \xi) \, e^{i\langle \xi, x'' - y'' \rangle} \, d\xi \tag{3.2}$$

*for almost all $x, y \in$ X.*

*Proof.* See Proposition 5 in [10]. □

The relation (3.2) between the kernel $\mathcal{K}_{G(\mathbf{L},\mathbf{T})}$ and the multiplier $G$, or rather its reparametrization $m$, involves a partial Fourier transform. This suggests that applying a suitable multiplication operator to the kernel may correspond to applying a differential operator to the multiplier. The presence of the Hermite expansion, however, makes things more complicated, and leads one to consider discrete difference operators as well as continuous derivatives on the spectral side. In order to make these observations precise, we introduce some notation.

For all $\ell \in \mathbb{Z}$, set $a_\ell = \sqrt{\ell(\ell - 1)}$ if $\ell > 0$ and $a_\ell = 0$ otherwise. For all $j \in \{1, \dots, d_1\}$, $k \in \{1, \dots, d_2\}$, $\rho \in \mathbb{Z}$ and $s \in \mathbb{N}$, define the following operators on functions $f : \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{C}$:

$$\tau_j f(n, \xi) = f(n + 2e_j, \xi),$$
$$\delta_j f(n, \xi) = f(n, \xi) - f(n - 2e_j, \xi),$$
$$N_{j,\rho,s} f(n, \xi) = \begin{cases} a_{n_j + 2\rho} f(n, \xi) & \text{if } s = 0, \\ N_{j,\rho,s-1} f(n, \xi) - N_{j,\rho-1,s-1} f(n, \xi) & \text{if } s > 0, \end{cases}$$
$$\partial_k f(n, \xi) = \frac{\partial}{\partial \xi_k} f(n, \xi).$$

Note that $\tau_j$ is invertible, and $\delta_j f = f - \tau_j^{-1} f$. We will also use the multi-index notation as follows:

$$\tau^\alpha = \tau_1^{\alpha_1} \cdots \tau_{d_1}^{\alpha_{d_1}}, \qquad \delta^\alpha = \delta_1^{\alpha_1} \cdots \delta_{d_1}^{\alpha_{d_1}}, \qquad \partial^\beta = \partial_1^{\beta_1} \cdots \partial_{d_2}^{\beta_{d_2}},$$

for all $\alpha \in \mathbb{N}^{d_1}$ and $\beta \in \mathbb{N}^{d_2}$; in fact, $\tau^\alpha$ is defined for all $\alpha \in \mathbb{Z}^{d_1}$. Inequalities between multi-indices, such as $\alpha \le \alpha'$, are to be understood componentwise. Moreover $|\cdot|_1$ will denote the 1-norm, that is, for all $t \in \mathbb{R}^d$, $|t|_1 = |t_1| + \cdots + |t_d|$.

For convenience, set $h_\ell = 0$ for all $\ell < 0$, and extend the definition of $\tilde{h}_n$ to all $n \in \mathbb{Z}^{d_1}$; hence $\tilde{h}_n = 0$ for all $n \in \mathbb{Z}^{d_1} \setminus \mathbb{N}^{d_1}$.

**Proposition 5.** *Let $G : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{C}$ be smooth and compactly supported in $\mathbb{R}^{d_1} \times (\mathbb{R}^{d_2} \setminus \{0\})$, and let $m(n,\xi)$ be defined by (3.1). For all $\beta \in \mathbb{N}^{d_2}$, we have*

$$\begin{aligned}&(x'' - y'')^\beta \, \mathcal{K}_{G(\mathbf{L},\mathbf{T})}(x,y) \\ &\quad = \int_{\mathbb{R}^{d_2}} \sum_{n \in \mathbb{Z}^{d_1}} \sum_{\iota \in I_\beta} \Theta_\iota(\xi) \, \partial^{\beta^\iota} \mathcal{N}_\iota \, \tau^{\tilde{\alpha}^\iota} \delta^{\alpha^\iota} m(n,\xi) \, \tilde{h}_{n+2r^\iota}(y',\xi) \, \tilde{h}_n(x',\xi) \, e^{i\langle \xi, x''-y''\rangle} \, d\xi\end{aligned}$$

*for almost all $x, y \in \mathrm{X}$, where $I_\beta$ is a finite set and, for all $\iota \in I_\beta$,*

i) *$\beta^\iota \in \mathbb{N}^{d_2}$ and $\beta^\iota \le \beta$;*

ii) *$\alpha^\iota, \tilde{\alpha}^\iota \in \mathbb{N}^{d_1}$ and $|\alpha^\iota|_1 + |\beta^\iota|_1 \le |\beta|_1$;*

iii) *if $|\beta|_1 > 0$ then $|\alpha^\iota|_1 + |\beta^\iota|_1 > 0$;*

iv) *$r^\iota \in \mathbb{Z}^{d_1}$ and $|r^\iota|_1 \le |\beta|_1$;*

v) *$\Theta_\iota$ is a smooth function on $\mathbb{R}^{d_2} \setminus \{0\}$, homogeneous of degree $|\beta^\iota|_1 - |\beta|_1$;*

vi) *$\mathcal{N}_\iota$ is a composition product of the form*

$$N_{1,\rho_1^1,s_1^1} \cdots N_{1,\rho_{u_1}^1,s_{u_1}^1} \cdots N_{d_1,\rho_1^{d_1},s_1^{d_1}} \cdots N_{d_1,\rho_{u_{d_1}}^{d_1},s_{u_{d_1}}^{d_1}} \tag{3.3}$$

*with $u_1 + \cdots + u_{d_1} \le |\beta|_1 - |\beta^\iota|_1$ and*

$$\begin{aligned} &s_1^j + \cdots + s_{u_j}^j = u_j - \alpha_j^\iota, \qquad s_l^j - |\beta|_1 \le \rho_l^j \le |\beta|_1, \\ &\max\{0, 1-\rho_1^j, \ldots, 1-\rho_{u_j}^j\} \ge \alpha_j^\iota - \tilde{\alpha}_j^\iota \end{aligned}$$

*for all $j \in \{1,\ldots,d_1\}$ and $l \in \{1,\ldots,u_j\}$.*

*Proof.* Because of (3.2), we are reduced to proving that

$$\begin{aligned}&\left(\frac{\partial}{\partial \xi}\right)^\beta \sum_{n \in \mathbb{Z}^{d_1}} m(n,\xi) \, \tilde{h}_n(y',\xi) \, \tilde{h}_n(x',\xi) \\ &\quad = \sum_{\iota \in I_\beta} \sum_{n \in \mathbb{Z}^{d_1}} \Theta_\iota(\xi) \, \partial^{\beta^\iota} \mathcal{N}_\iota \, \tau^{\tilde{\alpha}^\iota} \delta^{\alpha^\iota} m(n,\xi) \, \tilde{h}_{n+2r^\iota}(y',\xi) \, \tilde{h}_n(x',\xi),\end{aligned} \tag{3.4}$$

where $I_\beta$, $\beta^\iota$, $\alpha^\iota$, $\tilde{\alpha}^\iota$, $r^\iota$, $\Theta_\iota$, and $\mathcal{N}_\iota$ are as in the statement of Proposition 5.

This formula can be proved by induction on $|\beta|_1$. For $|\beta|_1 = 0$ it is trivial. For the inductive step, from well-known properties of the Hermite functions (see page 2 of [16]), we deduce

$$2th'_\ell(t) = a_\ell h_{\ell-2}(t) - a_{\ell+2}h_{\ell+2}(t) - h_\ell(t)$$

for all $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$. Correspondingly, for all $n, r \in \mathbb{Z}^{d_1}$, $x', y' \in \mathbb{R}^{d_1}$ and $\xi \in \mathbb{R}^{d_2} \setminus \{0\}$,

$$\begin{aligned}\frac{\partial}{\partial \xi_k}\left[\tilde{h}_{n+2r}(y',\xi)\,\tilde{h}_n(x',\xi)\right] = \frac{\xi_k}{4|\xi|^2}\sum_{j=1}^{d_1}\Big[&a_{n_j+2r_j}\tilde{h}_{n+2(r-e_j)}(y',\xi)\,\tilde{h}_n(x',\xi)\\ &- a_{n_j+2(r_j+1)}\tilde{h}_{n+2(r+e_j)}(y',\xi)\,\tilde{h}_n(x',\xi)\\ &+ a_{n_j}\tilde{h}_{n+2r}(y',\xi)\,\tilde{h}_{n-2e_j}(x',\xi)\\ &- a_{n_j+2}\tilde{h}_{n+2r}(y',\xi)\,\tilde{h}_{n+2e_j}(x',\xi)\Big].\end{aligned}$$

Hence, for all smooth $f : \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{C}$ compactly supported in $\mathbb{Z}^{d_1} \times (\mathbb{R}^{d_2} \setminus \{0\})$,

$$\begin{aligned}\frac{\partial}{\partial \xi_k}\sum_{n\in\mathbb{Z}^{d_1}} f(n,\xi)\,\tilde{h}_{n+2r}(y',\xi)\,\tilde{h}_n(x',\xi) = \sum_{n\in\mathbb{Z}^{d_1}}\Big[&\partial_k f(n,\xi)\,\tilde{h}_{n+2r}(y',\xi)\\ &+ \frac{\xi_k}{4|\xi|^2}\sum_{j=1}^{d_1} N_{j,1,0}\,\tau_j\,\delta_j\,f(n,\xi)\,\tilde{h}_{n+2(r+e_j)}(y',\xi)\\ &- \frac{\xi_k}{4|\xi|^2}\sum_{j=1}^{d_1}\epsilon_{r_j}\sum_{\rho=1-(r_j)_-}^{(r_j)_+} N_{j,\rho+1,1} f(n,\xi)\,\tilde{h}_{n+2(r+e_j)}(y',\xi)\\ &+ \frac{\xi_k}{4|\xi|^2}\sum_{j=1}^{d_1}\epsilon_{r_j}\sum_{\rho=1-(r_j)_-}^{(r_j)_+} N_{j,\rho,1} f(n,\xi)\,\tilde{h}_{n+2(r-e_j)}(y',\xi)\\ &+ \frac{\xi_k}{4|\xi|^2}\sum_{j=1}^{d_1} N_{j,0,0}\,\delta_j\,f(n,\xi)\,\tilde{h}_{n+2(r-e_j)}(y',\xi)\Big]\,\tilde{h}_n(x',\xi),\end{aligned} \tag{3.5}$$

where, for all $\ell \in \mathbb{Z}$,

$$\epsilon_\ell = \begin{cases}+1 & \text{if } \ell \geq 0,\\ -1 & \text{if } \ell < 0\end{cases} \quad \text{and} \quad (\ell)_\pm = \max\{\pm\ell, 0\}.$$

By taking the derivative $\partial/\partial\xi_k$ of both sides of (3.4), applying (3.5) to each summand on the right-hand side, and exploiting the commutation relations

$$\tau_j\,N_{l,\rho,s} = \begin{cases}N_{l,\rho+1,s}\,\tau_j & \text{if } j = l,\\ N_{l,\rho,s}\,\tau_j & \text{if } j \neq l,\end{cases} \qquad \delta_j\,N_{l,\rho,s} = \begin{cases}N_{l,\rho,s+1} + N_{l,\rho-1,s}\,\delta_j, & \text{if } j = l,\\ N_{l,\rho,s}\,\delta_j, & \text{if } j \neq l,\end{cases}$$

we obtain the analogue of (3.4) where $\beta$ is increased by 1 in the $k$th component. □

Plancherel's formula, together with the orthonormality of Hermite functions and the finiteness of the index set $I_\beta$, then yields the following estimate.

**Corollary 6.** *Under the hypotheses of Proposition 5, for all $\beta \in \mathbb{N}^{d_2}$ and almost all $y \in \mathrm{X}$,*

$$\int_{\mathrm{X}} \left|(x'' - y'')^\beta \; \mathcal{K}_{G(\mathbf{L},\mathbf{T})}(x,y)\right|^2 dx$$

$$\leq C_\beta \int_{\mathbb{R}^{d_2}} \sum_{n \in \mathbb{N}^{d_1}} \sum_{\iota \in I_\beta} |\xi|^{2|\beta^\iota|_1 - 2|\beta|_1} \left|\mathcal{N}_\iota \tau^{\tilde{\alpha}^\iota} \delta^{\alpha^\iota} \partial^{\beta^\iota} m(n,\xi)\right|^2 \tilde{h}^2_{n+2r^\iota}(y',\xi)\, d\xi. \tag{3.6}$$

## 4. From discrete to continuous

The next few lemmas will be of use in clarifying the meaning of the various terms appearing in the right-hand side of (3.6).

Note that, for all $\xi \in \mathbb{R}^{d_2}$, $\tau_j f(\cdot,\xi)$, $\delta_j f(\cdot,\xi)$, and $N_{j,\rho,s} f(\cdot,\xi)$ depend only on $f(\cdot,\xi)$. In other words, the operators $\tau_j$, $\delta_j$, $N_{j,\rho,s}$, and their compositions can be considered as operators on functions $\mathbb{Z}^{d_1} \to \mathbb{C}$.

**Lemma 7.** *Let $f : \mathbb{Z}^{d_1} \to \mathbb{C}$ have a smooth extension $\tilde{f} : \mathbb{R}^{d_1} \to \mathbb{C}$, and let $\alpha \in \mathbb{N}^{d_1}$ and $\tilde{\alpha} \in \mathbb{Z}^{d_1}$; then*

$$\tau^{\tilde{\alpha}} \delta^\alpha f(n) = 2^{|\alpha|_1} \int_{J_{\alpha,\tilde{\alpha}}} \partial^\alpha \tilde{f}(n-s)\, d\nu_{\alpha,\tilde{\alpha}}(s)$$

*for all $n \in \mathbb{Z}^{d_1}$, where $J_{\alpha,\tilde{\alpha}} = \prod_{j=1}^{d_1} [-2\tilde{\alpha}_j, 2\alpha_j - 2\tilde{\alpha}_j]$ and $\nu_{\alpha,\tilde{\alpha}}$ is a Borel probability measure on $J_{\alpha,\tilde{\alpha}}$. In particular*

$$|\tau^{\tilde{\alpha}} \delta^\alpha f(n)|^2 \leq 2^{2|\alpha|_1} \int_{J_{\alpha,\tilde{\alpha}}} |\partial^\alpha \tilde{f}(n-s)|^2\, d\nu_{\alpha,\tilde{\alpha}}(s)$$

*and, for all $n \in \mathbb{Z}^{d_1}$,*

$$|\tau^{\tilde{\alpha}} \delta^\alpha f(n)| \leq 2^{|\alpha|_1} \sup_{s \in J_{\alpha,\tilde{\alpha}}} |\partial^\alpha \tilde{f}(n-s)|$$

*Proof.* Iterated application of the fundamental theorem of integral calculus gives

$$\delta^\alpha f(n) = 2^{|\alpha|_1} \int_{[0,1]^{\alpha_1}} \cdots \int_{[0,1]^{\alpha_{d_1}}} \partial^\alpha \tilde{f}\big(n_1 - 2|s_1|_1, \ldots, n_{d_1} - 2|s_{d_1}|_1\big)\, ds_1 \ldots ds_{d_1}$$

and the conclusion follows by taking as $\nu_{\alpha,\tilde{\alpha}}$ the push-forward of the uniform distribution on $\prod_{j=1}^{d_1} [0,1]^{\alpha_j}$ via the map $(s_1, \ldots, s_{d_1}) \mapsto (2|s_1|_1 - 2\tilde{\alpha}_1, \ldots, 2|s_{d_1}|_1 - 2\tilde{\alpha}_{d_1})$, and by Hölder's inequality. □

**Lemma 8.** *Let $\mathcal{N}$ be the product (3.3), and let $f : \mathbb{Z}^{d_1} \to \mathbb{C}$. Then*

1) *$\mathcal{N} f(n) = 0$ for all $n \in \mathbb{Z}^{d_1}$ such that $n_j < 2\max\{-\infty, 1-\rho_1^j, \ldots, 1-\rho_{u_j}^j\}$ for at least one $j \in \{1, \ldots, d_1\}$, and*

2) *$|\mathcal{N} f(n)| \leq C_{\mathcal{N}} |f(n)| \prod_{j=1}^{d_1} (2|n_j|+1)^{u_j - (s_1^j + \cdots + s_{u_j}^j)}$ for all $n \in \mathbb{Z}^{d_1}$.*

*Proof.* It is sufficient to prove the conclusion in the case where the product $\mathcal{N}$ contains a single factor $N_{j,\rho,s}$.

Note that $N_{j,\rho,s}$ is a multiplication operator, with multiplier $\tau_j^\rho \, \delta_j^s \, w_j$, where $w_j(n) = a_{n_j}$. Since $a_\ell = 0$ when $\ell < 2$, inductively we obtain $\tau_j^\rho \, \delta_j^s \, w_j(n) = \delta_j^s \, w(n + 2\rho \, e_j) = 0$ when $n_j < 2(1-\rho)$, and part (1) follows.

The function $w_j : \mathbb{Z}^{d_1} \to \mathbb{C}$ can be extended to a smooth function $\tilde{w}_j : \mathbb{R}^{d_1} \to \mathbb{C}$ such that $\tilde{w}_j(t) = \sqrt{t_j(t_j - 1)}$ if $t_j > 3/2$, say, and $\tilde{w}(t) = 0$ if $t_j \leq 1$. By Leibniz's rule, if $t_j > 3/2$, then

$$\partial_j^s \tilde{w}(t) = \sum_{v=0}^{s} c_{s,v} \, t_j^{1/2-v} (t_j - 1)^{1/2-(s-v)}$$

for some constants $c_{s,v} \in \mathbb{R}$, and in particular $|\partial_j^s \, \tilde{w}(t)| \leq C_s t_j^{1-s}$ if $t_j > 3/2$. Lemma 7 then gives that

$$|\tau_j^\rho \, \delta_j^s \, w_j(n)| \leq C_s \sup_{2\rho - 2s \leq \theta \leq 2\rho} (n_j + \theta)^{1-s} \leq C_{\rho,s} (2|n_j| + 1)^{1-s}$$

for all $n$ with $n_j \geq 2(1 - \rho + s)$. With a possible increase of the constant, the inequality $|\tau_j^\rho \, \delta_j^s \, w(n)| \leq C_{\rho,s}(2|n_j| + 1)^{1-s}$ extends to all $n \in \mathbb{Z}^{d_1}$, and part 2) follows. □

For all $d \in \mathbb{N} \setminus \{0\}$, $\ell \in \mathbb{N}$, and $u \in \mathbb{R}^d$, set

$$H_{d,\ell}(u) = \sum_{\substack{n \in \mathbb{N}^d \\ |n|_1 = \ell}} h_{n_1}^2(u_1) \cdots h_{n_d}^2(u_d).$$

For the reader's convenience, we rewrite here the known bounds for the functions $H_{d,\ell}$ that will be used in the following (see Lemma 8 of [10] and references therein).

**Lemma 9.** *Let $d \in \mathbb{N} \setminus \{0\}$ and set $[\ell] = 2\ell + d$. If $d = 1$ then, for all $\ell \in \mathbb{N}$,*

$$H_{1,\ell}(u) \leq \begin{cases} C([\ell]^{1/3} + |u^2 - [\ell]|)^{-1/2} & \text{for all } u \in \mathbb{R}, \\ C \exp(-cu^2) & \text{when } u^2 \geq 2[\ell]. \end{cases} \tag{4.1}$$

*If $d \geq 2$ then, for all $\ell \in \mathbb{N}$,*

$$H_{d,\ell}(u) \leq \begin{cases} C_d [\ell]^{d/2-1} & \text{for all } u \in \mathbb{R}^d, \\ C_d \exp(-c_d |u|_\infty^2) & \text{when } |u|_\infty^2 \geq 2[\ell], \end{cases} \tag{4.2}$$

*where $|u|_\infty = \max\{|u_1|, \ldots, |u_d|\}$.*

The following lemma is a refined version of Lemma 9 in [10].

**Lemma 10.** *Let $d \in \mathbb{N} \setminus \{0\}$ and set $[\ell] = 2\ell + d$. Let $(b_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $]0, \infty[$ such that, for some $\kappa \in [1, \infty[$,*

$$\kappa^{-1} \leq b_\ell / [\ell] \leq \kappa$$

*for all $\ell \in \mathbb{N}$. In the case $d = 1$, suppose further that, for all $\ell \in \mathbb{N}$,*

$$|b_\ell - [\ell]| \leq \kappa [\ell]^{2/3}$$

*Then, for all $x \in ]0,\infty[$ and $u \in \mathbb{R}^d$,*

$$\sum_{\substack{\ell\in\mathbb{N}\\ [\ell]\le x}} H_{d,\ell}(b_\ell^{-1/2}u) \le C_{d,\kappa} \begin{cases} x^{d/2} & \textit{in any case,}\\ \exp(-|u|^2/(c_{d,\kappa}\,x)) & \textit{if } |u|\ge c_{d,\kappa}\,x, \end{cases} \tag{4.3}$$

*for some $c_{d,\kappa} \in [1,\infty[$.*

*Proof.* We can assume that $x \ge 1$, otherwise the left-hand side of (4.3) vanishes.

In order to exploit the bounds (4.1) and (4.2), we consider several cases.

First, in the case $|u|_\infty \ge x\sqrt{2\kappa}$, if $[\ell] \le x$, then $b_\ell \le \kappa x$, hence

$$|b_\ell^{-1/2}u|_\infty^2 \ge |u|_\infty^2/(\kappa x) \ge 2x \ge 2[\ell],$$

and therefore

$$\begin{aligned} \sum_{[\ell]\le x} H_{d,\ell}(b_\ell^{-1/2}u) &\le C_d x \exp\big(-c_d|u|_\infty^2/(\kappa x)\big)\\ &\le C_d \exp\big(-c_d|u|_\infty^2/(2\kappa x)\big) \sup_{t\ge1}\big(t\exp(-c_d t)\big). \end{aligned} \tag{4.4}$$

Thus the second inequality in (4.3) is proved (by a suitable choice of $c_{d,\kappa}$).

In the case $d > 1$, the first inequality in (4.3) is immediate because

$$\sum_{[\ell]\le x} H_{d,\ell}(b_\ell^{-1/2}u) \le C_d \sum_{[\ell]\le x} [\ell]^{d/2-1} \le C_d\, x^{d/2}.$$

In the case $d = 1$, we need instead to split the sum in (4.3) in several parts:

$$\sum_{[\ell]\le x} H_{1,\ell}(b_\ell^{-1/2}u) = \sum_{\substack{[\ell]\le x\\ [\ell]\le|u|/\sqrt{2\kappa}}} + \sum_{\substack{[\ell]\le x\\ |u|/\sqrt{2\kappa}<[\ell]<|u|\sqrt{2\kappa}}} + \sum_{\substack{[\ell]\le x\\ [\ell]\ge|u|\sqrt{2\kappa}}}.$$

The first and the last part are the easiest to control. In fact, the part where $[\ell] \le |u|/\sqrt{2\kappa}$ is controlled by a constant because of (4.4). Moreover, in the part where $|u|\sqrt{2\kappa} \le [\ell] \le x$, we have $u^2/b_\ell \le [\ell]/2$, hence

$$\sum_{|u|\sqrt{2\kappa}\le[\ell]\le x} H_{1,\ell}(b_\ell^{-1/2}u) \le C \sum_{[\ell]\le x} [\ell]^{-1/2} \le C\, x^{1/2}.$$

The middle part instead requires a further splitting:

$$\sum_{\substack{[\ell]\le x\\ |u|/\sqrt{2\kappa}<[\ell]<|u|\sqrt{2\kappa}}} = \sum_{\substack{[\ell]\le x\\ |u|/\sqrt{2\kappa}<[\ell]\\ [\ell]\le|u|-\kappa[\ell]^{2/3}}} + \sum_{\substack{[\ell]\le x\\ |u|/\sqrt{2\kappa}<[\ell]<|u|\sqrt{2\kappa}\\ |u|-\kappa[\ell]^{2/3}<[\ell]<|u|+\kappa[\ell]^{2/3}}} + \sum_{\substack{[\ell]\le x\\ |u|+\kappa[\ell]^{2/3}\le[\ell]\\ [\ell]<|u|\sqrt{2\kappa}}}.$$

In the part where $|u|/\sqrt{2\kappa} < [\ell] \le |u| - \kappa[\ell]^{2/3}$, we have $|u| \ge 1+\kappa$ and

$$[\ell] \le |u|-1, \quad b_\ell \le |u|, \quad 1/\sqrt{2\kappa} \le [\ell]/|u| < 1,$$

hence

$$\left|\frac{u^2}{b_\ell} - [\ell]\right| \ge |u|\Big(1-\frac{[\ell]}{|u|}\Big),$$

so this part of the sum is bounded from above by

$$C_\kappa \frac{x^{1/2}}{|u|} \sum_{|u|/\sqrt{2\kappa}<[\ell]\leq|u|-\kappa[\ell]^{2/3}} \Big(1-\frac{[\ell]}{|u|}\Big)^{-1/2} \leq C_\kappa\, x^{1/2} \int_{1/\sqrt{2\kappa}}^1 (1-t)^{-1/2}\,dt,$$

and the last integral is finite.

In the part where $|u|+\kappa[\ell]^{2/3} \leq [\ell] < |u|\sqrt{2\kappa}$, we have $|u| \geq 1/\sqrt{2\kappa}$ and

$$[\ell] \geq |u|+1, \quad b_\ell \geq |u|, \quad 1 < [\ell]/|u| \leq \sqrt{2\kappa},$$

hence

$$\Big|\frac{u^2}{b_\ell} - [\ell]\Big| \geq |u| \Big(\frac{[\ell]}{|u|} - 1\Big),$$

so this part of the sum is bounded from above by

$$C\frac{x^{1/2}}{|u|} \sum_{|u|+\kappa[\ell]^{2/3}\leq[\ell]<|u|\sqrt{2\kappa}} \Big(\frac{[\ell]}{|u|}-1\Big)^{-1/2} \leq C\, x^{1/2} \int_1^{\sqrt{2\kappa}} (t-1)^{-1/2}\,dt,$$

and the last integral is finite.

In the part where $|u|/\sqrt{2\kappa} < [\ell] < |u|\sqrt{2\kappa}$ and $|u|-\kappa[\ell]^{2/3} < [\ell] < |u|+\kappa[\ell]^{2/3}$ there are at most $\kappa(2\kappa)^{1/3}|u|^{2/3}$ summands, and moreover $|u| \leq x\sqrt{2\kappa}$, hence this part of the sum is bounded from above by

$$C_\kappa |u|^{2/3} |u|^{-1/6} \leq C_\kappa\, x^{1/2},$$

and we are done. □

We can now give a more explicit form for the right-hand side of (3.6), in terms of a Sobolev norm of the multiplier, in the case where we restrict to the functional calculus for the Grushin operator $L$ alone. In order to avoid divergent series, however, it is convenient to truncate at first the multiplier along the spectrum of $\mathbf{T}$.

**Lemma 11.** *Let $\chi \in C_c^\infty(]0,\infty[)$ be such that* $\operatorname{supp}\chi \subseteq [1/2,2]$. *Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that* $\operatorname{supp} f \subseteq K$ *for some compact set $K \subseteq ]0,\infty[$. For all $r \in [0,\infty[$ and $M \in [1,\infty[$, if $F_M : \mathbb{R}\times\mathbb{R}^{d_2} \to \mathbb{C}$ is defined by*

$$F_M(\lambda,\xi) = F(\lambda)\,\chi(\lambda/(M|\xi|)),$$

*then, for almost all $y \in \mathrm{X}$,*

$$\int_{\mathrm{X}} \Big| |x''-y''|^r \; \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\Big|^2\,dx \\ \leq C_{\chi,K,r} M^{2r-d_2} \Big(\chi_{[0,c_{K,r}]}(|y'|/M) + e^{-|y'|}\Big) \|F\|^2_{W_2^r}$$

*Proof.* Without loss of generality, we can restrict to the case $r \in \mathbb{N}$, since the remaining values of $r$ can be treated by interpolation.

It is then sufficient to prove

$$\int_{\mathrm{X}} \left|(x''-y'')^\beta \; \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\right|^2 dx \\ \leq C_{\chi,K,\beta} M^{2|\beta|_1-d_2} \Big(\chi_{[0,c_{K,\beta}]}(|y'|/M) + e^{-|y'|}\Big) \|F\|^2_{W_2^{|\beta|_1}}$$

for all $\beta \in \mathbb{N}^{d_2}$ and almost all $y \in \mathrm{X}$.

Set $\langle t \rangle = |2t + \tilde{1}|_1 = 2|t|_1 + d_1$ for all $t \in \mathbb{R}^{d_1}$. An estimate for the left-hand side of the previous inequality is given by Corollary 6, by taking $m(n,\xi) = F(|\xi|\langle n \rangle)\,\chi(\langle n \rangle/M)$ for $n \in \mathbb{N}^{d_1}$ and $m(n,\xi) = 0$ for $n \in \mathbb{Z}^{d_1} \setminus \mathbb{N}^{d_1}$. This estimate, combined with Lemma 8, gives

$$\int_{\mathrm{X}} \left|(x''-y'')^\beta \; \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\right|^2 dx \leq C_\beta \sum_{\iota \in I_\beta} \int_{\mathbb{R}^{d_2}} \sum_{n \geq \gamma^\iota} |\xi|^{2|\beta^\iota|_1 - 2|\beta|_1} \\ \times (2n_1+1)^{2\alpha_1^\iota} \cdots (2n_{d_1}+1)^{2\alpha_{d_1}^\iota} \left| \tau^{\tilde{\alpha}^\iota} \delta^{\alpha^\iota} \partial^{\beta^\iota} m(n,\xi) \right|^2 \tilde{h}^2_{n+2r^\iota}(y',\xi)\, d\xi,$$

where $\gamma^\iota := (\gamma^\iota_1, \dots, \gamma^\iota_{d_1})$ and $\gamma^\iota_j := 2\max\{0, 1-\rho^j_1, \dots, 1-\rho^j_{u_j}\} \geq 2(\alpha^\iota_j - \tilde{\alpha}^\iota_j)$ for all $j \in \{1, \dots, d_1\}$.

If $\tilde{m}$ is a smooth extension of $m$, then Lemma 7 gives

$$\int_{\mathrm{X}} \left|(x''-y'')^\beta \; \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\right|^2 dx \leq C_\beta \sum_{\iota \in I_\beta} \int_{J_\iota} \int_{\mathbb{R}^{d_2}} \sum_{n \geq \tilde{\gamma}^\iota} |\xi|^{2|\beta^\iota|_1 - 2|\beta|_1} \\ \times \langle n \rangle^{2|\alpha^\iota|_1} \left| \partial_t^{\alpha^\iota} \partial_\xi^{\beta^\iota} \tilde{m}(n-s,\xi) \right|^2 \tilde{h}^2_n(y',\xi)\, d\xi\, d\nu_\iota(s),$$

where $\tilde{\gamma}^\iota := (\tilde{\gamma}^\iota_1, \dots, \tilde{\gamma}^\iota_{d_1})$, $\tilde{\gamma}^\iota_j := \max\{0, \gamma^\iota_j + 2r^\iota_j\} \geq 2(r^\iota_j - \tilde{\alpha}^\iota_j + \alpha^\iota_j)$ for all $j \in \{1, \dots, d_1\}$, $J_\iota = \prod_{j=1}^{d_1} \left[2(r^\iota_j - \tilde{\alpha}^\iota_j), 2(r^\iota_j - \tilde{\alpha}^\iota_j + \alpha^\iota_j)\right]$, and $\nu_\iota$ is a probability measure on $J_\iota$. Note that the components of the first argument $n-s$ of $\tilde{m}$ on the right-hand side of the previous inequality are always nonnegative, since $n \geq \tilde{\gamma}^\iota$ and $s \in J_\iota$.

A smooth extension $\tilde{m}$ of $m$ is given by

$$\tilde{m}(t,\xi) = F\big(|\xi|(2t_1 + \cdots + 2t_{d_1} + d_1)\big)\, \chi\big((2t_1 + \cdots + 2t_{d_1} + d_1)/M\big)$$

for $\xi \in \mathbb{R}^{d_2} \setminus \{0\}$ and $t \in {]{-1/2}, \infty[}^{d_1}$. An inductive argument then shows that

$$\partial_\xi^{\beta^\iota} \partial_t^{\alpha^\iota} \tilde{m}(t,\xi) = \sum_{\substack{0 \leq a \leq |\alpha^\iota|_1 \\ 0 \leq b \leq |\beta^\iota|_1}} M^{a - |\alpha^\iota|_1} \chi^{(|\alpha^\iota|_1 - a)}(\langle t \rangle/M)\, \Psi_{\beta^\iota,a,b}(\xi)\, \langle t \rangle^b F^{(a+b)}(|\xi|\langle t \rangle)$$

for all $t \in [0,\infty[^{d_1}$, where the $\Psi_{\beta^\iota,a,b} : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ are smooth functions, homogeneous of degree $a + b - |\beta^\iota|_1$. Hence

$$|\partial_\xi^{\beta^\iota} \partial_t^{\alpha^\iota} \tilde{m}(t,\xi)|^2 \leq C_{\chi,\iota} \sum_{v=0}^{|\alpha^\iota|_1 + |\beta^\iota|_1} |\xi|^{2v - 2|\beta^\iota|_1} M^{2v - 2|\alpha^\iota|_1} \left| F^{(v)}(|\xi|\langle t \rangle) \right|^2 \tilde{\chi}(\langle t \rangle/M)$$

for all $t \in [0,\infty[^{d_1}$, where $\tilde{\chi}$ is the characteristic function of $[1/2, 2]$.

Therefore, since $|\alpha^\iota|_1 + |\beta^\iota|_1 \le |\beta|_1$ for all $\iota \in I_\beta$, we have

$$\int_X \left|(x''-y'')^\beta \ \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\right|^2 dx \le C_{\chi,\beta} \sum_{v=0}^{|\beta|_1} M^{2v} \sum_{\iota\in I_\beta} \times \sum_{\substack{n \ge \tilde\gamma^\iota \\ c_\iota^{-1} \le \langle n\rangle/M \le c_\iota}} \int_{J_\iota} \int_{\mathbb{R}^{d_2}} |\xi|^{2v-2|\beta|_1} \left|F^{(v)}(|\xi|\langle n-s\rangle)\right|^2 \tilde h_n^2(y',\xi)\, d\xi\, d\nu_\iota(s),$$

where $c_\iota \in [2,\infty]$ is chosen so that $2/c_\iota \le \langle n\rangle/\langle n-s\rangle \le c_\iota/2$ for all $n \ge \tilde\gamma^\iota, s \in J_\iota$.

If $k_\iota := \tilde\gamma_1^\iota + \cdots + \tilde\gamma_{d_1}^\iota$, $\tilde J_\iota$ is the interval in $\mathbb{R}$ which is the image of $J_\iota$ via the map $(s_1,\dots,s_{d_1}) \mapsto s_1 + \cdots + s_d$, and $\tilde\nu_\iota$ is the corresponding push-forward of $\nu_\iota$ on $\tilde J_\iota$, then $k_\iota \ge \max \tilde J_\iota$ and

$$\int_X \left|(x''-y'')^\beta \ \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\right|^2 dx \le C_{\chi,\beta} \sum_{v=0}^{|\beta|_1} \sum_{\iota\in I_\beta} \sum_{\substack{\ell \ge k_\iota \\ c_\iota^{-1} \le [\ell]/M \le c_\iota}} M^{2v} \times \int_{\tilde J_\iota} \int_{\mathbb{R}^{d_2}} |\xi|^{2v-2|\beta|_1} \left|F^{(v)}\big(|\xi|[\ell-s]\big)\right|^2 \sum_{n\,:\,|n|_1=\ell} \tilde h_n^2(y',\xi)\, d\xi\, d\tilde\nu_\iota(s),$$

where $[\ell] = 2\ell + d_1$. Note that $\sum_{n\,:\,|n|_1=\ell} \tilde h_n^2(y',\xi) = |\xi|^{d_1/2} H_{d_1,\ell}(|\xi|^{1/2} y')$, and that the integrand in $\xi \in \mathbb{R}^{d_2}$ depends only on $|\xi|$. Hence

$$\int_X \left|(x''-y'')^\beta \ \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\right|^2 dx \le C_{\chi,\beta} \sum_{v=0}^{|\beta|_1} \sum_{\iota\in I_\beta} \sum_{\substack{\ell \ge k_\iota \\ c_\iota^{-1} \le [\ell]/M \le c_\iota}} M^{2v} \times \int_{\tilde J_\iota} \int_0^\infty \lambda^{2v-2|\beta|_1+d_1/2+d_2} \left|F^{(v)}\big(\lambda[\ell-s]\big)\right|^2 H_{d_1,\ell}\big(\lambda^{1/2} y'\big) \,\frac{d\lambda}{\lambda}\, d\tilde\nu_\iota(s).$$

Note that $[\ell - s] \sim [\ell] \sim M$ in the domains of summation and integration on the right-hand side; rescaling the integral in $\lambda$, together with the fact that $\operatorname{supp} F \subseteq K$ and $K \subseteq ]0,\infty[$ is compact, then gives

$$\int_X \left|(x''-y'')^\beta \ \mathcal{K}_{F_M(L,\mathbf{T})}(x,y)\right|^2 dx \le C_{\chi,K,\beta} \sum_{v=0}^{|\beta|_1} \sum_{\iota\in I_\beta} \int_0^\infty |F^{(v)}(\lambda)|^2 \times M^{2|\beta|_1-d_2-d_1/2} \int_{\tilde J_\iota} \sum_{\substack{\ell \ge k_\iota \\ c_\iota^{-1} \le [\ell]/M \le c_\iota}} H_{d_1,\ell}\Big(\frac{\lambda^{1/2} y'}{[\ell-s]^{1/2}}\Big)\, d\tilde\nu_\iota(s)\, d\lambda.$$

On the other hand, from Lemma 10 we easily obtain

$$M^{-d_1/2} \sum_{\substack{\ell \ge k_\iota \\ c_\iota^{-1} \le [\ell]/M \le c_\iota}} H_{d_1,\ell}\Big(\frac{\lambda^{1/2} y'}{[\ell-s]^{1/2}}\Big) \le C_{K,\beta}\big(\chi_{[0,c_{K,\beta}]}(|y'|/M) + e^{-|y'|}\big),$$

uniformly in $\iota \in I_\beta$, $s \in \tilde J_\iota$, and $\lambda \in K$, by choosing $c_{K,\beta}$ sufficiently large, and we are done. □

Define the weight $w : \mathrm{X} \times \mathrm{X} \to [1, \infty[$ by

$$w(x, y) = 1 + \frac{|x'' - y''|}{1 + |y'|}.$$

**Proposition 12.** *Let $F : \mathbb{R} \to \mathbb{C}$ be smooth and such that $\operatorname{supp} F \subseteq K$ for some compact $K \subseteq ]0, \infty[$. For all $r \in [0, d_2/2[$, we have*

$$\operatorname*{ess\,sup}_{y \in \mathrm{X}} |B(y, 1)| \int_{\mathrm{X}} \left| w(x, y)^r \; \mathcal{K}_{F(L)}(x, y)\right|^2 dx \le C_{K,r} \|F\|^2_{W_2^r}.$$

*Proof.* Take $\chi \in C_c^\infty(]0, \infty[)$ such that $\operatorname{supp} \chi \subseteq [1/2, 2]$ and $\sum_{k \in \mathbb{Z}} \chi(2^{-k} t) = 1$ for all $t \in ]0, \infty[$. If $F_M$ is defined for all $M \in [1, \infty[$ as in Lemma 11, then

$$F(L) = \sum_{k \in \mathbb{N}} F_{2^k}(L, \mathbf{T})$$

(with convergence in the strong sense). Hence an estimate for $\mathcal{K}_{F(L)}$ can be obtained, via Minkowski's inequality, by summing the corresponding estimates for $\mathcal{K}_{F_{2^k}}(L, \mathbf{T})$ given by Lemma 11.

On the other hand, since $|B(y, 1)| \sim \max\{1, |y'|\}^{d_2}$ (see Proposition 3 in [10]), it is easily checked that

$$\begin{aligned} \sum_{k \in \mathbb{N}} 2^{k(r - d_2/2)} \Big(\chi_{[0, c_{K,r}]}(2^{-k} |y'|) + e^{-|y'|/2}\Big) &\le C_{K,r} \max\{1, |y'|\}^{r - d_2/2} \\ &\le C_{K,r} \frac{(1 + |y'|)^r}{|B(y, 1)|^{1/2}} \end{aligned}$$

when $r \in [0, d_2/2[$. Therefore from Lemma 11 we obtain that

$$|B(y, 1)| \int_{\mathrm{X}} \left| \left(\frac{|x'' - y''|}{1 + |y'|}\right)^r \; \mathcal{K}_{F(L)}(x, y) \right|^2 dx \le C_{K,r} \|F\|^2_{W_2^r}.$$

The conclusion follows by combining the last inequality with the corresponding one for $r = 0$. □

## 5. The multiplier theorems

Now we need some properties of the weight $w$.

**Lemma 13.** *For all $x, y \in \mathrm{X}$,*

$$w(x, y) \le C\big(1 + \varrho(x, y)\big)^2.$$

*Moreover, if $\alpha, r \in [0, \infty[$ satisfy $r < d_2/2$ and $\alpha + 2r > (d_1 + 2d_2)/2$, then, for all $y \in \mathrm{X}$,*

$$\int_{\mathrm{X}} w(x, y)^{-2r} \left(1 + \varrho(x, y)\right)^{-2\alpha} dx \le C_{\alpha,r} |B(y, 1)|.$$

*Proof.* Recall that $\varrho(x,y) \sim \min\{\varrho_1(x,y), \varrho_2(x,y)\}$, where

$$\varrho_1(x,y) = |x'-y'| + |x''-y''|^{1/2}, \qquad \varrho_2(x,y) = |x'-y'| + \frac{|x''-y''|}{|x'|+|y'|}, \tag{5.1}$$

while $|B(y,1)| \sim \max\{1,|y'|\}^{d_2}$ (see Proposition 3 in [10]).

The conclusion will then follow by proving that, for $i=1,2$,

$$w(x,y) \leq C\big(1+\varrho_i(x,y)\big)^2, \tag{5.2}$$

$$\int_X w(x,y)^{-2r}\,\big(1+\varrho_i(x,y)\big)^{-2\alpha}\,dx \leq C_{\alpha,r}(1+|y'|)^{d_2}. \tag{5.3}$$

As for (5.2), when $i=1$,

$$w(x,y) \leq \big(1+|x''-y''|^{1/2}\big)^2 \leq \big(1+\varrho_1(x,y)\big)^2,$$

whereas, when $i=2$,

$$w(x,y) = 1 + \frac{|x''-y''|}{|x'|+|y'|}\,\frac{|x'|+|y'|}{1+|y'|} \leq 1+\varrho_2(x,y)\,\big(2+|x'-y'|\big) \leq \big(1+\varrho_2(x,y)\big)^2.$$

To show (5.3), in the case $i=1$, since $\alpha > d_1/2 + (d_2-2r)$, we can decompose $\alpha = \alpha'+\alpha''$ so that $\alpha' > d_1/2 > 0$ and $\alpha'' > d_2 - 2r > 0$, and therefore

$$\begin{aligned}
&\int_X w(x,y)^{-2r}\big(1+\varrho_1(x,y)\big)^{-2\alpha}\,dx \\
&\qquad \leq \int_X \left(1+\frac{|x''|}{1+|y'|}\right)^{-2r} (1+|x'|)^{-2\alpha'}\,(1+|x''|)^{-\alpha''}\,dx \\
&\qquad \leq (1+|y'|)^{2r} \int_X (1+|x'|)^{-2\alpha'}\,(1+|x''|)^{-2r-\alpha''}\,dx;
\end{aligned}$$

the last integral is finite since $2\alpha' > d_1$ and $2r+\alpha'' > d_2$, and moreover $2r < d_2$.

In the case $i=2$, instead, since $\alpha - d_1/2 > d_2 - 2r$, we can choose $\alpha''$ so that $2\alpha'' \in \,]d_2-2r, \alpha - d_1/2[$; in particular $0 < \alpha'' < \alpha/2$, hence $\alpha' = \alpha - \alpha'' > \alpha/2 > 0$. Then

$$\begin{aligned}
&\int_X w(x,y)^{-2r}\,(1+\varrho_2(x,y))^{-2\alpha}\,dx \\
&\quad \leq C_{\alpha,r} \int_X \left(1+\frac{|x''|}{1+|y'|}\right)^{-2r} (1+|x'|)^{-2\alpha'} \left(1+\frac{|x''|}{1+|x'|+|y'|}\right)^{-2\alpha''} dx \\
&\quad \leq C_{\alpha,r} \int_X \left(1+\frac{|x'|}{1+|y'|}\right)^{2\alpha''} \left(1+\frac{|x''|}{1+|y'|}\right)^{-2r-2\alpha''} (1+|x'|)^{-2\alpha'}\,dx.
\end{aligned}$$

Since $2\alpha''+2r > d_2$, the integral in $x''$ converges, and moreover $2\alpha'' > 0$, hence the denominator $1+|y'|$ in the first factor can be discarded, and we obtain

$$\int_X w(x,y)^{-2r}\,\big(1+\varrho_2(x,y)\big)^{-2\alpha}\,dx \leq C_{\alpha,r}(1+|y'|)^{d_2} \int_{\mathbb{R}^{d_1}} (1+|x'|)^{-2\alpha'+2\alpha''}\,dx'.$$

Since $2\alpha'-2\alpha'' = 2(\alpha-2\alpha'') > d_1$, the integral in $x'$ converges too, and we are done. □

Via interpolation, we are now able to give a strengthened version of the standard weighted $L^2$ estimate that follows from the Gaussian heat kernel bounds for $L$ (see Proposition 11 in [10] and references therein).

**Proposition 14.** *Let $\alpha, \beta, r \in [0, \infty[$ be such that $r < d_2/2$ and $\beta > \alpha + r$. Let $K \subseteq ]0, \infty[$ be compact. For all smooth $F : \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq K$, we have*

$$\operatorname*{ess\,sup}_{y \in \mathrm{X}} |B(y,1)|^{1/2} \big\| w(\cdot, y)^r \left(1 + \varrho(\cdot, y)\right)^\alpha \mathcal{K}_{F(L)}(\cdot, y) \big\|_2 \, dx \leq C_{K,r,\alpha,\beta} \|F\|_{W_2^\beta}.$$

*Proof.* For $\alpha = 0$ and $\beta \geq r$, the inequality is given by Proposition 12.

On the other hand, for arbitrary $\alpha$, if $\beta > \alpha + 2r + 1/2$, then the inequality follows from Lemma 13 and Proposition 11 in [10].

The full range $\beta > \alpha + r$ is then recovered by interpolation (see Lemma 1.2 in [11] and Proposition 13 in [10]). □

We are finally able to prove the fundamental estimate, and consequently our theorems.

*Proof of Proposition 3.* Since the operator $L$ and the distance $\varrho$ are homogeneous with respect to the dilations $D_r$, it is not restrictive to assume that $R = 1$.

Let $r, \alpha' \in [0, \infty[$. For all $y \in \mathrm{X}$, Hölder's inequality gives

$$\big\| (1 + \varrho(\cdot, y))^\alpha \mathcal{K}_{F(L)}(\cdot, y) \big\|_1 \leq \left( \int_{\mathrm{X}} w(x,y)^{-2r} \left(1 + \varrho(x,y)\right)^{-2\alpha'} dx \right)^{1/2} \times \big\| w(\cdot, y)^r \left(1 + \varrho(\cdot, y)\right)^{\alpha + \alpha'} \mathcal{K}_{F(L)}(\cdot, y) \big\|_2.$$

The first factor on the right-hand side can be controlled by Lemma 13 if $r < d_2/2$ and $\alpha' + 2r > (d_1 + 2d_2)/2$,, while the second factor can be controlled by Proposition 14 if moreover $\beta > \alpha + \alpha' + r$.

Under our hypotheses, $\varepsilon := \beta - \alpha - (d_1 + d_2)/2 > 0$; therefore, if we choose $r \in ]d_2/2 - \varepsilon, d_2/2[$ and $\alpha' \in ]d_1/2 + d_2 - 2r, \beta - \alpha - r[$, then the above conditions are satisfied, and we are done. □

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