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Discrete Fourier restriction associated with Schrödinger equations

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Abstract. We present a novel proof on the discrete Fourier restriction. The proof recovers Bourgain's level set result for Strichartz estimates associated with Schrödinger equations on a torus. Some sharp estimates on $L^{2(d+2)/d}$ norms of certain exponential sums in higher dimensional cases are established. As an application, we show that some discrete multilinear maximal functions are bounded on $L^2(\mathbb{Z})$.

1. Introduction

We consider discrete Fourier restriction problems associated with Schrödinger equations. More precisely, for any given $N \in \mathbb{N}$, let $S_{d,N}$ stand for the set

$$\big\{(n_1, \ldots, n_d) \in \mathbb{Z}^d \,:\, |n_j| \leq N,\ 1 \leq j \leq d\big\}\,.$$

For $p > 1$, let $A_{p,d,N}$ represent the best constant satisfying

$$\sum_{\mathbf{n}\in S_{d,N}} \big|\,\widehat{f}(\mathbf{n}, |\mathbf{n}|^2)\big|^2 \leq A_{p,d,N}\, \|f\|_{p'}^2\,, \tag{1.1}$$

where $\mathbf{n} = (n_1, \ldots, n_d) \in S_{d,N}$, $|\mathbf{n}| = \sqrt{n_1^2 + \cdots + n_d^2}$, f is any $L^{p'}$-function on $\mathbb{T}^{d+1}$, $\widehat{f}$ stands for Fourier transform of periodic function f on $\mathbb{T}^{d+1}$, and $p' = p/(p-1)$.

A harmonic analysis method was introduced by Bourgain [1] to obtain

$$A_{p,d,N} \leq C\, N^{d-2(d+2)/p+\varepsilon} \quad \text{for} \quad p > \frac{2(d+4)}{d}\,. \tag{1.2}$$

In [1] Bourgain conjectured that

$$A_{p,d,N} \leq \begin{cases} C_p\, N^{d-2(d+2)/p+\varepsilon} & \text{for } \ p \geq 2(d+2)/d, \\ C_p & \text{for } \ 2 \leq p < 2(d+2)/d. \end{cases} \tag{1.3}$$

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Understanding of this conjecture is still incomplete. For instance, the desired upper bounds for $A_{5,1,N}$, $A_{3,2,N}$ or $A_{2(d+2)/d,d,N}$ for $d \geq 3$ have not been obtained. The most crucial estimate established by Bourgain in [1] is a certain (sharp) level set estimate. In this paper we give a novel proof of the level set estimate.

These problems arise from the study of periodic nonlinear Schrödinger equations:

$$\begin{aligned} \Delta_{\mathbf{x}} u + i\partial_t u + u|u|^{p-2} &= 0, \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}). \end{aligned} \tag{1.4}$$

Here $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{T}^d$, and $u(\mathbf{x}, t)$ is a function of $d+1$ variables which is periodic in space. The corresponding Strichartz estimate is the inequality yielding the best constant $K_{p,d,N}$ satisfying

$$\Big\| \sum_{\mathbf{n} \in S_{d,N}} a_{\mathbf{n}}\, e^{2\pi i (\mathbf{n}\cdot\mathbf{x} + |\mathbf{n}|^2 t)} \Big\|_{L^p(\mathbb{T}^{d+1})} \leq K_{p,d,N} \Big(\sum_{\mathbf{n}} |a_{\mathbf{n}}|^2 \Big)^{1/2}, \tag{1.5}$$

where $\{a_{\mathbf{n}}\}$ is a sequence of complex numbers. The restriction estimate (1.1) is essentially the Strichartz estimate because

$$K_{p,d,N} \sim \sqrt{A_{p,d,N}} \tag{1.6}$$

follows easily by duality.

The Duhamel principle allows us to represent a differential equation as an integral equation

$$u(\mathbf{x}, t) = e^{it\Delta} u_0(\mathbf{x}) + i \int_0^t e^{i(t-\tau)\Delta} \big(|u(\mathbf{x}, \tau)|^{p-2} u(\mathbf{x}, \tau) \big)\, d\tau\,.$$

Applying the Picard iteration and the Strichartz estimate (1.5), Bourgain in [1] obtained local (global) well-posedness of the Schrödinger equations (1.4). Hence, the discrete restriction problems are crucial in studying the dispersive equations on torus. Moreover, they are closely related to the Vinogradov mean value conjecture on exponential sums, which is very important in additive number theory.

Let us introduce Vinogradov's mean value in order to see more clearly the connection between additive number theory and discrete Fourier restriction. For any given polynomial $P(x, \alpha_1, \ldots, \alpha_d) = \sum_{j=1}^k \alpha_j\, x^j$ for $\alpha_1, \ldots, \alpha_k \in \mathbb{T}$, the mean value $J_k(N, b)$ is defined by

$$J_k(N, b) = \int_{\mathbb{T}^k} \Big| \sum_{n=1}^{N} e^{2\pi i P(n, \alpha_1, \ldots, \alpha_k)} \Big|^{2b} d\alpha_1 \cdots d\alpha_k\,.$$

The Vinogradov mean value conjecture addresses the following question. For positive integers k and b, is it true that

$$J_k(N, b) \leq C_{k,b,\varepsilon} \left(N^{b+\varepsilon} + N^{2b - k(k+1)/2 + \varepsilon} \right)? \tag{1.7}$$

Vinogradov invented a method (now called the Vinogradov method) to establish some partial results on the mean value conjecture, and then utilize these partial results for exponential sums to gain new pointwise estimates, which cannot be done via Weyl's classical squaring method. One of main points in Vinogradov's method is that pointwise estimates of the exponential sums follow from a suitable upper bound of the mean value. Although many brilliant mathematicians have devoted considerable time and energy to this conjecture, only the $k = 2$ case is completely settled. The conjecture is also answered affirmatively for cubic polynomials provided $b > 8$ by Hua's work [2], and for $b \geq k(k+1)$ by Wooley's very recent work [4].

In the language of discrete restriction, Vinogradov's mean value conjecture can be rephrased as asking whether the inequality

$$\sum_{n=1}^{N} \left|\widehat{f}(n,\ldots,n^k)\right|^2 \leq C\, N^{1-k(k+1)/p+\varepsilon}\ \|f\|_{p'}^2 \tag{1.8}$$

is true for $p \geq k(k+1)$. Of course, (1.8) is apparently harder. In fact, (1.8) implies the conjecture. Moreover, the conjecture only yields some partial results for (1.8). It would be very interesting if the equivalence of (1.7) and (1.8) could be established.

Because the difficulty of (1.8), we pose a relatively simple question here. Let $k \geq 3$ be a positive integer. Suppose $p \geq 2(k+1)$. Is it true that

$$\sum_{n=1}^{N} \left|\widehat{f}(n,n^k)\right|^2 \leq C\, N^{1-2(k+1)/p+\varepsilon}\ \|f\|_{p'}^2\ ? \tag{1.9}$$

This question essential seeks the Strichartz estimates associated with higher order dispersive equations. Bourgain's proof of (1.2) is based on three ingredients: Weyl's sum estimates, the Hardy–Littlewood circle method, and Tomas–Stein's restriction theorem. It is difficult to employ Bourgain's method for (1.9). Hence we are forced to seek a method that can be adjusted to handle higher order polynomials like $ax + bx^k$. This is our main motivation. In this paper, we present a different proof of (1.2). This paper is our first paper on discrete restriction. In subsequent papers, we will modify this method to obtain an affirmative answer to (1.9) for p large enough and give applications for the corresponding nonlinear dispersive equations.

Our first theorem is about weighted restriction estimates, which deal with the large p cases of (1.1). Moreover, there is no ε required in the upper bound that we obtain.

Theorem 1.1. *For any $\sigma > 0$, any $d \in \mathbb{N}$, and any $p > 4(d+2)/d$, there exists a constant C independent of N such that*

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} e^{-\sigma|\mathbf{n}|^2/N^2}\left|\widehat{f}(\mathbf{n}\,,|\mathbf{n}|^2)\right|^2 \leq C\, N^{d-2(d+2)/p}\ \|f\|_{p'}^2\,, \tag{1.10}$$

for all $f \in L^{p'}(\mathbb{T}^{d+1})$.

Theorem 1.1 yields (1.2) for large p immediately. The proof of Theorem 1.1 presented in Section 2 is very straightforward. The tool we use is the Hardy–Littlewood circle method. The decay factor $e^{-\sigma|\mathbf{n}|^2/N^2}$ makes it possible to calculate the L^p norm of the kernel restricted to major arcs or minor arcs.

For the small p cases, we need a new level set estimate, which implies Bourgain's level set estimate (see Corollary 1.3). Its proof relies on a decomposition of the kernel, which is a sum of a L^∞ function and a function with bounded Fourier transform (see Proposition 3.2).

Theorem 1.2. *Suppose that F is a periodic function on $\mathbb{T}^{d+1}$ given by*

$$F(\mathbf{x},t)=\sum_{\mathbf{n}\in S_{d,N}} a_{\mathbf{n}}\, e^{2\pi i\mathbf{n}\cdot\mathbf{x}}\, e^{2\pi i|\mathbf{n}|^2 t}\,, \tag{1.11}$$

where $\{a_{\mathbf{n}}\}$ is a sequence with $\sum_{\mathbf{n}}|a_{\mathbf{n}}|^2=1$ and $(\mathbf{x},t)\in\mathbb{T}^d\times\mathbb{T}$. For any $\lambda>0$, let

$$E_\lambda=\big\{(\mathbf{x},t)\in\mathbb{T}^{d+1}:|F(\mathbf{x},t)|>\lambda\big\}.$$

Then for any positive number Q satisfying $Q\geq N$,

$$\lambda^2\,|E_\lambda|^2\leq C_1\,Q^{d/2}\,|E_\lambda|^2+\frac{C_2\,N^\varepsilon}{Q}\,|E_\lambda| \tag{1.12}$$

holds for all λ. Here C_1 and C_2 are constants not depending on N and Q.

Applying Theorem 1.2, we can easily obtain the following corollaries, which were proved by Bourgain in [1] in a different way. The details appear in Section 3.

Corollary 1.3. *If $\lambda\geq CN^{d/4}$ for some suitably large constant C, then the level set defined in Theorem 1.2 satisfies*

$$|E_\lambda|\leq C_1\,N^\varepsilon\,\lambda^{-2(d+2)/d}\,.$$

Corollary 1.4.

$$K_{p,d,N}\leq C_\varepsilon\,N^{d/2-(d+2)/p+\varepsilon}\quad \textit{if } p>\frac{2(d+4)}{d}. \tag{1.13}$$

Remark 1.5. Corollary 1.4 clearly yields (1.2) because $K_{p,d,N}\sim\sqrt{A_{p,d,N}}$. Moreover, the tiny positive number ε in (1.13) can be removed. Clearly from Theorem 1.1, we see immediately that the ε is superfluous for large p. For $2(d+4)/d<p\leq 4(d+2)/d$, Bourgain in [1] succeeded in removing the ε via a delicate interpolation argument.

Moreover, Theorem 1.2 implies the following recurrence relation for $K_{p,d,N}$ in the sense of inequality.

Corollary 1.6. *For $p>2$, we have*

$$K^p_{p,d,N}\leq C\,N^d\,K^{p-2}_{p-2,d,N}+C\,N^{dp/2-d-2+\varepsilon}\,. \tag{1.14}$$

Here C is independent of N.

These three corollaries will be proved in Section 3. Developing the idea used in the proof of Theorem 1.2, we can get the following theorem.

Theorem 1.7. *Let $N_1, \dots, N_d \in \mathbb{N}$ and let $S_{N_1,\dots,N_d}$ be defined by*

$$S_{N_1,\dots,N_d}(\mathbf{x},t) = \sum_{\mathbf{n}\in S(N_1,\dots,N_d)} e^{2\pi i\mathbf{n}\cdot\mathbf{x}}\, e^{2\pi i|\mathbf{n}|^2 t}\,. \tag{1.15}$$

where $S(N_1,\dots,N_d)$ is given by

$$S(N_1,\dots,N_d) = \big\{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d \colon |n_j| \le N_j \ \text{ for all } j \in \{1,\dots,d\}\big\}. \tag{1.16}$$

For any $\varepsilon > 0$, there exists a constant C not depending on $N_1,\dots,N_d$ such that

$$\big\|S_{N_1,\dots,N_d}\big\|_{2(d+2)/d} \le C\big(N_1\cdots N_d\big)^{\frac{d}{2(d+2)}} \max\big\{N_1,\dots,N_d\big\}^{\frac{d}{d+2}+\varepsilon}\,. \tag{1.17}$$

Observe that if $N_1 = \cdots = N_d = N$, (1.17) implies that

$$\Big\| \sum_{\mathbf{n}\in S_{d,N}} e^{2\pi i\mathbf{n}\cdot\mathbf{x}}\, e^{2\pi i|\mathbf{n}|^2 t}\Big\|_{2(d+2)/d} \le N^{d/2+\varepsilon}\,, \tag{1.18}$$

that is,

$$\Big\| \sum_{\mathbf{n}\in S_{d,N}} a_{\mathbf{n}}\, e^{2\pi i\mathbf{n}\cdot\mathbf{x}}\, e^{2\pi i|\mathbf{n}|^2 t}\Big\|_{2(d+2)/d} \le N^{\varepsilon}\Big(\sum_{\mathbf{n}} |a_{\mathbf{n}}|^2\Big)^{1/2}, \tag{1.19}$$

provided $a_{\mathbf{n}} = 1$ for all $\mathbf{n}$. If the conditions $a_{\mathbf{n}} = 1$ for all $\mathbf{n}$ could be removed, then the Bourgain conjecture would be solved for all p not less than the critical index $2(d+2)/d$.

Theorem 1.7 has a direct application to some multilinear maximal functions, related to the maximal ergodic theorem, for instance, to the pointwise convergence of the unconventional bilinear average

$$N^{-1}\sum_{n=1}^{N} f_1(T^n)\, f_2(T^{n^2})\,,$$

where T is a measure-preserving transformation on a probability space $(X,\mathcal{A},\mu)$. This application is given in Section 5.

2. Large p cases

In this section we prove Theorem 1.1. All we need to employ is the Hardy–Littlewood circle method. Observe that for large p, $A_{p,d,N} \le CN^{d-2(d+2)/p}$ follows immediately upon noticing

$$\sum_{\mathbf{n}\in S_{d,N}} \big|\,\widehat{f}(\mathbf{n},|\mathbf{n}|^2)\big|^2 \le e^{\sigma d} \sum_{\mathbf{n}\in S_{d,N}} e^{-\frac{\sigma|\mathbf{n}|^2}{N^2}} \big|\,\widehat{f}(\mathbf{n},|\mathbf{n}|^2)\big|^2 \le e^{\sigma d} \sum_{\mathbf{n}\in \mathbb{Z}^d} e^{-\frac{\sigma|\mathbf{n}|^2}{N^2}} \big|\,\widehat{f}(\mathbf{n},|\mathbf{n}|^2)\big|^2.$$

Thus Theorem 1.1 yields the desired upper bounds of $A_{p,d,N}$ for large p cases. Here the decay factor $e^{-\sigma|\mathbf{n}|^2/N^2}$ will make our calculation much easier. The key idea is to decompose the circle into arcs (called major arcs and minor arcs) and then estimate the L^p norm of the corresponding kernel over each arcs.

First we present some technical lemmas. In order to introduce the major arcs, we should state the Dirichlet principle.

Lemma 2.1 (Dirichlet principle). *For any given $N \in \mathbb{N}$ and any $t \in (0,1]$, there exist $a, q \in \mathbb{N}$, $1 \le q \le N$, $1 \le a \le q$, and $(a,q) = 1$, such that $\big| t - a/q \big| \le 1/(Nq)$.*

This principle can be proved by utilizing the pigeonhole principle or by the Farey dissection of order N. For any integer q, define $\mathcal{P}_q$ by

$$\mathcal{P}_q = \left\{ a \in \mathbb{Z} : 1 \le a \le q, (a,q) = 1 \right\},$$

and for any $a \in \mathcal{P}_q$, define the interval $J_{a/q}$ by $J_{a/q} = \left(\frac{a}{q} - \frac{1}{Nq}, \frac{a}{q} + \frac{1}{Nq} \right)$. If $q < N/10$, the interval $J_{a/q}$ is called a major arc, otherwise it is called a minor arc. Clearly we can partition $(0,1]$ into a union of major arcs and minor arcs, that is,

$$(0,1] = \bigcup_{1 \le q \le N, a \in \mathcal{P}_q} J_{a/q} = \mathcal{M}_1 \cup \mathcal{M}_2 \,.$$

Here $\mathcal{M}_1$ is the union of all major arcs and $\mathcal{M}_2$ is the union of all minor arcs.

Lemma 2.2. *Let $\mathbf{1}_A$ denote the indicator function of a measurable set A. Then*

$$\Big\| \sum_{J \in \mathcal{M}_1} \mathbf{1}_J \Big\|_\infty + \Big\| \sum_{J \in \mathcal{M}_2} \mathbf{1}_J \Big\|_\infty \le 100 \,. \tag{2.1}$$

Proof. It is easy to see that all major arcs are disjoint. Thus it suffices to prove that

$$\Big\| \sum_{J \in \mathcal{M}_2} \mathbf{1}_J \Big\|_\infty \le 80 \,.$$

For any given minor arc J_{a_0/q_0}, let $\mathcal{Q}$ denote the collection of all rational numbers a/q such that each $J_{a/q}$ is a minor arc and there is a point common to J_{a_0/q_0} and all $J_{a/q}$'s. We prove that the cardinality of $\mathcal{Q}$ is less than 40. Notice that for any $a/q \in \mathcal{Q}$,

$$\Big| \frac{a_0}{q_0} - \frac{a}{q} \Big| < \frac{1}{Nq_0} + \frac{1}{Nq} \,.$$

This implies that $|a_0 q - a q_0| < 2$. Since $a_0 q - a q_0 \in \mathbb{Z}$, we conclude that either $a_0 q - a q_0 = -1$ or $a_0 q - a q_0 = 1$ if $a/q \ne a_0/q_0$. Hence if $a/q \ne a_0/q_0$, $a/q \in \mathcal{Q}$ must satisfy the diophantine equation $a_0 x - q_0 y = -1$ or $a_0 x - q_0 y = 1$ with $|x| \le N$. The general solution of the diophantine equation is $x = x_0 + q_0 k$ and $y = y_0 + a_0 k$ for all $k \in \mathbb{Z}$ and any given particular solution (x_0, y_0). Then $|kq_0| \le 2N$. By $q_0 \ge N/10$, we have $|k| \le 20$. Thus the number of solutions of either diophantine equation is no more than 40. This completes the proof. □

Remark 2.3. Lemma 2.2 is the finite overlap property of minor arcs. The reason why we use this lemma is that we try to only calculate the L^p norm of the kernel restricted to each arc. Of course, this is not necessarily needed. An alternative way, which is very classic, is to obtain the L^∞ norm for the kernel restricted to the union of minor arcs, and then to find the L^p norm of the kernel on each major arc.

Let K_σ be the kernel defined by

$$K_\sigma(\mathbf{x},t)=\sum_{\mathbf{n}\in\mathbb{Z}^d} e^{-\frac{\sigma|\mathbf{n}|^2}{N^2}}\, e^{2\pi i|\mathbf{n}|^2 t}\, e^{2\pi i\mathbf{n}\cdot\mathbf{x}}\,. \tag{2.2}$$

We set $K_{a/q}$ to be

$$K_{a/q}(\mathbf{x},t)=K_\sigma(\mathbf{x},t)\,\mathbf{1}_{J_{a/q}}(t)\,. \tag{2.3}$$

The following lemma gives an upper bound for the L^p norm of $K_{a/q}$.

Lemma 2.4. *For any integers $1\le q\le N$, $a\in\mathcal{P}_q$, and any $p>2(d+1)/d$,*

$$\big\|K_{a/q}\big\|_p\le \frac{C\,N^{d-(d+2)/p}}{q^{d/2-d/p}}\,. \tag{2.4}$$

Proof. For any given $t\in J_{a/q}$, let $\beta=t-a/q$ and write $\mathbf{n}=\mathbf{k}q+\mathbf{l}$. Here $\mathbf{l}\in\mathbb{Z}_q^d=\{(l_1,\dots,l_d):l_j\in\mathbb{Z}_q\}$. Then we have

$$K_\sigma(\mathbf{x},t)=\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{l}\in\mathbb{Z}_q^d} e^{-\frac{\sigma|\mathbf{k}q+\mathbf{l}|^2}{N^2}}\, e^{2\pi i(\mathbf{k}q+\mathbf{l})\cdot\mathbf{x}}\, e^{2\pi i|\mathbf{k}q+\mathbf{l}|^2(a/q+\beta)}\,.$$

Interchanging the sums, we represent the kernel as

$$K_\sigma(\mathbf{x},t)=\sum_{\mathbf{l}\in\mathbb{Z}_q^d} e^{2\pi i|\mathbf{l}|^2a/q}\sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-|\mathbf{k}q+\mathbf{l}|^2(\sigma/N^2-2\pi i\beta)}\, e^{2\pi i(\mathbf{k}q+\mathbf{l})\cdot\mathbf{x}}\,.$$

Applying the Poisson summation formula to the inner sum, we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-|\mathbf{k}q+\mathbf{l}|^2(\frac{\sigma}{N^2}-2\pi i\beta)}\, e^{2\pi i(\mathbf{k}q+\mathbf{l})\cdot\mathbf{x}}=\sum_{\mathbf{k}\in\mathbb{Z}^d}\Big(\frac{\sqrt{\pi}}{q\sqrt{\sigma/N^2-2\pi i\beta}}\Big)^d e^{2\pi i\frac{\mathbf{l}\cdot\mathbf{k}}{q}}\, e^{-\frac{\pi^2|\mathbf{x}-\mathbf{k}/q|^2}{\sigma/N^2-2\pi i\beta}}.$$

Henceforth, the kernel can be written as

$$K_\sigma(\mathbf{x},t)=\Big(\frac{\sqrt{\pi}}{q\sqrt{\sigma/N^2-2\pi i\beta}}\Big)^d\sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-\frac{\pi^2|\mathbf{x}-\mathbf{k}/q|^2}{\sigma/N^2-2\pi i\beta}}\sum_{\mathbf{l}\in\mathbb{Z}_q^d} e^{2\pi i|\mathbf{l}|^2a/q}\, e^{2\pi i\,\mathbf{l}\cdot\mathbf{k}/q}\,. \tag{2.5}$$

The upper bound of the Gauss sum implies that

$$\Big|\sum_{\mathbf{l}\in\mathbb{Z}_q^d} e^{2\pi i|\mathbf{l}|^2a/q}\, e^{2\pi i\,\mathbf{l}\cdot\mathbf{k}/q}\Big|\le(2q)^{d/2}\,.$$

Thus by inserting the absolute value, the kernel can be majorized by

$$\big|K_\sigma(\mathbf{x},t)\big|\le\frac{(2\pi)^{d/2}}{q^{d/2}\,(\sigma^2/N^4+4\pi^2\beta^2)^{d/4}}\sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-\frac{\pi^2|\mathbf{x}-\mathbf{k}/q|^2\sigma/N^2}{\sigma^2/N^4+4\pi^2\beta^2}}\,.$$

Integrating $|K_\sigma|^p$ on each arc $J_{a/q}$, we obtain that

$$\begin{aligned}\|K_{a/q}\|_p^p &\le \int_{|\beta|\le\frac{1}{Nq}}\int_{\mathbb{T}^d}\frac{(2\pi)^{dp/2}}{q^{dp/2}\big(\sigma^2/N^4+4\pi^2\beta^2\big)^{dp/4}}\Big|\sum_{\mathbf{k}\in\mathbb{Z}^d}e^{-\frac{\pi^2|\mathbf{x}-\mathbf{k}/q|^2\sigma/N^2}{\sigma^2/N^4+4\pi^2\beta^2}}\Big|^p\,d\mathbf{x}\,d\beta\\ &=\int_{|\beta|\le\frac{1}{Nq}}\frac{(2\pi)^{dp/2}}{q^{dp/2}\big(\sigma^2/N^4+4\pi^2\beta^2\big)^{dp/4}}\Big(\int_0^1\Big|\sum_{k\in\mathbb{Z}}e^{-\frac{\pi^2|x-k/q|^2\sigma/N^2}{\sigma^2/N^4+4\pi^2\beta^2}}\Big|^p\,dx\Big)^d\,d\beta\,.\end{aligned}$$

Notice that for $|\beta|\le 1/(Nq)$ and $q\le N$,

$$\frac{\frac{\sigma}{q^2N^2}}{\frac{\sigma^2}{N^4}+4\pi^2\beta^2}\ge C_\sigma\,.$$

This yields that

$$\sum_{k\in\mathbb{Z}}e^{-\frac{\pi^2|x-k/q|^2\sigma/N^2}{\sigma^2/N^4+4\pi^2\beta^2}}\le C_\sigma\,.$$

For $p>2(d+1)/d$, we estimate the L^p norm of $K_{a/q}$ by

$$\|K_{a/q}\|_p^p\le\int_{|\beta|\le\frac{1}{Nq}}\frac{(2\pi)^{dp/2}}{q^{dp/2}\big(\sigma^2/N^4+4\pi^2\beta^2\big)^{dp/4}}\Big(\int_0^1\sum_{k\in\mathbb{Z}}e^{-\frac{\pi^2|x-k/q|^2\sigma/N^2}{\sigma^2/N^4+4\pi^2\beta^2}}\,dx\Big)^d\,d\beta\,,$$

which can be bounded by

$$\int_{|\beta|\le\frac{1}{Nq}}\frac{C\,(2\pi)^{dp/2}N^d}{q^{dp/2}-d}\Big(\frac{\sigma^2}{N^4}+4\pi^2\beta^2\Big)^{dp/4-d/2}\,d\beta\le\frac{CN^{dp-d-2}}{q^{dp/2-d}}\,.$$

Therefore, the proof is finished. □

Lemma 2.5. *For $p>2(d+2)/d$,*

$$\|K_\sigma\|_p\le C_{p,\sigma}N^{d-(d+2)/p}\,. \tag{2.6}$$

Proof. By Lemmas 2.2 and 2.4, we have that

$$\|K_\sigma\|_p^p\le C\sum_{q=1}^N\sum_{a\in\mathcal{P}_q}\|K_{a/q}\|_p^p\le C\sum_{q=1}^N\sum_{a\in\mathcal{P}_q}\frac{N^{dp-d-2}}{q^{dp/2-d}}\le CN^{dp-d-2}\,,$$

which yields Lemma 2.5. □

We now return to the proof of Theorem 1.1. Indeed, observe that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d}e^{-\sigma|\mathbf{n}|^2/N^2}\big|\widehat{f}(\mathbf{n},|\mathbf{n}|^2)\big|^2=\big\langle K_\sigma*f,f\big\rangle\,.$$

Applying Hölder's inequality and then the Hausdorff–Young convolution inequality, we get

$$\big\langle K_\sigma*f,f\big\rangle\le\|K_\sigma\|_{p/2}\|f\|_{p'}^2\,.$$

Since $p>4(d+2)/d$, we can use Lemma 2.5 to conclude Theorem 1.1.

3. Level set estimates

In this section, we prove Theorem 1.2. Theorem 1.2 can be utilized for handling the small p cases. First, we state an arithmetic result.

Lemma 3.1. *For any integer $Q \geq 1$, any integer $n \neq 0$, and any $\varepsilon > 0$,*

$$\sum_{Q \leq q < 2Q} \Big| \sum_{a \in \mathcal{P}_q} e^{2\pi i \frac{a}{q} n} \Big| \leq C_\varepsilon \, d(n, Q) \, Q^{1+\varepsilon} \, .$$

Here $d(n, Q)$ denotes the number of divisors of n less than Q and C_ε is a constant depending on neither Q nor n.

Lemma 3.1 can be proved by observing that the arithmetic function defined by $f(q) = \sum_{a \in \mathcal{P}_q} e^{2\pi i \frac{a}{q} n}$ is multiplicative, and then using the prime factorization for q to conclude the proof. The details can be found in [1].

The next proposition is crucial to our proof.

Proposition 3.2. *For any given positive number Q with $N \leq Q \leq N^2$, the kernel K_σ given by (2.2) can be decomposed as $K_{1,Q} + K_{2,Q}$ where*

$$\|K_{1,Q}\|_\infty \leq C_1 \, Q^{d/2} \tag{3.1}$$

$$\big\|\widehat{K_{2,Q}}\big\|_\infty \leq \frac{C_2 \, N^\varepsilon}{Q} \, . \tag{3.2}$$

Here the constants C_1 and C_2 do not depend on Q and N.

Proof. We can assume that Q is an integer, since otherwise we can take the integer part of Q. For a standard bump function φ supported in $[1/200, 1/100]$, we set

$$\Phi(t) = \sum_{Q \leq q < 2Q} \sum_{a \in \mathcal{P}_q} \varphi\Big(\frac{t - a/q}{1/q^2}\Big) \, . \tag{3.3}$$

Clearly Φ is supported in $[0, 1]$. We can extend Φ periodically to other intervals to obtain a periodic function on $\mathbb{T}$. We continue to denote this extended periodic function by Φ. Then it is easy to see that

$$\widehat{\Phi}(0) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{\mathcal{F}_\mathbb{R}\varphi(0)}{q^2} = \sum_{q \sim Q} \frac{\phi(q)}{q^2} \, \mathcal{F}_\mathbb{R} \, \varphi(0) \tag{3.4}$$

is a constant independent of Q. Here ϕ is Euler's totient function, and $\mathcal{F}_\mathbb{R}$ denotes the Fourier transform of a function on $\mathbb{R}$. Also we have

$$\widehat{\Phi}(k) = \sum_{q \sim Q} \sum_{a \in \mathcal{P}_q} \frac{1}{q^2} \, e^{-2\pi i \frac{a}{q} k} \, \mathcal{F}_\mathbb{R} \, \varphi(k/q^2) \, . \tag{3.5}$$

We define

$$K_{1,Q}(\mathbf{x}, t) = \frac{1}{\widehat{\Phi}(0)} \, K_\sigma(\mathbf{x}, t) \, \Phi(t), \quad \text{and} \quad K_{2,Q} = K_\sigma - K_{1,Q} \, .$$

We prove (3.2) first. Write Φ as its Fourier series to get

$$K_{2,Q}(\mathbf{x},t) = -\frac{1}{\widehat{\Phi}(0)}\sum_{k\neq 0}\widehat{\Phi}(k)\, e^{2\pi ikt}\, K_\sigma(\mathbf{x},t)\,.$$

Thus its Fourier coefficient is

$$\widehat{K_{2,Q}}(\mathbf{n},n_{d+1}) = -\frac{e^{-\sigma|\mathbf{n}|^2/N^2}}{\widehat{\Phi}(0)}\sum_{k\neq 0}\widehat{\Phi}(k)\,\mathbf{1}_{\{n_{d+1}=|\mathbf{n}|^2+k\}}(k)\,.$$

Here $\mathbf{n}\in\mathbb{Z}^d$ and $n_{d+1}\in\mathbb{Z}$. This implies that $\widehat{K_{2,Q}}(\mathbf{n},n_{d+1}) = 0$ if $n_{d+1}=|\mathbf{n}|^2$, and if $n_{d+1}\neq|\mathbf{n}|^2$,

$$\widehat{K_{2,Q}}(\mathbf{n},n_{d+1}) = -\frac{e^{-\sigma|\mathbf{n}|^2/N^2}}{\widehat{\Phi}(0)}\,\widehat{\Phi}\big(n_{d+1}-|\mathbf{n}|^2\big)\,.$$

Applying (3.5) and Lemma 3.1, we estimate $\widehat{K_{2,Q}}(\mathbf{n},n_{d+1})$ by

$$\big|\,\widehat{K_{2,Q}}(\mathbf{n},n_{d+1})\big| \le \frac{CN^\varepsilon}{Q}\,,$$

since $N\le Q\le N^2$. Hence we obtain (3.2).

We now prove (3.1). Observe that the intervals $[\frac{a}{q}+\frac{1}{200q^2},\frac{a}{q}+\frac{1}{100q^2}]$ are pairwise disjoint. Thus we can fix $q\sim Q$ and $a\in\mathcal{P}_q$ and try to obtain the upper bound of $K_{1,Q}$ restricted to $[\frac{a}{q}+\frac{1}{200q^2},\frac{a}{q}+\frac{1}{100q^2}]$. Let $\beta = t-\frac{a}{q}$. Hence we have $|\beta|\sim 1/q^2$ for $t\in[\frac{a}{q}+\frac{1}{200q^2},\frac{a}{q}+\frac{1}{100q^2}]$. As we did in the previous section, by the Poisson summation formula, we have

$$K_\sigma(\mathbf{x},t) = \Big(\frac{\sqrt{\pi}}{q\sqrt{\sigma/N^2-2\pi i\beta}}\Big)^d \sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-\frac{\pi^2|\mathbf{x}-\mathbf{k}/q|^2}{\sigma/N^2-2\pi i\beta}} \sum_{\mathbf{l}\in\mathbb{Z}_q^d} e^{2\pi i|\mathbf{l}|^2a/q}\, e^{2\pi i\,\mathbf{l}\cdot\mathbf{k}/q}\,.$$

For $|\beta|\sim 1/q^2$, we estimate

$$\big|K_\sigma(\mathbf{x},t)\big| \le \frac{C}{q^{d/2}\big((\sigma/N^2)^2+\beta^2\big)^{d/4}}\sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-\pi^2\frac{|\mathbf{k}/q-\mathbf{x}|^2}{(\sigma/N^2)^2+\beta^2}\frac{\sigma}{N^2}}\,,$$

which is bounded by

$$\frac{CN^d}{q^{d/2}}\sum_{\mathbf{k}\in\mathbb{Z}^d} e^{-\pi^2\frac{N^2}{\sigma}|\mathbf{k}/q-\mathbf{x}|^2} \le C_\sigma\, q^{d/2} \le C_\sigma\, Q^{d/2}\,.$$

This implies (3.1). Therefore we complete the proof. □

We now start to prove Theorem 1.2. For the function F and the level set E_λ given in Theorem 1.2, we define f by

$$f(\mathbf{x},t) = \frac{F(\mathbf{x},t)}{|F(\mathbf{x},t)|}\,\mathbf{1}_{E_\lambda}(\mathbf{x},t)\,.$$

Clearly

$$\lambda\,|E_\lambda| \le \int_{\mathbb{T}^{d+1}} \overline{F(\mathbf{x},t)}\, f(\mathbf{x},t)\, d\mathbf{x}\, dt\,.$$

By the definition of F, we get

$$\lambda\,|E_\lambda| \le \sum_{\mathbf{n}\in S_{d,N}} \overline{a_{\mathbf{n}}}\; \widehat{f}(\mathbf{n},|\mathbf{n}|^2)\,.$$

Using the Cauchy–Schwarz inequality, we have

$$\lambda^2\,|E_\lambda|^2 \le \sum_{\mathbf{n}\in S_{d,N}} \big|\widehat{f}(\mathbf{n},|\mathbf{n}|^2)\big|^2\,.$$

The right hand side is bounded by

$$e^{\sigma d}\sum_{\mathbf{n}} e^{-\sigma|\mathbf{n}|^2/N^2}\big|\,\widehat{f}(\mathbf{n},|\mathbf{n}|^2)\big|^2 = e^{\sigma d}\big\langle K_\sigma * f, f\big\rangle\,.$$

For any Q with $N \le Q \le N^2$, we employ Proposition 3.2 to decompose the kernel K_σ. Then we have

$$\lambda^2\,|E_\lambda|^2 \le C_\sigma\big|\big\langle K_{1,Q} * f, f\big\rangle\big| + C_\sigma\big|\big\langle K_{2,Q} * f, f\big\rangle\big|\,.$$

From (3.1) and (3.2), we then obtain

$$\lambda^2|E_\lambda|^2 \le C_1\,Q^{d/2}\,\|f\|_1^2 + \frac{C_2\,N^\varepsilon}{Q}\,\|f\|_2^2 \le C_1\,Q^{d/2}\,|E_\lambda|^2 + \frac{C_2\,N^\varepsilon}{Q}\,|E_\lambda|\,.$$

The case $Q \ge N^2$ is trivial since the level set E_λ is empty if $\lambda > CN^{d/2}$. This completes the proof of Theorem 1.2.

We now start to prove Corollary 1.3 by using Theorem 1.2. We should take $Q^{d/2} = \frac{1}{2C_1}\lambda^2$, where C_1 is the constant in (1.12). Since $Q \ge N$, we need to restrict to $\lambda > \sqrt{2C_1}\,N^{d/4}$. Then $|E_\lambda| \le CN^\varepsilon\lambda^{-2(d+2)/d}$ follows immediately from (1.12). This completes the proof of Corollary 1.3.

To prove Corollary 1.4, write

$$\|F\|_p^p = C_p\int_0^\infty \lambda^{p-1}|E_\lambda|\,d\lambda\,,$$

which equals

$$C_p\int_0^{CN^{d/4}} \lambda^{p-1}\,|E_\lambda|\,d\lambda + C_p\int_{CN^{d/4}}^\infty \lambda^{p-1}\,|E_\lambda|\,d\lambda\,.$$

Using the trivial estimate $|E_\lambda| \le C\lambda^{-2}$ for the first term and employing Corollary 1.3 for the second term, we then obtain, for $p > 2(d+4)/d$,

$$\|F\|_p^p \le C\,N^{dp/2-(d+2)+\varepsilon}$$

as desired. The proof of Corollary 1.4 is completed.

We now prove Corollary 1.6. Multiply (1.12) by λ^{p-3} to get, for $N \le Q$,

$$\lambda^{p-1}\,|E_\lambda| \le C_1\, Q^{d/2}\, \lambda^{p-3}\,|E_\lambda| + \frac{C_2\, N^{\varepsilon}}{Q}\, \lambda^{p-3}\,. \tag{3.6}$$

Integrating (3.6) over λ from 0 to $CN^{d/2}$, we obtain that

$$\|F\|_p^p \le C_1\, Q^{d/2}\, \|F\|_{p-2}^{p-2} + C_2\, \frac{N^{dp/2-d+\varepsilon}}{Q}\,. \tag{3.7}$$

Taking $Q = N^2$, we then have

$$\|F\|_p^p \le C_1\, N^d\, K_{p-2,d,N}^{p-2} + C_2\, N^{dp/2-d-2+\varepsilon}\,. \tag{3.8}$$

This finishes the proof of Corollary 1.6.

4. Proof of Theorem 1.7

In this section, we prove Theorem 1.7 by developing an idea similar to that used in Section 3. We introduce a level set G_λ for any $\lambda > 0$ by setting

$$G_\lambda = \left\{(\mathbf{x}, t) \in \mathbb{T}^d \times \mathbb{T} : |S_{N_1,\ldots,N_d}(\mathbf{x}, t)| > \lambda\right\}\,. \tag{4.1}$$

As in Section 3, let $f = \mathbf{1}_{G_\lambda} S_{N_1,\ldots,N_d}/|S_{N_1,\ldots,N_d}|$. We then have

$$\lambda\,|G_\lambda| \le \sum_{\mathbf{n}\in S(N_1,\ldots,N_d)} \widehat{f}(\mathbf{n}, \mathbf{n}^2) = \left\langle f_{N_1,\ldots,N_d}, S_{N_1,\ldots,N_d}\right\rangle, \tag{4.2}$$

where $f_{N_1,\ldots,N_d}$ is a rectangular Fourier partial sum defined by

$$f_{N_1,\ldots,N_d}(\mathbf{x}, t) = \sum_{\substack{\mathbf{n}\in S(N_1,\ldots,N_d)\\ |n_{d+1}|\le d\max\{N_1,\ldots,N_d\}^2}} \widehat{f}(\mathbf{n}, n_{d+1})\, e^{2\pi \mathbf{n}\cdot\mathbf{x}}\, e^{2\pi i n_{d+1} t}\,. \tag{4.3}$$

Here, unlike what we did in Section 3, we do not use the Cauchy–Schwarz inequality to estimate the right-hand side of (4.2). We actually need to get a decomposition of $S_{N_1,\ldots,N_d}$. Before we state this decomposition, we include a famous result on Weyl's sums.

Lemma 4.1. *Suppose t is a real number satisfying*

$$\left|t - \frac{a}{q}\right| \le \frac{1}{q^2}\,.$$

Here a and q are relatively prime integers. Then

$$\left|\sum_{n=1}^{N} e^{2\pi i(tn^2+xn)}\right| \le C\max\left\{\frac{N}{\sqrt{q}}, \sqrt{N\log q}, \sqrt{q\log q}\right\}. \tag{4.4}$$

The proof can be accomplished by Weyl's squaring method. See [2] or [3] for details.

Lemma 4.2. *For any real number Q with $\max_{1\le j\le d} N_j \le Q \le \max_{1\le j\le d} N_j^2$, the function $S_{N_1,\dots,N_d}$ defined in (1.15) can be written as a sum of $S_{1,Q}$ and $S_{2,Q}$, where $S_{1,Q}$ satisfies*

$$\|S_{1,Q}\|_\infty \le C\, Q^{d/2}\,(\log Q)^{d/2} \tag{4.5}$$

and $S_{2,Q}$ satisfies

$$\big\|\widehat{S_{2,Q}}\big\|_\infty \le \frac{C\,\max\{N_1,\dots,N_d\}^\varepsilon}{Q}\,. \tag{4.6}$$

Here the constant C is independent of $N_1,\dots,N_d$ and Q.

Proof. Let Φ be the function defined in (3.3). We then obtain

$$S_{N_1,\dots,N_d} = S_{1,Q} + S_{2,Q}\,, \tag{4.7}$$

where $S_{1,Q}$ is given by

$$S_{1,Q}(\mathbf{x},t) = \frac{1}{\widehat{\Phi}(0)}\, S_{N_1,\dots,N_d}(\mathbf{x},t)\,\Phi(t) \tag{4.8}$$

and $S_{2,Q}$ is

$$S_{2,Q} = S_{N_1,\dots,N_d} - S_{1,Q}\,. \tag{4.9}$$

(4.5) follows immediately from (4.4). Notice that

$$S_{2,Q}(\mathbf{x},t) = -\frac{1}{\widehat{\Phi}(0)} \sum_{k\ne 0} \widehat{\Phi}(k)\, e^{2\pi i k t}\, S_{N_1,\dots,N_d}(\mathbf{x},t)\,.$$

The inequality (4.6) follows by using Lemma 3.1, as in the proof of (3.2). $\square$

We now return to the proof of Theorem 1.7. From (4.2) and Lemma 4.2, the level set G_λ satisfies

$$\lambda\,|G_\lambda| \le \big|\big\langle f_{N_1,\dots,N_d}, S_{1,Q}\big\rangle\big| + \big|\big\langle f_{N_1,\dots,N_d}, S_{2,Q}\big\rangle\big|\,, \tag{4.10}$$

which can be bounded by

$$C\Big(Q^{d/2}(\log Q)^{d/2}\,\big\|f_{N_1,\dots,N_d}\big\|_1 + \sum_{\substack{\mathbf{n}\in S(N_1,\dots,N_d)\\ |n_{d+1}|\le d\max\{N_1,\dots,N_d\}^2}} \Big|\widehat{S_{2,Q}}(\mathbf{n},n_{d+1})\,\widehat{f}(\mathbf{n},n_{d+1})\Big|\Big).$$

Thus from the fact that the L^1 norm of Dirichlet kernel D_N is comparable to $\log N$, (4.6), and Cauchy–Schwarz inequality, we have

$$\lambda\,|G_\lambda| \le C\,Q^{d/2}\,(\log Q)^{2d}|G_\lambda| + \frac{C(N_1\cdots N_d)^{1/2}\max\{N_1,\dots,N_d\}^{1+\varepsilon}}{Q}\,|G_\lambda|^{1/2}\,. \tag{4.11}$$

For $\lambda \geq C \max\{N_1, \dots, N_d\}^{d/2+\varepsilon}$, take Q to be a number satisfying

$$Q^{d/2} \max\{N_1, \dots, N_d\}^{\varepsilon} = \lambda,$$

and then Lemma 4.2 yields

$$|G_\lambda| \leq \frac{C\, N_1 \cdots N_d \max\{N_1, \dots, N_d\}^{2+\varepsilon}}{\lambda^{2(d+2)/d}}\,. \tag{4.12}$$

Notice that

$$\big\|S_{N_1,\dots,N_d}\big\|_2 \sim \big(N_1 \cdots N_d\big)^{1/2}\,. \tag{4.13}$$

Thus, for $\lambda < C \max\{N_1, \dots, N_d\}^{d/2+\varepsilon}$, we have

$$|G_\lambda| \leq \frac{C\, N_1 \cdots N_d}{\lambda^2} \leq \frac{C\, N_1 \cdots N_d \max\{N_1, \dots, N_d\}^{2+\varepsilon}}{\lambda^{2(d+2)/d}}\,. \tag{4.14}$$

Hence (4.12) holds for all $\lambda > 0$.

We now estimate the $L^{2(d+2)/d}$ norm of $S_{N_1,\dots,N_d}$ by

$$\begin{aligned} &\big\|S_{N_1,\dots,N_d}\big\|_{2(d+2)/d}^{2(d+2)/d} \\ &\quad \leq C \int_1^{2^d N_1 \cdots N_d} \lambda^{2(d+2)/d-1}\, |G_\lambda|\, d\lambda + C \int_0^1 \lambda^{2(d+2)/d-1}\, |G_\lambda|\, d\lambda\,. \end{aligned} \tag{4.15}$$

Since (4.12) holds for all $\lambda > 0$, the first term on the right-hand side of (4.15) can be bounded by $C\, N_1 \cdots N_d \max\{N_1, \dots, N_d\}^{2+\varepsilon}$. The second term is clearly bounded by C because G_λ is a set with finite measure. Putting both estimates together, we get

$$\big\|S_{N_1,\dots,N_d}\big\|_{2(d+2)/d}^{2(d+2)/d} \leq C\, N_1 \cdots N_d \max\{N_1, \dots, N_d\}^{2+\varepsilon}\,, \tag{4.16}$$

as desired. Therefore the proof is complete.

5. Estimates of multilinear maximal functions

In this section, we provide an application of Theorem 1.7.

Definition 5.1. Let $d \in \mathbb{N}$ and $K \in \{1, \dots, d\}$. A subset S of $\mathbb{N}^d$ is called K-admissible if for every element $(n_1, \dots, n_d) \in S$, there exist $n_{i_1}, \dots, n_{i_K}$ such that

- $i_1 < i_2 < \cdots < i_K$ and $i_1, \dots, i_K \in \{1, \dots, d\}$;
- $\max\{n_1, \dots, n_d\} \leq C \min\{n_{i_1}, \dots, n_{i_K}\}$.

Here the constant C is independent of $(n_1, \dots, n_d)$.

Theorem 5.2. *Let $d, M_1, \ldots, M_d \in \mathbb{N}$, let $K \in \{1, \ldots, d\}$, and let $A_{M_1,\ldots,M_d}$ be the multilinear operator defined by setting $A_{M_1,\ldots,M_d}(f_1, \ldots, f_{d+1})(n)$ to be*

$$\frac{1}{M_1 \cdots M_d} \sum_{m_1=1}^{M_1} \cdots \sum_{m_d=1}^{M_d} f_1(n-m_1) \cdots f_d(n-m_d)\, f_{d+1}\big(n-(m_1^2+\cdots+m_d^2)\big)\,. \tag{5.1}$$

Here $n \in \mathbb{Z}$. Suppose T^* is the maximal function given by

$$T^*(f_1, \ldots, f_{d+1})(n) = \sup_{(M_1,\ldots,M_d)\in S_K} \big|A_{M_1,\ldots,M_d}(f_1, \ldots, f_{d+1})(n)\big|\,. \tag{5.2}$$

Here S_K is any K-admissible subset of $\mathbb{N}^d$. Then, if K satisfies

$$K > \frac{2d}{d+4}\,, \tag{5.3}$$

we have

$$\big\|T^*(f_1, \ldots, f_{d+1})\big\|_{L^2(\mathbb{Z})} \le C \prod_{j=1}^{d+1} \|f_j\|_{L^2(\mathbb{Z})}\,. \tag{5.4}$$

Here $L^2(\mathbb{Z})$ stands for the L^2 norm associated with counting measure on $\mathbb{Z}$, and C is independent of the f_j but may depend on K and d.

Remark 5.3. Notice that $2d/(d+4) < 1$ for $d = 1, 2, 3$. Thus the condition (5.3) is superfluous in Theorem 5.2 for $d = 1, 2, 3$. Thus for $d = 1, 2, 3$, the set S_K in Theorem 5.2 can be replaced by $\mathbb{N}^d$ because $\mathbb{N}^d$ is 1-admissible according to Definition 5.1. It is very possible that, for $d \ge 4$, the condition (5.3) on K is redundant too. A delicate analysis involving the circle method should be utilized in order to remove (5.3) for the $d \ge 4$ cases. We do not discuss this in this paper.

Remark 5.4. It is natural to ask whether the inequality

$$\big\|T^*(f_1, \ldots, f_{d+1})\big\|_{L^{\frac{2}{d+1}}(\mathbb{Z})} \le C \prod_{j=1}^{d+1} \|f_j\|_{L^2(\mathbb{Z})} \tag{5.5}$$

holds. This seems to be difficult but also interesting. So far we are only able to establish the boundedness of T^* from $L^2 \times \cdots \times L^2$ to L^p for $p > 2/(d+1)$ by an interpolation argument and Theorem 5.2.

To prove Theorem 5.2, we first introduce a simple multilinear estimate.

Lemma 5.5. *Let $M \in \mathbb{N}$ and let $F_1, \ldots, F_{M+1}$ be periodic functions on $\mathbb{T}$. Let $T(F_1, \ldots, F_{M+1})$ be the multilinear operator given by*

$$T(F_1, \ldots, F_{M+1})(x_1, \ldots, x_M) = F_1(x_1) \cdots F_M(x_M)\, F_{M+1}(x_1 + \cdots + x_M)\,, \tag{5.6}$$

for $(x_1, \ldots, x_M) \in \mathbb{T}^M$. If $1 \le p \le 2M/(M+1)$,

$$\big\|T(F_1, \ldots F_{M+1})\big\|_{L^p(\mathbb{T}^M)} \le \prod_{j=1}^{M+1} \|F_j\|_{L^2(\mathbb{T})}\,. \tag{5.7}$$

Proof. We only need to prove the case when $p = 2M/(M+1)$, since the other cases follow easily by the Hölder inequality. By a change of variables, we get

$$\big\|T(F_1, \dots F_{M+1})\big\|_{L^p(\mathbb{T}^M)} \le \|F_i\|_\infty \prod_{\substack{j\neq i \\ j\in\{1,\dots,M+1\}}} \|F_j\|_p\,, \tag{5.8}$$

for any $i \in \{1, \dots, M+1\}$. Now define $\alpha_1, \dots, \alpha_{M+1} \in \mathbb{Q}^{M+1}$ by

$$\alpha_1 = \Big(0, \frac{1}{p}, \dots, \frac{1}{p}\Big), \quad \alpha_2 = \Big(\frac{1}{p}, 0, \frac{1}{p}, \dots, \frac{1}{p}\Big), \;\dots\;, \;\alpha_{M+1} = \Big(\frac{1}{p}, \dots, \frac{1}{p}, 0\Big).$$

Clearly for $p = 2M/(M+1)$, we have

$$\Big(\frac{1}{2}, \dots, \frac{1}{2}\Big) = \frac{1}{M+1}\left(\alpha_1 + \cdots + \alpha_{M+1}\right). \tag{5.9}$$

Thus $(1/2, \dots, 1/2)$ is in the convex hull of $\alpha_1, \dots, \alpha_{M+1}$. The inequality (5.7) follows immediately by interpolation. □

To finish the proof of Theorem 5.2, we need the following proposition.

Proposition 5.6. *Let $d \in \mathbb{N}$, $1 \le K \le d$, and $M_{K+1}, \dots, M_d \in \mathbb{N}$. Define $A_{M,M_{K+1},\dots,M_d}$ by setting $A_{M,M_{K+1},\dots,M_d}(f_1, \dots, f_{d+1})(n)$ to be*

$$\begin{aligned} &\frac{1}{M^K M_{K+1}\cdots M_d}\Big(\prod_{j=1}^{K}\sum_{m_j=1}^{M} f_j(n-m_j)\Big) \\ &\qquad \times \Big(\prod_{j=K+1}^{d}\sum_{m_j=1}^{M_j} f_j(n-m_j)\Big) f_{d+1}\big(n-(m_1^2+\cdots+m_d^2)\big). \end{aligned} \tag{5.10}$$

Suppose that $M \ge C\max\{M_{K+1}, \dots, M_d\}$. Then we have

$$\begin{aligned} &\big\|A_{M,M_{K+1},\dots,M_d}(f_1, \dots, f_{d+1})\big\|_{L^2(\mathbb{Z})} \\ &\qquad \le C\,(M_{K+1}\cdots M_d)^{-\frac{d+4}{2(d+2)}}\, M^{\frac{-(d+4)K+2d}{2(d+2)}+\varepsilon}\prod_{j=1}^{d+1}\|f_j\|_{L^2(\mathbb{Z})}\,. \end{aligned} \tag{5.11}$$

Proof. By duality, it is sufficient to prove that, for any $f_{d+2} \in L^2(\mathbb{Z})$,

$$\begin{aligned} &\sum_n A_{M,M_{K+1},\dots,M_d}(n) f_{d+2}(n) \\ &\qquad \le C\big(M_{K+1}\cdots M_d\big)^{-\frac{d+4}{2(d+2)}} M^{\frac{-(d+4)K+2d}{2(d+2)}+\varepsilon}\prod_{j=1}^{d+2}\|f_j\|_{L^2(\mathbb{Z})}\,. \end{aligned} \tag{5.12}$$

Now define F_j for any $j \in \{1, \dots, d+2\}$ by

$$F_j(x) = \sum_n f_j(n)\, e^{2\pi i n x}\,. \tag{5.13}$$

Then the left-hand side of (5.12) can be represented by

$$\frac{1}{M^K M_{K+1}\cdots M_d}\int_{\mathbb{T}^{d+1}}\prod_{j=1}^{d+1}F_j(x_j)\,F_{d+2}(x_1+\cdots+x_{d+1}) \times S(x_1,\dots,x_{d+1})\,dx_1\cdots dx_{d+1}\,. \tag{5.14}$$

Here $S(x_1,\dots,x_{d+1})$ is given by

$$S(x_1,\dots,x_{d+1}) = \sum_{m_1=1}^{M}\cdots\sum_{m_K=1}^{M}\sum_{m_{K+1}=1}^{M_{K+1}}\cdots\sum_{m_d=1}^{M_d} e^{2\pi i(m_1x_1+\cdots+m_dx_d)}\,e^{2\pi i(m_1^2+\cdots+m_d^2)x_{d+1}}\,. \tag{5.15}$$

Utilizing Theorem 1.7, we have

$$\|S\|_{2(d+2)/d}\le C\,(M_{K+1}\cdots M_d)^{\frac{d}{2(d+2)}}\,M^{\frac{dK}{2(d+2)}+\frac{d}{d+2}+\varepsilon}\,.$$

Then the Hölder inequality yields that

$$(5.14)\le C\,\|T(F_1,\dots,F_{d+2})\|_{\frac{2(d+2)}{d+4}}\,(M_{K+1}\cdots M_d)^{-\frac{d+4}{2(d+2)}}\,M^{\frac{dK}{2(d+2)}+\frac{d}{d+2}-K+\varepsilon}\,.$$

Since $2(d+2)/(d+4)\le 2(d+1)/(d+2)$, we can apply Lemma 5.5 to obtain

$$(5.14)\le C\,(M_{K+1}\cdots M_d)^{-\frac{d+4}{2(d+2)}}\,M^{\frac{-(d+4)K+2d}{2(d+2)}+\varepsilon}\prod_{j=1}^{d+2}\|F_j\|_{L^2(\mathbb{T})}\,. \tag{5.16}$$

□

We now prove Theorem 5.2. Since S_K is K-admissible, without loss of generality, we assume that $M_1=\cdots=M_K=M$ and $M\ge C\max\{M_{K+1},\dots,M_d\}$. Moreover, we may also assume that M is dyadic. Hence we only need to consider $\tilde{T}^*(f_1,\dots,f_{d+1})$ given by

$$\tilde{T}^*(f_1,\dots,f_{d+1})=\sup_{M,M_{K+1},\dots,M_d}\left|A_{M,M_{K+1},\dots,M_d}(f_1,\dots,f_{d+1})\right|. \tag{5.17}$$

Clearly we have

$$\left|\tilde{T}^*(f_1,\dots,f_{d+1})\right|\le\Big(\sum_{M,M_{K+1},\dots,M_d}\left|A_{M,M_{K+1},\dots,M_d}(f_1,\dots,f_{d+1})\right|^2\Big)^{1/2}.$$

Taking the L^2 norm of both sides, we then get

$$\left\|\tilde{T}^*(f_1,\dots,f_{d+1})\right\|_{L^2(\mathbb{Z})}\le\Big(\sum_{M,M_{K+1},\dots,M_d}\left\|A_{M,M_{K+1},\dots,M_d}(f_1,\dots,f_{d+1})\right\|^2_{L^2(\mathbb{Z})}\Big)^{1/2}. \tag{5.18}$$

Employing Proposition 5.6, we estimate $\|\tilde{T}^*(f_1,\dots,f_{d+1})\|_{L^2(\mathbb{Z})}$ by

$$\Big(\sum_{M,M_{K+1},\dots,M_d}C\,(M_{K+1}\cdots M_d)^{-\frac{d+4}{(d+2)}}\,M^{\frac{-(d+4)K+2d}{(d+2)}+\varepsilon}\Big)^{1/2}\prod_{j=1}^{d+1}\|f_j\|_{L^2(\mathbb{Z})}\,, \tag{5.19}$$

which is bounded by

$$C \prod_{j=1}^{d+1} \|f_j\|_{L^2(\mathbb{Z})} ,$$

since $K > 2d/(d+4)$ implies $((d+4)K - 2d)/(d+2) > 0$. This completes the proof of Theorem 5.2.

Also we are able to obtain an L^2 estimate for the corresponding bilinear Hilbert transform.

Theorem 5.7. *Let K be a function on $\mathbb{Z}$ satisfying*

$$|K(n)| \le \frac{C}{|n|} \tag{5.20}$$

for $n \neq 0$. Let $T(f_1, f_2)$ be defined by

$$T(f_1, f_2)(n) = \sum_{m \neq 0} K(m)\, f_1(n-m)\, f_2(n-m^2) , \tag{5.21}$$

for Schwartz functions $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$. Then we have

$$\big\|T(f_1, f_2)\big\|_{L^2(\mathbb{Z})} \le C \,\big\|f_1\big\|_{L^2(\mathbb{Z})}\big\|f_2\big\|_{L^2(\mathbb{Z})} . \tag{5.22}$$

Proof. For any dyadic number $M \ge 1$, define $T_M(f_1, f_2)$ by

$$T_M(f_1, f_2)(n) = \frac{1}{M} \sum_{m \sim M} \big| f_1(n-m) f_2(n-m^2)\big| . \tag{5.23}$$

Apply Proposition 5.6 to get

$$\big\| T_M(f_1, f_2)\big\|_{L^2(\mathbb{Z})} \le M^{-1/2+\varepsilon} \big\| f_1\big\|_{L^2(\mathbb{Z})}\big\| f_2\big\|_{L^2(\mathbb{Z})} . \tag{5.24}$$

Then (5.22) follows from (5.24). □

Remark 5.8. If the kernel K in Theorem 5.7 has some cancellation condition, then $T(f_1, f_2)$ could be a bounded operator from $L^2 \times L^2$ to L^1. This problem is still open and seems to be challenging.

6. Estimate for $K_{p,d,N}$ when p is even

In this section, we give a bound on $K_{p,d,N}$ when p is even. The idea is not new, and it is used often in the field of number theory. For the sake of self-containedness, we include it here. By using it and an arithmetic argument, one can get sharp estimates, up to a factor of N^ε, for $K_{6,1,N}$, $K_{4,2,N}$, etc. See [1] for details.

Proposition 6.1. *If $p > 0$ is an even integer, then we have*

$$K_{p,d,N}^{p} \leq \sup_{(\mathbf{l},m)\in S_{d,pN/2}\times\{1,\dots,pN^2/2\}} e^{2\pi\varepsilon m}\,\mathcal{F}_{\mathbb{T}^d\times\mathbb{T}}\big(F^{p/2}(\cdot,\cdot+i\varepsilon)\big)\,(\mathbf{l},m)\,. \tag{6.1}$$

Here $\mathcal{F}_{\mathbb{T}^d\times\mathbb{T}}$ is the Fourier transform of functions on $\mathbb{T}^d\times\mathbb{T}$, ε is any positive number, and F is given by

$$F(\mathbf{x},z) = \sum_{\mathbf{n}\in\mathbb{Z}^d} e^{2\pi i z|\mathbf{n}|^2+2\pi i\mathbf{x}\cdot\mathbf{n}}\,. \tag{6.2}$$

Proof. Let $k = p/2$. A direct calculation yields

$$\int_{\mathbb{T}^{d+1}}\Big|\sum_{\mathbf{n}\in S_{d,N}} a_{\mathbf{n}}\,e^{2\pi i(\mathbf{n}\cdot\mathbf{x}+|\mathbf{n}|^2t)}\Big|^{2k} d\mathbf{x}\,dt = \sum_{(\mathbf{n}_1,\dots,\mathbf{n}_k,\mathbf{m}_1,\dots,\mathbf{m}_k)\in S_{d,N,k}} a_{\mathbf{n}_1}\dots a_{\mathbf{n}_k}\,\overline{a_{\mathbf{m}_1}}\dots\overline{a_{\mathbf{m}_k}}\,. \tag{6.3}$$

Here $S_{d,N,k}$ is given by

$$S_{d,N,k}=\Big\{\big(\mathbf{n}_1,\dots,\mathbf{n}_k,\mathbf{m}_1,\dots,\mathbf{m}_k\big)\in S_{d,N}^{p}:\sum_{j=1}^{k}\mathbf{n}_j=\sum_{j=1}^{k}\mathbf{m}_j\,,\sum_{j=1}^{k}|\mathbf{n}_j|^2=\sum_{j=1}^{k}|\mathbf{m}_j|^2\Big\}.$$

For any $\mathbf{l}\in S_{d,kN}$ and any positive integer $m \leq kN^2$, we set

$$S_k(\mathbf{l},m) = \Big\{\big(\mathbf{n}_1,\dots,\mathbf{n}_k\big)\in S_{d,N}^{k}:\sum_{j=1}^{k}\mathbf{n}_j=\mathbf{l},\ \sum_{j=1}^{k}|\mathbf{n}_j|^2=m\Big\}.$$

Now we can estimate (6.3) by

$$\sum_{\mathbf{l}\in S_{d,kN}}\sum_{m=1}^{kN^2}\Big|\sum_{(\mathbf{n}_1,\cdots,\mathbf{n}_k)\in S_k(\mathbf{l},m)} a_{\mathbf{n}_1}\cdots a_{\mathbf{n}_k}\Big|^2. \tag{6.4}$$

Utilizing the Cauchy–Schwarz inequality and the fact that $\{S_k(\mathbf{l},m)\}$ forms a partition of $S_{d,N}^k$, we dominate (6.4) by

$$\max_{\mathbf{l}\in S_{d,kN},1\leq m\leq kN^2}\big|S_k(\mathbf{l},m)\big|\,\Big(\sum_{\mathbf{n}}|a_{\mathbf{n}}|^2\Big)^k, \tag{6.5}$$

where $|S_k(\mathbf{l},m)|$ denotes the cardinality of $S_k(\mathbf{l},m)$.

Employing the elementary fact that $\int_0^1 e^{2\pi in\theta}d\theta = 0$ if $n\neq 0$ and $\int_0^1 e^{2\pi in\theta}d\theta = 1$ if $n = 0$, for any $\mathbf{l}\in S_{d,kN}$ and any positive integer $m \leq kN^2$ we can estimate $|S_k(\mathbf{l},m)|$ by

$$\sum_{(\mathbf{n}_1,\dots,\mathbf{n}_k)\in S_{d,N}^k}\int_0^1 e^{2\pi it(\sum_{j=1}^k|\mathbf{n}_j|^2-m)}\,dt\int_{\mathbb{T}^d} e^{2\pi i\sum_{j=1}^k\mathbf{x}\cdot\mathbf{n}_j}e^{-2\pi i\mathbf{x}\cdot\mathbf{l}}\,d\mathbf{x}\,, \tag{6.6}$$

which equals

(6.7)
$$\sum_{(\mathbf{n}_1,\dots,\mathbf{n}_k)\in S_{d,N}^k} e^{2\pi\varepsilon m}\int_0^1 e^{2\pi i(t+i\varepsilon)\sum_{j=1}^k |\mathbf{n}_j|^2}\, e^{-2\pi i m t}dt \int_{\mathbb{T}^d} e^{2\pi i \sum_{j=1}^k \mathbf{x}\cdot\mathbf{n}_j}\, e^{-2\pi i \mathbf{x}\cdot\mathbf{l}}\, d\mathbf{x}\,,$$

for any real number ε. This term can also be written as

$$e^{2\pi\varepsilon m}\int_{\mathbb{T}^d\times\mathbb{T}} \Big(\sum_{\mathbf{n}\in S_{d,N}} e^{2\pi i(t+i\varepsilon)|\mathbf{n}|^2}\, e^{2\pi i \mathbf{x}\cdot\mathbf{n}}\Big)^k e^{-2\pi i\mathbf{x}\cdot\mathbf{l}}\, e^{-2\pi i m t}\, d\mathbf{x}\, dt\,. \tag{6.8}$$

Notice that we may replace $S_{d,N}$ by $\mathbb{Z}^d$ in (6.6), (6.7), and (6.8) to make the upper bounds larger. Thus, by the definition of F in (6.2), we dominate $|S_k(\mathbf{l},m)|$ by

$$e^{2\pi\varepsilon m}\int_{\mathbb{T}^d\times\mathbb{T}} \big(F(\mathbf{x},t+i\varepsilon)\big)^k e^{-2\pi i\mathbf{x}\cdot\mathbf{l}}\, e^{-2\pi i m t}\, d\mathbf{x}\, dt\,. \tag{6.9}$$

This finishes the proof of Proposition 6.1. □

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