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Linear multifractional stable motion: fine path properties

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Abstract. For at least a decade, there has been considerable interest in applied and theoretical issues related to multifractional random models. Nonetheless, only a few results are known in the framework of heavy-tailed stable distributions. In this framework, a paradigmatic example is the linear multifractional stable motion (LMSM) $\{Y(t)\colon t\in\mathbb{R}\}$. Stoev and Taqqu [30], [29] introduced LMSM by replacing the constant Hurst parameter of classical linear fractional stable motion (LFSM) by a deterministic function $H(\cdot)$ depending on the time variable t. The main goal of our article is to make a comprehensive study of the local and asymptotic behavior of $\{Y(t) : t\in\mathbb{R}\}$. To this end, one needs to derive fine path properties of $\{X(u,v)\colon (u,v)\in\mathbb{R}\times(1/\alpha,1)\}$, the field generating the process (i.e., one has $Y(t)=X(t,H(t))$ for all $t\in\mathbb{R}$). This leads us to introduce random wavelet series representations of $\{X(u,v)\colon (u,v)\in\mathbb{R}\times(1/\alpha,1)\}$ as well as of all its pathwise partial derivatives of any order with respect to v. Then our strategy consists in using wavelet methods which are reminiscent of those in [2], [5]. Among other things, we solve a conjecture of Stoev and Taqqu concerning the existence for LMSM of a version with almost surely continuous paths; moreover we significantly improve Theorem 4.1 in [29], which provides some bounds for the local Hölder exponent (in other words, the uniform pointwise Hölder exponent) of LMSM. Namely, we obtain a quasi-optimal global modulus of continuity, and also an optimal local one. It is worth noticing that, even in the quite classical case of LFSM, the optimal local modulus of continuity provides a new result which was previously not known.

1. Introduction

For at least a decade, there has been considerable interest in applied and theoretical issues related to multifractional random models (see, for instance, [4], [3], [1], [6], [9], [8], [7], [11], [17], [16], [14], [15], [18], [19], [21], [20], [22], [25], [30], [29], [28], [32]).

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These fractal nonstationary increments stochastic processes/fields, are natural extensions of the well-known fractional Brownian motion (FBM). They have richer path behavior than FBM and they are more widely applicable because their local properties, typically the index governing self-similarity as well as the degree of path roughness, can be controlled via a nonconstant functional Hurst parameter and thus are allowed to change with location. In the Gaussian case, and more generally when all moments are finite, many results concerning the path behavior of such random models have been derived in the literature; but much less is known in the framework of heavy-tailed stable distributions. A paradigmatic example of a multifractional process in such a setting, is the so-called linear multifractional stable motion (LMSM), which was introduced by Stoev and Taqqu in [30], [29]. According to these two authors (see page 1086 in [30]): "a LMSM model is a good candidate to adequately describe some features of traffic traces on telecommunication networks, typically changes in operating regimes and burstiness (the presence of rare but extremely busy periods of activity)."

In order to define LMSM precisely, we fix some notation to be used throughout the article:

- Recall that heaviness of the tail of a stable distribution is governed by a constant parameter belonging to the open interval $(0, 2)$, usually denoted by α; the smaller α is, the heavier the tail is. In the present article, we always assume that $\alpha \in (1, 2)$, since it has been shown in [30] that this assumption is actually a necessary condition for the paths of LMSM to be, with probability 1, continuous functions.
- $H(\cdot)$ denotes an arbitrary deterministic continuous function defined on $\mathbb{R}$ and with values in an arbitrary fixed compact interval $[\underline{H}, \overline{H}] \subset (1/\alpha, 1)$. Similar to the constant Hurst parameter of FBM, this function will be an essential parameter for LMSM.
- $Z_\alpha(ds)$ is an independently scattered strictly α-stable ($\mathcal{S}t\alpha\mathcal{S}$) random measure on $\mathbb{R}$, with Lebesgue measure as its control measure and an arbitrary Borel function $\beta(\cdot) : \mathbb{R} \to [-1, 1]$ as its skewness intensity. Much information on such random measures and the corresponding stochastic integrals can be found in the book [27].

LMSMs are generated by the $\mathcal{S}t\alpha\mathcal{S}$ random field $\widetilde{X} = \{\widetilde{X}(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$, defined for all (u, v) as the stochastic integral

$$\widetilde{X}(u, v) = \int_{\mathbb{R}} \left\{ (u - s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha} \right\} Z_\alpha(ds), \tag{1.1}$$

with the convention that, for all real numbers x and κ,

$$(x)_+^\kappa := \begin{cases} x^\kappa, & \text{if } x \in (0, +\infty), \\ 0, & \text{if } x \in (-\infty, 0]. \end{cases} \tag{1.2}$$

Actually, $\widetilde{Y} = \{\widetilde{Y}(t) : t \in \mathbb{R}\}$, the LMSM of functional Hurst parameter $H(\cdot)$, is defined for every $t \in \mathbb{R}$, as

$$\widetilde{Y}(t) = \widetilde{X}(t, H(t)). \tag{1.3}$$

Observe that, if $\beta(\cdot)$ is assumed to be a constant function, then for each fixed $v \in (1/\alpha, 1)$, the process $\widetilde{X}(\cdot, v) := \{\widetilde{X}(u, v) : (u, v) \in \mathbb{R}\}$ is the usual linear fractional stable motion (LFSM) of Hurst parameter v; therefore LMSM reduces to the latter process when one also assumes $H(\cdot)$ to be a constant function. We note in passing that LFSM and harmonizable fractional stable motion (HFSM) are two classical self-similar stable processes with stationary increments. They regarded as the most two natural extensions of FBM to the setting of heavy-tailed distributions. Also, we note that, in contrast with moving average and harmonizable representations of FBM in the Gaussian framework, the path behavior of LFSM is considerably more irregular and more complex than that of HFSM. We refer to [27], [33], [13] for a detailed presentation of these two processes, as well as other classical examples of stable processes. Before ending this paragraph, let us mention that a harmonizable multifractional stable process, which extends HFSM and thus behaves very differently from LMSM, has been introduced quite recently in [11].

The main goal of our paper is to make a comprehensive study of the local and asymptotic behavior of LMSM, under the quite general condition that its parameter $H(\cdot)$ is an arbitrary deterministic continuous function with values in an arbitrary fixed compact interval $[\underline{H}, \overline{H}] \subset (1/\alpha, 1)$. This study mainly relies on wavelet methods which are reminiscent of those in [2], [5]. As we will explain more precisely soon, among other things, we significantly improve two earlier results of Stoev and Taqqu [29], concerning continuity and path behavior of LMSM. On the other hand, even in the quite classical case of LFSM (in other words, in the particular case where $\beta(\cdot)$ and $H(\cdot)$ are constant), the determination of the optimal lower bound for the power of the logarithmic factor in a local modulus of continuity has so far been an open problem. Corollary 5.6 and Theorem 7.1 in the present article solve it, in the more general case of LMSM, by showing that $1/\alpha$ is in fact this optimal lower bound.

We give now the precise statements of the two results in [29] we have just mentioned.

1. **Theorem 3.2 in [29]** (the existence for LMSM of a version whose paths are, with probability 1, Hölder continuous functions). *Let $I' \subset I$ be arbitrary nonempty bounded subintervals of $\mathbb{R}$, which are respectively closed and open; suppose that $1/\alpha < H(t) < 1$, for each $t \in I$ and that, for all $t', t'' \in I$,*

$$\big|H(t') - H(t'')\big| \le c\,|t' - t''|^{\rho}, \quad \text{with } 1/\alpha < \rho, \tag{1.4}$$

where the constants c and ρ do not depend on t' and t''. Then the LMSM $\{\widetilde{Y}(t) : t \in \mathbb{R}\}$ *has a version $\{Y(t) : t \in \mathbb{R}\}$, whose paths are, with probability 1, continuous functions on I. Moreover, they are Hölder functions on I', with a uniform Hölder exponent (see (8.1)) $\rho_Y^{\mathrm{unif}}(I')$ satisfying*

$$\rho_Y^{\mathrm{unif}}(I') \ge \Big(\rho \wedge \min_{t \in I'} H(t)\Big) - 1/\alpha.$$

2. **Theorem 4.1 in [29]** (local Hölder exponent –in other words, uniform pointwise Hölder exponent– of LMSM). *Assume that $H(\cdot)$ is continuous, with*

values in $(1/\alpha, 1)$ *and satisfies,* $\rho_H^{\mathrm{unif}}(t) > 1/\alpha$ *for all* $t \in \mathbb{R}$*, where* $\rho_H^{\mathrm{unif}}(t)$ *denotes the local Hölder exponent* (*see* (8.3)) *of* $H(\cdot)$ *at* t*. Then,* $\rho_Y^{\mathrm{unif}}(t_0)$*, the local Hölder exponent of the* LMSM $\{Y(t) : t \in \mathbb{R}\}$ *at an arbitrary point* $t_0 \neq 0$*, can be almost surely bounded, in the following way:*

$$\rho_H^{\mathrm{unif}}(t_0) \wedge H(t_0) - 1/\alpha \leq \rho_Y^{\mathrm{unif}}(t_0) \leq \rho_H(t_0) \wedge H(t_0). \tag{1.5}$$

Here $\rho_H(t_0)$ *denotes the pointwise Hölder exponent at* t_0 (*see, e.g., Definition* 4.1 *in* [29]) *of the function* $H(\cdot)$*.*

The first of the two previous theorems and the first inequality in (1.5) have been derived in [29] by using the strong version of Kolmogorov's continuity criterion provided by Theorem 3.3.16 in [31], for instance. On the other hand, the proof given in [29] for the second inequality in (1.5) relies mainly on the inequality $\rho_Y^{\mathrm{unif}}(t_0) \leq \rho_Y(t_0)$ as well as relations (4.11) and (4.12) in [29]. Using a different strategy, namely wavelet methods which are reminiscent of those in [2], [5], in the present work, we have been able to improve Theorems 3.2 and 4.1 in [29]. More precisely:

1. The condition (1.4) seems to be too strong if one is only interested in the existence of a version of LMSM with almost surely continuous paths. Namely, in their Remark 1 on page 166 in [29], Stoev and Taqqu have conjectured that such a version should exist as long as $H(\cdot)$ is a continuous function with values in $(1/\alpha, 1)$. This conjecture is solved in our article. To solve it, we construct $X = \{X(u, v) \colon (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ a version with almost surely continuous paths, of the field $\{\widetilde{X}(u, v) \colon (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ which generates LMSMs. In fact $\{X(u, v) \colon (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ is obtained as a random series of functions, resulting from the decomposition of the kernel in (1.1) into a Daubechies wavelet basis (see Theorem 2.1). Thus, denoting by $\{Y(t) \colon t \in \mathbb{R}\}$ the version of LMSM defined for each $t \in \mathbb{R}$ by $Y(t) := X(t, H(t))$, it is clear that the paths of the process $\{Y(t) \colon t \in \mathbb{R}\}$ are continuous with probability 1, as long as $H(\cdot)$ is a continuous function on the real line and with values in $(1/\alpha, 1)$. Observe that at this stage, we do not need to restrict the range of $H(\cdot)$ to the compact interval $[\underline{H}, \overline{H}]$.

2. Theorem 8.1 in the present article shows that, almost surely, for any $t_0 \in \mathbb{R}$ satisfying $\rho_H^{\mathrm{unif}}(t_0) > 1/\alpha$, one has, $\rho_Y^{\mathrm{unif}}(t_0) = H(t_0) - 1/\alpha$. Observe that the exceptional negligible event on which this equality fails to be true does not depend on t_0. Also observe that this equality remains valid even in the case where $t_0 = 0$.

The rest of the paper is structured in the following way. Section 2 is devoted to the construction of the version $\{X(u, v) \colon (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ of the field $\{\widetilde{X}(u, v) \colon (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ which generates LMSMs. As we have already pointed out, this version is in fact a random series of functions, resulting from the decomposition of the kernel in (1.1) into a Daubechies wavelet basis. In Section 3, we show that this series and all its term-by-term pathwise partial derivatives of any order with respect to v are convergent in a very strong sense: with probability 1, in the space $\mathcal{E}_\gamma(a,b,M) := \mathcal{C}^1\big([a,b],\mathcal{C}^\gamma([-M,M],\mathbb{R})\big)$, where the real numbers $M>0$, $0<1/\alpha<a<b<1$ and $0\le\gamma<a-1/\alpha$ are arbitrary and fixed, and where $\mathcal{C}^\lambda(I,\mathbb{B})$ denotes the space of λ-Hölder functions defined on an interval I and with values in the Banach space $\mathbb{B}$. Notice that an important consequence of this result is that, for each $q\in\mathbb{Z}_+$, a typical path of the field $\{(\partial_v^q X)(u,v) : (u,v)\in\mathbb{R}\times(1/\alpha,1)\}$ belongs to $\mathcal{E}_\gamma(a,b,M)$. Thus, not only is such a path a continuous function but also it has much better properties than continuity. In Section 4, fine path properties of the field $\{(\partial_v^q X)(u,v)\colon (u,v)\in\mathbb{R}\times(1/\alpha,1)\}$ are derived using wavelet methods. Namely we determine a global modulus of continuity on the rectangle $[-M,M]\times[a,b]$; also we give an upper bound for $\big|(\partial_v^q X)(u,v)\big|$, on the domain $(u,v)\in\mathbb{R}\times[a,b]$. These two results are used in Section 5, to obtain global and local moduli of continuity for the LMSM $\{Y(t)\colon t\in\mathbb{R}\}$. The optimality of some of these moduli of continuity is discussed in Sections 6 and 7; under some Hölder conditions on $H(\cdot)$, it turns out that the global modulus is quasi-optimal (it provides, up to a logarithmic factor, a sharp estimate of the behavior of $\{Y(t)\colon t\in\mathbb{R}\}$, on an arbitrary fixed compact interval) and the local modulus is optimal (it provides, without any logarithmic gap, a sharp estimate of the behavior of $\{Y(t)\colon t\in\mathbb{R}\}$ on a neighborhood of an arbitrary fixed point). In Section 8, by making use of the quasi-optimality of the global modulus of continuity of LMSM, we determine its local Hölder exponent. Finally, some technical lemmas as well as their proofs are given in Section 9 (the appendix).

2. Wavelet series version of the field generating LMSMs

Let $\widetilde{X}=\{\widetilde{X}(u,v) : (u,v)\in\mathbb{R}\times(1/\alpha,1)\}$ be the $\mathcal{S}t\alpha\mathcal{S}$ stochastic field introduced in (1.1). The goal of this section is to construct a version of $\widetilde{X}$, denoted by X, which is defined as a random wavelet series. We note in passing that random wavelet series version of LFSM and other self-similar stable fields with stationary increments have been introduced in [12].

First, we fix some notation related to wavelets that will be used extensively throughout the article:

- The real-valued function ψ defined on the real line, is a 3 times continuously differentiable compactly supported Daubechies mother wavelet [10], [24], [23]. Observe that ψ has $Q\ge 15$ vanishing moments:

$$\int_{\mathbb{R}} t^m\psi(t)\,dt=0,\quad \text{for all } m=0,\dots,Q-1,\quad \text{and}\quad \int_{\mathbb{R}} t^Q\psi(t)\,dt\neq 0. \tag{2.1}$$

  The fact that ψ is a compactly supported function will play a crucial role. For the sake of convenience, we assume that R is a fixed real number strictly bigger than 1, such that

$$\operatorname{supp}\psi\subseteq[-R,R]. \tag{2.2}$$

- The real-valued function Ψ is defined for all $(x,v) \in \mathbb{R} \times (1/\alpha, 1)$ as,

$$\Psi(x,v) := \int_{\mathbb{R}} (s)_+^{v-1/\alpha} \psi(x-s)\, ds = \int_{\mathbb{R}} (x-s)_+^{v-1/\alpha} \psi(s)\, ds. \tag{2.3}$$

  Recall that $(\cdot)_+^{v-1/\alpha}$ is defined in (1.2). Denoting by Γ the usual Gamma function

$$\Gamma(u) := \int_0^{+\infty} t^{u-1}\, e^{-t}\, dt, \quad \text{for all } u \in (0, +\infty),$$

  and denoting, for each fixed v, by $\widehat{\Psi}(\cdot, v)$ the Fourier transform of the function $\Psi(\cdot, v)$,

$$\widehat{\Psi}(\xi, v) := \int_{\mathbb{R}} e^{-i\xi x}\, \Psi(x,v)\, dx, \quad \text{for all } \xi \in \mathbb{R},$$

  one has

$$\widehat{\Psi}(\xi, v) = \Gamma(v+1-1/\alpha) \frac{e^{-i\,\mathrm{sgn}(\xi)(v+1-1/\alpha)\frac{\pi}{2}}\, \widehat{\psi}(\xi)}{|\xi|^{v+1-1/\alpha}}, \quad \text{for all } \xi \in \mathbb{R} \setminus \{0\}. \tag{2.4}$$

  This equality can be obtained by using a result in [26] concerning Fourier transforms of left-sided fractional derivatives.

- $\{\epsilon_{j,k} : (j,k) \in \mathbb{Z}^2\}$ is the sequence of real-valued $\mathcal{S}t\alpha\mathcal{S}$ random variables defined by

$$\epsilon_{j,k} := 2^{j/\alpha} \int_{\mathbb{R}} \psi(2^j s - k)\, Z_\alpha(ds). \tag{2.5}$$

Now we are ready to state the main result of this section.

Theorem 2.1. *Let Ψ be the function defined in (2.3), let $\{\epsilon_{j,k} : (j,k) \in \mathbb{Z}^2\}$ be the sequence of real-valued $\mathcal{S}t\alpha\mathcal{S}$ random variables defined in (2.5), and let Ω_0^* be the event of probability 1 introduced in Lemma 2.3 below. The following two results hold.*

(i) *For all fixed $\omega \in \Omega_0^*$ and $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, one has*

$$\sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv} \big|\epsilon_{j,k}(\omega)\big| \, \big|\Psi(2^j u - k, v) - \Psi(-k, v)\big| < \infty. \tag{2.6}$$

Therefore, the series

$$\sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv} \epsilon_{j,k}(\omega) \big(\Psi(2^j u - k, v) - \Psi(-k, v)\big), \tag{2.7}$$

converges to a finite limit which does not depend on the way the terms of the series are ordered; this limit is denoted by $X(u,v,\omega)$. Moreover for each $\omega \notin \Omega_0^*$ and every $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, one sets $X(u,v,\omega) = 0$.

(ii) *The field* $\{X(u,v)\colon (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ *is a version of the* $\mathcal{S}t\alpha\mathcal{S}$ *field* $\{\widetilde{X}(u,v) : (u,v) \in \mathbb{R} \times (1/\alpha, 1)\}$ *defined in* (1.1).

In order to prove Theorem 2.1, we need some preliminary results.

Remark 2.2. (i) The scale parameter $\|\epsilon_{j,k}\|_\alpha$ of $\epsilon_{j,k}$ does not depend on (j,k), since standard computations show

$$\|\epsilon_{j,k}\|_\alpha = \|\epsilon_{0,0}\|_\alpha = \Big\{ \int_{\mathbb{R}} |\psi(t)|^\alpha \, dt \Big\}^{1/\alpha}. \tag{2.8}$$

(ii) The skewness parameter of $\epsilon_{j,k}$ is denoted by $\beta_{j,k}$ and is given by

$$\beta_{j,k} = \|\epsilon_{0,0}\|_\alpha^{-\alpha} \int_{\mathbb{R}} \psi^{<\alpha>}(x)\, \beta(2^{-j}x + 2^{-j}k)\, dx,$$

where $z^{<\alpha>} := |z|^\alpha \operatorname{sgn}(z)$ for all $z \in \mathbb{R}$, and where $\beta(\cdot)$ is the skewness intensity function of the $\mathcal{S}t\alpha\mathcal{S}$ measure $Z_\alpha(ds)$. Notice that, when this function is constant, then the random variables $\epsilon_{j,k}$ are identically distributed, since, in addition to having the same scale parameter, they have the same skewness parameter.

(iii) Combining Properties 1.2.15 and 1.2.13 on page 16 of [27] with the fact that $\|\epsilon_{j,k}\|_\alpha$ does not vanish and does not depend on (j,k), it follows that there exist two constants $0 < c' \le c''$, not depending on (j,k), such that one has, for all real numbers $x \ge 1$,

$$c'\, x^{-\alpha} \le \mathbb{P}\big(|\epsilon_{j,k}| > x\big) \le c''\, x^{-\alpha}. \tag{2.9}$$

(iv) In view of (2.5), (2.2), and the fact that $Z_\alpha(ds)$ is independently scattered, for each fixed integers $p > 2R$ and $j \in \mathbb{Z}$, one has that $\{\epsilon_{j,pq} : q \in \mathbb{Z}\}$ is a sequence of independent random variables.

The following lemma, which was derived in [2], gives rather sharp estimates of the asymptotic behavior of the sequence $\big\{|\epsilon_{j,k}| \,:\, (j,k) \in \mathbb{Z}^2\big\}$. It can be proved by showing that for every fixed real number $\eta > 0$, one has

$$\mathbb{E}\Bigg(\sum_{(j,k)\in\mathbb{Z}^2} \mathbb{1}_{\big\{|\epsilon_{j,k}| > (1+|j|)^{1/\alpha+\eta}(1+|k|)^{1/\alpha} \log^{1/\alpha+\eta}(2+|k|)\big\}} \Bigg) < \infty.$$

This follows easily from the second inequality in (2.9).

Lemma 2.3 ([2]). *There exists an event of probability* 1*, denoted by* Ω_0^**, such that for every fixed real number* $\eta > 0$*, one has, for all* $\omega \in \Omega_0^*$ *and for each* $(j,k) \in \mathbb{Z}^2$*,*

$$\begin{aligned} \big|\epsilon_{j,k}(\omega)\big| &\le C(\omega)\, \big(1+|j|\big)^{1/\alpha+\eta} \big(1+|k|\big)^{1/\alpha} \log^{1/\alpha+\eta}\big(2+|k|\big) \\ &\le C'(\omega)\, \big(3+|j|\big)^{1/\alpha+\eta} \big(3+|k|\big)^{1/\alpha+\eta}, \end{aligned} \tag{2.10}$$

where C *and* C' *are positive and finite random variables depending only on* η*.*

The following proposition, which shows that the function Ψ and its partial derivatives of any order, have nice smoothness and localization properties, will also play an important role throughout this article.

Proposition 2.4. *The function Ψ satisfies the following two properties.*

(i) *For all $(p,q) \in \{0,1,2,3\} \times \mathbb{Z}_+$ and $(x,v) \in \mathbb{R} \times (1/\alpha, 1)$, the partial derivative $(\partial_x^p \partial_v^q \Psi)(x,v)$ exists and is given by*

$$\begin{aligned}(\partial_x^p \partial_v^q \Psi)(x,v) &= \int_{\mathbb{R}} (s)_+^{v-1/\alpha} \log^q((s)_+)\, \psi^{(p)}(x-s)\, ds \\ &= \int_{\mathbb{R}} (x-s)_+^{v-1/\alpha} \log^q((x-s)_+)\, \psi^{(p)}(s)\, ds, \qquad (2.11)\end{aligned}$$

where $\psi^{(p)}$ is the derivative of ψ of order p and $0 \log^q(0) := 0$. Moreover the function $\partial_x^p \partial_v^q \Psi$ is continuous on $\mathbb{R} \times (1/\alpha, 1)$.

(ii) *For each $(p,q) \in \{0,1,2,3\} \times \mathbb{Z}_+$ and for every real numbers a and b satisfying $1 > b > a > 1/\alpha$, the function $\partial_x^p \partial_v^q \Psi$ is well localized in the variable x uniformly in the variable $v \in [a,b]$; namely one has*

$$\sup_{(x,v)\in\mathbb{R}\times[a,b]} (3+|x|)^2 \left|(\partial_x^p \partial_v^q \Psi)(x,v)\right| < \infty. \qquad (2.12)$$

Proof. First we show (i). In view of (2.3), the function Ψ can be expressed, for all $(x,v) \in \mathbb{R} \times (1/\alpha, 1)$, as

$$\Psi(x,v) = \int_{\mathbb{R}} L(x,v,s)\, ds.$$

where $L(x,v,s) := (s)_+^{v-1/\alpha} \psi(x-s)$. Also observe that for all $(p,q) \in \{0,\dots,3\} \times \mathbb{Z}_+$ and $(x,v,s) \in \mathbb{R} \times (1/\alpha,1) \times \mathbb{R}$, the partial derivative $(\partial_x^p \partial_v^q L)(x,v,s)$ exists and is given by

$$(\partial_x^p \partial_v^q L)(x,v,s) = (s)_+^{v-1/\alpha} \log^q((s)_+)\, \psi^{(p)}(x-s). \qquad (2.13)$$

Therefore, to show that the partial derivative $(\partial_x^p \partial_v^q \Psi)(x,v)$ exists and is given by (2.11), it is sufficient to prove that for all real numbers M, a, and b, satisfying

$$M > 0 \quad \text{and} \quad 1/\alpha < a < b < 1, \qquad (2.14)$$

one has

$$\int_{\mathbb{R}} \sup_{(x,v)\in[-M,M]\times[a,b]} \left|(\partial_x^p \partial_v^q L)(x,v,s)\right| ds < \infty. \qquad (2.15)$$

This is true, since (2.13), (2.2), and (2.14) imply that

$$\begin{aligned}&\int_{\mathbb{R}} \sup_{(x,v)\in[-M,M]\times[a,b]} \left|(\partial_x^p \partial_v^q L)(x,v,s)\right| ds \\ &\le \|\psi^{(p)}\|_{L^\infty(\mathbb{R})} \int_{-M-R}^{M+R} \left((s)_+^{a-1/\alpha} + (s)_+^{b-1/\alpha}\right) \left|\log((s)_+)\right|^q ds < \infty. \qquad (2.16)\end{aligned}$$

Finally, observe that it follows from (2.11), (2.13), (2.15), and the dominated convergence theorem, that for all $(p,q) \in \{0,\dots,3\} \times \mathbb{Z}_+$, the function $\partial_x^p \partial_v^q \Psi$ is continuous on $\mathbb{R} \times (1/\alpha, 1)$.

We show part (ii) of the proposition. The relations (2.2) and (2.11) imply that, for all $(p,q) \in \{0,\dots,3\} \times \mathbb{Z}_+$ and for each $(x,v) \in (-\infty,-R) \times (1/\alpha,1)$, one has

$$(\partial_x^p \partial_v^q \Psi)(x,v) = 0. \tag{2.17}$$

Combining (2.17) with the fact $\partial_x^p \partial_v^q \Psi$ is continuous on the compact set $[-R,2R] \times [a,b]$, it follows that

$$\sup_{(x,v)\in(-\infty,2R]\times[a,b]} (3+|x|)^2 \left|(\partial_x^p \partial_v^q \Psi)(x,v)\right| < \infty.$$

Therefore, there remains to show

$$\sup_{(x,v)\in(2R,+\infty)\times[a,b]} (3+x)^2 \left|(\partial_x^p \partial_v^q \Psi)(x,v)\right| < \infty. \tag{2.18}$$

In view of (2.11) and (2.2), one has, for each $(x,v) \in (2R,+\infty) \times [a,b]$,

$$(\partial_x^p \partial_v^q \Psi)(x,v) = \int_{-R}^{R} K_q(x,v,s)\, \psi^{(p)}(s)\, ds,$$

where

$$K_q(x,v,s) := (x-s)^{v-1/\alpha} \log^q(x-s).$$

For each $l \in \{1,2,3\}$ and real number s, one sets

$$\psi^{(p-l)}(s) = \int_{-\infty}^{s} \psi^{(p+1-l)}(t)\, dt;$$

observe that, in view of (2.1) and (2.2), the supports of these three functions are included in $[-R,R]$. Thus, integrating three times by parts, one gets that

$$(\partial_x^p \partial_v^q \Psi)(x,v) = -\int_{-R}^{R} (\partial_s^3 K_q)(x,v,s)\, \psi^{(p-3)}(s)\, ds. \tag{2.19}$$

Next, standard computations show that there is a constant $c_{q,\alpha} > 0$, depending only on q and α, such that for all $(x,v,s) \in (2R,+\infty) \times [a,b] \times [-R,R]$, one has

$$\left|(\partial_s^3 K_q)(x,v,s)\right| \le c_{q,\alpha}\, (x-s)^{-2} \le 4c_{q,\alpha}\, x^{-2}. \tag{2.20}$$

Finally, combining (2.19) and (2.20), one obtains (2.18). □

Now we are prepared to prove Theorem 2.1.

Proof of Theorem 2.1, *Part* (i). Let $\omega \in \Omega_0^*$ and $(u,v) \in \mathbb{R} \times (1/\alpha,1)$ be arbitrary and fixed. Also, we assume that η is an arbitrarily small fixed positive real number. By using the triangle inequality, (2.12) (in which one takes $p=q=0$ and a and b

such that $v \in [a,b]$), and (2.10), it follows that for all fixed $j \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^j u-k,v)-\Psi(-k,v)\big| \\
&\quad \le C_1(\omega)\big(3+j\big)^{1/\alpha+\eta}\sum_{k\in\mathbb{Z}}\Big(\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|2^ju-k|\big)^2}+\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|k|\big)^2}\Big) \\
&\quad \le C_2(\omega)\big(3+j\big)^{1/\alpha+\eta}\big(3+2^j|u|\big)^{1/\alpha+\eta} \\
&\qquad \times \sum_{k\in\mathbb{Z}}\Big(\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|2^ju-[2^ju]-k|\big)^2}+\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|k|\big)^2}\Big), \qquad (2.21)
\end{aligned}$$

where $[2^ju]$ denotes the integer part of 2^ju and where $C_1(\omega)$ and $C_2(\omega)$ are two finite constants not depending on j and u. Then, noticing that

$$\sup_{x\in[0,1]}\Big\{\sum_{k\in\mathbb{Z}}\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(3+|x-k|\big)^2}\Big\} \le \sum_{k\in\mathbb{Z}}\frac{\big(3+|k|\big)^{1/\alpha+\eta}}{\big(2+|k|\big)^2}<\infty, \qquad (2.22)$$

it follows from (2.21) that

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}}2^{-jv}\,\big|\epsilon_{j,k}(\omega)\big|\,\big|\Psi(2^ju-k,v)-\Psi(-k,v)\big|<\infty. \qquad (2.23)$$

We now prove that

$$\sum_{j\in\mathbb{Z}_-}\sum_{k\in\mathbb{Z}}2^{-jv}\,\big|\epsilon_{j,k}(\omega)\big|\,\big|\Psi(2^ju-k,v)-\Psi(-k,v)\big|<\infty. \qquad (2.24)$$

Applying the mean value theorem, one has for all $(j,k)\in\mathbb{Z}_-\times\mathbb{Z}$,

$$\Psi(2^ju-k,v)-\Psi(-k,v)=2^ju(\partial_x\Psi)(\nu-k,v), \qquad (2.25)$$

where $\nu\in[-2^j|u|,2^j|u|]\subseteq[-|u|,|u|]$. Putting together (2.25), (2.10) and (2.12) (in which one takes $p=1$, $q=0$, and a and b such that $v\in[a,b]$), one obtains that,

$$\begin{aligned}
&\sum_{|k|\le|u|}\big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^ju-k,v)-\Psi(-k,v)\big| \\
&\le C_3(\omega)|u|\big(2|u|+1\big)\big(3+|u|\big)^{1/\alpha+\eta}\Big(\sup_{x\in\mathbb{R}}\big|(\partial_x\Psi)(x,v)\big|\Big)2^j\big(3+|j|\big)^{1/\alpha+\eta} \qquad (2.26)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{|k|>|u|}\big|\epsilon_{j,k}(\omega)\big|\big|\Psi(2^ju-k,v)-\Psi(-k,v)\big| \\
&\quad\le C_4(\omega)|u|\big(3+|u|\big)^{1/\alpha+\eta}\Big(\sum_{k\in\mathbb{Z}}\big(3+|k|\big)^{1/\alpha+\eta-2}\Big)2^j\big(3+|j|\big)^{1/\alpha+\eta}, \qquad (2.27)
\end{aligned}$$

where $C_3(\omega)$ and $C_4(\omega)$ are two positive finite constants not depending on j and u. Next combining (2.26) and (2.27) with the fact that $v\in(1/\alpha,1)$, one gets (2.24). Finally (2.23) and (2.24) show that (2.6) holds. □

Proof of Theorem 2.1, *Part* (ii). For all $(j,k) \in \mathbb{Z}^2$ and any $s \in \mathbb{R}$, we set

$$\psi_{j,k}(s) = 2^{j/\alpha}\, \psi(2^j s - k), \tag{2.28}$$

where ψ is the Daubechies mother wavelet introduced at the very beginning of this section. Observe that the sequence $\{\psi_{j,k} : (j,k) \in \mathbb{Z}^2\}$ forms an unconditional basis of $L^\alpha(\mathbb{R})$ and the sequence $\{2^{j(1/2-1/\alpha)}\psi_{j,k} : (j,k) \in \mathbb{Z}^2\}$ is an orthonormal basis of $L^2(\mathbb{R})$ (see [24], [23]). Therefore, noticing that, for any fixed $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, the function $s \mapsto (u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha}$ belongs to $L^\alpha(\mathbb{R}) \cap L^2(\mathbb{R})$, it follows that

$$(u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha} = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} w_{j,k}(u,v)\, \psi_{j,k}(s), \tag{2.29}$$

where

$$\begin{aligned} w_{j,k}(u,v) &:= 2^{j(1-1/\alpha)} \int_{\mathbb{R}} \left\{ (u-s)_+^{v-1/\alpha} - (-s)_+^{v-1/\alpha} \right\} \psi(2^j s - k)\, ds \\ &= 2^{-jv} \left\{ \Psi(2^j u - k, v) - \Psi(-k, v) \right\}, \end{aligned} \tag{2.30}$$

and where the convergence of the series, as a function of s, holds in $L^\alpha(\mathbb{R})$ as well as in $L^2(\mathbb{R})$. Observe that the limit of the series does not depend on the way its terms are ordered. Next, using (2.29), (2.30), (1.1), a classical property of the stochastic integral $\int_{\mathbb{R}} (\cdot) Z_\alpha(ds)$, (2.28), and (2.5), we get that the random series

$$\sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv}\, \epsilon_{j,k} \left(\Psi(2^j u - k, v) - \Psi(-k, v) \right),$$

converges in probability to the random variable $\widetilde{X}(u,v)$. Observe that the terms of this series can be ordered in an arbitrary way. Finally, combining this result with Part (i) of Theorem 2.1, we obtain that the random variables $\widetilde{X}(u,v)$ and $X(u,v)$ are equal almost surely. □

3. Convergence of the wavelet series in Hölder spaces

The goal of this section is to show that when the terms of the series in (2.7), viewed as a random series of functions of the variable (u,v), are ordered in an appropriate way, then not only does this series converge almost surely for every fixed $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, but also it, as well as its term-by-term pathwise partial derivatives of any order with respect to v, converges almost surely in some Hölder spaces. First we define these spaces precisely.

Definition 3.1. Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space and let $\mathcal{K}$ be a subset of $\mathbb{R}$. For every $\gamma \in [0,1]$, the Banach space of γ-Hölder functions from $\mathcal{K}$ to $\mathbb{B}$ is denoted by $\mathcal{C}^\gamma(\mathcal{K}, \mathbb{B})$ and defined by

$$\mathcal{C}^\gamma(\mathcal{K}, \mathbb{B}) := \left\{ f \colon \mathcal{K} \to \mathbb{B} \,:\, \mathcal{N}_\gamma(f) < \infty \right\},$$

where

$$\mathcal{N}_\gamma(f) := \sup_{x\in\mathcal{K}} \|f(x)\| + \sup_{x,y\in\mathcal{K}} \frac{\|f(x)-f(y)\|}{|x-y|^\gamma}$$

is the natural norm on this space. In the definition of $\mathcal{N}_\gamma(f)$, we understand that $0/0 = 0$. Also notice that $\mathcal{C}^1(\mathcal{K},\mathbb{B})$ is usually called the space of the Lipschitz functions from $\mathcal{K}$ to $\mathbb{B}$.

Definition 3.2. Let γ, M, a, and b be arbitrary fixed real numbers satisfying $\gamma \in [0,1]$, $M > 0$, and $a < b$. We denote by $\mathcal{E}_\gamma(a,b,M)$ the Banach space

$$\mathcal{E}_\gamma(a,b,M) := \mathcal{C}^1\big([a,b],\mathcal{C}^\gamma([-M,M],\mathbb{R})\big)$$

of the Lipschitz functions defined on $[a,b]$ and with values in the Hölder space $\mathcal{C}^\gamma([-M,M],\mathbb{R})$. Observe that each function f belonging to $\mathcal{E}_\gamma(a,b,M)$ can be viewed as a bivariate real-valued function $(u,v) \mapsto f(u,v) := \big(f(v)\big)(u)$ on the rectangle $[-M,M]\times[a,b]$; moreover, the natural norm on $\mathcal{E}_\gamma(a,b,M)$ is equivalent to the norm $|||\,.\,|||$ defined by

$$\begin{aligned} |||f||| := &\sup_{(u,v)\in[-M,M]\times[a,b]} |f(u,v)| \\ &+ \sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \frac{\big|(\Delta^{1,\cdot}_{u_1-u_2} f)(u_2,v)\big|}{|u_1-u_2|^\gamma} \\ &+ \sup_{(u,v_1,v_2)\in[-M,M]\times[a,b]^2} \frac{\big|(\Delta^{\cdot,1}_{v_1-v_2} f)(u,v_2)\big|}{|v_1-v_2|} \\ &+ \sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \frac{\big|(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)} f)(u_2,v_2)\big|}{|u_1-u_2|^\gamma |v_1-v_2|}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &(\Delta^{1,\cdot}_{u_1-u_2} f)(u_2,v) := f(u_1,v) - f(u_2,v), \\ &(\Delta^{\cdot,1}_{v_1-v_2} f)(u,v_2) := f(u,v_1) - f(u,v_2), \\ &(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)} f)(u_2,v_2) := f(u_1,v_1) - f(u_1,v_2) - f(u_2,v_1) + f(u_2,v_2). \end{aligned} \tag{3.2}$$

In (3.1), we understand that $0/0 = 0$.

Now we are ready to state the main result of this section.

Theorem 3.3. *We use the same notation as in Theorem 2.1. The following two results hold for all $\omega \in \Omega_0^*$, the event of probability 1 introduced in Lemma 2.3.*

(i) *For each fixed $u \in \mathbb{R}$, the function $X(u,\cdot,\omega) : v \mapsto X(u,v,\omega)$ is infinitely differentiable on $(1/\alpha,1)$; its derivative of order $q \in \mathbb{Z}_+$ at all $v \in (1/\alpha,1)$ is given by*

$$\begin{aligned} (\partial_v^q X)(u,v,\omega) = &\sum_{p=0}^{q} \binom{q}{p} \big(-\log 2\big)^p \\ &\times \sum_{(j,k)\in\mathbb{Z}^2} j^p\, 2^{-jv}\, \epsilon_{j,k}(\omega)\, \Big(\big(\partial_v^{q-p}\Psi\big)(2^j u - k, v) - \big(\partial_v^{q-p}\Psi\big)(-k,v)\Big), \end{aligned} \tag{3.3}$$

where $0^0 := 1$*, for every fixed* (u, v) *the series is absolutely convergent (its terms can therefore be ordered in an arbitrary way), and* $\binom{q}{p}$ *denotes the binomial coefficient.*

(ii) *For each fixed* $q \in \mathbb{Z}_+$ *and* $M, a, b \in \mathbb{R}$ *satisfying* $M > 0$ *and* $1/\alpha < a < b < 1$*, the function* $(\partial_v^q X)(\cdot, \cdot, \omega) : (u, v) \mapsto (\partial_v^q X)(u, v, \omega)$ *belongs to the space* $\mathcal{E}_\gamma(a, b, M)$ *for all* $\gamma \in [0, a - 1/\alpha)$.

The proof of Theorem 3.3 relies mainly on the following proposition.

Proposition 3.4. *Let* M *be an arbitrary and fixed positive real number. For every* $n \in \mathbb{Z}_+$*, denote by* $X_{M,n} = \{X_{M,n}(u, v) \colon (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ *the* $\mathcal{S}t\alpha\mathcal{S}$ *random field defined for every* $(u, v) \in \mathbb{R} \times (1/\alpha, 1)$*, as the finite sum,*

$$X_{M,n}(u, v) = \sum_{(j,k) \in D_{M,n}} 2^{-jv} \, \epsilon_{j,k} \left(\Psi(2^j u - k, v) - \Psi(-k, v) \right), \tag{3.4}$$

where

$$D_{M,n} := \left\{ (j, k) \in \mathbb{Z}^2 : |j| \leq n \text{ and } |k| \leq M 2^{n+1} \right\}. \tag{3.5}$$

Then, the following three results hold.

(i) *For all fixed* $\omega \in \Omega$ *(the underlying probability space) and* $u \in \mathbb{R}$*, the function* $X_{M,n}(u, \cdot, \omega) \colon v \mapsto X_{M,n}(u, v, \omega)$ *is infinitely differentiable on* $(1/\alpha, 1)$*; its derivative of order* $q \in \mathbb{Z}_+$ *at a point* $v \in (1/\alpha, 1)$ *is denoted by* $(\partial_v^q X_{M,n})(u, v, \omega)$.

(ii) *For all fixed* $\omega \in \Omega$*,* $q, n \in \mathbb{Z}_+$*, and* $a, b \in \mathbb{R}$ *satisfying* $1/\alpha < a < b < 1$*, the function* $(\partial_v^q X_{M,n})(\cdot, \cdot, \omega)$ *belongs to the Banach space* $\mathcal{E}_1(a, b, M)$.

(iii) *For each fixed* $\omega \in \Omega_0^*$*,* $q \in \mathbb{Z}_+$*, and* $a, b, \gamma \in \mathbb{R}$ *satisfying* $1/\alpha < a < b < 1$ *and* $0 \leq \gamma < a - 1/\alpha$*,* $\left((\partial_v^q X_{M,n})(\cdot, \cdot, \omega)\right)_{n \in \mathbb{Z}_+}$ *is a Cauchy sequence in the Banach space* $\mathcal{E}_\gamma(a, b, M)$.

Proof. Parts (i) and (ii) of Proposition 3.4 are more or less straightforward consequences of Proposition 2.4. In view of Definition 3.2, Part (iii) of Proposition 3.4 results from the following four lemmas. □

Lemma 3.5. *Let* M*,* a*, and* b *be real numbers satisfying* $M > 0$ *and* $1/\alpha < a < b < 1$*. For all fixed* $q \in \mathbb{Z}_+$ *and* $\omega \in \Omega_0^*$*, when* n *goes to infinity,*

$$\left| \left(\partial_v^q X_{M,n+l}\right)(u, v, \omega) - \left(\partial_v^q X_{M,n}\right)(u, v, \omega) \right| \tag{3.6}$$

converges to 0*, uniformly in* $(u, v) \in [-M, M] \times [a, b]$ *and in* $l \in \mathbb{Z}_+$.

Lemma 3.6. *Let* M*,* a*,* b*, and* γ *be real numbers satisfying* $M > 0$*,* $1/\alpha < a < b < 1$ *and* $\gamma < a - 1/\alpha$*. For all fixed* $q \in \mathbb{Z}_+$ *and* $\omega \in \Omega_0^*$*, when* n *goes to infinity,*

$$\frac{\left| \left(\Delta^{1,\cdot}_{u_1 - u_2} \left(\partial_v^q X_{M,n+l}\right)\right)(u_2, v, \omega) - \left(\Delta^{1,\cdot}_{u_1 - u_2} \left(\partial_v^q X_{M,n}\right)\right)(u_2, v, \omega) \right|}{|u_1 - u_2|^\gamma} \tag{3.7}$$

converges to 0 *uniformly in* $(u_1, u_2, v) \in [-M, M]^2 \times [a, b]$ *and in* $l \in \mathbb{Z}_+$.

Lemma 3.7. *Let M, a, and b be real numbers satisfying $M > 0$ and $1/\alpha < a < b < 1$. For all fixed $q \in \mathbb{Z}_+$ and $\omega \in \Omega_0^*$, when n goes to infinity,*

$$\frac{\left|\left(\Delta^{\cdot,1}_{v_1-v_2}\left(\partial_v^q X_{M,n+l}\right)\right)(u,v_2,\omega) - \left(\Delta^{\cdot,1}_{v_1-v_2}\left(\partial_v^q X_{M,n}\right)\right)(u,v_2,\omega)\right|}{|v_1-v_2|} \tag{3.8}$$

converges to 0 uniformly in $(u, v_1, v_2) \in [-M, M] \times [a, b]^2$ and in $l \in \mathbb{Z}_+$.

Lemma 3.8. *Let M, a, b, and γ be real numbers satisfying $M > 0$, $1/\alpha < a < b < 1$ and $\gamma < a - 1/\alpha$. For all fixed $q \in \mathbb{Z}_+$ and $\omega \in \Omega_0^*$, when n goes to infinity,*

$$\frac{\left|\left(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\left(\partial_v^q X_{M,n+l}\right)\right)(u_2,v_2,\omega) - \left(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\left(\partial_v^q X_{M,n}\right)\right)(u_2,v_2,\omega)\right|}{|u_1-u_2|^\gamma |v_1-v_2|} \tag{3.9}$$

converges to 0 uniformly in $(u_1, u_2, v_1, v_2) \in [-M, M]^2 \times [a, b]^2$ and in $l \in \mathbb{Z}_+$.

The proofs of the previous four lemmas are quite similar, so we will only give that of Lemma 3.8.

Proof. In view of the convention that $0/0 = 0$, it is no restriction to assume that $u_1 \neq u_2$ and $v_1 \neq v_2$. By using (3.4), (3.2), and the Leibniz formula, one can rewrite (3.9) as

$$\frac{\left|\sum_{p=0}^q \binom{q}{p}\left(-\log 2\right)^p \sum_{(j,k)\in D_{M,n+l}\setminus D_{M,n}} j^p \epsilon_{j,k}(\omega)\left(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\Theta^{q-p}_{j,k}\right)(u_2,v_2)\right|}{|u_1-u_2|^\gamma |v_1-v_2|}, \tag{3.10}$$

where for all $(u, v) \in \mathbb{R} \times (1/\alpha, 1)$,

$$\Theta^{q-p}_{j,k}(u,v) = 2^{-jv}\left(\partial_v^{q-p}\Psi\right)(2^j u - k, v). \tag{3.11}$$

In the sequel, we denote by $D^c_{M,n}$ the set defined as $D^c_{M,n} = \{(j,k) \in \mathbb{Z}^2 : (j,k) \notin D_{M,n}\}$; recall that $D_{M,n}$ has been introduced in (3.5). Using (3.10), the Taylor formula with respect to the variable v, (3.2), and the triangle inequality, one obtains that

$$\begin{aligned}&\frac{\left|\left(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\left(\partial_v^q X_{n+l}\right)\right)(u_2,v_2,\omega) - \left(\Delta^{1,1}_{(u_1-u_2,v_1-v_2)}\left(\partial_v^q X_{n}\right)\right)(u_2,v_2,\omega)\right|}{|u_1-u_2|^\gamma |v_1-v_2|}\\ &\qquad\le G^{1,q}_{M,n}(u_1,u_2,v_2,\omega) + |v_1-v_2| G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega),\end{aligned}$$

where

$$\begin{aligned}&G^{1,q}_{M,n}(u_1,u_2,v_2,\omega)\\ &:= \sum_{p=0}^q \binom{q}{p}(\log 2)^p \sum_{(j,k)\in D^c_{M,n}} |j|^p \, |\epsilon_{j,k}(\omega)| \, \frac{\left|\left(\Delta^{1,\cdot}_{u_1-u_2}\left(\partial_v \Theta^{q-p}_{j,k}\right)\right)(u_2,v_2)\right|}{|u_1-u_2|^\gamma}\end{aligned} \tag{3.12}$$

and

$$G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega) := \sum_{p=0}^{q} \binom{q}{p} (\log 2)^p \times \sum_{(j,k)\in D^c_{M,n}} |j|^p |\epsilon_{j,k}(\omega)| \frac{\left| \int_0^1 (1-s)\big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v^2 \Theta^{q-p}_{j,k}\big)\big)(u_2, v_2+s(v_1-v_2))\,ds \right|}{|u_1-u_2|^{\gamma}}. \tag{3.13}$$

Thus, to prove the lemma, it suffices to show that

$$G^{1,q}_{M,n}(u_1,u_2,v_2,\omega) \quad \text{and} \quad G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega)$$

converge to 0, when $n \to +\infty$, uniformly in $(u_1,u_2,v_1,v_2) \in [-M,M]^2 \times [a,b]^2$.

First, we study $G^{1,q}_{M,n}(u_1,u_2,v_2,\omega)$. It follows from (3.11) that

$$\begin{aligned}(\partial_v \Theta^{q-p}_{j,k})(u,v) \\ = 2^{-jv}\big(\partial_v^{q+1-p}\Psi\big)(2^j u-k,v) - (\log 2)\, j\, 2^{-jv}\big(\partial_v^{q-p}\Psi\big)(2^j u-k,v).\end{aligned} \tag{3.14}$$

Next, using (3.14), (3.2), the triangle inequality, Lemma 2.3, (9.1) and (9.2), one has

$$\begin{aligned}
&\sum_{(j,k)\in D^c_{M,n}} |j|^p |\epsilon_{j,k}(\omega)| \frac{\left|\big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v \Theta^{q-p}_{j,k}\big)\big)(u_2,v_2)\right|}{|u_1-u_2|^{\gamma}} \\
&\le \sum_{(j,k)\in D^c_{M,n}} 2^{-jv_2} |j|^p |\epsilon_{j,k}(\omega)| \frac{\left|\big(\partial_v^{q+1-p}\Psi\big)(2^j u_1-k,v_2) - \big(\partial_v^{q+1-p}\Psi\big)(2^j u_2-k,v_2)\right|}{|u_1-u_2|^{\gamma}} \\
&+ (\log 2) \sum_{(j,k)\in D^c_{M,n}} 2^{-jv_2} |j|^{p+1} |\epsilon_{j,k}(\omega)| \frac{\left|\big(\partial_v^{q-p}\Psi\big)(2^j u_1-k,v_2) - \big(\partial_v^{q-p}\Psi\big)(2^j u_2-k,v_2)\right|}{|u_1-u_2|^{\gamma}} \\
&\le C_1(\omega)\Big(A_n\big(u_1,u_2,v_2;M,\gamma,\eta,p,\partial_v^{q+1-p}\Psi\big) + B_n\big(u_1,u_2,v_2;M,\gamma,\eta,p,\partial_v^{q+1-p}\Psi\big)\Big) \\
&\quad + C_1(\omega)(\log 2)\Big(A_n\big(u_1,u_2,v_2;M,\gamma,\eta,p+1,\partial_v^{q-p}\Psi\big) \\
&\qquad + B_n\big(u_1,u_2,v_2;M,\gamma,\eta,p+1,\partial_v^{q-p}\Psi\big)\Big),
\end{aligned}$$

where C_1 denotes the random variable C' introduced in Lemma 2.3. Lemma 9.1 and (3.12) imply that, when $n \to +\infty$, $G^{1,q}_{M,n}(u_1,u_2,v_2,\omega)$ converges to 0, uniformly in $(u_1,u_2,v_1,v_2) \in [-M,M]^2 \times [a,b]^2$.

Now we study $G^{2,q}_{M,n}(u_1,u_2,v_1,v_2,\omega)$. It follows from (3.14) that

$$\begin{aligned}(\partial_v^2 \Theta^{q-p}_{j,k})(u,v) &= 2^{-jv}\big(\partial_v^{q+2-p}\Psi\big)(2^j u-k,v) - 2(\log 2) j\, 2^{-jv}\big(\partial_v^{q+1-p}\Psi\big)(2^j u-k,v) \\ &\quad + (\log 2)^2 j^2\, 2^{-jv}\big(\partial_v^{q-p}\Psi\big)(2^j u-k,v).\end{aligned} \tag{3.15}$$

Next, using (3.15), (3.2), the triangle inequality, Lemma 2.3, (9.1) and (9.2), one has

$$\sum_{(j,k)\in D_{M,n}^c} |j|^p |\epsilon_{j,k}(\omega)| \frac{\left|\int_0^1 (1-s)\big(\Delta^{1,\cdot}_{u_1-u_2}\big(\partial_v^2\Theta_{j,k}^{q-p}\big)\big)(u_2,v_2+s(v_1-v_2))\,ds\right|}{|u_1-u_2|^\gamma}$$
$$\le C_1(\omega)\int_0^1 \Big(A_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p,\partial_v^{q+2-p}\Psi\big)+ B_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p,\partial_v^{q+2-p}\Psi\big)\Big)\,ds$$
$$+C_2(\omega)\int_0^1 \Big(A_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+1,\partial_v^{q+1-p}\Psi\big)+ B_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+1,\partial_v^{q+1-p}\big)\Big)\,ds$$
$$+C_2(\omega)\int_0^1 \Big(A_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+2,\partial_v^{q-p}\Psi\big)+ B_n\big(u_1,u_2,v_2+s(v_1-v_2);M,\gamma,\eta,p+2,\partial_v^{q-p}\Psi\big)\Big)\,ds,$$

where $C_2(\omega) = (2\log 2)C_1(\omega)$. Then Lemma 9.1 and (3.13) imply that, when $n \to +\infty$, $G_{M,n}^{2,q}(u_1,u_2,v_1,v_2,\omega)$ converges to 0, uniformly in $(u_1,u_2,v_1,v_2) \in [-M,M]^2 \times [a,b]^2$. □

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let $\omega \in \Omega_0^*$ be arbitrary and fixed. First we show that part (i) of the theorem holds. By using Lemma 2.3, Proposition 2.4, and a method similar to the one used to derive (2.6), we can prove that, for all fixed $q \in \mathbb{N}$ and $(u,v) \in \mathbb{R} \times (1/\alpha, 1)$, one has

$$\sum_{p=0}^{q} \binom{q}{p} \big(\log 2\big)^p \times \sum_{(j,k)\in\mathbb{Z}^2} |j|^p\, 2^{-jv} \big|\epsilon_{j,k}(\omega)\big|\, \big|\big(\partial_v^{q-p}\Psi\big)(2^j u-k,v) - \big(\partial_v^{q-p}\Psi\big)(-k,v)\big| < \infty.$$

Therefore, the series

$$\sum_{p=0}^{q} \binom{q}{p} \big(-\log 2\big)^p \sum_{(j,k)\in\mathbb{Z}^2} j^p\, 2^{-jv} \epsilon_{j,k}(\omega) \Big(\big(\partial_v^{q-p}\Psi\big)(2^j u-k,v) - \big(\partial_v^{q-p}\Psi\big)(-k,v)\Big)$$

is convergent, and its finite limit, denoted by $\check{X}^{(q)}(u,v,\omega)$, does not depend on the way the terms of the series are ordered. Now we assume that $u \in \mathbb{R}$ is arbitrary and fixed and that the variable v belongs to an arbitrary fixed compact interval $[a,b]$ contained in $(1/\alpha, 1)$. We denote by M an arbitrary fixed positive real number such that $u \in [-M,M]$. In view of part (i) of Theorem 2.1, part (iii) of Proposition 3.4, and (3.1), when n goes to infinity, the following two results are satisfied:

- the function $v \mapsto X_{M,n}(u,v,\omega)$ converges to the function $v \mapsto X(u,v,\omega)$, uniformly in $v \in [a,b]$;
- for each fixed $q \in \mathbb{N}$, the function $v \mapsto (\partial_v^q X_{M,n})(u,v,\omega)$ converges to the function $v \mapsto \check{X}^{(q)}(u,v,\omega)$, uniformly in $v \in [a,b]$.

The latter two results imply that $v \mapsto X(u,v,\omega)$ is infinitely differentiable on $[a,b]$ and one has, for all $q \in \mathbb{N}$ and $v \in [a,b]$,

$$(\partial_v^q X)(u,v,\omega) = \lim_{n\to+\infty} (\partial_v^q X_{M,n})(u,v,\omega) = \check{X}^{(q)}(u,v,\omega); \tag{3.16}$$

these equalities mean that (3.3) is satisfied. Thus, it remains to show that part (ii) of the theorem holds. In fact, the equality $X(u,v,\omega) = \lim_{n\to+\infty} X_{M,n}(u,v,\omega)$, (3.16), and part (iii) of Proposition 3.4 imply that this is indeed the case. □

Before ending this section, we stress that for each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, and $M,a,b \in \mathbb{R}$ satisfying $M > 0$ and $1/\alpha < a < b < 1$, a global modulus of continuity for the function $u \mapsto (\partial_v^q X)(u,v,\omega)$ on the interval $[-M,M]$, uniformly in $v \in [a,b]$, can be derived using part (ii) of Theorem 3.3; similarly there can be derived a global modulus of continuity for the function $v \mapsto (\partial_v^q X)(u,v,\omega)$ on the interval $[a,b]$, uniformly in $u \in [-M,M]$. More precisely, in view of Definition 3.2, a straightforward consequence of part (ii) of Theorem 3.3 is the following:

Corollary 3.9. *For each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, and $M,a,b,\eta \in \mathbb{R}$ satisfying $M > 0$, $1/\alpha < a < b < 1$, and $\eta > 0$, one has*

$$\sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v,\omega) - \big(\partial_v^q X\big)(u_2,v,\omega)\big|}{|u_1-u_2|^{a-1/\alpha-\eta}} \right\} < \infty, \tag{3.17}$$

and

$$\sup_{(u,v_1,v_2)\in[-M,M]\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u,v_1,\omega) - \big(\partial_v^q X\big)(u,v_2,\omega)\big|}{|v_1-v_2|} \right\} < \infty. \tag{3.18}$$

4. Fine path properties of the field generating LMSMs

The main two goals of this section are the following:

- to give an improved version of the global modulus of continuity (3.17);
- to derive an upper bound for $\big|\big(\partial_v^q X\big)(u,v,\omega)\big|$, for all $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, $v \in [a,b] \subset (1/\alpha, 1)$, and $u \in \mathbb{R}$.

More precisely, we will show the following two results.

Proposition 4.1. *For each fixed* $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, *and* $M, a, b, \eta \in \mathbb{R}$ *satisfying* $M > 0$, $1/\alpha < a < b < 1$, *and* $\eta > 0$, *one has*

$$\sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v,\omega) - \big(\partial_v^q X\big)(u_2,v,\omega)\big|}{|u_1-u_2|^{v-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}} \right\}$$

$$\leq \sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \Bigg\{ \sum_{p=0}^{q} \binom{q}{p} \big(\log 2\big)^p$$

$$\times \frac{\sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv} \big|\epsilon_{j,k}(\omega)\big| \,\big|\big(\partial_v^{q-p}\Psi\big)(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(2^j u_2 - k, v)\big|}{|u_1-u_2|^{v-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}} \Bigg\}$$

$$< \infty. \tag{4.1}$$

Proposition 4.2. *For each fixed* $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, *and* $a, b, \eta \in \mathbb{R}$ *satisfying* $1/\alpha < a < b < 1$, *and* $\eta > 0$, *one has*

$$\sup_{(u,v)\in\mathbb{R}\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u,v,\omega)\big|}{|u|^v\big(1+\big|\log|u|\big|\big)^{q+1/\alpha+\eta}} \right\}$$

$$\leq \sup_{(u,v)\in\mathbb{R}\times[a,b]} \Bigg\{ \sum_{p=0}^{q} \binom{q}{p} \big(\log 2\big)^p$$

$$\times \left\{ \frac{\sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv} \big|\epsilon_{j,k}(\omega)\big| \,\big|\big(\partial_v^{q-p}\Psi\big)(2^j u - k, v) - \big(\partial_v^{q-p}\Psi\big)(-k, v)\big|}{|u|^v\big(1+\big|\log|u|\big|\big)^{q+1/\alpha+\eta}} \right\}$$

$$< \infty. \tag{4.2}$$

The proofs of Propositions 4.1 and 4.2 are, to a certain extent, inspired by that of Theorem 1 in [2].

Proof of Proposition 4.1. Let $(u_1, u_2, v) \in [-M, M]^2 \times [a, b]$ be arbitrary and fixed; in the sequel we assume that $u_1 \neq u_2$. Observe that, in view of (2.12), there is a constant $c_1 > 0$, not depending on (u_1, u_2, v), such that for all $p \in \{0, \ldots, q\}$ and $(j, k) \in \mathbb{Z}^2$, one has

$$\begin{aligned} &\big|\big(\partial_v^{q-p}\Psi\big)(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(2^j u_2 - k, v)\big| \\ &\qquad \leq c_1 \Big(\big(3 + |2^j u_1 - k|\big)^{-2} + \big(3 + |2^j u_2 - k|\big)^{-2}\Big). \end{aligned} \tag{4.3}$$

Also notice that

$$\big|\big(\partial_v^{q-p}\Psi\big)(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(2^j u_2 - k, v)\big|$$

can be bounded more sharply when the condition

$$2^j |u_1 - u_2| \leq 1 \tag{4.4}$$

holds, namely using the mean value theorem and (2.12), one has

$$\begin{aligned}
&\big|\big(\partial_v^{q-p}\Psi\big)\,(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(2^j u_2 - k, v)\big| \\
&\quad\le 2^j|u_1-u_2| \sup_{(u,v)\in[u_1\wedge u_2, u_1\vee u_2]\times[a,b]} \big|\big(\partial_x\partial_v^{q-p}\Psi\big)(2^j u - k, v)\big| \\
&\quad\le c_1\, 2^j|u_1-u_2| \sup_{u\in[u_1\wedge u_2, u_1\vee u_2]} \big(3+|2^j u - k|\big)^{-2} \\
&\quad\le c_1\, 2^j|u_1-u_2|\big(2+|2^j u_1 - k|\big)^{-2}, \qquad (4.5)
\end{aligned}$$

where the last inequality results from the triangle inequality and (4.4). Denote by $j_0 > -\log_2(4M)$ the unique integer satisfying

$$2^{-1} < 2^{j_0}|u_1-u_2| \le 1. \qquad (4.6)$$

Then, the first inequality in (2.10), (4.3), and (4.5), entail that, for all $\eta>0$ and $\omega\in\Omega_0^*$,

$$\begin{aligned}
&\sum_{(j,k)\in\mathbb{Z}^2} |j|^p 2^{-jv}\big|\epsilon_{j,k}(\omega)\big|\,\big|\big(\partial_v^{q-p}\Psi\big)(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(u_2 - k, v)\big| \\
&\quad\le C(\omega) \sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv}(1+|j|)^{p+1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|) \\
&\qquad\qquad\times\big|\big(\partial_v^{q-p}\Psi\big)(2^j u_1 - k, v) - \big(\partial_v^{q-p}\Psi\big)(2^j u_2 - k, v)\big| \\
&\quad\le C(\omega)\, c_1\,\big(\check{A}_{j_0}(u_1,v)|u_1-u_2| + \check{B}_{j_0}(u_1,u_2,v)\big), \qquad (4.7)
\end{aligned}$$

where the random variable C was introduced in Lemma 2.3 and where for each $J\in\mathbb{Z}$, $(y_1,y_2)\in\mathbb{R}^2$, and $v\in[a,b]$,

$$\begin{aligned}
\check{A}_J(y_1,v) &:= \sum_{j\le J}\sum_{k\in\mathbb{Z}} 2^{j(1-v)}(1+|j|)^{p+1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|) \\
&\qquad\times\big(2+|2^j y_1 - k|\big)^{-2} \qquad (4.8)
\end{aligned}$$

and

$$\begin{aligned}
\check{B}_J(y_1,y_2,v) &:= \sum_{j> J}\sum_{k\in\mathbb{Z}} 2^{-jv}(1+|j|)^{p+1/\alpha+\eta}(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|) \\
&\qquad\times\Big(\big(3+|2^j y_1 - k|\big)^{-2} + \big(3+|2^j y_2 - k|\big)^{-2}\Big). \qquad (4.9)
\end{aligned}$$

Now we give an appropriate upper bound for $\check{A}_{j_0}(u_1,v)$. Assume that $j\le j_0$; using Lemma 9.4 (in which one takes $\theta=1/\alpha$, $\zeta=1/\alpha+\eta$, and $u=2^j u_1$) and the inequality $|u_1|\le M$, one obtains that

$$\sum_{k\in\mathbb{Z}} \frac{(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}(2+|k|)}{\big(2+|2^j\, u_1 - k|\big)^2} \le c_2\, 2^{j_0/\alpha}\,(1+|j_0|)^{1/\alpha+\eta},$$

where c_2 is a constant depending only on M, α, and η.

Next, it follows from this inequality, (4.8), and Lemma 9.3 (in which one takes $\theta = 1 - v$, $\theta_0 = 1 - b$, $\lambda = p + 1/\alpha + \eta$, $n_0 = -\infty$, and $n_1 = j_0$) that

$$\begin{aligned} \check{A}_{j_0}(u_1, v) &\le c_2\, 2^{j_0/\alpha} \left(1 + |j_0|\right)^{1/\alpha+\eta} \sum_{j \le j_0} 2^{j(1-v)} (1 + |j|)^{p+1/\alpha+\eta} \\ &\le c_3\, 2^{j_0(1-v+1/\alpha)} \left(1 + |j_0|\right)^{p+2/\alpha+2\eta} \\ &\le c_4\, |u_1 - u_2|^{v-1/\alpha-1} \left(1 + \big|\log|u_1 - u_2|\big|\right)^{p+2/\alpha+2\eta}, \end{aligned} \tag{4.10}$$

where the last inequality results from (4.6), and where c_3 and c_4 are two constants not depending on (u_1, u_2, v). Now we give an upper bound for $\check{B}_{j_0}(u_1, u_2, v)$. In view of (4.9), this quantity can be expressed as,

$$\check{B}_{j_0}(u_1, u_2, v) = T_{j_0}(u_1, v) + T_{j_0}(u_2, v), \tag{4.11}$$

where, for each $J \in \mathbb{Z}$, $y \in \mathbb{R}$, and $v \in [a, b]$,

$$T_J(y, v) := \sum_{j > J} \sum_{k \in \mathbb{Z}} 2^{-jv} \frac{(1 + |j|)^{p+1/\alpha+\eta} (1 + |k|)^{1/\alpha} \log^{1/\alpha+\eta}(2 + |k|)}{\left(3 + |2^j y - k|\right)^2}. \tag{4.12}$$

Assume that $j > j_0$ and that $x \in \{u_1, u_2\}$; using Lemma 9.4 (in which one takes $\theta = 1/\alpha$, $\zeta = 1/\alpha + \eta$, and $u = 2^j x$) and the inequality $|x| \le M$, one gets that

$$\sum_{k \in \mathbb{Z}} \frac{(1 + |k|)^{1/\alpha} \log^{1/\alpha+\eta}(2 + |k|)}{\left(2 + |2^j x - k|\right)^2} \le c_2\, 2^{j/\alpha} \left(1 + |j|\right)^{1/\alpha+\eta}.$$

Next, in view of (4.12), it follows from this inequality and Lemma 9.3 (in which one takes $\theta = v - 1/\alpha$, $\theta_0 = a - 1/\alpha$, $\lambda = p + 2/\alpha + 2\eta$, $n_0 = j_0 + 1$, and $n_1 = +\infty$) that

$$\begin{aligned} T_{j_0}(x, v) &\le c_2 \sum_{j > j_0} 2^{-j(v-1/\alpha)} (1 + |j|)^{p+2/\alpha+2\eta} \le c_5\, 2^{-j_0(v-1/\alpha)} (1 + |j_0|)^{p+2/\alpha+2\eta} \\ &\le c_6\, |u_1 - u_2|^{v-1/\alpha} \left(1 + \big|\log|u_1 - u_2|\big|\right)^{p+2/\alpha+2\eta}, \end{aligned} \tag{4.13}$$

where the last inequality results from (4.6), and where c_5 and c_6 are two constants not depending on (x, v). Next, (4.13) and (4.11) imply that

$$\check{B}_{j_0}(u_1, u_2, v) \le 2c_6\, |u_1 - u_2|^{v-1/\alpha} \left(1 + \big|\log|u_1 - u_2|\big|\right)^{p+2/\alpha+2\eta}. \tag{4.14}$$

Next, combining (4.10), (4.14), and (4.7), one obtains that, for all $\eta > 0$ and $\omega \in \Omega_0^*$,

$$\begin{aligned} &\sum_{(j,k) \in \mathbb{Z}^2} |j|^p\, 2^{-jv} \left|\epsilon_{j,k}(\omega)\right| \left| \left(\partial_v^{q-p}\Psi\right)(2^j u_1 - k, v) - \left(\partial_v^{q-p}\Psi\right)(u_2 - k, v) \right| \\ &\quad \le C(\omega)\, c_7\, |u_1 - u_2|^{v-1/\alpha} \left(1 + \big|\log|u_1 - u_2|\big|\right)^{p+2/\alpha+2\eta}, \end{aligned} \tag{4.15}$$

where c_7 is a constant not depending on (u_1, u_2, v). Finally, (3.3), the triangle inequality, and (4.15) entail that (4.1) holds. □

Proof of Proposition 4.2. Let $(u,v)\in\mathbb{R}\times[a,b]$ be arbitrary and fixed. In all the sequel we assume that $u\neq 0$.

Observe that, in view of (2.12), there is a constant $c_1>0$, not depending on (u,v), such that for all $p\in\{0,\dots,q\}$ and $(j,k)\in\mathbb{Z}^2$, one has,

$$\big|\big(\partial_v^{q-p}\Psi\big)(2^ju-k,v)-\big(\partial_v^{q-p}\Psi\big)(-k,v)\big|\le c_1\Big(\big(3+|2^ju-k|\big)^{-2}+\big(3+|k|\big)^{-2}\Big). \tag{4.16}$$

Also notice that $|(\partial_v^{q-p}\Psi)(2^ju-k,v)-(\partial_v^{q-p}\Psi)(-k,v)|$ can be bounded more sharply when the condition

$$2^j|u|\le 1 \tag{4.17}$$

holds, namely, using the mean value theorem and (2.12), one has

$$\begin{aligned}\big|\big(\partial_v^{q-p}\Psi\big)(2^ju-k,v)&-\big(\partial_v^{q-p}\Psi\big)(-k,v)\big|\\ &\le 2^j\,|u|\sup_{y\in[u\wedge 0,u\vee 0]}\big|\big(\partial_x\partial_v^{q-p}\Psi\big)(2^jy-k,v)\big|\\ &\le c_1\,2^j\,|u|\sup_{y\in[u\wedge 0,u\vee 0]}\big(3+|2^jy-k|\big)^{-2}\le c_1 2^j|u|\big(2+|k|\big)^{-2},\end{aligned} \tag{4.18}$$

where the last inequality results from the triangle inequality and (4.17). Denote by $j_1\in\mathbb{Z}$ the unique integer satisfying

$$2^{-1}<2^{j_1}|u|\le 1. \tag{4.19}$$

The first inequality in (2.10), (4.16), and (4.18) entail that, for all $\eta>0$ and $\omega\in\Omega_0^*$,

$$\begin{aligned}\sum_{(j,k)\in\mathbb{Z}^2}&|j|^p\,2^{-jv}\,\big|\epsilon_{j,k}(\omega)\big|\,\big|\big(\partial_v^{q-p}\Psi\big)(2^ju-k,v)-\big(\partial_v^{q-p}\Psi\big)(-k,v)\big|\\ &\le C(\omega)\sum_{(j,k)\in\mathbb{Z}^2}2^{-jv}\big(1+|j|\big)^{p+1/\alpha+\eta}\big(1+|k|\big)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+|k|\big)\\ &\qquad\qquad\times\big|\big(\partial_v^{q-p}\Psi\big)(2^ju-k,v)-\big(\partial_v^{q-p}\Psi\big)(-k,v)\big|\\ &\le C(\omega)\,c_1\,\big(|u|\check{A}_{j_1}(0,v)+\check{B}_{j_1}(u,0,v)\big),\end{aligned} \tag{4.20}$$

where the random variable C was introduced in Lemma 2.3, and where $\check{A}_{j_1}(0,v)$ and $\check{B}_{j_1}(u,0,v)$ are defined respectively by (4.8) and (4.9).

Now we give an upper bound for $\check{A}_{j_1}(0,v)$. Observe that

$$c_2:=\sum_{k\in\mathbb{Z}}\frac{(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+|k|\big)}{\big(2+|k|\big)^2}<\infty.$$

Thus, (4.8) and Lemma 9.3 (in which one takes $\theta=1-v$, $\theta_0=1-b$, $\lambda=p+1/\alpha+\eta$, $n_0=-\infty$, and $n_1=j_1$) imply that

$$\begin{aligned}\check{A}_{j_1}(0,v)=c_2\sum_{j\le j_1}2^{j(1-v)}\big(1+|j|\big)^{p+1/\alpha+\eta}&\le c_3\,2^{j_1(1-v)}\,\big(1+|j_1|\big)^{p+1/\alpha+\eta}\\ &\le c_4\,|u|^{v-1}\,\big(1+\big|\log|u|\big|\big)^{p+1/\alpha+\eta},\end{aligned} \tag{4.21}$$

where the last inequality results from (4.19) and where c_3 and c_4 are two constants not depending on (u,v).

Now we give an upper bound for $\check{B}_{j_1}(u,0,v)$. In view of (4.9), this quantity can be expressed as

$$\check{B}_{j_1}(u,0,v) := T_{j_1}(u,v) + T_{j_1}(0,v), \tag{4.22}$$

where $T_{j_1}(u,v)$ and $T_{j_1}(0,v)$ are defined by (4.12).

Assume that $j > j_1$ and that $x \in \{u, 0\}$; it follows from Lemma 9.4, in which one takes $\theta = 1/\alpha$ and $\zeta = 1/\alpha + \eta$, that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{(1+|k|)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+|k|\big)}{\big(3+|2^j x - k|\big)^2} &\le c_5\,\big(1+2^j|x|\big)^{1/\alpha}\log^{1/\alpha+\eta}\big(2+2^j|x|\big)\\ &\le c_6\, 2^{(j-j_1)/\alpha}\,(1+j-j_1)^{1/\alpha+\eta},\end{aligned}$$

where the last inequality results from (4.19) and where c_5 and c_6 are two constants not depending on x, v, j, and j_1. Therefore, in view of (4.12), one obtains that

$$T_{j_1}(x,v) \le c_6 \sum_{j>j_1} 2^{-jv}2^{(j-j_1)/\alpha}\big(1+|j|\big)^{p+1/\alpha+\eta}\big(1+j-j_1\big)^{1/\alpha+\eta}. \tag{4.23}$$

Next, taking $l = j - j_1$ on the right-hand side of (4.23), it follows that

$$\begin{aligned} T_{j_1}(x,v) &\le c_6 \sum_{l=1}^{+\infty} 2^{-j_1 v}2^{-l(v-1/\alpha)}(1+l)^{1/\alpha+\eta}\big(1+|l+j_1|\big)^{p+1/\alpha+\eta}\\ &\le c_7\, 2^{-j_1 v}\sum_{l=1}^{+\infty} 2^{-l(v-1/\alpha)}(1+l)^{1/\alpha+\eta}\Big(\big(1+l\big)^{p+1/\alpha+\eta} + \big(1+|j_1|\big)^{p+1/\alpha+\eta}\Big)\\ &\le c_7\, 2^{-j_1 v}\sum_{l=1}^{+\infty} 2^{-l(a-1/\alpha)}(1+l)^{1/\alpha+\eta}\Big(\big(1+l\big)^{p+1/\alpha+\eta} + \big(1+|j_1|\big)^{p+1/\alpha+\eta}\Big)\\ &\le c_8\, 2^{-j_1 v}\,\big(1+|j_1|\big)^{p+1/\alpha+\eta} \le c_9\,|u|^v\,\big(1+\big|\log|u|\big|\big)^{p+1/\alpha+\eta}, \end{aligned} \tag{4.24}$$

where the last inequality results from (4.19) and where the constants c_7, c_8, and c_9 do not depend on x, v, and j_1. Next, (4.22) and (4.24) imply that

$$\check{B}_{j_1}(u,0,v) \le 2c_9\,|u|^v\,\big(1+\big|\log|u|\big|\big)^{p+1/\alpha+\eta}. \tag{4.25}$$

Next, combining (4.20), (4.21), and (4.25), one gets that

$$\begin{aligned} &\sum_{(j,k)\in\mathbb{Z}^2} |j|^p\, 2^{-jv}\,\big|\epsilon_{j,k}(\omega)\big|\,\big|\big(\partial_v^{q-p}\Psi\big)(2^j u - k, v) - \big(\partial_v^{q-p}\Psi\big)(-k,v)\big|\\ &\qquad \le C(\omega)\, c_{10}\,|u|^v\,\big(1+\big|\log|u|\big|\big)^{p+1/\alpha+\eta}, \end{aligned} \tag{4.26}$$

where c_{10} is a constant not depending on (u,v). Finally, (3.3), the triangle inequality and (4.26) entail that (4.2) holds. □

Before ending this section, we stress that, thanks to (3.18) and (4.1), for each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, and $M, a, b \in \mathbb{R}$ satisfying $M > 0$ and $1/\alpha < a < b < 1$, one can derive a global modulus of continuity for the function $(u,v) \mapsto (\partial_v^q X)(u,v,\omega)$ on the rectangle $[-M,M] \times [a,b]$. More precisely, the following result holds.

Corollary 4.3. *For each fixed $\omega \in \Omega_0^*$, $q \in \mathbb{Z}_+$, and $M,a,b,\eta \in \mathbb{R}$ satisfying $M>0$, $1/\alpha < a < b < 1$, and $\eta > 0$, one has*

$$\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1,\omega) - \big(\partial_v^q X\big)(u_2,v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} < \infty. \tag{4.27}$$

Proof. For each $(u_1,u_2,v_1,v_2) \in [-M,M]^2 \times [a,b]^2$, one sets,

$$f(u_1,u_2,v_1,v_2) := \frac{\big|\big(\partial_v^q X\big)(u_1,v_1,\omega) - \big(\partial_v^q X\big)(u_2,v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|},$$

with the convention $0/0=0$. Using the fact that $f(u_1,u_2,v_1,v_2) = f(u_2,u_1,v_2,v_1)$, it follows that

$$\begin{aligned} &\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1,\omega) - \big(\partial_v^q X\big)(u_2,v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} \\ &= \sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1\vee v_2,\omega) - \big(\partial_v^q X\big)(u_2,v_1\wedge v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\}. \end{aligned} \tag{4.28}$$

Moreover, using the triangle inequality, and the inequality, for all $(u_1,u_2,v_1,v_2) \in [-M,M]^2 \times [a,b]^2$,

$$\begin{aligned} &\max\left\{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}, |v_1-v_2|\right\} \\ &\qquad \le |u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|, \end{aligned}$$

one gets that

$$\begin{aligned} &\sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1\vee v_2,\omega) - \big(\partial_v^q X\big)(u_2,v_1\wedge v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} \\ &\le \sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v_1\vee v_2,\omega) - \big(\partial_v^q X\big)(u_2,v_1\vee v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} \\ &+ \sup_{(u_1,u_2,v_1,v_2)\in[-M,M]^2\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_2,v_1\vee v_2,\omega) - \big(\partial_v^q X\big)(u_2,v_1\wedge v_2,\omega)\big|}{|u_1-u_2|^{v_1\vee v_2-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta} + |v_1-v_2|} \right\} \\ &\le \sup_{(u_1,u_2,v)\in[-M,M]^2\times[a,b]} \left\{ \frac{\big|\big(\partial_v^q X\big)(u_1,v,\omega) - \big(\partial_v^q X\big)(u_2,v,\omega)\big|}{|u_1-u_2|^{v-1/\alpha}\big(1+\big|\log|u_1-u_2|\big|\big)^{q+2/\alpha+\eta}} \right\} \\ &+ \sup_{(u,v_1,v_2)\in[-M,M]\times[a,b]^2} \left\{ \frac{\big|\big(\partial_v^q X\big)(u,v_1,\omega) - \big(\partial_v^q X\big)(u,v_2,\omega)\big|}{|v_1-v_2|} \right\}. \end{aligned} \tag{4.29}$$

Finally, combining (4.28), (4.29), (3.18), and (4.1), one obtains (4.27). □

5. Global and local moduli of continuity for LMSM

From now until the end of the article, LMSM is identified with its version $\{Y(t) : t \in \mathbb{R}\}$, defined for all $t \in \mathbb{R}$, by,

$$Y(t) = X(t, H(t)), \tag{5.1}$$

where $\{X(u, v) : (u, v) \in \mathbb{R} \times (1/\alpha, 1)\}$ is the $\mathcal{S}\alpha\mathcal{S}$ field introduced in Theorem 2.1. Recall that $H(\cdot)$ denotes an arbitrary continuous function defined on the real line and with values in a compact interval $[\underline{H}, \overline{H}] \subset (1/\alpha, 1)$.

First we determine a global modulus of continuity for $\{Y(t) : t \in \mathbb{R}\}$ on an arbitrary nonempty compact interval; it is no restriction to assume that this interval has the form $[-M, M]$, where M is an arbitrary positive real number.

Theorem 5.1. *Let Ω_0^* be the event of probability 1 introduced in Lemma 2.3. Then for each $\omega \in \Omega_0^*$ and for all positive real numbers M and η, one has*
(5.2)

$$\sup_{(t,s)\in[-M,M]^2} \left\{ \frac{\big|\big(Y(t,\omega) - Y(s,\omega)\big|}{|t-s|^{H(t)\vee H(s)-1/\alpha}\big(1 + \big|\log|t-s|\big|\big)^{2/\alpha+\eta} + \big|H(t) - H(s)\big|} \right\} < \infty.$$

Proof. The claim follows easily from (5.1) and Corollary 4.3, in which one takes $q = 0$, $a = \min_{x\in[-M,M]} H(x)$, and $b = \max_{x\in[-M,M]} H(x)$. □

Remark 5.2. (i) Theorem 5.1 remains valid under the weaker condition that $H(\cdot)$ is a continuous function on the real line with values in the open interval $(1/\alpha, 1)$; indeed, even in this case, $H\big([-M, M]\big)$ is a compact interval contained in $(1/\alpha, 1)$.

(ii) A straightforward consequence of Theorem 5.1 is that LMSM has a version with almost surely continuous paths, as soon as its functional Hurst parameter $H(\cdot)$ is a continuous function with values in $(1/\alpha, 1)$. This proves the conjecture made by Stoev and Taqqu in Remark 1 on page 166 of [29].

The following corollary follows easily from Theorem 5.1.

Corollary 5.3. (i) *Assume that for some real numbers $M_1 < M_2$, one has, for each $\eta > 0$,*

$$\sup_{(t,s)\in[M_1,M_2]^2} \frac{\big|H(t) - H(s)\big|}{|t-s|^{H(t)\vee H(s)-1/\alpha}\big(1 + \big|\log|t-s|\big|\big)^{2/\alpha+\eta}} < \infty. \tag{5.3}$$

Then it follows that, for all $\omega \in \Omega_0^*$ and $\eta > 0$,

$$\sup_{(t,s)\in[M_1,M_2]^2} \left\{ \frac{\big|Y(t,\omega) - Y(s,\omega)\big|}{|t-s|^{H(t)\vee H(s)-1/\alpha}\big(1 + \big|\log|t-s|\big|\big)^{2/\alpha+\eta}} \right\} < \infty. \tag{5.4}$$

(ii) *Assume that for some real numbers $M_1 < M_2$, one has, for each $\eta > 0$,*

$$\sup_{(t,s)\in[M_1,M_2]^2} \frac{\big|H(t) - H(s)\big|}{|t-s|^{\min_{x\in[M_1,M_2]} H(x)-1/\alpha}\big(1 + \big|\log|t-s|\big|\big)^{2/\alpha+\eta}} < \infty. \tag{5.5}$$

Then it follows that, for all $\omega \in \Omega_0^*$ and $\eta > 0$,

$$\sup_{(t,s)\in[M_1,M_2]^2} \left\{ \frac{\big|Y(t,\omega) - Y(s,\omega)\big|}{|t-s|^{\min_{x\in[M_1,M_2]} H(x)-1/\alpha}\big(1 + \big|\log|t-s|\big|\big)^{2/\alpha+\eta}} \right\} < \infty. \tag{5.6}$$

Remark 5.4. (i) Condition (5.3) is satisfied if

$$H(\cdot) \in \mathcal{C}^{\max_{x\in[M_1,M_2]} H(x)-1/\alpha}\big([M_1,M_2],\mathbb{R}\big).$$

(ii) Condition (5.5) is satisfied if

$$H(\cdot) \in \mathcal{C}^{\min_{x\in[M_1,M_2]} H(x)-1/\alpha}\big([M_1,M_2],\mathbb{R}\big).$$

Now we determine a local modulus of continuity for $\{Y(t) : t \in \mathbb{R}\}$.

Theorem 5.5. *Assume that the skewness intensity function $\beta(\cdot)$ of the $\mathcal{S}t\alpha\mathcal{S}$ measure $Z_\alpha(ds)$ is constant. Let $t_0 \in \mathbb{R}$ be arbitrary and fixed. Then, one has, almost surely, for all positive real numbers M and η,*

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|Y(t) - Y(t_0)\big|}{|t-t_0|^{H(t_0)}\big(1 + \big|\log|t-t_0|\big|\big)^{1/\alpha+\eta} + \big|H(t) - H(t_0)\big|} \right\} < \infty. \tag{5.7}$$

Proof. First observe that for any fixed $t_0 \in \mathbb{R}$, the process $\big\{X(t,H(t_0)) : t \in \mathbb{R}\big\}$ has stationary increments since it is a LFSM with Hurst parameter $H(t_0)$. Hence, the processes $\big\{X(t,H(t_0)) - X(t_0,H(t_0)) : t \in \mathbb{R}\big\}$ and $\big\{X(t-t_0,H(t_0)) : t \in \mathbb{R}\big\}$ have the same finite-dimensional distributions. Therefore, using their path continuity, and the fact that the set of dyadic numbers in $[-M,M]$ is dense in $[-M,M]$, it follows that the random variables

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t_0)) - X(t_0,H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1 + \big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\}$$

and

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t-t_0,H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1 + \big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\}$$

are equal in law. Thus, taking $q = 0$ and a, b such that $H(t_0) \in [a,b]$ in Proposition 4.2, one gets that, almost surely,

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t_0)) - X(t_0,H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1 + \big|\log|t-t_0|\big|\big)^{1/\alpha+\eta}} \right\} < \infty. \tag{5.8}$$

On the other hand, taking $q = 0$, $a = \underline{H} := \inf_{x\in\mathbb{R}} H(x)$, and $b = \overline{H} := \sup_{x\in\mathbb{R}} H(x)$ in (3.18), one obtains that

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t)) - X(t,H(t_0))\big|}{\big|H(t) - H(t_0)\big|} \right\} < \infty. \tag{5.9}$$

Finally, combining (5.1), (5.8), and (5.9), it follows that (5.7) holds. □

The following result is a straightforward consequence of Theorem 5.5.

Corollary 5.6. *Assume that the skewness intensity function $\beta(\cdot)$ of the $\mathcal{S}t\alpha\mathcal{S}$ measure $Z_\alpha(ds)$ is constant. Also assume that $t_0 \in \mathbb{R}$ is such that, for each $\eta > 0$, one has, for all $t \in \mathbb{R}$,*

$$\big|H(t) - H(t_0)\big| \le c\,|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha+\eta}, \tag{5.10}$$

where $c > 0$ is a constant depending only on t_0 and η. Then, one has, almost surely, for each positive real numbers M and η,

$$\sup_{t\in[-M,M]}\left\{\frac{\big|Y(t) - Y(t_0)\big|}{|t - t_0|^{H(t_0)}\big(1 + \big|\log|t - t_0|\big|\big)^{1/\alpha+\eta}}\right\} < \infty. \tag{5.11}$$

6. Quasi-optimality of the global modulus of continuity for LMSM

The goal of this section is to show that, under some conditions a bit stronger than (5.5), the global modulus of continuity given in (5.6) is quasi-optimal. More precisely:

Theorem 6.1. *Assume that $M_1 < M_2$ are two arbitrary fixed real numbers such that the condition*

$(\mathcal{A})$: $H(\cdot)$ belongs to the Hölder space $\mathcal{C}^{\gamma_*}\big([M_1, M_2], \mathbb{R}\big)$ for some

$$\gamma_* \in \Big(\min_{x\in[M_1,M_2]} H(x) - 1/\alpha, 1\Big],$$

is satisfied. Set

$$\rho := \sup\Big\{\theta \in \mathbb{R}_+ : \exists\, t_0 \in [M_1, M_2] \text{ s.t. } H(t_0) = \min_{x\in[M_1,M_2]} H(x) \text{ and } \sup_{t\in[M_1,M_2]} |H(t) - H(t_0)|/|t - t_0|^{\theta} < \infty\Big\} \tag{6.1}$$

and

$$\tau := \frac{1 + 2\alpha^{-1}}{\alpha\rho - 1}, \tag{6.2}$$

with the convention that $\tau := 0$ when $\rho = +\infty$. Assume that

$$\alpha\rho > 1. \tag{6.3}$$

Then τ is a well-defined nonnegative real number, and one has, almost surely, for all $\eta > 0$,

$$\sup_{(t,s)\in[M_1,M_2]^2}\left\{\frac{|Y(t) - Y(s)|}{|t - s|^{\min_{x\in[M_1,M_2]} H(x) - 1/\alpha}\big(1 + \big|\log|t - s|\big|\big)^{-\tau-\eta}}\right\} = \infty. \tag{6.4}$$

Remark 6.2. Notice that the conditions $(\mathcal{A})$ and (6.3) are satisfied when $H(\cdot)$ belongs to the Hölder space $\mathcal{C}^\gamma\big([M_1, M_2], \mathbb{R}\big)$, for some $\gamma > 1/\alpha$.

In order to prove Theorem 6.1, we need some preliminary results. First we introduce $\widetilde{\Psi}$ the real-valued deterministic continuous function defined, for all $(x, v) \in \mathbb{R} \times (1/\alpha, 1)$, by

$$\widetilde{\Psi}(x, v) := \frac{1}{\Gamma(v + 1 - 1/\alpha)\,\Gamma(1/\alpha - v + 1)} \int_{\mathbb{R}} (s - x)_+^{1/\alpha - v}\, \psi^{(2)}(s)\, ds, \tag{6.5}$$

where $\psi^{(2)}$ is the second derivative of the Daubechies mother wavelet ψ defined at the very beginning of Section 2, and where Γ is the usual Gamma function. Also, recall the definition of $(\cdot)_+^{1/\alpha - v}$ given in (1.2). By using a result in [26] concerning Fourier transforms of right-sided fractional derivatives, one has for each $(\xi, v) \in \mathbb{R} \times (1/\alpha, 1)$,

$$\widehat{\widetilde{\Psi}}(\xi, v) = \frac{1}{\Gamma(v + 1 - 1/\alpha)} |\xi|^{v + 1 - 1/\alpha}\, e^{-i\,\mathrm{sgn}(\xi)\,(v + 1 - 1/\alpha)\pi/2}\, \widehat{\psi}(\xi), \tag{6.6}$$

where $\widehat{\widetilde{\Psi}}(\cdot, v)$ denotes the Fourier transform of the function $\widetilde{\Psi}(\cdot, v)$. We now give some useful properties of the function $\widetilde{\Psi}$.

Proposition 6.3. *The function $\widetilde{\Psi}$ has the following three properties.*

(i) *For all real numbers a and b such that $1 > b > a > 1/\alpha$, the function $\widetilde{\Psi}$ is well-localized in the variable x, uniformly in the variable $v \in [a, b]$. Namely, one has*

$$\sup_{(x,v) \in \mathbb{R} \times [a,b]} (3 + |x|)^2 \left|\widetilde{\Psi}(x, v)\right| < \infty. \tag{6.7}$$

(ii) *For any fixed $v \in \big(1/\alpha, 1\big)$, the first moment of the function $\widetilde{\Psi}(\cdot, v)$ vanishes, which means that*

$$\int_{\mathbb{R}} \widetilde{\Psi}(x, v)\, dx = 0. \tag{6.8}$$

(iii) *Let Ψ be the function defined in (2.3). Then, for each fixed $v \in \big(1/\alpha, 1\big)$, the system of functions $\big\{2^{j/2}\Psi(2^j \cdot - k, v) : (j, k) \in \mathbb{Z}^2\big\}$ and $\big\{2^{j/2}\widetilde{\Psi}(2^j \cdot - k, v) : (j, k) \in \mathbb{Z}^2\big\}$ is biorthogonal. This means that, for any $j \in \mathbb{Z}$, $j' \in \mathbb{Z}$, $k \in \mathbb{Z}$, and $k' \in \mathbb{Z}$, one has,*

$$2^{(j + j')/2} \int_{\mathbb{R}} \Psi(2^j t - k, v)\, \widetilde{\Psi}(2^{j'} t - k', v)\, dt = \delta_{(j,k;j',k')}, \tag{6.9}$$

where $\delta_{(j,k;j',k')} = 1$ if $(j, k) = (j', k')$ and equals 0 otherwise.

Proof. Part (i) can be obtained by using the fact that

$$\sup_{v\in[a,b]}\Big\{\frac{1}{\Gamma(v+1-1/\alpha)\,\Gamma(1/\alpha-v+1)}\Big\}<\infty$$

and a method similar to the one used in the proof of part (ii) of Proposition 2.4. In view of the definition of a Fourier transform, taking $\xi = 0$ in (6.6) one gets (ii). Now we prove (iii). Using the Parseval formula, (2.4), and (6.6), one obtains, for all $(j,k)\in\mathbb{Z}^2$,

$$\begin{aligned}
&\int_{\mathbb{R}} 2^{j/2}\Psi(2^j t-k,v)2^{j'/2}\widetilde{\Psi}(2^{j'}t-k',v)\,dt\\
&\quad= 2^{-(j+j')/2}(2\pi)^{-1}\int_{\mathbb{R}} e^{-i\xi(k/2^j-k'/2^{j'})}\widehat{\Psi}(2^{-j}\xi,v)\overline{\widehat{\widetilde{\Psi}}(2^{-j'}\xi,v)}d\xi\\
&\quad= 2^{-(j+j')/2+(j-j')(v+1-1/\alpha)}(2\pi)^{-1}\int_{\mathbb{R}} e^{-i\xi(k/2^j-k'/2^{j'})}\widehat{\psi}(2^{-j}\xi)\overline{\widehat{\psi}(2^{-j'}\xi)}d\xi\\
&\quad= 2^{(j-j')(v+1-1/\alpha)}\int_{\mathbb{R}} 2^{j/2}\psi(2^j t-k)2^{j'/2}\psi(2^{j'}t-k')\,dt=\delta_{(j,k;j',k')},
\end{aligned}$$

where the last equality results from the fact that $\big\{2^{j/2}\psi(2^j\cdot -k):(j,k)\in\mathbb{Z}^2\big\}$ is an orthonormal basis of $L^2(\mathbb{R})$. □

In the rest of this section, $M_1<M_2$ denote two arbitrary real numbers such that the conditions $(\mathcal{A})$ and (6.3) hold. For the sake of simplicity, we set

$$H_* := \min_{x\in[M_1,M_2]} H(x). \tag{6.10}$$

Lemma 6.4. *Let Ω_0^* be the event of probability 1 defined in Lemma 2.3 and let $\big\{g_{j,k}:(j,k)\in\mathbb{N}\times\mathbb{Z}\big\}$ be the sequence of the random variables defined on Ω_0^* by*

$$g_{j,k}=2^{j(1+H_*)}\int_{\mathbb{R}} Y(t)\,\widetilde{\Psi}(2^j t-k,H_*)\,dt. \tag{6.11}$$

Assume that there exists $\omega_0\in\Omega_0^*$, $\tau_0>\tau$, and $\eta_0>0$ such that

$$\sup_{(t,s)\in[M_1,M_2]^2}\frac{\big|Y(t,\omega_0)-Y(s,\omega_0)\big|}{|t-s|^{H_*-1/\alpha}\big(1+\big|\log|t-s|\big|\big)^{-\tau_0-\eta_0}}<\infty. \tag{6.12}$$

Then one has

$$\begin{aligned}
&\limsup_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\big\{\big|g_{j,k}(\omega_0)\big|: k\in\mathbb{Z} \text{ and } M_1+2^{-j/(2\alpha)}\le k/2^j\le M_2-2^{-j/(2\alpha)}\big\}\\
&\qquad=0.
\end{aligned}\tag{6.13}$$

Remark 6.5. Notice that (6.7) (in which one takes a and b such that $H_*\in[a,b]$), Proposition 4.2 (in which one takes $q=0$, $a=\underline{H}:=\inf_{x\in\mathbb{R}}H(x)$, $b=\overline{H}:=\sup_{x\in\mathbb{R}}H(x)$, and η an arbitrary positive real number) and the relation (5.1) imply that the random variables $g_{j,k}$ are well-defined and finite on Ω_0^*.

Proof. In all that follows, we assume that $j \in \mathbb{N}$ and $k \in \mathbb{Z}$ are arbitrary and satisfy

$$M_1 + 2^{-j/(2\alpha)} \le \frac{k}{2^j} \le M_2 - 2^{-j/(2\alpha)}. \tag{6.14}$$

It follows from (6.11) and (6.8), in which one takes $v = H_*$, that

$$g_{j,k}(\omega_0) = 2^{j(1+H_*)} \int_{\mathbb{R}} \big(Y(t,\omega_0) - Y(k2^{-j},\omega_0)\big)\, \widetilde{\Psi}(2^j t - k, H_*)\, dt. \tag{6.15}$$

In order to bound conveniently $\big|g_{j,k}(\omega_0)\big|$, we split $\mathbb{R}$ into the three disjoint subdomains

$$\mathcal{B}_1 := [M_1, M_2],\ \mathcal{B}_2 := [-2M_0, 2M_0]\backslash[M_1, M_2] \text{ and } \mathcal{B}_3 := \mathbb{R}\backslash[-2M_0, 2M_0], \tag{6.16}$$

where $M_0 := |M_1| + |M_2|$. Therefore (6.15) implies that

$$\big|g_{j,k}(\omega_0)\big| \le \sum_{l=1}^{3} A^l_{j,k}(\omega_0), \tag{6.17}$$

where, for each $l \in \{1,2,3\}$, one has set,

$$A^l_{j,k}(\omega_0) = 2^{j(1+H_*)} \int_{\mathcal{B}_l} \big|Y(t,\omega_0) - Y(k2^{-j},\omega_0)\big|\, \big|\widetilde{\Psi}(2^j t - k, H_*)\big|\, dt. \tag{6.18}$$

First, we show that (6.13) holds when $|g_{j,k}(\omega_0)|$ is replaced by $A^1_{j,k}(\omega_0)$. The relation (6.12) and the change of variable $u = 2^j t - k$, yield

$$\begin{aligned}
A^1_{j,k}(\omega_0) &\le C_1(\omega_0)\, 2^{j(1+H_*)} \int_{\mathcal{B}_1} |t - k2^{-j}|^{H_*-1/\alpha}\big(1 + \big|\log|t-k2^{-j}|\big|\big)^{-\tau_0-\eta_0} \times \\
&\qquad\qquad |\widetilde{\Psi}(2^j t - k, H_*)|\, dt \\
&\le C_1(\omega_0)\, 2^{j(1+H_*)} \int_{\mathbb{R}} |t - k2^{-j}|^{H_*-1/\alpha}\big(1 + \big|\log|t-k2^{-j}|\big|\big)^{-\tau_0-\eta_0} \times \\
&\qquad\qquad |\widetilde{\Psi}(2^j t - k, H_*)|\, dt \\
&= C_1(\omega_0)\, 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_*-1/\alpha}\big(1 + \big|\log|2^{-j}u|\big|\big)^{-\tau_0-\eta_0} |\widetilde{\Psi}(u, H_*)|\, du \\
&= C_1(\omega_0)\, j^{-\tau_0-\eta_0} 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_*-1/\alpha}\Big(\frac{1}{j} + \Big|\log(2) - \frac{\log|u|}{j}\Big|\Big)^{-\tau_0-\eta_0} |\widetilde{\Psi}(u, H_*)|\, du,
\end{aligned} \tag{6.19}$$

where

$$C_1(\omega_0) := \sup_{(t,s)\in\mathcal{B}_1^2} \frac{\big|Y(t,\omega_0) - Y(s,\omega_0)\big|}{|t-s|^{H_*-1/\alpha}\big(1 + \big|\log|t-s|\big|\big)^{-\tau_0-\eta_0}} < \infty.$$

Now we show that

$$\sup_{j\ge 1}\int_{\mathbb{R}} |u|^{H_*-1/\alpha}\Big(\frac{1}{j}+\Big|\log(2)-\frac{\log|u|}{j}\Big|\Big)^{-\tau_0-\eta_0}|\widetilde{\Psi}(u,H_*)|\,du<\infty. \tag{6.20}$$

In view of (6.7) and the inequality

$$\Big(\frac{1}{j}+\Big|\log(2)-\frac{\log|u|}{j}\Big|\Big)^{-\tau_0-\eta_0}\le\Big(\frac{\log 2}{2}\Big)^{-\tau_0-\eta_0},$$

which holds for all real number u satisfying $|u|\le 2^{j/2}$, one gets, for some constants $c_2,\dots,c_5$ and all integers $j\ge 1$, that

$$\begin{aligned}
&\int_{\mathbb{R}} |u|^{H_*-1/\alpha}\Big(\frac{1}{j}+\Big|\log(2)-\frac{\log|u|}{j}\Big|\Big)^{-\tau_0-\eta_0}|\widetilde{\Psi}(u,H_*)|\,du\\
&\le c_2\, j^{\tau_0+\eta_0}\int_{|u|>2^{j/2}}\frac{|u|^{H_*-1/\alpha}}{(3+|u|)^2}du+c_2\Big(\frac{\log 2}{2}\Big)^{-\tau_0-\eta_0}\int_{|u|\le 2^{j/2}}\frac{|u|^{H_*-1/\alpha}}{(3+|u|)^2}\,du\\
&\le 2c_2\, j^{\tau_0+\eta_0}\int_{u>2^{j/2}}\frac{u^{H_*-1/\alpha}}{(3+u)^2}du+c_3\int_{\mathbb{R}}\frac{|u|^{H_*-1/\alpha}}{(3+|u|)^2}du\\
&\le c_4\, j^{\tau_0+\eta_0}\,\frac{2^{-j/2(1+1/\alpha-H_*)}}{1+1/\alpha-H_*}+c_5,
\end{aligned}$$

which shows that (6.20) is satisfied. Next, (6.19) and (6.20) entail that

$$\limsup_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\big\{A^1_{j,k}(\omega_0):k\in\mathbb{Z}\text{ and }M_1+2^{-\frac{j}{2\alpha}}\le k/2^j\le M_2-2^{-\frac{j}{2\alpha}}\big\}=0. \tag{6.21}$$

Next, we prove that (6.13) holds when $\big|g_{j,k}(\omega_0)\big|$ is replaced by $A^2_{j,k}(\omega_0)$. Set

$$C_6(\omega_0):=\sup_{t\in[-2M_0,2M_0]}\big|Y(t,\omega_0)\big|<\infty. \tag{6.22}$$

Observe that $C_6(\omega_0)$ is finite, since the function $t\mapsto Y(t,\omega_0)$ is continuous on the compact interval $[-2M_0,2M_0]$. Also, observe that, in view of (6.14) and (6.16), one has that, for all $t\in\mathcal{B}_2$,

$$|2^jt-k|>2^{j(1-1/(2\alpha))}.$$

Therefore, it follows from (6.7), that, for each $t\in\mathcal{B}_2$,

$$|\widetilde{\Psi}(2^jt-k,H_*)|\le c_7\,2^{-j(2-1/\alpha)}, \tag{6.23}$$

where c_7 is a constant not depending on t, j, and k. Combining (6.18), (6.22), and (6.23), one gets that

$$A^2_{j,k}(\omega_0)\le C_8(\omega_0)\,2^{-j(1-H_*-1/\alpha)},$$

where $C_8(\omega_0)$ is a constant not depending on j and k. This last inequality and the inequality $H_* < 1$, imply that

(6.24)
$$\limsup_{j\to+\infty} j^{\tau_0} 2^{-j/\alpha} \max\left\{A^2_{j,k}(\omega_0) : k\in\mathbb{Z} \text{ and } M_1+2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2-2^{-\frac{j}{2\alpha}}\right\} = 0.$$

Next, we prove that (6.13) holds when $\left|g_{j,k}(\omega_0)\right|$ is replaced by $A^3_{j,k}(\omega_0)$. Observe that, by using the triangle inequality, (6.14), and (6.16), one has, for each $t\in\mathcal{B}_3$,

$$|2^j t - k| = 2^j\left|t-\frac{k}{2^j}\right| \ge 2^j\left(|t|-\frac{|k|}{2^j}\right) > 2^j\left(|t|-M_0\right) > 2^{j-1}|t|.$$

Therefore, it follows from (6.7), that, for each $t\in\mathcal{B}_3$,

$$|\widetilde{\Psi}(2^j t-k, H_*)| \le c_9\, 2^{-2j}\, |t|^{-2}, \tag{6.25}$$

where c_9 is a constant not depending on t, j, and k. On the other hand, using (5.1) and Proposition 4.2, in the case where $q=0$, $a=\underline{H}:=\inf_{x\in\mathbb{R}} H(x)$, and $b=\overline{H}:=\sup_{x\in\mathbb{R}} H(x)$, one obtains that, for any fixed $\eta>0$, and for each $t\in\mathcal{B}_3$,

$$|Y(t,\omega_0)| \le C_{10}(\omega_0)\, |t|^{\overline{H}} \left(1+\left|\log|t|\right|\right)^{1/\alpha+\eta},$$

where $C_{10}(\omega_0)$ is a positive finite constant not depending on t. Next, combining the last inequality with (6.14) and (6.22), one gets that, for all $j\in\mathbb{N}$ and $k\in\mathbb{Z}$ satisfying (6.14), and for each $t\in\mathcal{B}_3$, one has,

$$|Y(t,\omega_0)-Y(k2^{-j},\omega_0)| \le C_{11}(\omega_0)\, |t|^{\overline{H}} \left(1+\left|\log|t|\right|\right)^{1/\alpha+\eta}, \tag{6.26}$$

where $C_{11}(\omega_0)$ is a constant not depending on j, k, and t. Next, (6.18), (6.25), and (6.26), yield

$$A^3_{j,k}(\omega_0) \le C_{12}(\omega_0)\, 2^{-(1-H_*)j},$$

where $C_{12}(\omega_0)$ is a constant not depending on j and k. Moreover, this last inequality implies that

(6.27)
$$\limsup_{j\to+\infty} j^{\tau_0} 2^{-j/\alpha} \max\left\{A^3_{j,k}(\omega_0) : k\in\mathbb{Z} \text{ and } M_1+2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2-2^{-\frac{j}{2\alpha}}\right\} = 0.$$

Finally, combining (6.17), (6.21), (6.24), and (6.27), it follows that (6.13) holds. □

Lemma 6.6. *Let Ω_0^* be the event of probability 1 defined in Lemma 2.3 and let $\{\widetilde{g}_{j,k} : (j,k)\in\mathbb{N}\times\mathbb{Z}\}$ be the sequence of random variables defined on Ω_0^* by*

$$\widetilde{g}_{j,k} = 2^{j(1+H_*)}\int_{\mathbb{R}} X(t, H(k2^{-j}))\, \widetilde{\Psi}(2^j t-k, H_*)\, dt. \tag{6.28}$$

Assume that $H(\cdot)$ satisfies condition $(\mathcal{A})$. Then one has

$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)} \max\Big\{\left|g_{j,k}(\omega)-\widetilde{g}_{j,k}(\omega)\right| : k\in\mathbb{Z} \text{ and } M_1+2^{-j/(2\alpha)} \le k/2^j \le M_2-2^{-j/(2\alpha)}\Big\} = 0, \tag{6.29}$$

for each $\omega\in\Omega_0^*$ and all $\theta\in[0,\min\{\gamma_*+1/\alpha-H_*, 1-H_*\})$.

Remark 6.7. Notice that (6.7) (in which one takes a and b such that $H_* \in [a,b]$) and Proposition 4.2 (in which one takes $q=0$, $a=\underline{H}$, $b=\overline{H}$, and η to be an arbitrary positive real number) imply that the random variables $\widetilde{g}_{j,k}$ are well-defined and finite on Ω_0^*.

Proof. In what follows, we assume that $j\in\mathbb{N}$ and $k\in\mathbb{Z}$ are arbitrary and satisfy (6.14). Using (5.1), (6.11), and (6.28), one has

$$|g_{j,k}(\omega)-\widetilde{g}_{j,k}(\omega)| \le \sum_{l=1}^{3} L_{j,k}^{l}(\omega), \tag{6.30}$$

where for all $l\in\{1,2,3\}$,

$$L_{j,k}^{l}(\omega) = 2^{j(1+H_*)}\int_{\mathcal{B}_l} |X(t,H(t),\omega)-X(t,H(k2^{-j}),\omega)|\,|\widetilde{\Psi}(2^jt-k,H_*)|\,dt; \tag{6.31}$$

the sets $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ were defined in (6.16). Now we prove that (6.29) holds when $\big|g_{j,k}(\omega)-\widetilde{g}_{j,k}(\omega)\big|$ is replaced by $L_{j,k}^{1}(\omega)$. It follows from the definition of $\mathcal{B}_1$, (3.18) (in which one takes $q=0$, $M=M_0$, $a=\underline{H}$, and $b=\overline{H}$), (6.14), condition $(\mathcal{A})$, and the change of variable $u=2^jt-k$, that

$$\begin{aligned} L_{j,k}^{1}(\omega) &\le C_1(\omega)\,2^{j(1+H_*)}\int_{\mathcal{B}_1} |H(t)-H(k2^{-j})|\,|\widetilde{\Psi}(2^jt-k,H_*)|\,dt \\ &\le C_2(\omega)\,2^{j(1+H_*)}\int_{\mathcal{B}_1} |t-k2^{-j}|^{\gamma_*}\,|\widetilde{\Psi}(2^jt-k,H_*)|\,dt \\ &\le C_2(\omega)\,2^{j(1+H_*)}\int_{\mathbb{R}} |t-k2^{-j}|^{\gamma_*}\,|\widetilde{\Psi}(2^jt-k,H_*)|\,dt \\ &= C_2(\omega)\,2^{jH_*}\int_{\mathbb{R}} |2^{-j}u|^{\gamma_*}\,|\widetilde{\Psi}(u,H_*)|\,du \le C_3(\omega)\,2^{j(H_*-\gamma_*)}, \end{aligned} \tag{6.32}$$

where the positive and finite constants $C_1(\omega)$, $C_2(\omega)$, and $C_3(\omega)$ do not depend on j and k. Then, using (6.32) and the inequality $\theta<\gamma_*+1/\alpha-H_*$, one gets

$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)}\max\Big\{L_{j,k}^{1}(\omega) : k\in\mathbb{Z} \text{ and } M_1+2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2-2^{-\frac{j}{2\alpha}}\Big\}=0. \tag{6.33}$$

Next, we prove that (6.29) holds when $\big|g_{j,k}(\omega)-\widetilde{g}_{j,k}(\omega)\big|$ is replaced by $L_{j,k}^{2}(\omega)$. Set

$$C_4(\omega) := \sup_{(u,v)\in[-2M_0,2M_0]\times[\underline{H},\overline{H}]} \big|X(u,v,\omega)\big| < \infty. \tag{6.34}$$

Observe that $C_4(\omega)$ is finite, since the function $(u,v)\mapsto X(u,v,\omega)$ is continuous on the compact rectangle $[-2M_0,2M_0]\times[\underline{H},\overline{H}]$. Combining (6.31), (6.34), and (6.23), one obtains that

$$L_{j,k}^{2}(\omega) \le C_5(\omega)\,2^{-j(1-H_*-1/\alpha)}, \tag{6.35}$$

where $C_5(\omega)$ is a constant not depending on j and k. Then, using (6.35) and the inequality $\theta < 1 - H_*$, it follows that

(6.36)
$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)} \max\Big\{L^2_{j,k}(\omega) : k \in \mathbb{Z} \text{ and } M_1 + 2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2 - 2^{-\frac{j}{2\alpha}}\Big\} = 0.$$

Next, we prove that (6.29) holds, when $\big|g_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)\big|$ is replaced by $L^3_{j,k}(\omega)$. Setting $q = 0$, $a = \underline{H}$, and $b = \overline{H}$ in Proposition 4.2, one gets that, for any fixed $\eta > 0$ and for each $t \in \mathcal{B}_3$,

$$|X(t, H(t), \omega) - X(t, H(k2^{-j}), \omega)| \le C_6(\omega)\, |t|^{\overline{H}} \left(1 + \big|\log|t|\big|\right)^{1/\alpha+\eta},$$

where $C_6(\omega)$ is a constant not depending on t and (j,k). Next, combining this last inequality with (6.31) and (6.25), it follows that

$$L^3_{j,k}(\omega) \le C_7(\omega)\, 2^{-(1-H_*)j}, \tag{6.37}$$

where $C_7(\omega)$ is a constant not depending on j and k. Then, using (6.37) and the inequality $\theta < 1 - H_*$, it follows that

(6.38)
$$\limsup_{j\to+\infty} 2^{j(\theta-1/\alpha)} \max\Big\{L^3_{j,k}(\omega) : k \in \mathbb{Z} \text{ and } M_1 + 2^{-\frac{j}{2\alpha}} \le k/2^j \le M_2 - 2^{-\frac{j}{2\alpha}}\Big\} = 0.$$

Finally, combining (6.30), (6.33), (6.36), and (6.38), it follows that (6.29) holds. □

Proposition 6.8. *Let Ω_0^* be the event of probability 1 defined in Lemma 2.3. Then for all $\omega \in \Omega_0^*$, $v \in (1/\alpha, 1)$ and $(j,k) \in \mathbb{Z}^2$, one has*

$$2^{j(1+v)} \int_{\mathbb{R}} X(t, v, \omega) \widetilde{\Psi}(2^j t - k, v)\, dt = \epsilon_{j,k}(\omega), \tag{6.39}$$

where $\epsilon_{j,k}$ is the random variable defined in (2.5).

Proof. First observe that, by using (6.7) and (4.2) in which one takes $q = 0$ and a and b such that $v \in [a,b]$, it follows that, for all $\omega \in \Omega_0^*$ and $(j,k) \in \mathbb{Z}^2$,

$$2^{j(1+v)} \Bigg(\sum_{(j',k')\in\mathbb{Z}^2} 2^{-j'v} |\epsilon_{j',k'}(\omega)| \big|\Psi(2^{j'} \cdot - k', v) - \Psi(-k', v)\big| \Bigg) \times \big|\widetilde{\Psi}(2^j \cdot - k, v)\big| \in L^1_t(\mathbb{R}).$$

Therefore we can apply the dominated convergence theorem, and we obtain, in view of part (i) of Theorem 2.1 that

$$\begin{aligned} &2^{j(1+v)} \int_{\mathbb{R}} X(t, v, \omega) \widetilde{\Psi}(2^j t - k, v)\, dt \\ &= 2^{j(1+v)} \sum_{(j',k')\in\mathbb{Z}^2} 2^{-j'v} \epsilon_{j',k'}(\omega) \int_{\mathbb{R}} \big(\Psi(2^{j'} t - k', v) - \Psi(-k', v)\big) \widetilde{\Psi}(2^j t - k, v)\, dt. \end{aligned}$$

Finally, combining this equality with (ii) and (iii) of Proposition 6.3, one gets (6.39). □

Remark 6.9. Let τ and ρ be as in Theorem 6.1. Also suppose that (6.3) holds. We denote by τ_0 an arbitrary real number such that $\tau_0 > \tau \geq 0$.

(i) One has

$$\frac{1+2\alpha^{-1}+\tau_0}{\rho} < \alpha\tau_0. \tag{6.40}$$

(ii) Denote by $d(\tau_0)$ and $e(\tau_0)$ the positive real numbers defined by

$$d(\tau_0) := \frac{2}{3}\Big(\frac{1+2\alpha^{-1}+\tau_0}{\rho}\Big) + \frac{1}{3}(\alpha\tau_0) \text{ and } e(\tau_0) := \frac{1}{3}\Big(\frac{1+2\alpha^{-1}+\tau_0}{\rho}\Big) + \frac{2}{3}(\alpha\tau_0). \tag{6.41}$$

Then,

$$\frac{1+2\alpha^{-1}+\tau_0}{\rho} < d(\tau_0) < e(\tau_0) < \alpha\tau_0. \tag{6.42}$$

(iii) For any fixed $t_0 \in [M_1, M_2]$ and $j \in \mathbb{N}$, denote by $D_j(t_0,\tau_0)$ the set of indices, defined by

$$D_j(t_0,\tau_0) := \big\{k \in \mathbb{Z}\colon k2^{-j} \in [M_1,M_2] \text{ and } j^{-e(\tau_0)} \leq |t_0 - k2^{-j}| \leq j^{-d(\tau_0)}\big\}. \tag{6.43}$$

Then, for all large enough j, the set $D_j(t_0,\tau_0)$ is nonempty and satisfies

$$D_j(t_0,\tau_0) \subseteq \big\{k \in \mathbb{Z} : M_1 + 2^{-j/(2\alpha)} \leq k/2^j \leq M_2 - 2^{-j/(2\alpha)}\big\}. \tag{6.44}$$

Proof. Observe that, in view of (6.2), one has

$$\frac{1+2\alpha^{-1}+\tau}{\rho} = \alpha\tau;$$

therefore, (6.3) implies that (i) holds. Part (ii) follows easily from (6.40) and (6.41). Now we prove (iii). For the sake of simplicity, we set $d = d(\tau_0)$ and $e = e(\tau_0)$. Observe that, since $\lim_{j\to+\infty} 2^j(j^{-d} - j^{-e}) = +\infty$, the set $D_j(t_0,\tau_0)$ is nonempty for all large enough j. Let $j \geq 1$ and k be arbitrary integers such that j is large enough and $k \in D_j(t_0,\tau_0)$.

In order to show that these integers satisfy (6.14), we study three cases: $t_0 \in (M_1, M_2)$, $t_0 = M_1$, and $t_0 = M_2$.

First suppose that $M_1 < t_0 < M_2$, i.e., $\min\{t_0 - M_1, M_2 - t_0\} > 0$. Then, in view of the fact that j is large enough, one can assume that $j^{-d} + 2^{-j/(2\alpha)} \leq \min\{t_0 - M_1, M_2 - t_0\}$; this inequality and the inequality $|t_0 - k2^{-j}| \leq j^{-d}$ imply that (6.14) holds.

Now assume that $t_0 = M_1$. It follows from the equality $|t_0 - k2^{-j}| = k2^{-j} - M_1$ and the inequalities $j^{-e} \leq |t_0 - k2^{-j}| \leq j^{-d}$, that $M_1 + j^{-e} \leq k2^{-j} \leq M_1 + j^{-d}$. Moreover, in view of the fact that j is large enough, one can assume $M_1 + j^{-e} \geq M_1 + 2^{-j/(2\alpha)}$ and $M_1 + j^{-d} \leq M_2 - 2^{-j/(2\alpha)}$; thus (6.14) holds. Finally, the case where $t_0 = M_2$, can be treated in a manner similar to the case $t_0 = M_1$. □

Lemma 6.10. *Let τ be as in Theorem 6.1. Also suppose that (6.3) holds. We denote by τ_0 an arbitrary fixed real number such that $\tau_0 > \tau \geq 0$. Then, for all $t_0 \in [M_1, M_2]$, there exists $\Omega^*_{1,\tau_0}(t_0)$ an event of probability 1 (which a priori depends on τ_0 and t_0) contained in Ω^*_0 (recall that this event was defined in Lemma 2.3), such that, for each $\omega \in \Omega^*_{1,\tau_0}(t_0)$, one has*

$$\liminf_{j\to+\infty} j^{\tau_0}\, 2^{-j/\alpha} \max\big\{\big|\epsilon_{j,k}(\omega)\big| : k \in D_j(t_0,\tau_0)\big\} > 0, \tag{6.45}$$

where the $\epsilon_{j,k}$ are the random variables defined in (2.5) and where $D_j(t_0,\tau_0)$ is the set defined in (6.43).

Proof. Let p be a fixed integer such that $p > 2R$ (see (2.2) for the definition of R). We assume that j is sufficiently large that

$$\begin{aligned} \overline{D_j}(t_0,\tau_0) &:= \big\{q \in \mathbb{Z} \,:\, pq \in D_j(t_0,\tau_0)\big\} \\ &= \big\{q \in \mathbb{Z} \,:\, pq/2^j \in [M_1,M_2] \text{ and } j^{-e(\tau_0)} \leq |pq\,2^{-j} - t_0| \leq j^{-d(\tau_0)}\big\}. \end{aligned} \tag{6.46}$$

is nonempty. From now on, for the sake of simplicity, $d(\tau_0)$ and $e(\tau_0)$ are denoted by d and e. Notice that, since j is large enough, the cardinality of $\overline{D_j}(t_0,\tau_0)$ satisfies

$$c_1\, j^{-d}\, 2^j \leq \operatorname{card}\big(\overline{D_j}(t_0,\tau_0)\big) \leq c_2\, j^{-d}\, 2^j, \tag{6.47}$$

where c_1 and c_2 are positive constants not depending on j. We denote by Γ_j the event defined by

$$\Gamma_j := \big\{\omega \in \Omega^*_0 \,:\, \max\big\{\big|\epsilon_{j,k}(\omega)\big| : k \in D_j(t_0,\tau_0)\big\} \leq j^{-\tau_0}\, 2^{j/\alpha}\big\}. \tag{6.48}$$

Now we give an upper bound for the probability $\mathbb{P}(\Gamma_j)$. Since j is large enough, it is no restriction to suppose that $j^{-\tau_0} 2^{j/\alpha} \geq 1$ and that $c_3 j^{\alpha\tau_0} 2^{-j} < 1$, where c_3 is the positive constant c' in (2.9). Next, using (6.48), (6.46), (iv) of Remark 2.2, (2.9), and the first inequality in (6.47), one obtains that

$$\begin{aligned} \mathbb{P}(\Gamma_j) &\leq \mathbb{P}\Big(\bigcap_{q\in\overline{D_j}(t_0,\tau_0)} \big\{\big|\epsilon_{j,pq}\big| \leq j^{-\tau_0} 2^{j/\alpha}\big\}\Big) = \prod_{q\in\overline{D_j}(t_0,\tau_0)} \mathbb{P}\big(\big|\epsilon_{j,pq}\big| \leq j^{-\tau_0}\, 2^{j/\alpha}\big) \\ &= \prod_{q\in\overline{D_j}(t_0,\tau_0)} \big(1 - \mathbb{P}\big(|\epsilon_{j,pq}| > j^{-\tau_0}\, 2^{j/\alpha}\big)\big) \leq \big(1 - c_3\, j^{\alpha\tau_0}\, 2^{-j}\big)^{c_1 j^{-d} 2^j}. \end{aligned} \tag{6.49}$$

Moreover, the inequality $\log(1-x) \leq -x$ for all $x \in [0,1)$, allows us to prove that

$$\begin{aligned} \big(1 - c_3\, j^{\alpha\tau_0}\, 2^{-j}\big)^{c_1 j^{-d} 2^j} &:= \exp\big(c_1\, j^{-d}\, 2^j \log(1 - c_3\, j^{\alpha\tau_0}\, 2^{-j})\big) \\ &\leq \exp\big(-c_1\, c_3\, j^{\alpha\tau_0 - d}\big). \end{aligned} \tag{6.50}$$

Finally, combining (6.42), (6.49), and (6.50), one gets that

$$\sum_{j\in\mathbb{N}} \mathbb{P}(\Gamma_j) < \infty.$$

Thus, the Borel–Cantelli lemma implies that (6.45) holds. □

Lemma 6.11. *Let τ be as in Theorem 6.1. Also suppose that (6.3) holds. We denote by τ_0 an arbitrary fixed real number such that $\tau_0 > \tau \geq 0$. Then there exists $t_0 \in [M_1, M_2]$ (a priori t_0 depends on τ_0) such that, for all $\omega \in \Omega_0^*$ (the event of probability 1 defined in Lemma 2.3), one has*

$$\limsup_{j\to+\infty} j^{\tau_0} 2^{-j/\alpha} \max\Big\{\big|\widetilde{g}_{j,k}(\omega) - \epsilon_{j,k}(\omega)\big| : k \in D_j(t_0,\tau_0)\Big\} = 0. \tag{6.51}$$

Recall that the random variables $\widetilde{g}_{j,k}$ and $\epsilon_{j,k}$ were defined in (6.28) and (2.5). Also recall that the set $D_j(t_0,\tau_0)$ was defined in (6.43).

Proof. Let ρ be as in (6.1). Assume that $\rho_0 \in (1/\alpha, \rho)$ is arbitrary and such that

$$\frac{1+2\alpha^{-1}+\tau_0}{\rho} < \frac{1+2\alpha^{-1}+\tau_0}{\rho_0} < d(\tau_0) < e(\tau_0) < \alpha\tau_0, \tag{6.52}$$

where $d(\tau_0)$ and $e(\tau_0)$ are defined in (6.41). Then, in view of (6.1) and (6.10), there exists $t_0 \in [M_1, M_2]$, that satisfies

$$\begin{cases} H(t_0) = H_* \\ \sup_{t\in[M_1,M_2]} \frac{|H(t)-H(t_0)|}{|t-t_0|^{\rho_0}} < \infty. \end{cases} \tag{6.53}$$

In what follows, we suppose that j is a sufficiently large integer. Thus the set $D_j(t_0,\tau_0)$ is nonempty and (6.44) holds. Also we suppose that $k \in D_j(t_0,\tau_0)$ is arbitrary. Using (6.39), in which one takes $v = H_*$, (6.28), and the equality, for each fixed $t \in \mathbb{R}$,

$$\begin{aligned} &X(t, H(k2^{-j}),\omega) - X(t,H_*,\omega) \\ &\quad = \big(H(k2^{-j}) - H_*\big)\int_0^1 \big(\partial_v X\big)\big(t, H_* + \theta(H(k2^{-j})-H_*),\omega\big)d\theta, \end{aligned}$$

one gets that

$$\begin{aligned} \widetilde{g}_{j,k}(\omega) - \epsilon_{j,k}(\omega) &= 2^{j(1+H_*)}\int_{\mathbb{R}} \big(X(t,H(k2^{-j}),\omega) - X(t,H_*,\omega)\big)\widetilde{\Psi}(2^j t - k, H_*)\,dt \\ &= 2^{j(1+H_*)}\big(H(k2^{-j})-H_*\big)\int_{\mathbb{R}}\int_0^1 \big(\partial_v X\big)\big(t, H_*+\theta(H(k2^{-j})-H_*),\omega\big) \\ &\qquad\qquad \times \widetilde{\Psi}(2^j t - k, H_*)\,d\theta\,dt. \end{aligned}$$

Therefore, it follows from (6.8), in which one takes $v = H_*$, that

$$|\epsilon_{j,k}(\omega) - \widetilde{g}_{j,k}(\omega)| \leq |H(k2^{-j}) - H_*| \sum_{l=1}^{3} F_{j,k}^l(\omega), \tag{6.54}$$

where, for each $l \in \{1,2,3\}$,

$$\begin{aligned} F_{j,k}^l(\omega) = 2^{j(1+H_*)}\int_{\mathcal{B}_l}\int_0^1 |\widetilde{\Psi}(2^j t-k,H_*)|\,\big|\big(\partial_v X\big)\big(t,H_*+\theta(H(k2^{-j})-H_*),\omega\big) \\ - \big(\partial_v X\big)\big(k2^{-j}, H_*+\theta(H(k2^{-j})-H_*),\omega\big)\big|\,d\theta\,dt. \end{aligned} \tag{6.55}$$

The sets $\mathcal{B}_l$ were defined in (6.16). Observe that, in view of (6.53) and (6.43), one has

$$|H(k2^{-j}) - H_*| \le c_1\, j^{-d(\tau_0)\rho_0}, \tag{6.56}$$

where c_1 is a constant not depending on j and k. Also observe that, in view of (6.52), there exists η_1, an arbitrarily small positive real number, such that

$$\frac{1 + 2\alpha^{-1} + \tau_0 + \eta_1}{\rho_0} < d(\tau_0). \tag{6.57}$$

Now we prove that (6.51) holds when $\left|\widetilde{g}_{j,k}(\omega) - \epsilon_{j,k}(\omega)\right|$ is replaced by $|H(k2^{-j}) - H_*|F^1_{j,k}(\omega)$. Using Proposition 4.1 (in which one takes $q = 1$, $M = M_0$, $a = \underline{H}$, $b = \overline{H}$, and $\eta = \eta_1$), the inequality $H(k2^{-j}) \ge H_*$, and the fact that $k2^{-j} \in \mathcal{B}_1 \subset [-M_0, M_0]$, one gets that

$$\begin{aligned}
F^1_{j,k}(\omega) &\le C_2(\omega)\, 2^{j(1+H_*)} \int_{\mathcal{B}_1} \int_0^1 |\widetilde{\Psi}(2^j t - k, H_*)| \\
&\qquad \times |t - k2^{-j}|^{H_* - 1/\alpha + \theta(H(k2^{-j}) - H_*)} \left(1 + \left|\log|t - k2^{-j}|\right|\right)^{1+2/\alpha+\eta_1} d\theta\, dt \\
&\le C_2(\omega)\, 2^{j(1+H_*)} \int_{\mathcal{B}_1} |\widetilde{\Psi}(2^j t - k, H_*)||t - k2^{-j}|^{H_* - 1/\alpha} \\
&\qquad \times \left(1 + \left|\log|t - k2^{-j}|\right|\right)^{1+2/\alpha+\eta_1} \left\{\int_0^1 |t - k2^{-j}|^{\theta(H(k2^{-j}) - H_*)} d\theta\right\} dt \\
&\le C_3(\omega)\, 2^{j(1+H_*)} \int_{\mathbb{R}} |t - k2^{-j}|^{H_* - 1/\alpha} \left(1 + \left|\log|t - k2^{-j}|\right|\right)^{1+2/\alpha+\eta_1} \\
&\qquad\qquad \times |\widetilde{\Psi}(2^j t - k, H_*)|\, dt,
\end{aligned}$$

where $C_2(\omega)$ is a constant not depending on j and k and where

$$C_3(\omega) = \left(1 + 2M_0\right)^{\overline{H} - H_*} C_2(\omega).$$

Then, setting $u = 2^j t - k$ in the last integral, one obtains that

$$\begin{aligned}
F^1_{j,k}(\omega) &\le C_3(\omega)\, 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_* - 1/\alpha} \left(1 + \left|\log|2^{-j} u|\right|\right)^{1+2/\alpha+\eta_1} |\widetilde{\Psi}(u, H_*)|\, du \\
&\le C_4(\omega) 2^{j/\alpha} \int_{\mathbb{R}} |u|^{H_* - 1/\alpha} \left(j^{1+2/\alpha+\eta_1} + \left(1 + \left|\log|u|\right|\right)^{1+2/\alpha+\eta_1}\right) |\widetilde{\Psi}(u, H_*)|\, du \\
&\le C_5(\omega)\, j^{1+2/\alpha+\eta_1}\, 2^{j/\alpha},
\end{aligned} \tag{6.58}$$

where $C_4(\omega)$ and $C_5(\omega)$ are constants not depending on j and k. Combining (6.56), (6.57), and (6.58), it follows that

$$\limsup_{j \to +\infty} j^{\tau_0}\, 2^{-j/\alpha} \max\left\{|H(k2^{-j}) - H_*|F^1_{j,k}(\omega) : k \in D_j(t_0, \tau_0)\right\} = 0. \tag{6.59}$$

Now we prove that (6.51) holds when $\left|\widetilde{g}_{j,k}(\omega)-\epsilon_{j,k}(\omega)\right|$ is replaced by $|H(k2^{-j})-H_*|F^2_{j,k}(\omega)$. We set

$$(6.60)\qquad C_6(\omega):=\sup_{(u,v)\in[-2M_0,2M_0]\times[\underline{H},\overline{H}]}\left|(\partial_v X)(u,v,\omega)\right|<\infty.$$

Observe that $C_6(\omega)$ is finite, since the function $(u,v)\mapsto(\partial_v X)(u,v,\omega)$ is continuous on the compact rectangle $[-2M_0,2M_0]\times[\underline{H},\overline{H}]$. Combining (6.55), (6.60), (6.44), and (6.23), one obtains that

$$(6.61)\qquad F^2_{j,k}(\omega)\le C_7(\omega)\,2^{-j(1-H_*-1/\alpha)},$$

where $C_7(\omega)$ is a constant not depending on j and k. Then, using (6.61), the fact that $H(\cdot)$ is a bounded function, and the inequality $0<1-H_*$, it follows that

$$(6.62)\qquad \limsup_{j\to+\infty} j^{\tau_0}\,2^{-j/\alpha}\max\left\{|H(k2^{-j})-H_*|F^2_{j,k}(\omega):k\in D_j(t_0,\tau_0)\right\}=0.$$

Now we prove that (6.51) holds when $\left|\widetilde{g}_{j,k}(\omega)-\epsilon_{j,k}(\omega)\right|$ is replaced by $|H(k2^{-j})-H_*|F^3_{j,k}(\omega)$. Setting $q=1$, $a=\underline{H}$, and $b=\overline{H}$ in Proposition 4.2, one gets , in view of (6.60), that, for any fixed $\eta>0$, for each $t\in\mathcal{B}_3$, and for all $\theta\in[0,1]$,

$$\begin{aligned}&\left|\big(\partial_v X\big)\big(t,H_*+\theta(H(k2^{-j})-H_*),\omega\big)-\big(\partial_v X\big)\big(k2^{-j},H_*+\theta(H(k2^{-j})-H_*),\omega\big)\right|\\&\qquad\le C_8(\omega)\,|t|^{\overline{H}}\,\big(1+\big|\log|t|\big|\big)^{1+1/\alpha+\eta},\end{aligned}$$

where $C_8(\omega)$ is a constant not depending on t, θ, and (j,k). Next, combining this last inequality with (6.55) and (6.25), it follows that,

$$(6.63)\qquad F^3_{j,k}(\omega_0)\le C_9(\omega)\,2^{-(1-H_*)j},$$

where $C_9(\omega)$ is a constant not depending on j and k. Then, using (6.63), the fact that $H(\cdot)$ is a bounded function, and the inequality $0<1-H_*$, it follows that

$$(6.64)\qquad \limsup_{j\to+\infty} j^{\tau_0}\,2^{-j/\alpha}\max\left\{|H(k2^{-j})-H_*|F^3_{j,k}(\omega):k\in D_j(t_0,\tau_0)\right\}=0.$$

Finally, combining (6.54), (6.59), (6.62), and (6.64), it follows that (6.51) holds. □

Lemma 6.12. *Let τ be as in Theorem 6.1. Also suppose that the conditions ($\mathcal{A}$) and (6.3) hold. Let τ_0 be an arbitrary fixed real number such that $\tau_0>\tau\ge 0$. Then there exists Ω^*_{2,τ_0} an event of probability 1 (which a priori depends on τ_0) contained in Ω^*_0 (recall that this event was defined in Lemma 2.3), such that, for each $\omega\in\Omega^*_{2,\tau_0}$, one has*

$$(6.65)\qquad \liminf_{j\to+\infty} j^{\tau_0}2^{-j/\alpha}\max\left\{\left|g_{j,k}(\omega)\right|:k\in\mathbb{Z}\text{ and }M_1+2^{-\frac{j}{2\alpha}}\le k/2^j\le M_2-2^{-\frac{j}{2\alpha}}\right\}>0,$$

where the $g_{j,k}$ are the random variables defined in (6.11).

Proof. Combining (6.44) and Lemmas 6.11, 6.10, and 6.6 one gets the lemma. □

Now, we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Denote by Ω_3^* the event of probability 1 defined by

$$\Omega_3^* := \bigcap_{\tau_0 \in \mathbb{Q} \text{ and } \tau_0 > \tau} \Omega_{2,\tau_0}^*.$$

The events Ω_{2,τ_0}^* were defined in Lemma 6.12. It is clear that (6.65) holds for all $\omega \in \Omega_3^*$ and for all real $\tau_0 > \tau \geq 0$. Therefore, it follows from Lemma 6.4, that for each $\omega \in \Omega_3^*$, $\tau_0 > \tau$, and $\eta_0 > 0$,

$$\sup_{(t,s)\in[M_1,M_2]^2} \frac{\big|Y(t,\omega) - Y(s,\omega)\big|}{|t-s|^{H_*-1/\alpha}\big(1 + \big|\log|t-s|\big|\big)^{-\tau_0-\eta_0}} = \infty.$$

Then, in view of (6.10), one gets the theorem. □

7. Optimality of the local modulus of continuity for LMSM

The goal of this section is to show that under a condition a bit stronger than (5.10), the local modulus of continuity given in (5.11) is optimal. More precisely:

Theorem 7.1. *Let M be a positive real number. Assume that $t_0 \in (-M, M)$ satisfies*

$$\big|H(t) - H(t_0)\big| \leq c\,|t-t_0|^{H(t_0)}\,\big(1 + \big|\log|t-t_0|\big|\big)^{1/\alpha} \tag{7.1}$$

for some constant $c > 0$ and all $t \in \mathbb{R}$. Then one has, almost surely,

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|Y(t) - Y(t_0)\big|}{|t-t_0|^{H(t_0)}\big(1 + \big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\} = \infty. \tag{7.2}$$

Remark 7.2. We mention that, even in the quite classical case of LFSM (in other words, in the particular case where the functional parameter $H(\cdot)$ of LMSM is constant), the determination of the optimal lower bound for the power of the logarithmic factor in a local modulus of continuity has so far been an open problem. Corollary 5.6 and Theorem 7.1 in the present article solve it, in the more general case of LMSM, by showing that $1/\alpha$ is in fact this optimal lower bound.

The proof of Theorem 7.1 relies on (3.18), in which one takes $q = 0$. Also, more importantly, it relies on the following proposition.

Proposition 7.3. *Let M be a positive real number. For all $t_0 \in (-M, M)$, one has, almost surely,*

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t_0)) - X(t_0,H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1 + \big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\} = \infty. \tag{7.3}$$

In order to show that Proposition 7.3 holds, we need to introduce some additional notation. Also we need to derive some preliminary results. Let m_0 be the positive integer defined by

$$m_0 := \left[\log_2(3R+2)\right] + 1. \tag{7.4}$$

Recall that R is a fixed real number strictly bigger than 1, such that (2.2) holds. For all $j \in \mathbb{N}$, one sets

$$r(j,m_0) := jm_0 \quad \text{and} \quad l(j,m_0) := \left[2^{r(j,m_0)}t_0 + R + 2\right]. \tag{7.5}$$

Observe that the inequalities

$$(R+1)2^{-r(j,m_0)} < l(j,m_0)2^{-r(j,m_0)} - t_0 < (R+1)2^{1-r(j,m_0)} < 4/5 \tag{7.6}$$

hold. One denotes by $\check{\epsilon}_j$ the $\mathcal{S}t\alpha\mathcal{S}$ random variable

$$\check{\epsilon}_j := \epsilon_{r(j,m_0),l(j,m_0)}. \tag{7.7}$$

In other words, $\check{\epsilon}_j$ is defined through (2.5) in which j and k are replaced, respectively, by $r(j,m_0)$ and $l(j,m_0)$.

Lemma 7.4. *The $\mathcal{S}t\alpha\mathcal{S}$ random variables $\check{\epsilon}_j$, $j \in \mathbb{N}$, are independent and they all have the same scale parameter. Namely, for each $j \in \mathbb{N}$,*

$$\|\check{\epsilon}_j\|_\alpha = \left\{\int_{\mathbb{R}} |\psi(t)|^\alpha \, dt\right\}^{1/\alpha}. \tag{7.8}$$

Proof. First, observe that (7.8) is a straightforward consequence of (7.7) and (2.8). Now we prove that the random variables $\check{\epsilon}_j$, for $j \in \mathbb{N}$, are independent. Notice that (2.2) entails that

$$\begin{aligned}
&\operatorname{supp}\psi\left(2^{r(j,m_0)}\cdot - l(j,m_0)\right)\\
&\qquad \subseteq \left[l(j,m_0)2^{-r(j,m_0)} - R\,2^{-r(j,m_0)}, l(j,m_0)\,2^{-r(j,m_0)} + R\,2^{-r(j,m_0)}\right].
\end{aligned}$$

Therefore, in view of (2.5) and the fact that the $\mathcal{S}t\alpha\mathcal{S}$ random measure $Z_\alpha(ds)$ is independently scattered, it suffices to show that the intervals

$$\left[l(j,m_0)2^{-r(j,m_0)} - R2^{-r(j,m_0)}, l(j,m_0)2^{-r(j,m_0)} + R2^{-r(j,m_0)}\right], \quad j \in \mathbb{N},$$

are disjoint. This can be obtained by proving that the inequality

$$R2^{-r(j,m_0)} + R2^{-r(j+p,m_0)} < \left|l(j,m_0)2^{-r(j,m_0)} - l(j+p,m_0)2^{-r(j+p,m_0)}\right| \tag{7.9}$$

holds for all $(j,p) \in \mathbb{N}^2$. By using the triangle inequality, (7.6), and the first equality in (7.5), one has

$$\begin{aligned}
&\left|l(j,m_0)\,2^{-r(j,m_0)} - l(j+p,m_0)\,2^{-r(j+p,m_0)}\right|\\
&\qquad \ge \left|l(j,m_0)\,2^{-r(j,m_0)} - t_0\right| - \left|l(j+p,m_0)\,2^{-r(j+p,m_0)} - t_0\right|\\
&\qquad > (R+1)\,2^{-r(j,m_0)} - (R+1)\,2^{1-r(j+p,m_0)} = (R+1)\,2^{-jm_0}\left(1 - 2^{1-pm_0}\right)\\
&\qquad \ge (R+1)\,2^{-jm_0}\left(1 - 2^{1-m_0}\right). \qquad (7.10)
\end{aligned}$$

On the other hand, the first equality in (7.5) implies that

$$(7.11)\quad R\,2^{-r(j,m_0)} + R\,2^{-r(j+p,m_0)} = R\,2^{-jm_0}\big(1+2^{-pm_0}\big) \le R\,2^{-jm_0}\big(1+2^{-m_0}\big).$$

Next, notice that (7.4) implies that $2^{-m_0} < (3R+2)^{-1}$ and consequently that

$$(7.12)\qquad R\big(1+2^{-m_0}\big) < \frac{3R(R+1)}{3R+2} < (R+1)\big(1-2^{1-m_0}\big).$$

Finally, combining (7.10), (7.11), and (7.12), one gets (7.9) □

Lemma 7.5. *One has, almost surely,*

$$(7.13)\qquad \limsup_{j\to+\infty} \frac{|\check{\epsilon}_j|}{j^{1/\alpha}\log^{1/\alpha}(j)} \ge 1.$$

Proof. Notice that, in view of Lemma 7.4, the events $\{|\check{\epsilon}_j| > j^{1/\alpha}\log^{1/\alpha}(j)\}$, $j\in\mathbb{N}$, are independent. Moreover, (7.7) and the first inequality in (2.9) imply that

$$\sum_{j=2}^{+\infty} \mathbb{P}\big(|\check{\epsilon}_j| > j^{1/\alpha}\,\log^{1/\alpha}(j)\big) \ge c' \sum_{j=2}^{+\infty} j^{-1}\log^{-1}(j) = +\infty.$$

Thus, applying the second Borel–Cantelli lemma, one gets (7.13). □

Lemma 7.6. *Let Ω_0^* be the event of probability 1 defined in Lemma 2.3. Assume that, for some $t_0\in(-M,M)$ and $\omega_0\in\Omega_0^*$, one has*

$$(7.14)\qquad \sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t_0),\omega_0) - X(t_0,H(t_0),\omega_0)\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\} < \infty.$$

Then, it follows that

$$(7.15)\qquad \limsup_{j\to+\infty} \frac{|\check{\epsilon}_j(\omega_0)|}{j^{1/\alpha}} < \infty.$$

Proof. First notice that (7.7), (6.39) (in which one takes $v = H(t_0)$), (6.8), and the change of variable $x = t - l_j 2^{-r_j}$, imply that

$$\begin{aligned}
(7.16)\quad & \check{\epsilon}_j(\omega_0)\\
&= 2^{r_j(1+H(t_0))}\int_{\mathbb{R}} \big(X(t,H(t_0),\omega_0) - X(t_0,H(t_0),\omega_0)\big)\widetilde{\Psi}(2^{r_j}t - l_j, H(t_0))\,dt\\
&= 2^{r_j(1+H(t_0))}\int_{\mathbb{R}} \big(X\big(x+l_j 2^{-r_j},H(t_0),\omega_0\big) - X\big(t_0,H(t_0),\omega_0\big)\big)\,\widetilde{\Psi}(2^{r_j}x, H(t_0))\,dx,
\end{aligned}$$

where, for the sake of simplicity, we have set $r_j = r(j,m_0)$ and $l_j = l(j,m_0)$. Let $s_* := |t_0| + 2$. Observe that, in view of (7.6), one has

$$(7.17)\qquad \forall\, x\in\mathbb{R},\ |x|\ge s_* \implies \big|x + l_j 2^{-r_j}\big| \ge 1.$$

Also, observe that (7.16) entails that

$$\big|\check{\epsilon}_j(\omega_0)\big| \le S_j + Z_j, \tag{7.18}$$

where

$$S_j = 2^{r_j(1+H(t_0))} \times \int_{|x|<s_*} \Big|X\big(x + l_j 2^{-r_j}, H(t_0), \omega_0\big) - X\big(t_0, H(t_0), \omega_0\big)\Big| \big|\widetilde{\Psi}(2^{r_j}x, H(t_0))\big|\, dx \tag{7.19}$$

and

$$Z_j = 2^{r_j(1+H(t_0))} \times \int_{|x|\ge s_*} \Big|X\big(x + l_j 2^{-r_j}, H(t_0), \omega_0\big) - X\big(t_0, H(t_0), \omega_0\big)\Big| \big|\widetilde{\Psi}(2^{r_j}x, H(t_0))\big|\, dx. \tag{7.20}$$

Now we bound S_j from above. Notice that the fact that $t \mapsto X(t, H(t_0), \omega_0)$ is a continuous function on $\mathbb{R}$ entails that (7.14) remains valid when $[-M, M]$ is replaced by any other compact interval. Also notice that, in view of (7.6) when $|x| < s_*$,

$$x + l_j 2^{-r_j} \text{ belongs to the compact interval } \big[-s_* - |t_0| - 4/5, s_* + |t_0| + 4/5\big].$$

Thus, using (7.14), in which M is replaced by $s_* + |t_0| + 4/5$, one gets that,

$$\begin{aligned} S_j &\le C_1(\omega_0) 2^{r_j(1+H(t_0))} \int_{|x|<s_*} |\nu_j + x|^{H(t_0)} \big(1 + \big|\log|\nu_j + x|\big|\big)^{1/\alpha} \\ &\qquad\qquad \times \big|\widetilde{\Psi}(2^{r_j}x, H(t_0))\big|\, dx \\ &\le C_1(\omega_0) 2^{r_j(1+H(t_0))} \int_{\mathbb{R}} |\nu_j + x|^{H(t_0)} \big(1 + \big|\log|\nu_j + x|\big|\big)^{1/\alpha} \big|\widetilde{\Psi}(2^{r_j}x, H(t_0))\big|\, dx, \end{aligned} \tag{7.21}$$

where, $C_1(\omega_0)$ is a constant not depending on j, and

$$\nu_j := l_j\, 2^{-r_j} - t_0. \tag{7.22}$$

Observe that (7.6) implies that

$$R + 1 < 2^{r_j}\, \nu_j < 2R + 2. \tag{7.23}$$

For the sake of convenience, we set

$$c_2 := \sup_{y \in \mathbb{R}} (3 + |y|)^2 \big|\widetilde{\Psi}(y, H(t_0))\big| < \infty. \tag{7.24}$$

Observe that the inequality in (7.24) results from (6.7). Next, making the change of variable $u = x/\nu_j$ in (7.21), and using the triangle inequality, (7.23), (7.24), (7.22), the last two inequalities in (7.6), and the the first equality in (7.5), it follows that

$$S_j \le C_1(\omega_0)\, 2^{r_j(1+H(t_0))}\nu_j \int_{\mathbb{R}} |\nu_j+\nu_j u|^{H(t_0)}\big(1+\big|\log|\nu_j+\nu_j u|\big|\big)^{1/\alpha} \times \big|\widetilde{\Psi}(2^{r_j}\nu_j u, H(t_0))\big|du$$

$$= C_1(\omega_0)\, 2^{r_j(1+H(t_0))}\nu_j^{1+H(t_0)} \int_{\mathbb{R}} |1+u|^{H(t_0)}\big(1+\big|\log(\nu_j)+\log|1+u|\big|\big)^{1/\alpha} \times \big|\widetilde{\Psi}(2^{r_j}\nu_j u, H(t_0))\big|du$$

$$\le C_3(\omega_0)\,\big(2^{r_j}\nu_j\big)^{1+H(t_0)}\,\big|\log(\nu_j)\big|^{1/\alpha} \le C_4(\omega_0)\, j^{1/\alpha} \tag{7.25}$$

where

$$C_3(\omega_0) := c_2\,\big(\log(5/4)\big)^{-1/\alpha} C_1(\omega_0)\int_{\mathbb{R}} \frac{\big|2+|\log|1+u||\big|^{1/\alpha}}{(3+|u|)^{2-H(t_0)}}\,du < \infty,$$

and $C_4(\omega_0) = C_3(\omega_0)\big(2R+2\big)^{1+H(t_0)} m_0^{1/\alpha}$. Now we find an upper bound for Z_j. Using (7.20), (7.24), and the triangle inequality, one obtains that

$$Z_j \le c_2 2^{-r_j(1-H(t_0))}\int_{|x|\ge s_*} \big|X\big(x+l_j 2^{-r_j}, H(t_0),\omega_0\big)\big|\, x^{-2}\,dx + C_5(\omega_0)2^{-r_j(1-H(t_0))}, \tag{7.26}$$

where

$$C_5(\omega_0) := c_2\,\big|X\big(t_0, H(t_0),\omega_0\big)\big| \int_{|x|\ge s_*} x^{-2}\,dx < \infty.$$

Next, observe that (7.17) and (7.6) imply that, for all real x that satisfies $|x| \ge s_*$, and for each $j \in \mathbb{N}$, one has

$$1 \le |x + l_j\, 2^{-r_j}| \le |x| + |t_0| + 1.$$

Thus, taking $q = 0$ in (4.2), a and b such that $H(t_0) \in [a, b]$, and η an arbitrary fixed positive real number, it follows that

$$\big|X\big(x+l_j 2^{-r_j}, H(t_0),\omega_0\big)\big| \le C_6(\omega_0)\big(|x|+|t_0|+1\big)^{H(t_0)}\big(1+\log\big(|x|+|t_0|+1\big)\big)^{1/\alpha+\eta}, \tag{7.27}$$

where the finite constant $C_6(\omega_0)$ does not depend on x and j. Next, combining (7.26) with (7.27), one gets that

$$Z_j \le C_7(\omega_0)\, 2^{-r_j(1-H(t_0))}, \tag{7.28}$$

where

$$C_7(\omega_0) := C_5(\omega_0) + c_2\int_{|x|\ge s_*} \big(|x|+|t_0|+1\big)^{H(t_0)}\big(1+\log\big(|x|+|t_0|+1\big)\big)^{1/\alpha+\eta}\, x^{-2}\,dx$$

is a finite constant. Finally, combining (7.18), (7.25), (7.28), and the first equality in (7.5), one obtains (7.15). □

Now, we are ready to prove Proposition 7.3 and Theorem 7.1.

Proof of Proposition 7.3. The proposition is a straightforward consequence of Lemmas 7.5 and 7.6. □

Proof of Theorem 7.1. Using (5.1) and the triangle inequality, one has, for all $t \in [-M, M]$,

$$\big|X(t,H(t_0)) - X(t_0,H(t_0))\big| \le \big|Y(t) - Y(t_0)\big| + \big|X(t,H(t)) - X(t,H(t_0))\big|,$$

and, as a consequence,

$$\begin{aligned}\sup_{t\in[-M,M]} &\left\{ \frac{\big|X(t,H(t_0)) - X(t_0,H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\} \\ &\le \sup_{t\in[-M,M]} \left\{ \frac{\big|Y(t) - Y(t_0)\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\} \\ &\quad + \sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t)) - X(t,H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\}.\end{aligned}$$

Thus, in view of (7.3), in order to show that (7.2) holds, it is suffices to prove that

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t)) - X(t,H(t_0))\big|}{|t-t_0|^{H(t_0)}\big(1+\big|\log|t-t_0|\big|\big)^{1/\alpha}} \right\} < \infty. \tag{7.29}$$

Taking $q = 0$ in (3.18) , $a = \underline{H} := \inf_{x\in\mathbb{R}} H(x)$, and $b := \overline{H} := \sup_{x\in\mathbb{R}} H(x)$, one gets that

$$\sup_{t\in[-M,M]} \left\{ \frac{\big|X(t,H(t)) - X(t,H(t_0))\big|}{|H(t) - H(t_0)|} \right\} < \infty. \tag{7.30}$$

Finally, combining (7.1) with (7.30), it follows that (7.29) holds. □

8. Local Hölder exponent of LMSM

The goal of this section is to determine the local Hölder exponent of a typical path of LMSM. First we recall, in a general framework, the definition of this exponent.

Let f be an arbitrary deterministic real-valued continuous function defined on the real line. The critical global Hölder regularity of f, over an arbitrary nonempty compact interval $[M_1, M_2]$, can be measured through

$$\rho_f^{\mathrm{unif}}([M_1,M_2]) := \sup\left\{\rho \ge 0 : \sup_{s',s''\in[M_1,M_2]} \frac{|f(s') - f(s'')|}{|s'-s''|^{\rho}} < \infty\right\}, \tag{8.1}$$

the uniform (or global) Hölder exponent of f over $[M_1, M_2]$. Observe that one has

$$\rho_f^{\mathrm{unif}}([M_1',M_2']) \ge \rho_f^{\mathrm{unif}}([M_1,M_2]) \tag{8.2}$$

when $[M_1', M_2'] \subseteq [M_1, M_2]$. The local Hölder regularity of f in a neighborhood of some point $t_0 \in \mathbb{R}$ can be measured through

$$\rho_f^{\mathrm{unif}}(t_0) := \sup\left\{\rho_f^{\mathrm{unif}}\big([M_1, M_2]\big) : M_1 \in \mathbb{R},\ M_2 \in \mathbb{R} \text{ and } M_1 < t_0 < M_2\right\}, \tag{8.3}$$

the local Hölder exponent of f at t_0. Notice that this exponent is sometimes called the uniform pointwise Hölder exponent of f at t_0 (see [29]).

Let $t \mapsto Y(t,\omega)$ be a continuous path of the LMSM $\{Y(t) : t \in \mathbb{R}\}$. The uniform Hölder exponent of $t \mapsto Y(t,\omega)$ over $[M_1, M_2]$ is denoted by $\rho_Y^{\mathrm{unif}}\big([M_1, M_2], \omega\big)$. The local Hölder exponent of $t \mapsto Y(t,\omega)$ at t_0 is denoted by $\rho_Y^{\mathrm{unif}}(t_0, \omega)$.

Thanks to (ii) of Corollary 5.3 and Theorem 6.1, under some Hölder condition on $H(\cdot)$, one can, almost surely for $t_0 \in \mathbb{R}$, completely determine $\rho_Y^{\mathrm{unif}}(t_0, \omega)$. More precisely:

Theorem 8.1. *There is an event Ω_4^* of probability 1 (not depending on t_0) such that, for all $\omega \in \Omega_4^*$ and for each $t_0 \in \mathbb{R}$ satisfying*

$$\rho_H^{\mathrm{unif}}(t_0) > 1/\alpha, \tag{8.4}$$

one has

$$\rho_Y^{\mathrm{unif}}(t_0, \omega) = H(t_0) - \frac{1}{\alpha}\,. \tag{8.5}$$

Notice that Theorem 8.1 is more precise than Theorem 4.1 in [29].

Proof. The theorem does not make sense if there is no $t_0 \in \mathbb{R}$ which satisfies (8.4), so, in what follows, we assume that (8.4) is satisfied for some $t_0 \in \mathbb{R}$. In view of (8.2) and (8.3), this assumption implies that the set

$$\Lambda := \left\{(\mu_1, \mu_2) \in \mathbb{Q}^2 : \mu_1 < \mu_2 \ \text{ and } \ \rho_H^{\mathrm{unif}}\big([\mu_1, \mu_2]\big) > 1/\alpha\right\},$$

is nonempty. Next, observe that, (8.1), (ii) of Corollary 5.3, Theorem 6.1, and Remark 6.2, imply that, for all $(\mu_1, \mu_2) \in \Lambda$, one has, almost surely,

$$\rho_Y^{\mathrm{unif}}\big([\mu_1, \mu_2]\big) = \min_{x \in [\mu_1, \mu_2]} H(x) - \frac{1}{\alpha}\,. \tag{8.6}$$

Moreover, the fact that Λ is countable entails that (8.6) holds on Ω_4^*, an event of probability 1 which does not depend on (μ_1, μ_2). Also, observe that, for each $t_0 \in \mathbb{R}$ which satisfies (8.4), one has, for all $\omega \in \Omega_4^*$,

$$\rho_Y^{\mathrm{unif}}(t_0, \omega) = \sup\left\{\rho_Y^{\mathrm{unif}}\big([\mu_1, \mu_2]\big) : (\mu_1, \mu_2) \in \Lambda \text{ and } \mu_1 < t_0 < \mu_2\right\}. \tag{8.7}$$

The equality (8.7) can be obtained by using (8.2), (8.3), and the density of the rational numbers in the real numbers. Finally, since $H(\cdot)$ is a continuous function, combining (8.6) with (8.7), one gets (8.5). □

9. Appendix

The following technical lemma plays a crucial role in the proof of (iii) of Proposition 3.4 as well as in the proofs of other important results in our article.

Lemma 9.1. *Let $(p,q) \in \{0,1,2\} \times \mathbb{Z}_+$. Set $\phi := \partial_x^p \partial_v^q \Psi$, where Ψ is the function defined in (2.3). Let M, ν, a, b, and κ be real numbers satisfying $M > 0$, $1 > b > a > 1/\alpha$, $a - 1/\alpha > \kappa$, and $a - 1/\alpha - \kappa > \nu \geq 0$. Finally, let i be a nonnegative integer. For all $n \in \mathbb{Z}_+$ and $(t,s,v) \in \mathbb{R}^2 \times (1/\alpha, 1)$ we set*

$$A_n(t,s,v;M,\kappa,\nu,i,\phi) := \sum_{|j|\leq n} \sum_{|k|>M2^{n+1}} 2^{-jv} \frac{\left|\phi(2^j t - k, v) - \phi(2^j s - k, v)\right|}{|t-s|^\kappa} (3+|j|)^{i+1/\alpha+\nu} (3+|k|)^{1/\alpha+\nu} \tag{9.1}$$

and

$$B_n(t,s,v;M,\kappa,\nu,i,\phi) := \sum_{|j|\geq n+1} \sum_{k\in\mathbb{Z}} 2^{-jv} \frac{\left|\phi(2^j t - k, v) - \phi(2^j s - k, v)\right|}{|t-s|^\kappa} (3+|j|)^{i+1/\alpha+\nu} (3+|k|)^{1/\alpha+\nu}, \tag{9.2}$$

with the convention that $A_n(t,t,v;M,\kappa,\nu,i,\phi) = B_n(t,t,v;M,\kappa,\nu,i,\phi) = 0$ for any $t \in \mathbb{R}$. Then $A_n(t,s,v;M,\kappa,\nu,i,\phi)$ and $B_n(t,s,v;M,\kappa,\nu,i,\phi)$ converge to 0, when n goes to $+\infty$, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$.

In order to prove Lemma 9.1, we need some preliminary results.

Lemma 9.2. *For all real numbers $\xi > 0$ and $M > 0$, there exists a constant $c > 0$ such that, for each integer $n \geq 0$,*

$$\sum_{k > M2^{n+1}} (1+k)^{-1-\xi} \leq c\, 2^{-n\xi}.$$

Proof. Clearly, one has, for any integer $k \geq 1$,

$$(1+k)^{-1-\xi} \leq \int_{k-1}^{k} (1+x)^{-1-\xi}\, dx.$$

Therefore,

$$\sum_{k > M2^{n+1}} (1+k)^{-1-\xi} \leq \int_{M2^{n+1}-1}^{+\infty} (1+x)^{-1-\xi}\, dx = \xi^{-1} M^{-\xi} 2^{-(n+1)\xi}.$$

□

Lemma 9.3. *Fix $\lambda \in \mathbb{R}$ and $\theta_0 > 0$. Set*

$$c := \sum_{m=0}^{+\infty} 2^{-m\theta_0} (1+m)^{|\lambda|} < +\infty.$$

Then for any real number θ such that $|\theta| \geq \theta_0$ and each $n_0, n_1 \in \{0, \pm 1, \dots, \pm\infty\}$ satisfying $n_0 < n_1$, one has

$$\sum_{n=n_0}^{n_1} 2^{n\theta}(1+|n|)^{\lambda} \leq c \begin{cases} 2^{n_0\theta}\,(1+|n_0|)^{\lambda} & \text{if } \theta < 0 \\ 2^{n_1\theta}\,(1+|n_1|)^{\lambda} & \text{if } \theta > 0, \end{cases} \tag{9.3}$$

with the conventions $2^{-\infty}(1+\infty)^{\lambda} = 0$ and $2^{+\infty}(1+\infty)^{\lambda} = +\infty$.

Proof. First, notice that the lemma clearly holds in the following three cases:

- $n_0 = -\infty$ and $n_1 = +\infty$;
- $n_0 = -\infty$ and $\theta < 0$;
- $n_1 = +\infty$ and $\theta > 0$.

Indeed, in these three cases (9.3) becomes $+\infty \leq +\infty$.

We study the case where $\theta < 0$ and $-\infty < n_0 < n_1 \leq +\infty$. The case where $\theta > 0$ and $-\infty \leq n_0 < n_1 < +\infty$ can be treated similarly. One has

$$\begin{aligned}\sum_{n=n_0}^{n_1} 2^{n\theta}(1+|n|)^{\lambda} &\leq \sum_{m=0}^{+\infty} 2^{(m+n_0)\theta}(1+|m+n_0|)^{\lambda} \\ &= 2^{n_0\theta}\,(1+|n_0|)^{\lambda} \sum_{m=0}^{+\infty} 2^{m\theta} \left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\lambda} \\ &\leq 2^{n_0\theta}\,(1+|n_0|)^{\lambda} \sum_{m=0}^{+\infty} 2^{-m\theta_0} \left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\lambda}.\end{aligned}$$

There remains to show

$$\sum_{m=0}^{+\infty} 2^{-m\theta_0} \left(\frac{1+|m+n_0|}{1+|n_0|}\right)^{\lambda} \leq c := \sum_{m=0}^{+\infty} 2^{-m\theta_0}\,(1+m)^{|\lambda|}. \tag{9.4}$$

In fact, (9.4) can be obtained by proving that for every integer $m \geq 0$, one has

$$\frac{1}{1+m} \leq \frac{1+|m+n_0|}{1+|n_0|} \leq 1+m. \tag{9.5}$$

Clearly the second inequality in (9.5) is satisfied. We divide the proof of the first into three cases:

- if $n_0 \geq 0$, one gets $\frac{1+|m+n_0|}{1+|n_0|} = \frac{1+m+n_0}{1+n_0} = 1 + \frac{m}{1+n_0} \geq 1 \geq \frac{1}{1+m}$;
- if $n_0 < 0$ and $m \geq -n_0 = |n_0|$, then $\frac{1+|m+n_0|}{1+|n_0|} \geq \frac{1}{1+|n_0|} \geq \frac{1}{1+m}$;
- if $n_0 < 0$ and $m < -n_0 = |n_0|$, then

$$\frac{1+|m+n_0|}{1+|n_0|} = \frac{1-m+|n_0|}{1+|n_0|} = 1 - \frac{m}{1+|n_0|} \geq 1 - \frac{m}{1+m} = \frac{1}{1+m}.$$

□

The following lemma is a more or less classical result. We refer for instance to [2] for its proof.

Lemma 9.4 ([2]). *For all fixed real numbers $\theta \in [0,1)$ and $\zeta \geq 0$, there exists a constant $c > 0$ such that for any $u \in \mathbb{R}$, one has*

$$\sum_{k\in\mathbb{Z}} \frac{(1+|k|)^{\theta}\,\log^{\zeta}(2+|k|)}{(2+|u-k|)^2} \leq c\,(1+|u|)^{\theta}\,\log^{\zeta}(2+|u|).$$

Now, we are prepared to prove Lemma 9.1.

Proof of Lemma 9.1. Let $t, s \in [-M, M]$. It is no restriction to assume that $s \neq t$. We denote by $j_0 > -\log_2(2M) - 1$ the unique integer such that

$$2^{-j_0-1} < |t-s| \leq 2^{-j_0}. \tag{9.6}$$

From now on, for the sake of simplicity we set

$$A_n(t,s,v) := A_n(t,s,v; M, \kappa, \nu, i, \phi) \quad \text{and} \quad B_n(t,s,v) := B_n(t,s,v; M, \kappa, \nu, i, \phi).$$

First we prove that, when $n \to +\infty$, $A_n(t,s,v)$ converges to 0, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$. In what follows, we assume that j is an arbitrary integer satisfying $|j| \leq n$. We need a suitable upper bound on the quantity

$$A_n^j(t,s,v) := \sum_{|k|>M2^{n+1}} \frac{\left|\phi(2^j t - k, v) - \phi(2^j s - k, v)\right|}{|t-s|^{\kappa}}\,(3+|k|)^{1/\alpha+\nu}. \tag{9.7}$$

To this end, we consider the cases $j \leq j_0$ and $j \geq j_0 + 1$ separately. First, we suppose that

$$j \leq j_0. \tag{9.8}$$

Using the mean value theorem, (2.12), (9.6), and (9.8), one obtains that

$$\begin{aligned} \left|\phi(2^j t - k, v) - \phi(2^j s - k, v)\right| &\leq c_1\,2^j\,|t-s|\,\sup_{u\in I}(3+|u|)^{-2} \\ &\leq c_1\,2^j\,|t-s|\,(2+|2^j t - k|)^{-2}, \end{aligned} \tag{9.9}$$

where I denotes the compact interval with endpoints $2^j t - k$ and $2^j s - k$. Notice that, in view of (9.6) and (9.8), the length of I is at most 1; this is why the last inequality holds. Next, (9.9) and (9.7) entail that

$$A_n^j(t,s,v) \leq c_1\,2^j\,|t-s|^{1-\kappa} \sum_{|k|>M2^{n+1}} (3+|k|)^{1/\alpha+\nu}\,(2+|2^j t-k|)^{-2}. \tag{9.10}$$

Moreover, using the inequalities $|t| \leq M$, $|j| \leq n$ and $|k| > M2^{n+1}$, one gets

$$\begin{aligned} (3+|k|)^{1/\alpha+\nu}\,(2+|2^j t-k|)^{-2} &\leq (3+|k|)^{1/\alpha+\nu}\,(2+|k|-2^j M)^{-2} \\ &\leq c_2\,(1+|k|)^{-(2-1/\alpha-\nu)}. \end{aligned} \tag{9.11}$$

Combining (9.10) and (9.11), one obtains that

$$A_n^j(t,s,v) \le c_3\, 2^j\, |t-s|^{1-\kappa} \sum_{|k|>M2^{n+1}} (1+|k|)^{-(2-1/\alpha-\nu)}.$$

Then Lemma 9.2 (in which one takes $\xi = 1-1/\alpha-\nu$) and the relation (9.6) imply that

$$A_n^j(t,s,v) \le c_4\, 2^{j_0(\kappa-1)+j-n(1-1/\alpha-\nu)}. \tag{9.12}$$

Now we study the second case where

$$j_0+1 \le j. \tag{9.13}$$

It follows from (9.7), (9.6), and (9.13) that

$$A_n^j(t,s,v) \le 2^{j\kappa} \sum_{|k|>M2^{n+1}} \left\{ \left|\phi(2^j t-k,v)\right| + \left|\phi(2^j s-k,v)\right| \right\} (3+|k|)^{1/\alpha+\nu}. \tag{9.14}$$

Moreover, using (2.12) and the fact that $|j| \le n$, one has, for all $(u,v) \in [-M,M]\times[a,b]$ and $k \in \mathbb{Z}$ satisfying $|k| > M2^{n+1}$,

$$\begin{aligned} \left|\phi(2^j u-k,v)\right| &\le c_5\,(3+|2^j u-k|)^{-2} \le c_5\,(3+|k|-2^j|u|)^{-2} \\ &\le c_5\,(3+|k|-2^n M)^{-2} \le c_6\,(3+|k|)^{-2}. \end{aligned} \tag{9.15}$$

Combining (9.15) with (9.14) one gets that

$$A_n^j(t,s,v) \le c_6\, 2^{j\kappa+1} \sum_{|k|>M2^{n+1}} (3+|k|)^{-(2-1/\alpha-\nu)}.$$

Thus, it follows from Lemma 9.2 (in which one takes $\xi = 1-1/\alpha-\nu$) that

$$A_n^j(t,s,v) \le c_7\, 2^{j\kappa-n(1-1/\alpha-\nu)}. \tag{9.16}$$

Combining (9.1), (9.7), (9.12), and (9.16) one obtains that

$$\begin{aligned} &A_n(t,s,v) \le c_8\, 2^{-n(1-1/\alpha-\nu)} \\ &\times \Big(2^{j_0(\kappa-1)} \sum_{j=-\infty}^{j_0} 2^{j(1-v)}(3+|j|)^{i+1/\alpha+\nu} + \sum_{j=j_0+1}^{+\infty} 2^{j(\kappa-v)}(3+|j|)^{i+1/\alpha+\nu} \Big). \end{aligned} \tag{9.17}$$

Next, using Lemma 9.3 with $n_0 = -\infty$, $n_1 = j_0$, $\theta = 1-v > 0$, $\theta_0 = 1-b$, and $\lambda = i+1/\alpha+\nu$ one gets that

$$\sum_{j=-\infty}^{j_0} 2^{j(1-v)}\,(3+|j|)^{i+1/\alpha+\nu} \le c_9\, 2^{j_0(1-v)}\,(1+|j_0|)^{i+1/\alpha+\nu}, \tag{9.18}$$

and, using again the same lemma with $n_0 = j_0 + 1$, $n_1 = +\infty$, $\theta = \kappa - v < 0$, $\theta_0 = 1/\alpha$, and $\lambda = i + 1/\alpha + \nu$, one obtains that

$$\sum_{j=j_0+1}^{+\infty} 2^{j(\kappa-v)} (3 + |j|)^{i+1/\alpha+\nu} \le c_{10}\, 2^{j_0(\kappa-v)} (1 + |j_0|)^{i+1/\alpha+\nu}. \tag{9.19}$$

Combining (9.17), (9.18), (9.19), the inequality $v - \kappa \ge 1/\alpha$ and the inequality $j_0 > -\log_2(2M) - 1$, one obtains that

$$A_n(s,t,v) \le c_{11} 2^{-j_0(v-\kappa)} (1+|j_0|)^{i+1/\alpha+\nu} 2^{-n(1-1/\alpha-\nu)} \le c_{12} 2^{-n(1-1/\alpha-\nu)}, \tag{9.20}$$

where

$$c_{12} := c_{11} \sup\left\{2^{-j/\alpha}(1+|j|)^{i+1/\alpha+\nu} \;:\; j \in \mathbb{Z} \text{ and } j > -\log_2(2M) - 1\right\} < +\infty.$$

The last inequality in (9.20) implies that, when $n \to +\infty$, $A_n(t,s,v)$ converges to 0, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$.

Henceforth our goal is to prove that $B_n(t,s,v)$ converges to 0 uniformly in t, s, and v, when n goes to infinity. In what follows j is an arbitrary integer satisfying $|j| \ge n+1$. First we derive a suitable upper bound for the quantity

$$B^j(t,s,v) := \sum_{k\in\mathbb{Z}} \frac{\left|\phi(2^j t - k, v) - \phi(2^j s - k, v)\right|}{|t-s|^\kappa} (3 + |k|)^{1/\alpha+\nu}. \tag{9.21}$$

As above, we distinguish two cases: $j \le j_0$ and $j \ge j_0 + 1$. First, we suppose that (9.8) is satisfied. Similarly to (9.10), one has that

$$B^j(t,s,v) \le c_{13}\, 2^j\, |t-s|^{1-\kappa} \sum_{k\in\mathbb{Z}} (3+|k|)^{1/\alpha+\nu} (2 + |2^j t - k|)^{-2}.$$

Then, using (9.6), Lemma 9.4 (in which we take $\theta = 1/\alpha + \nu$ and $\zeta = 0$), and the fact that $|t| \le M$, one obtains that

$$B^j(t,s,v) \le c_{14}\, 2^{j+j_0(\kappa-1)} (1+2^j)^{1/\alpha+\nu}. \tag{9.22}$$

Now suppose that (9.13) is satisfied. By using this relation, (9.6), the triangle inequality, (2.12), Lemma 9.4 (in which one takes $\theta = 1/\alpha+\nu$ and $\zeta = 0$), and the fact that $t, s \in [-M, M]$, one gets that

$$\begin{aligned} B^j(t,s,v) &\le 2^{j\kappa} \sum_{k\in\mathbb{Z}} (3+|k|)^{1/\alpha+\nu} \left\{\left|\phi(2^j t - k, v)\right| + \left|\phi(2^j s - k, v)\right|\right\} \\ &\le c_{15}\, 2^{j\kappa} \sum_{k\in\mathbb{Z}} (1+|k|)^{1/\alpha+\nu} \left\{(2+|2^j t - k|)^{-2} + (2 + |2^j s - k|)^{-2}\right\} \\ &\le c_{16}\, 2^{j\kappa} \left\{(1+2^j|t|)^{1/\alpha+\nu} + (1+2^j|s|)^{1/\alpha+\nu}\right\} \le c_{17}\, 2^{j(\kappa+1/\alpha+\nu)}. \end{aligned} \tag{9.23}$$

It is no restriction to assume that $n \geq \log_2(M)+2$. Then, in view of the inequality $j_0 > -\log_2(M)-2$, one has that $-n-1 < j_0$ and thus (9.22) entails that

$$\begin{aligned}\sum_{j=-\infty}^{-n-1} & 2^{-jv}\,(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v)\\ &\leq c_{14} \sum_{j=-\infty}^{-n-1} 2^{j(1-v)+j_0(\kappa-1)}\,(1+2^j)^{1/\alpha+\nu}\,(3+|j|)^{i+1/\alpha+\nu}\\ &\leq c_{18}\, 2^{j_0(\kappa-1)} \sum_{j=-\infty}^{-n-1} 2^{j(1-v)}\,(3+|j|)^{i+1/\alpha+\nu}. \end{aligned}\tag{9.24}$$

Next, using Lemma 9.3 (in which one takes $n_0=-\infty$, $n_1=-n-1$, $\theta=1-v$, $\theta_0=1-b$, and $\lambda=i+1/\alpha+\nu$), the inequality $2^{j_0(\kappa-1)} < (4M)^{1-\kappa}$, and the inequality $v\leq b$, one gets that

$$\sum_{j=-\infty}^{-n-1} 2^{-jv}\,(3+|j|)^{i+1/\alpha+\nu}\, B^j(t,s,v) \leq c_{19}\, 2^{-n(1-b)}(4+n)^{i+1/\alpha+\nu}. \tag{9.25}$$

Now we find a suitable upper bound for $\sum_{j\geq n+1} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v)$. First we assume that $j_0 \geq n+1$. Then, using (9.22), one has that

$$\begin{aligned}\sum_{j=n+1}^{j_0} & 2^{-jv}\,(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v)\\ &\leq c_{14}\, 2^{j_0(\kappa-1)} \sum_{j=-\infty}^{j_0} 2^{j(1-v)}(3+|j|)^{i+1/\alpha+\nu}(1+2^j)^{1/\alpha+\nu}\\ &\leq c_{20}\, 2^{j_0(\kappa-1+1/\alpha+\nu)} \sum_{j=-\infty}^{j_0} 2^{j(1-v)}(3+|j|)^{i+1/\alpha+\nu}\\ &\leq c_{21} 2^{-j_0(a-1/\alpha-\kappa-\nu)}(3+|j_0|)^{i+1/\alpha+\nu} \leq c_{22} 2^{-n(a-1/\alpha-\kappa-\nu)}(3+n)^{i+1/\alpha+\nu}. \end{aligned}\tag{9.26}$$

Observe that the third inequality in (9.26) follows from Lemma 9.3 (in which we take $n_0=-\infty$, $n_1=j_0$, $\theta=1-v$, $\theta_0=1-b$, and $\lambda=i+1/\alpha+\nu$) as well as from the inequality $v\geq a$. Also observe that the last inequality in (9.26) results from the fact that the function $x\mapsto 2^{-x(a-1/\alpha-\kappa-\nu)}(3+x)^{i+1/\alpha+\nu}$ is continuous on $\mathbb{R}_+$ and decreasing for large enough x.

On the other hand, by making use of (9.23), one has that

$$\begin{aligned}\sum_{j=j_0+1}^{+\infty} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v) &\leq c_{17} \sum_{j=j_0+1}^{+\infty} 2^{j(\kappa+1/\alpha+\nu-v)}\,(3+|j|)^{i+1/\alpha+\nu}\\ &\leq c_{23}\, 2^{-j_0(a-1/\alpha-\kappa-\nu)}\,(3+|j_0|)^{i+1/\alpha+\nu}\\ &\leq c_{24}\, 2^{-n(a-1/\alpha-\kappa-\nu)}\,(3+n)^{i+1/\alpha+\nu}. \end{aligned}\tag{9.27}$$

Observe that the second inequality in (9.27) follows from Lemma 9.3 (in which we take $n_0 = j_0 + 1$, $n_1 = +\infty$, $\theta = \kappa + 1/\alpha + \nu - v$, $\theta_0 = a - 1/\alpha - \kappa - \nu$, and $\lambda = i+1/\alpha+\nu$) as well as from the inequality $v \geq a$. Also observe that the last inequality in (9.27) results from the fact that the function $x \mapsto 2^{-x(a-1/\alpha-\kappa-\nu)}(3+x)^{i+1/\alpha+\nu}$ is continuous on $\mathbb{R}_+$ and decreasing for large enough x.

Combining (9.26) with (9.27), it follows that, in the case where $j_0 \geq n+1$,

$$\sum_{j=n+1}^{+\infty} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v) \leq c_{25}\, 2^{-n(a-1/\alpha-\kappa-\nu)}(3+n)^{i+1/\alpha+\nu}. \tag{9.28}$$

Now we assume that $j_0 < n+1$. Then, by making use of (9.23), one has that

$$\begin{aligned}\sum_{j=n+1}^{+\infty} 2^{-jv}(3+|j|)^{i+1/\alpha+\nu} B^j(t,s,v) &\leq c_{17} \sum_{j=n+1}^{+\infty} 2^{j(\kappa+1/\alpha+\nu-v)}(3+|j|)^{i+1/\alpha+\nu}\\ &\leq c_{26}\, 2^{-n(a-1/\alpha-\kappa-\nu)}\,(3+n)^{i+1/\alpha+\nu},\end{aligned} \tag{9.29}$$

where the last inequality follows from Lemma 9.3 (in which we take $n_0 = n+1$, $n_1 = +\infty$, $\theta = \kappa + 1/\alpha + \nu - v$, $\theta_0 = a - 1/\alpha - \kappa - \nu$, and $\lambda = i + 1/\alpha + \nu$) as well as from the inequality $v \geq a$.

Finally, (9.2), (9.21), (9.25), (9.28), and (9.29) imply that

$$B_n(t,s,v) \leq c_{27}\left(2^{-n(1-b)} + 2^{-n(a-1/\alpha-\kappa-\nu)}\right)(4+n)^{i+1/\alpha+\nu} \tag{9.30}$$

for all $n \geq \log_2(M) + 2$. This, in turn, entails that, when $n \to +\infty$, $B_n(t,s,v)$ converges to 0, uniformly in $(t,s,v) \in [-M,M]^2 \times [a,b]$. □

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