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Centers of algebras associated to higher-rank graphs

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Abstract. The Kumjian–Pask algebras are path algebras associated to higher-rank graphs, and generalize the Leavitt path algebras. We study the center of a simple Kumjian–Pask algebra and characterize commutative Kumjian–Pask algebras.

1. Introduction

Let E be a directed graph and let $\mathbb{F}$ be a field. The Leavitt path algebras $L_{\mathbb{F}}(E)$ of E over $\mathbb{F}$ were first introduced in [1] and [2], and have been widely studied since then. Many of the properties of a Leavitt path algebra can be inferred from properties of the graph, and for this reason provide a convenient way to construct examples of algebras with a particular set of attributes. The Leavitt path algebras are the algebraic analogues of the graph C^*-algebras associated to E. In [11], Tomforde constructed an analogous Leavitt path algebra $L_R(E)$ over a commutative ring R with 1, and introduced more techniques from the graph C^*-algebra setting to study it.

In [3], Aranda Pino, Clark, an Huef, and Raeburn generalized Tomforde's construction and associated to a higher-rank graph Λ a graded algebra $\mathrm{KP}_R(\Lambda)$ called the *Kumjian–Pask algebra*. Example 7.1 of [3] shows that the class of Kumjian–Pask algebras over a field is strictly larger than the class of Leavitt path algebras over that field.

The center of a simple Leavitt path algebra has been studied in [4] and, for a nonsimple algebra, in [6]. In this paper we initiate the study of the center of a Kumjian–Pask algebra. In the motivational section (§3) we work over $\mathbb{C}$ and show how the embedding of $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ in the C^*-algebra of Λ can be used together with the Dauns–Hofmann theorem to deduce that the center of a simple Kumjian–Pask algebra is either $\{0\}$ or isomorphic to $\mathbb{C}$. More generally, it follows from Theorem 4.7, that the center of a "basically simple" (see page 1394) Kumjian–Pask algebra $\mathrm{KP}_R(\Lambda)$ is either zero or is isomorphic to the underlying ring R.

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Thus our Theorem 4.7 generalizes the analogous theorem for Leavitt path algebras over a field (Theorem 4.2 in [4]), but our proof techniques are very different and more informative. Indeed, the Kumjian–Pask algebra is basically simple if and only if the graph Λ is cofinal and aperiodic, and our proofs show explicitly which of these properties of the graph are needed to infer various properties of elements in the center.

In Proposition 5.3 we show that a Kumjian–Pask algebra of a k-graph Λ is commutative if and only it is a direct sum of rings of Laurent polynomials in k-indeterminates, and this holds if and only if Λ is a disjoint union of copies of the category $\mathbb{N}^k$. This generalizes Proposition 2.7 of [4].

2. Preliminaries

We view $\mathbb{N}^k$ as a category with one object and the composition given by addition. We call a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ a k-graph if there exists a functor $d : \Lambda \to \mathbb{N}^k$, with the *unique factorization property*: given $m, n \in \mathbb{N}^k$ and $\lambda \in \Lambda$, if $d(\lambda) = m + n$ then there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m, d(\nu) = n$ and $\lambda = \mu\nu$. The functor d is called the *degree functor* and $d(\lambda)$ is called the *degree of* λ. Using the unique factorization property, we identify the set of objects Λ^0 with the set of morphisms of degree 0, that is, $\Lambda^0 = \{\lambda \in \Lambda : d(\lambda) = 0\}$. Then, for $n \in \mathbb{N}^k$, we set $\Lambda^n := d^{-1}(n)$, and call Λ^n the paths of shape n in Λ and Λ^0 the vertices of Λ. A path $\lambda \in \Lambda$ is *closed* if $r(\lambda) = s(\lambda)$.

For $V, W \subset \Lambda^0$, we set $V\Lambda := \{\lambda \in \Lambda : r(\lambda) \in V\}$, $\Lambda W := \{\lambda \in \Lambda : s(\lambda) \in W\}$ and $V\Lambda W := V\Lambda \cap \Lambda W$; the sets $V\Lambda^n, \Lambda^n W$ and $V\Lambda^n W$ are defined similarly. For simplicity we write $v\Lambda$ for $\{v\}\Lambda$.

A k-graph Λ is *row-finite* if $|v\Lambda^n| < \infty$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ and has *no sources* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. We assume throughout that Λ is a row-finite k-graph with no sources.

Let $m, n \in \mathbb{N}^k$. We write $m \leq n$ if $m_i \leq n_i$ for all $1 \leq i \leq k$ and write $m \vee n$ for the k-tuple with ith entry $\max\{m_i, n_i\}$. Following Lemma 3.2 (iv) in [10], we say that a k-graph Λ is *aperiodic* if for every $v \in \Lambda^0$ and $m \neq n \in \mathbb{N}^k$ there exists $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda\big(m, m + d(\lambda) - (m \vee n)\big) \neq \lambda\big(n, n + d(\lambda) - (m \vee n)\big).$$

This formulation of aperiodicity is equivalent to the original one from Definition 4.3 in [8] when Λ is a row-finite graph with no sources, but is often more convenient since it only involves finite paths.

Let $\Omega_k := \{(m, n) \in \mathbb{N}^k : m \leq n\}$. As in Definition 2.1 of [8], we define an *infinite path* in Λ to be a degree-preserving functor $x : \Omega_k \to \Lambda$, and denote the set of infinite paths by Λ^∞. As in Definition 4.1 of [8], we say Λ is *cofinal* if for every infinite path x and every vertex v there exists $m \in \mathbb{N}^k$ such that $v\Lambda x(m) \neq \emptyset$.

For each $\lambda \in \Lambda$ we introduce a *ghost path* λ^*; for $v \in \Lambda^0$ we take $v^* = v$. We write $G(\Lambda)$ for the set of ghost paths and $G(\Lambda^{\neq 0})$ if we exclude the vertices.

Let R be a commutative ring with 1. Following Definition 3.1 of [3], a *Kumjian–Pask Λ-family* (P, S) in an R-algebra A consists of functions $P : \Lambda^0 \to A$ and $S : \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \to A$ such that

(KP1) $\{P_v : v \in \Lambda^0\}$ is a set of mutually orthogonal idempotents;

(KP2) for $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu) = s(\lambda)$,

$$S_\lambda S_\mu = S_{\lambda\mu}, \qquad P_{r(\lambda)} S_\lambda = S_\lambda = S_\lambda P_{s(\lambda)},$$
$$S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}, \qquad P_{s(\lambda)} S_{\lambda^*} = S_{\lambda^*} = S_{\lambda^*} P_{r(\lambda)};$$

(KP3) for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $d(\lambda) = d(\mu)$, we have $S_{\lambda^*} S_\mu = \delta_{\lambda,\mu} P_{s(\lambda)}$;

(KP4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k \setminus \{0\}$, we have $P_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*}$.

By Theorem 3.4 of [3] there exists an R-algebra $\mathrm{KP}_R(\Lambda)$, generated by a nonzero Kumjian–Pask Λ-family (p, s), with the following universal property: whenever (Q, T) is a Kumjian–Pask Λ-family in an R-algebra A, then there is a unique R-algebra homomorphism $\pi_{Q,T} : \mathrm{KP}_R(\Lambda) \to A$ such that

$$\pi_{Q,T}(p_v) = Q_v, \ \pi_{Q,T}(s_\lambda) = T_\lambda \text{ and } \pi_{Q,T}(s_{\mu^*}) = T_{\mu^*} \text{ for } v \in \Lambda^0 \text{ and } \lambda, \mu \in \Lambda^{\neq 0}.$$

Also by Theorem 3.4 of [3], the subgroups

$$\mathrm{KP}_R(\Lambda)_n := \mathrm{span}_R \left\{ s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda \text{ and } d(\lambda) - d(\mu) = n \right\} \quad (n \in \mathbb{Z}^k)$$

give a $\mathbb{Z}^k$-grading of $\mathrm{KP}_R(\Lambda)$. Let S be a $\mathbb{Z}^k$-graded ring; then by the graded-uniqueness theorem ([3], Theorem 4.1), a graded homomorphism $\pi\colon \mathrm{KP}_R(\Lambda) \to S$ such that $\pi(rp_v) \neq 0$ for nonzero $r \in R$ is injective.

We will often write elements $a \in \mathrm{KP}_R(\Lambda) \setminus \{0\}$ in the *normal form* of Lemma 4.2 in [3]: there exists $m \in \mathbb{N}^k$ and a finite $F \subset \Lambda \times \Lambda^m$ such that $a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$ where $r_{\alpha,\beta} \in R \setminus \{0\}$ and $s(\alpha) = s(\beta)$.

3. Motivation

When A is a simple C^*-algebra (over $\mathbb{C}$, of course), it follows from the Dauns–Hofmann theorem (see, for example, Theorem A.34 in [9]) that the center $Z(A)$ of A is isomorphic to $\mathbb{C}$ if A has an identity and is $\{0\}$ otherwise. Let Λ be a row-finite k-graph without sources. In this short section we deduce that the center of a simple Kumjian–Pask algebra $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ is either isomorphic to $\mathbb{C}$ or is $\{0\}$.

Lemma 3.1. *Suppose A is a simple C^*-algebra. If A has an identity, then $z \mapsto z1_A$ is an isomorphism of $\mathbb{C}$ onto the center $Z(A)$ of A. If A has no identity, then $Z(A) = \{0\}$.*

Proof. Since A is simple, the primitive ideal space $\mathrm{Prim}\, A$ of A is a singleton $\{\star\}$, and $f \mapsto f(\star)$ is an isomorphism of the algebra $C_b(\mathrm{Prim}\, A)$ of continuous bounded functions from $\mathrm{Prim}\, A$ onto $\mathbb{C}$. By the Dauns–Hofmann theorem, $C_b(\mathrm{Prim}\, A)$ is isomorphic to the center $Z(M(A))$ of the multiplier algebra $M(A)$ of A. Putting the two isomorphisms together gives an isomorphism $z \mapsto z1_{M(A)}$ of $\mathbb{C}$ onto $Z(M(A))$.

Now suppose that A has an identity. Then $M(A) = A$, and it follows from the first paragraph that $Z(A)$ is isomorphic to $\mathbb{C}$.

Next suppose that A does not have an identity. Let $a \in Z(A)$, and let $\{u_\lambda\}$ be an approximate identity in A and let $m \in M(A)$. Then $ma = \lim(mu_\lambda)a = a\lim(mu_\lambda) = am$. Thus $Z(A) \subset Z(M(A))$. Now $Z(A) \subset Z(M(A)) \cap A = \{z1_{M(A)} : z \in \mathbb{C}\} \cap A = \{0\}$. □

Lemma 3.2. *Let D be a dense subalgebra of a C^*-algebra A. Then $Z(A) \cap D = Z(D)$.*

Proof. Trivially, $Z(A) \cap D \subset Z(D)$. To see the reverse inclusion, let $a \in Z(D)$. Let $b \in A$ and choose $\{d_\lambda\} \subset D$ such that $d_\lambda \to b$. Then $ba = \lim_\lambda d_\lambda a = \lim_\lambda ad_\lambda = ab$. Now $a \in Z(A) \cap Z(D) \subset Z(A) \cap D$, and hence $Z(A) \cap D = Z(D)$. □

By Theorem 6.1 of [3], $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ is simple if and only if Λ is cofinal and aperiodic, so in the next corollary $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ is simple. Also, $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ has an identity if and only if Λ^0 is finite (see Lemma 4.6 below).

Corollary 3.3. *Suppose that Λ is a row-finite, cofinal, aperiodic k-graph with no sources. If Λ^0 is finite, then $z \mapsto z1_{\mathrm{KP}_{\mathbb{C}}(\Lambda)}$ is an isomorphism of $\mathbb{C}$ onto the center $Z(\mathrm{KP}_{\mathbb{C}}(\Lambda))$ of $\mathrm{KP}_{\mathbb{C}}(\Lambda)$. If Λ^0 is infinite, then $\mathrm{KP}_{\mathbb{C}}(\Lambda) = \{0\}$.*

Proof. Let (p, s) be a generating Kumjian–Pask Λ-family for $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ and (q, t) a generating Cuntz–Krieger Λ-family for $C^*(\Lambda)$. Then (q, t) is a Kumjian–Pask Λ-family in $C^*(\Lambda)$, and the universal property of $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ gives a $*$-homomorphism $\pi_{q,t} : \mathrm{KP}_{\mathbb{C}}(\Lambda) \to C^*(\Lambda)$ which takes $s_\mu s_{\nu^*}$ to $t_\mu t_\nu^*$. It follows from the graded-uniqueness theorem that $\pi_{q,t}$ is a $*$-isomorphism onto a dense $*$-subalgebra of $C^*(\Lambda)$ (see Proposition 7.3 of [3]). Since Λ is aperiodic and cofinal, $C^*(\Lambda)$ is simple by Theorem 3.1 of [10].

Now suppose that Λ^0 is finite. Then $\mathrm{KP}_{\mathbb{C}}(\Lambda)$ has identity $1_{\mathrm{KP}_{\mathbb{C}}(\Lambda)} = \sum_{v\in\Lambda^0} p_v$ and $C^*(\Lambda)$ has identity $1_{C^*(\Lambda)} = \sum_{v\in\Lambda^0} q_v$, and $\pi_{q,t}$ is unital. By Lemma 3.1, $Z(C^*(\Lambda)) = \{z1_{C^*(\Lambda)} : z \in \mathbb{C}\}$. By Lemma 3.2,

$$Z\big(\pi_{q,t}(\mathrm{KP}_{\mathbb{C}}(\Lambda))\big) = Z\big(C^*(\Lambda)\big) \cap \pi_{q,t}\big(\mathrm{KP}_{\mathbb{C}}(\Lambda)\big) = \{z1_{C^*(\Lambda)} : z \in \mathbb{C}\}.$$

Since $\pi_{q,t}$ is unital, $Z(\mathrm{KP}_{\mathbb{C}}(\Lambda))$ is isomorphic to $\mathbb{C}$ as claimed.

Next suppose that Λ^0 is infinite. Then $Z(C^*(\Lambda)) = \{0\}$ and $Z\big(\pi_{q,t}(\mathrm{KP}_{\mathbb{C}}(\Lambda))\big) = Z(C^*(\Lambda)) \cap \pi_{q,t}(\mathrm{KP}_{\mathbb{C}}(\Lambda)) = \{0\}$, giving $\mathrm{KP}_{\mathbb{C}}(\Lambda) = \{0\}$. □

4. The center of a Kumjian–Pask algebra

Our goal is to extend Corollary 3.3 to Kumjian–Pask algebras over arbitrary commutative rings. Throughout R is a commutative ring with 1 and Λ is a row-finite k-graph with no sources.

We will need Lemma 4.1 several times. For notational convenience, for $v \in \Lambda^0$, s_v or s_{v^*} means p_v. In particular, taking $m = 0$ in Lemma 4.1 below shows the set $\{s_\alpha : \alpha \in \Lambda\}$ is linearly independent.

Lemma 4.1. *Let $m \in \mathbb{N}^k$. Then $\{s_\alpha s_{\beta^*} : s(\alpha) = s(\beta)$ and $d(\beta) = m\}$ is a linearly independent subset of $\mathrm{KP}_R(\Lambda)$.*

Proof. Let F be a finite subset of $\{(\alpha,\beta) \in \Lambda \times \Lambda^m : s(\alpha) = s(\beta)\}$, and suppose that $\sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*} = 0$. Fix $(\sigma,\tau) \in F$. Since all the β have degree m, using (KP3) twice we obtain

$$0 = s_{\sigma^*}\Big(\sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}\Big) s_\tau = r_{\sigma,\tau} p_{s(\sigma)} + \sum_{(\alpha,\beta)\in F\setminus\{(\sigma,\tau)\}} r_{\alpha,\tau} s_{\sigma^*} s_\alpha s_{\beta^*} s_\tau$$

$$= r_{\sigma,\tau} p_{s(\sigma)} + \sum_{\substack{(\alpha,\tau)\in F\\ \alpha\neq\sigma}} r_{\alpha,\tau} s_{\sigma^*} s_\alpha. \tag{4.1}$$

If $d(\sigma) = d(\alpha)$ and $\sigma \neq \alpha$, then $s_{\sigma^*} s_\alpha = 0$ by (KP3). If $d(\sigma) \neq d(\alpha)$ then, by Lemma 3.1 of [3], $s_{\sigma^*} s_\alpha$ is a linear combination of $s_\mu s_{\nu^*}$ where $d(\mu) - d(\nu) = d(\alpha) - d(\sigma)$. It follows that the 0-graded component of (4.1) is $r_{\sigma,\tau} p_{s(\sigma)}$. Thus $0 = r_{\sigma,\tau} p_{s(\sigma)}$. But $p_{s(\sigma)} \neq 0$, by Theorem 3.4 of [3]. Hence $r_{\sigma,\tau} = 0$. Since $(\sigma,\tau) \in F$ was arbitrary, it follows that $\{s_\alpha s_{\beta^*} : s(\alpha) = s(\beta)$ and $d(\beta) = m\}$ is linearly independent. □

The next lemma describes properties of elements in the center of $\mathrm{KP}_R(\Lambda)$.

Lemma 4.2. *Let $a \in Z(\mathrm{KP}_R(\Lambda)) \setminus \{0\}$ be in normal form $\sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$.*

1. *If $(\sigma,\tau) \in F$, then $r(\sigma) = r(\tau)$.*
2. *Let $W = \{v \in \Lambda^0 : \exists (\alpha,\beta) \in F$ with $v = r(\beta)\}$. If $\mu \in \Lambda W$, then $r(\mu) \in W$.*
3. *If $(\sigma,\tau) \in F$, then there exists $(\alpha,\beta) \in F$ such that $r(\alpha) = r(\beta) = s(\sigma) = s(\tau)$.*
4. *There exists $l \in \mathbb{N} \setminus \{0\}$ and $\{(\alpha_i,\beta_i)\}_{i=1}^l \subset F$ such that $\beta_1 \cdots \beta_l$ is a closed path in Λ.*

Proof. 1) Let $(\sigma,\tau) \in F$. By Lemma 2.3 of [5], we have $0 \neq s_{\sigma^*} a s_\tau$. Since $a \in Z(\mathrm{KP}_R(\Lambda))$,

$$0 \neq s_{\sigma^*} p_{r(\sigma)} a p_{r(\tau)} s_\tau = s_{\sigma^*} a p_{r(\sigma)} p_{r(\tau)} s_\tau = \delta_{r(\sigma),r(\tau)} s_{\sigma^*} a s_\tau.$$

Hence $r(\sigma) = r(\tau)$.

2) Aiming at a contradiction, assume there exists $\mu \in \Lambda W$ such that $r(\mu) \notin W$. Then $p_v p_{r(\mu)} = 0$ for all $v \in W$. Thus

$$a p_{r(\mu)} = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*} p_{r(\beta)} p_{r(\mu)} = 0.$$

Since $a \in Z(\mathrm{KP}_R(\Lambda))$ we get $s_\mu a = a s_\mu = a p_{r(\mu)} s_\mu = 0$. Since $s(\mu) \in W$, there exists $(\alpha',\beta') \in F$ with $r(\beta') = s(\mu)$. Then $r(\alpha') = s(\mu)$ also by 1). Thus $S := \{(\alpha,\beta) \in F : s(\mu) = r(\alpha)\}$ is nonempty, and

$$0 = s_\mu a = \sum_{(\alpha,\beta)\in S} r_{\alpha,\beta} s_{\mu\alpha} s_{\beta^*}.$$

But $\{s_{\mu\alpha}s_{\beta^*}\colon(\alpha,\beta)\in S\}$ is linearly independent, by Lemma 4.1, and hence $r_{\alpha,\beta}=0$ for all $(\alpha,\beta)\in S$. This contradicts the given normal form for a.

3) Let $(\sigma,\tau)\in F$. Then $s(\sigma)=s(\tau)$ by definition of normal form. By Lemma 2.3 in [5] we have $s_{\sigma^*}as_\tau\neq 0$. Since $a\in Z(\mathrm{KP}_R(\Lambda))$,

$$0\neq s_{\sigma^*}as_\tau=as_{\sigma^*}s_\tau=\sum_{(\alpha,\beta)\in F}r_{\alpha,\beta}s_\alpha s_{\beta^*}s_{\sigma^*}s_\tau=\sum_{\substack{(\alpha,\beta)\in F\\ r(\beta)=s(\sigma)}}r_{\alpha,\beta}s_\alpha s_{(\sigma\beta)^*}s_\tau. \tag{4.2}$$

In particular, the set $\{(\alpha,\beta)\in F : r(\beta)=s(\sigma)\}$ is nonempty. So there exists $(\alpha',\beta')\in F$ such that $r(\beta')=s(\sigma)$. Since $r(\alpha')=r(\beta')$ from 1), we are done.

4) Let $M=|F|+1$. Using (3) there exists a path $\beta_1\dots\beta_M$ such that, for $1\leq i\leq M$, there exists $\alpha_i\in\Lambda$ with $(\alpha_i,\beta_i)\in F$. Since $M>|F|$, there exists $i<j\in\{1,\dots,M\}$ such that $\beta_i=\beta_j$. Then $\beta_i\dots\beta_{j-1}$ is a closed path. □

The next corollary follows from Lemma 4.2 (4).

Corollary 4.3. *Let Λ be a row-finite k-graph with no sources and R a commutative ring with 1. If Λ has no closed paths then the center $Z(\mathrm{KP}_R(\Lambda))=\{0\}$.*

The next lemma describes the elements of the center of $\mathrm{KP}_R(\Lambda)$ when Λ is cofinal.

Lemma 4.4. *Suppose that Λ is cofinal. Let $a\in Z(\mathrm{KP}_R(\Lambda))\setminus\{0\}$ be in normal form $\sum_{(\alpha,\beta)\in F}r_{\alpha,\beta}s_\alpha s_{\beta^*}$. Then $\{v\in\Lambda^0:\exists(\alpha,\beta)\in F \text{ with } v=r(\beta)\}=\Lambda^0$.*

Proof. Write $W:=\{v\in\Lambda^0:\exists(\alpha,\beta)\in F \text{ with } v=r(\beta)\}$. By definition of normal form, there exists $m\in\mathbb{N}^k$ such that $F\subset\Lambda\times\Lambda^m$. Let $n=m\vee(1,1,\dots,1)$. By (KP4), for each $(\alpha,\beta)\in F$, we have $s_\alpha s_{\beta^*}=\sum_{\mu\in s(\alpha)\Lambda^{n-m}}s_{(\alpha\mu)}s_{(\beta\mu)^*}$. By "reshaping" each pair of paths in F in this way, collecting like terms, and dropping those with zero coefficients, we see that there exist $G\subset\Lambda\times\Lambda^n$ and $r'_{\gamma,\eta}\in R\setminus\{0\}$ such that $a=\sum_{(\gamma,\eta)\in G}r'_{\gamma,\eta}s_\gamma s_{\eta^*}$ is also in normal form. By construction, $W'=\{v\in\Lambda^0:\exists(\gamma,\eta)\in G \text{ with } v=r(\eta)\}\subset W$.

Let $v\in\Lambda^0$. Using Lemma 4.2 (4), there exists $\{(\gamma_i,\eta_i)\}_{i=1}^l\subset G$ such that $\eta_1\cdots\eta_l$ is a closed path. Since $d(\eta_i)\geq(1,\dots,1)$ for all i, $x:=\eta_1\cdots\eta_l\eta_1\cdots\eta_l\eta_1\cdots$ is an infinite path. By cofinality, there exist $q\in\mathbb{N}^k$ and $\nu\in v\Lambda x(q)$. By the definition of x, there exist $q'\geq q$ and j such that $x(q')=r(\eta_j)$. Let $\lambda=x(q,q')$. Then $\nu\lambda\in v\Lambda W'$. By Lemma 4.2 (2), $v=r(\nu\lambda)\in W'$ as well. Thus $W'=\Lambda^0$, and since $W'\subset W$ we have $W=\Lambda^0$. □

The next lemma describes the elements of the center of $\mathrm{KP}_R(\Lambda)$ when Λ is aperiodic.

Lemma 4.5. *Suppose that Λ is aperiodic and $a\in Z(\mathrm{KP}_R(\Lambda))\setminus\{0\}$. Then there exist $n\in\mathbb{N}^k$ and $G\subset\Lambda^n$ such that $a=\sum_{\alpha\in G}r_\alpha s_\alpha s_{\alpha^*}$ is in normal form.*

Proof. Suppose $a \in Z(\mathrm{KP}_R(\Lambda)) \setminus \{0\}$ with $a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$ in normal form. By definition there exists $n \in \mathbb{N}^k$ such that $F \subset \Lambda \times \Lambda^n$. Let $(\sigma,\tau) \in F$. From Lemma 2.3 in [5] we know that $s_{\sigma^*} a s_\tau \neq 0$. Let $m = \vee_{(\alpha,\beta)\in F}(d(\alpha) \vee d(\beta))$. Since Λ is aperiodic, by Lemma 6.2 of [7] there exists $\lambda \in s(\sigma)\Lambda$ with $d(\lambda) \geq m$ such that

$$\left.\begin{array}{l} \alpha,\beta \in \Lambda s(\sigma),\ d(\alpha), d(\beta) \leq m \\ \text{and}\ \ \alpha\lambda(0,d(\lambda)) = \beta\lambda(0,d(\lambda)) \end{array}\right\} \implies \alpha = \beta. \tag{4.3}$$

The same argument as in Proposition 4.9 of [3] now shows that $s_{\lambda^*} s_{\sigma^*} a s_\tau s_\lambda \neq 0$. Since $a \in Z(\mathrm{KP}_R(\Lambda))$, $0 \neq s_{\lambda^*} s_{\sigma^*} a s_\tau s_\lambda = a s_{\lambda^*} s_{\sigma^*} s_\tau s_\lambda = a s_{(\sigma\lambda)^*} s_{\tau\lambda}$. Thus

$$\begin{aligned} 0 \neq s_{(\sigma\lambda)^*} s_{\tau\lambda} &= s_{\sigma\lambda(d(\lambda),d(\lambda)+d(\sigma))^*} s_{\sigma\lambda(0,d(\lambda))^*} s_{\tau\lambda(0,d(\lambda))} s_{\tau\lambda(d(\lambda),d(\lambda)+d(\tau))} \\ &= \delta_{\sigma,\tau} s_{\sigma\lambda(d(\lambda),d(\lambda)+d(\sigma))^*} s_{\tau\lambda(d(\lambda),d(\lambda)+d(\tau))} \end{aligned}$$

using (4.3). Thus $\sigma = \tau$.

Since $(\sigma,\tau) \in F$ was arbitrary we have $\alpha = \beta$ for all $(\alpha,\beta) \in F$. Let $G = \{\alpha \in \Lambda\colon (\alpha,\alpha) \in F\}$ and write r_α for $r_{\alpha,\alpha}$. Note $G \subset \Lambda^n$ because $F \subset \Lambda \times \Lambda^n$. Thus $a = \sum_{\alpha\in G} r_\alpha s_\alpha s_{\alpha^*}$ in normal form as desired. □

Our main theorem (Theorem 4.7) has two cases: Λ^0 finite and infinite.

Lemma 4.6. $\mathrm{KP}_R(\Lambda)$ *has an identity if and only if* Λ^0 *is finite.*

Proof. If Λ^0 is finite, then $\sum_{v\in\Lambda^0} p_v$ is an identity for $\mathrm{KP}_R(\Lambda)$. Conversely, assume that $\mathrm{KP}_R(\Lambda)$ has an identity $1_{\mathrm{KP}_R(\Lambda)}$. Aiming at a contradiction, suppose that Λ^0 is infinite. Write $1_{\mathrm{KP}_R(\Lambda)}$ in normal form $\sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$. Since F is finite, so is $W := \{v \in \Lambda^0 : \exists (\alpha,\beta) \in F \ \text{ with } \ v = r(\beta)\}$. Thus there exists $w \in \Lambda^0 \setminus W$. Now $p_w = 1_{\mathrm{KP}_R(\Lambda)} p_w = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*} p_w = 0$ because $w \neq r(\beta)$ for any of the β. This contradiction shows that Λ^0 must be finite. □

Theorem 4.7. *Let* Λ *be a row-finite* k*-graph with no sources and let* R *be a commutative ring with* 1*.*

1. *Suppose* Λ *is aperiodic and cofinal, and that* Λ^0 *is finite. Then* $Z(\mathrm{KP}_R(\Lambda)) = R1_{\mathrm{KP}_R(\Lambda)}$*.*
2. *Suppose that* Λ *is cofinal and* Λ^0 *is infinite. Then* $Z(\mathrm{KP}_R(\Lambda)) = \{0\}$*.*

Proof. 1) Suppose Λ is aperiodic and cofinal, and that Λ^0 is finite. Let $a \in Z(\mathrm{KP}_R(\Lambda)) \setminus \{0\}$. Since Λ is aperiodic, by Lemma 4.5 there exist $n \in \mathbb{N}^k$ and $G \subset \Lambda^n$ such that $a = \sum_{\alpha\in G} r_\alpha s_\alpha s_{\alpha^*}$ is in normal form. Since Λ is row-finite and Λ^0 is finite, Λ^n is finite.

We claim that $G = \Lambda^n$. Aiming at a contradiction, suppose that $G \neq \Lambda^n$, and let $\lambda \in \Lambda^n \setminus G$. Then $a s_\lambda = 0$ by (KP3). Since $a \in Z(\mathrm{KP}_R(\Lambda))$,

$$0 = a s_\lambda = s_\lambda a = \sum_{\substack{\alpha\in G \\ r(\alpha)=s(\lambda)}} r_\alpha s_{\lambda\alpha} s_{\alpha^*}.$$

Since Λ is cofinal, $\{r(\alpha) : \alpha \in G\} = \Lambda^0$ by Lemma 4.4. Thus $S = \{\alpha \in G : r(\alpha) = s(\lambda)\} \neq \emptyset$. But $\{s_{\lambda\alpha}s_{\alpha^*} : \alpha \in S\}$ is linearly independent by Lemma 4.1. Thus $r_\alpha = 0$ for $\alpha \in S$, contradicting our choice of $\{r_\alpha\}$. It follows that $G = \Lambda^n$ as claimed, and that

$$a = \sum_{\alpha\in\Lambda^n} r_\alpha s_\alpha s_{\alpha^*}.$$

Next we claim that $r_\mu = r_\nu$ for all $\mu, \nu \in \Lambda^n$. Let $\mu, \nu \in \Lambda^n$. Let $x \in s(\mu)\Lambda^\infty$. Since Λ is cofinal, there exist $m \in \mathbb{N}^k$ and $\gamma \in s(\nu)\Lambda s(x(m))$. Set $\eta = x(0, m)$. Now

$$\begin{aligned} r_\mu s_{\nu\gamma}s_{(\mu\eta)^*} &= r_\mu s_{\nu\gamma}s_{\eta^*}s_{\mu^*} = s_{\nu\gamma}s_{\eta^*}\sum_{\alpha\in\Lambda^n} r_\mu s_{\mu^*}s_\alpha s_{\alpha^*} \\ &= s_{\nu\gamma}s_{\eta^*}s_{\mu^*}\sum_{\alpha\in\Lambda^n} r_\alpha s_\alpha s_{\alpha^*} = s_{\nu\gamma}s_{(\mu\eta)^*}a \\ &= a s_{\nu\gamma}s_{(\mu\eta)^*} = \sum_{\alpha\in\Lambda^n} r_\alpha s_\alpha s_{\alpha^*}s_\nu s_\gamma s_{(\mu\eta)^*} = r_\nu s_{\nu\gamma}s_{(\mu\eta)^*}. \end{aligned}$$

Since $s_{\nu\gamma}s_{(\mu\eta)^*} \neq 0$ this implies $r_\mu = r_\nu$. Let $r = r_\mu$. Now

$$a = \sum_{\alpha\in\Lambda^n} r s_\alpha s_{\alpha^*} = \sum_{v\in\Lambda^0}\sum_{\alpha\in v\Lambda^n} r s_\alpha s_{\alpha^*} = r\sum_{v\in\Lambda^0} p_v = r1_{\mathrm{KP}_R(\Lambda)}$$

as desired.

2) Suppose there exists $a \in Z(\mathrm{KP}_R(\Lambda))\setminus\{0\}$. Write $a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta}s_\alpha s_{\beta^*}$ in normal form. Then Lemma 4.4 gives that $\{v\in\Lambda^0 : \exists(\alpha,\beta)\in F \text{ with } v = r(\beta)\} = \Lambda^0$, contradicting that F is finite. □

The simplicity of $C^*(\Lambda)$ played an important role in §3. To reconcile this with Theorem 4.7, recall from [11] that an ideal $I \in \mathrm{KP}_R(\Lambda)$ is *basic* if $rp_v \in I$ for $r \in R\setminus\{0\}$ then $p_v \in I$, and that $\mathrm{KP}_R(\Lambda)$ is *basically simple* if its only basic ideals are $\{0\}$ and $\mathrm{KP}_R(\Lambda)$. By Theorem 5.14 of [3], $\mathrm{KP}_R(\Lambda)$ is basically simple if and only if Λ is cofinal and aperiodic (and by Theorem 6.1 of [3], $\mathrm{KP}_R(\Lambda)$ is simple if and only if R is a field and Λ is cofinal and aperiodic). Thus Theorem 4.7 is in the spirit of Corollary 3.3.

5. Commutative Kumjian–Pask algebras

We view $\mathbb{N}^k$ as a category with one object $\star$ and composition given by addition, and use $\{e_i\}_{i=1}^k$ to denote the standard basis of $\mathbb{N}^k$.

Example 5.1. Let $d : \mathbb{N}^k \to \mathbb{N}^k$ be the identity map. Then $(\mathbb{N}^k, d)$ is a k-graph. By Example 7.1 in [3], $\mathrm{KP}_R(\mathbb{N}^k)$ is commutative with identity $p_\star$, and $\mathrm{KP}_R(\mathbb{N}^k)$ is isomorphic to the ring of Laurent polynomials $R[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}]$ in k commuting indeterminates.

Lemma 5.2. *Suppose $\Lambda = \Lambda_1 \bigsqcup \Lambda_2$ is a disjoint union of two k-graphs. Then $\mathrm{KP}_R(\Lambda) = \mathrm{KP}_R(\Lambda_1) \oplus \mathrm{KP}_R(\Lambda_2)$.*

Proof. For each $i \in \{1,2\}$, let (q^i, t^i) be the generating Kumjian–Pask Λ_i-family of $\mathrm{KP}_R(\Lambda_i)$, and let (p,s) be the generating Kumjian–Pask Λ-family of $\mathrm{KP}_R(\Lambda)$. Restricting (p,s) to Λ_i gives a Λ_i-family in $\mathrm{KP}_R(\Lambda)$, and hence the universal property for $\mathrm{KP}_R(\Lambda_i)$ gives a homomorphism $\pi^i_{p,s} : \mathrm{KP}_R(\Lambda_i) \to \mathrm{KP}_R(\Lambda)$ such that $\pi^i_{p,s} \circ (q^i, t^i) = (p,s)$. Each $\pi^i_{p,s}$ is graded, and the graded uniqueness theorem (Theorem 4.1 in [3]) implies that $\pi^i_{p,s}$ is injective.

We now identify $\mathrm{KP}_R(\Lambda_i)$ with its image in $\mathrm{KP}_R(\Lambda)$. If $\mu \in \Lambda_1$ and $\lambda \in \Lambda_2$, then $s_\mu s_\lambda = s_\mu p_{s(\mu)} p_{r(\lambda)} s_\lambda = 0$. Similarly $s_\lambda s_\mu, s_{\mu^*} s_{\lambda^*}, s_{\lambda^*} s_{\mu^*}, s_\lambda s_{\mu^*}, s_{\mu^*} s_\lambda, s_{\lambda^*} s_\mu$, and $s_\mu s_{\lambda^*}$ are all zero. Thus $\mathrm{KP}_R(\Lambda_1)\,\mathrm{KP}_R(\Lambda_2) = 0 = \mathrm{KP}_R(\Lambda_2)\,\mathrm{KP}_R(\Lambda_1)$, and the internal direct sum $\mathrm{KP}_R(\Lambda_1) \oplus \mathrm{KP}_R(\Lambda_2)$ is a subalgebra of $\mathrm{KP}_R(\Lambda)$. Finally, $\mathrm{KP}_R(\Lambda_1) \oplus \mathrm{KP}_R(\Lambda_2)$ is all of $\mathrm{KP}_R(\Lambda)$ since the former contains all the generators of the latter. This gives the result. □

Proposition 5.3. *Let Λ be a row-finite k-graph with no sources and R a commutative ring with 1. Then the following conditions are equivalent:*

1. $\mathrm{KP}_R(\Lambda)$ *is commutative;*
2. $r = s$ *on* Λ *and* $r|_{\Lambda^n}$ *is injective;*
3. $\Lambda \cong \bigsqcup_{v \in \Lambda^0} \mathbb{N}^k$*;*
4. $\mathrm{KP}_R(\Lambda) \cong \bigoplus_{v \in \Lambda^0} R[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}]$.

Proof. 1) ⇒ 2) Suppose that $\mathrm{KP}_R(\Lambda)$ is commutative. Aiming at a contradiction, suppose there exists $\lambda \in \Lambda$ such that $s(\lambda) \neq r(\lambda)$. Then $s_{\lambda^*} s_\lambda = s_\lambda s_{\lambda^*}$, and

$$p_{s(\lambda)} = p^2_{s(\lambda)} = p_{s(\lambda)} s_{\lambda^*} s_\lambda = p_{s(\lambda)} s_\lambda s_{\lambda^*} = p_{s(\lambda)} p_{r(\lambda)} s_\lambda s_{\lambda^*} = 0.$$

But $p_v \neq 0$ for all $v \in \Lambda^0$ by Theorem 3.4 of [3]. This contradiction gives $r = s$.

Next, suppose $\lambda, \mu \in \Lambda^n$ with $\lambda \neq \mu$. Aiming at a contradiction, suppose that $r(\lambda) = r(\mu)$. Since $r = s$, we have $r(\lambda) = s(\lambda) = s(\mu) = r(\mu)$. Then

$$s_\lambda = p_{r(\lambda)} s_\lambda = p_{s(\lambda)} s_\lambda = p_{s(\mu)} s_\lambda = s_{\mu^*} s_\mu s_\lambda = s_{\mu^*} s_\lambda s_\mu = 0$$

by (KP3). Now $p_{s(\lambda)} = 0$, contradicting that $p_v \neq 0$ for all $v \in \Lambda^0$ by Theorem 3.4 of [3]. Thus r is injective on Λ^n.

2) ⇒ 3) Assume that $r = s$ on Λ and that $r|_{\Lambda^n}$ is injective. Since $r = s$, the set $\{v\Lambda v\}_{v \in \Lambda^0}$ is a partition of Λ. Since r is injective on Λ^{e_i}, the subgraph $v\Lambda^{e_i} v$ has a single vertex v and a single edge f^v_i. Thus $f^v_i \mapsto e_i$ defines a graph isomorphism $v\Lambda v \to \mathbb{N}^k$. Hence $\Lambda = \bigsqcup_{v \in \Lambda^0} v\Lambda v \cong \bigsqcup_{v \in \Lambda^0} \mathbb{N}^k$.

3) ⇒ 4) Assume that $\Lambda \cong \bigsqcup_{v \in \Lambda^0} \mathbb{N}^k$. By Lemma 5.2, $\mathrm{KP}_R(\Lambda)$ is isomorphic to $\bigoplus \mathrm{KP}_R(\mathbb{N}^k)$, and by Example 5.1 each $\mathrm{KP}_R(\mathbb{N}^k)$ is isomorphic to

$$R[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}].$$

4) ⇒ 1) This follows since $\bigoplus R[x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}]$ is commutative. □

References

[1] Abrams, G. and Aranda Pino, G.: The Leavitt path algebra of a graph. *J. Algebra* **293** (2005), no. 2, 319–334.

[2] Ara, P., Moreno, M. A. and Pardo, E.: Nonstable K-theory for graph algebras. *Algebr. Represent. Theory* **10** (2007), no. 2, 157–178.

[3] Aranda Pino, G., Clark, J., an Huef, A. and Raeburn, I.: Kumjian–Pask algebras of higher-rank graphs. *Trans. Amer. Math. Soc.* **365** (2013), no. 7, 3613–3641.

[4] Aranda Pino, G. and Crow, K.: The center of a Leavitt path algebra. *Rev. Mat. Iberoam.* **27** (2011), no. 2, 621–644.

[5] Brown, J. H. and an Huef, A.: The socle and semisimplicity of a Kumjian–Pask algebra. To appear in *Comm. Algebra*.

[6] Corrales García, M. G., Martín Barquero, D., Martín González, C., Siles Molina, M. and Solanilla Hernández, J. F.: Centers of path algebras, Cohn and Leavitt path algebras. ArXiv: 1209.4375v1.

[7] Hazlewood, R., Raeburn, I., Sims, A. and Webster, S. B. G.: Remarks on some fundamental results about higher-rank graphs and their C^*-algebras. *Proc. Edinb. Math. Soc. (2)* **56** (2013), no. 2, 575–597.

[8] Kumjian, A. and Pask, D.: Higher rank graph C^*-algebras. *New York J. Math.* **6** (2000), 1–20.

[9] Raeburn, I. and Williams, D. P.: *Morita equivalence and continuous-trace C^*-algebras*. Mathematical Surveys and Monographs 60, American Mathematical Society, Providence, RI, 1998.

[10] Robertson, D. I. and Sims, A.: Simplicity of C^*-algebras associated to higher-rank graphs. *Bull. Lond. Math. Soc.* **39** (2007), no. 2, 337–344.

[11] Tomforde, M.: Leavitt path algebras with coefficients in a commutative ring. *J. Pure Appl. Algebra* **215** (2011), no. 4, 471–484.

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