Rev. Mat. Iberoam. **30** (2014), no. 4, 1397–1411
DOI 10.4171/RMI/819

The geometry of the dyadic maximal operator

Eleftherios Nikolidakis

Abstract. We prove a sharp integral inequality which connects the dyadic maximal operator with the Hardy operator. We also give some applications of this inequality.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$\mathcal{M}_d\,\phi(x) = \sup\Big\{\frac{1}{|Q|}\int_Q |\phi(u)|\,du : x\in Q, Q\subseteq \mathbb{R}^n \text{ is a dyadic cube}\Big\} \tag{1.1}$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$, where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, for $N = 0,1,2,\dots$. As is well known it satisfies the weak type (1,1) inequality

$$\big|\{x\in\mathbb{R}^n : \mathcal{M}_d\,\phi(x) > \lambda\}\big| \le \frac{1}{\lambda}\int_{\{\mathcal{M}_d\,\phi>\lambda\}} |\phi(u)|\,du, \tag{1.2}$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$. The inequality (1.2) easily implies the following L^p inequality:

$$\big\|\mathcal{M}_d\,\phi\big\|_p \le \frac{p}{p-1}\,\big\|\phi\big\|_p. \tag{1.3}$$

It is easy to see that the weak type inequality (1.2) is the best possible, while (1.3) is also sharp. (See [1] and [2] for general martingales, and [19] for dyadic ones). One approach to studying the dyadic maximal operator is by refining the above inequalities. Concerning (1.2), some refinements have been made in [7], [11], [12], and [13], while for (1.3) the Bellman function of this operator has been explicitly computed in [5]. It is defined in the following way: for every f, F and L such that

Mathematics Subject Classification (2010): Primary 42B25; Secondary 42B99.
Keywords: Dyadic, maximal, rearrangement.

$0 < f^p \leq F$, $L \geq f$, the Bellman function of three variables associated to the dyadic maximal operator is defined by

$$S_p(f,F,L) = \sup\Big\{\frac{1}{|Q|}\int_Q \big(\mathcal{M}_d\,\phi\big)^p : \frac{1}{|Q|}\int_Q \phi(u)\,du = f,$$
$$\frac{1}{|Q|}\int_Q \phi(u)^p\,du = F,\ \sup_{R:Q\subseteq R}\frac{1}{|R|}\int_R \phi(u)\,du = L\Big\}, \tag{1.4}$$

where Q is a fixed dyadic cube, R runs over all dyadic cubes containing Q, and ϕ is nonnegative in $L^p(Q)$. Actually the above calculations have been made in a more general setting. More precisely, we define, for a nonatomic probability measure space (X,μ) and a tree $\mathcal{T}$, the dyadic maximal operator associated to $\mathcal{T}$ by

$$\mathcal{M}_{\mathcal{T}}\,\phi(x) = \sup\Big\{\frac{1}{\mu(I)}\int_I |\phi|\,d\mu : x \in I \in \mathcal{T}\Big\}, \tag{1.5}$$

for every $\phi \in L^1(X,\mu)$. In fact, the inequalities (1.2) and (1.3) remain true and sharp even in this setting. Then the corresponding main Bellman function of two variables is defined by

$$B_p(f,F) = \sup\Big\{\int_X \big(\mathcal{M}_{\mathcal{T}}\,\phi\big)^p\,d\mu : \phi \geq 0,\ \int_X \phi\,d\mu = f,\ \int_X \phi^p\,d\mu = F\Big\}, \tag{1.6}$$

for $0 < f^p \leq F$. It is proved in [5] that (1.6) equals

$$B_p(f,F) = F\,\omega_p\,(f^p/F)^p, \quad \text{where} \quad \omega_p : [0,1] \to \Big[1,\frac{p}{p-1}\Big]$$

denotes the inverse function H_p^{-1} of H_p, which is defined by $H_p(z) = -(p-1)z^p + pz^{p-1}$, for $z \in [1, p/(p-1)]$. As an immediate result we have that $B_p(f,F)$ is independent of the tree $\mathcal{T}$ and the measure space (X,μ).

Actually using $B_p(f,F)$ we can compute the Bellman function of three variables defined by

$$B_p(f,F,k) = \sup\Big\{\int_K \big(\mathcal{M}_{\mathcal{T}}\,\phi\big)^p\,d\mu : \phi \geq 0,\ \int_X \phi\,d\mu = f,\ \int_X \phi^p\,d\mu = F,$$
$$K \text{ measurable subset of } X \text{ with } \mu(K) = k\Big\}, \tag{1.7}$$

for $0 < f^p \leq F$ and $k \in (0,1]$. Using (1.6) one can also find the exact value of (1.4). Bellman functions arise in several problems in harmonic analysis. Such problems (including the dyadic Carleson imbedding theorem and weighted inequalities) are described in [9] (see also [8] and [10]) and also connections to stochastic optimal control are provided, from which it follows that the corresponding Bellman functions satisfy certain nonlinear second-order PDEs. The exact evaluation of a Bellman function is a difficult task which is connected with the deeper structure of the corresponding harmonic analysis problem. Several Bellman functions have been computed (see [1], [2], [5], [8], [15], [16], [17], and [18]).

Recently L. Slavin, A. Stokolos, and V. Vasyunin ([14]) linked some cases of the Bellman function computation to solving certain PDEs of Monge–Ampère type, and in this way they obtained an alternative proof of the results in [5] for the

Bellman functions related to the dyadic maximal operator. Also in [18] a Bellman function more general than the one related to the dyadic Carleson imbedding theorem has been evaluated precisely using the Monge–Ampère equation approach. Also the Bellman functions of the dyadic maximal operator have been evaluated in [6] in connection with Kolmogorov's inequality.

In [4] still more general Bellman functions have been computed, such as

$$T_{p,G}(f,F,k) = \sup\Big\{ \int_K G\big(\mathcal{M}_{\mathcal{T}}\phi\big)\,d\mu : \phi \geq 0,\ \int_X \phi\,d\mu = f,\ \int_X \phi^p\,d\mu = F,$$
$$K \text{ measurable subset of } X \text{ with } \mu(K) = k\Big\}, \tag{1.8}$$

where G is a suitable nonnegative increasing convex function on $[0,+\infty)$. For example one can use $G(x) = x^q$, with $1 < q < p$. The approach for evaluating (1.8) is by proving a symmetrization principle, namely that for suitable G as above there holds

$$T_{p,G}(f,F,k) = \sup\Big\{ \int_0^k G\Big(\frac{1}{u}\int_0^u r(t)dt\Big)du :\ r \geq 0,\ r \text{ non increasing on } [0,1],$$
$$\text{and } \int_0^1 r(u)du = f,\ \int_0^1 r^p(u)du = F\Big\}. \tag{1.9}$$

Equation (1.9) is important and is the tool for finding the exact value of $T_{p,G}(f,F,k)$ as is done in [4].

In this paper we prove a sharp integral inequality which connects the dyadic operator with the Hardy operator in a direct way. More precisely we consider nonincreasing integrable functions $g, h : (0,1] \to \mathbb{R}^+$ and a nondecreasing function $G : [0,+\infty) \to [0,+\infty)$. We prove the following.

Theorem 1.1. *For any $k \in (0,1]$,*

$$\sup\Big\{ \int_K G\big[(\mathcal{M}_{\mathcal{T}}\phi)^*\big]h(t)\,dt,\ \phi^* = g,$$
$$K \textit{ is a measurable subset of } [0,1] \textit{ with } |K| = k\Big\}$$
$$= \int_0^k G\Big(\frac{1}{t}\int_0^t g(u)\,du\Big)h(t)\,dt. \tag{1.10}$$

An immediate consequence of the above theorem is the following.

Corollary 1.2. *With the notation of Theorem 1.1 we have that, for any $p > 0$ and nonincreasing $g : (0,1] \to \mathbb{R}^+$,*

$$\sup\Big\{ \int_X \big(\mathcal{M}_{\mathcal{T}}\phi\big)^p\,d\mu : \phi^* = g\Big\} = \int_0^1 \Big(\frac{1}{t}\int_0^t g(u)\,du\Big)^p dt.$$

It is obvious that Theorem 1.1 implies the symmetrization principle mentioned above. Additionally we describe some applications of Theorem 1.1. First of all it is interesting to see what happens if in (1.8) we set $G(x) = x^q$ and replace the L^p-norm of ϕ by its $L^{p,\infty}$-quasi norm $\|\cdot\|_{p,\infty}$ defined by

$$\|\phi\|_{p,\infty} = \sup\left\{\mu\big(\{\phi \geq \lambda\}\big)^{1/p} \cdot \lambda : \lambda > 0\right\}. \tag{1.11}$$

More precisely using Theorem 1.1 we can evaluate

$$\begin{aligned} \Delta(f,F,k) = \sup\Big\{ \int_K (\mathcal{M}_\mathcal{T}\,\phi)^q \,d\mu : \phi \geq 0,\ \int_X \phi\,d\mu = f,\ \|\phi\|_{p,\infty} = F,\\ K \text{ measurable subset of } X \text{ with } \mu(K) = k\Big\}, \end{aligned} \tag{1.12}$$

for every $0 < f \leq \frac{p}{p-1}\,F$, $k \in [0,1]$ and $1 < q < p$.

Second, it is known by [11] that the inequality

$$\big\|\mathcal{M}_\mathcal{T}\,\phi\big\|_{p,\infty} \leq \frac{p}{p-1}\,\big\|\phi\big\|_{p,\infty}, \tag{1.13}$$

is the best possible and is independent of the L^1 and L^q-norm of ϕ, for any fixed q such that $1 < q < p$. In Section 1.1 of [3] there is introduced a norm on $L^{p,\infty}$, equivalent to $\|\cdot\|_{p,\infty}$. This is given by

$$|||\phi|||_{p,\infty} = \sup\left\{\mu(E)^{-1+1/p}\int_E |\phi|\,d\mu : E \text{ measurable subset of } X \text{ with } \mu(E) > 0\right\} \tag{1.14}$$

and it is easily proved that there holds

$$\|\phi\|_{p,\infty} \leq |||\phi|||_{p,\infty} \leq \frac{p}{p-1}\,\|\phi\|_{p,\infty}. \tag{1.15}$$

As a second application we prove that the inequality

$$|||\mathcal{M}_\mathcal{T}\,\phi|||_{p,\infty} \leq \left(\frac{p}{p-1}\right)^2 \|\phi\|_{p,\infty}, \tag{1.16}$$

is best possible and is independent of the L^1-norm of ϕ. Finally we prove that the inequality $\|\mathcal{M}_\mathcal{T}\phi\|_{L^{p,q}} \leq \frac{p}{p-1}\|\phi\|_{L^{p,q}}$ is the best possible for $q < p$ where $\|\cdot\|_{L^{p,q}}$ stands for the Lorentz quasi-norm on $L^{p,q}$ given by

$$\big\|\phi\big\|_{L^{p,q}} \equiv \big\|\phi\big\|_{p,q} = \left(\int_0^1 \left[\phi^*(t)\,t^{1/p}\right]^q \frac{dt}{t}\right)^{1/q}. \tag{1.17}$$

2. Preliminaries

Let (X,μ) be a nonatomic probability measure space.

Definition 2.1. A set $\mathcal{T}$ of measurable subsets of X will be called a *tree* if the following conditions are satisfied:

1. $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.

2. For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that
   (a) the elements of $C(I)$ are disjoint subsets of I;
   (b) $I = \bigcup C(I)$.
3. $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{ X \}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} C(I)$.
4. We have that
$$\lim_{m \to \infty} \sup_{I \in \mathcal{T}_{(m)}} \mu(I) = 0.$$

Examples of trees are given in [5]. The most well known is the one given by the family of all dyadic subcubes of $[0,1]^m$.

The following has been proved in [5].

Lemma 2.2. *For every $I \in \mathcal{T}$ and every a such that $0 < a < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of disjoint subsets of I such that*
$$\mu\Big(\bigcup_{J \in \mathcal{F}(I)} J \Big) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1-a)\,\mu(I).$$

We will need also the following fact.

Lemma 2.3. *Let $\phi : (X,\mu) \to \mathbb{R}^+$ and let $(A_j)_j$ be a measurable partition of X such that $\mu(A_j) > 0$ for all j. Then if $\int_X \phi\, d\mu = f$ there exists a rearrangement of ϕ, say h ($h^* = \phi^*$), such that $\frac{1}{\mu(A_j)} \int_{A_j} h\, d\mu = f$ for every j.*

Proof. We set $\phi^* = g : [0,1] \to \mathbb{R}^+$. First we find a measurable set $B_1 \subseteq [0,1]$ such that

$$|B_1| = \mu(A_1) \quad \text{and} \quad \frac{1}{|B_1|} \int_{B_1} g(u)\, du = f. \tag{2.1}$$

Obviously

$$\frac{1}{\mu(A_1)} \int_0^{\mu(A_1)} g(u)\, du \geq f \geq \frac{1}{\mu(A_1)} \int_{1-\mu(A_1)}^1 g(u)\, du. \tag{2.2}$$

As a result there exists r such that $0 < r$, $r + \mu(A_1) < 1$ and $\frac{1}{\mu(A_1)} \int_r^{r+\mu(A_1)} g(u)\, du = f$. We just need to set $B_1 = [r, r + \mu(A_1)]$. Then (2.1) is obviously satisfied. Now we define $h_1 : A_1 \to \mathbb{R}^+$ by $(h_1)^* = (g/B_1)^*$. This is a function defined on $(0, \mu(A_1))$. Then it is obvious that $\frac{1}{\mu(A_1)} \int_{A_1} h_1 = f$. We then continue in the same way for the space $X \setminus A_1$ and inductively complete the proof of Lemma 2.3. □

Now given a tree $\mathcal{T}$ on (X,μ) we define the associated dyadic maximal operator by
$$\mathcal{M}_{\mathcal{T}}\, \phi(x) = \sup \Big\{ \frac{1}{\mu(I)} \int_I |\phi|\, d\mu : x \in I \in \mathcal{T} \Big\}.$$

3. Main theorem

Suppose we are given non increasing integrable functions $g, h : (0,1] \to \mathbb{R}^+$. Also let $G : [0,+\infty) \to [0,+\infty)$ be a nondecreasing function.

Lemma 3.1. *Let $k \in (0,1]$ and let K be a measurable subset of $(0,1]$ such that $|K| = k$. Then with the notation above there holds*

$$\int_K G\big[\big(\mathcal{M}_\mathcal{T}\,\phi\big)^*\big]h(t)\,dt \le \int_0^k G\Big(\frac{1}{t}\int_0^t g(u)\,du\Big)h(t)\,dt$$

for every $\phi \in L^1(X,\mu)$ such that $\phi^* = g$.

Proof. Let v be the Borel measure on $(0,1]$ defined by $v(A) = \int_A h(t)\,dt$, for every Borel $A \subseteq (0,1]$, and set $I = \int_K G[(\mathcal{M}_\mathcal{T}\,\phi)^*]\,dv(t)$. Then

$$I = \int_{\lambda=0}^{+\infty} v\big(\big\{t \in K : (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \ge \lambda\big\}\big)\,dG(\lambda).$$

Let $f = \int_X \phi\,d\mu$. For $0 < \lambda \le f$ we obviously have

$$v\big(\big\{t \in K : (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \ge \lambda\big\}\big) = v(K), \quad \text{since} \quad (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \ge f, \quad \forall\, t \in [0,1].$$

Then $I = II + III$, where $II = v(K)\,[G(f) - G(0)]$ and

$$III = \int_{\lambda=f}^{+\infty} v\big(\big\{t \in K : (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \ge \lambda\big\}\big)\,dG(\lambda).$$

Obviously $II \le [G(f) - G(0)]\int_0^k h(u)\,du$. Additionally

$$v(\{t \in K : (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \ge \lambda\}) \le v(\{t \in (0,k] : (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \ge \lambda\})$$

since h and $(\mathcal{M}_\mathcal{T}\,\phi)^*$ are nonincreasing and $|K| = k$.

As a consequence,

$$III \le \int_{\lambda=f}^{+\infty} v(\{t \in (0,k] : (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \ge \lambda\})\,dG(\lambda).$$

Fix $\lambda > f$ and let $E_\lambda = \{\mathcal{M}_\mathcal{T}\,\phi \ge \lambda\}$. Then there exists a pairwise disjoint family, $(I_j)_j$, of elements of $\mathcal{T}$, such that

$$\frac{1}{\mu(I_j)}\int_{I_j} \phi\,d\mu \ge \lambda, \quad \text{and} \quad E_\lambda = \bigcup I_j. \tag{3.1}$$

In fact we just need to consider the family $(I_j)_j$ of elements of $\mathcal{T}$ maximal with respect to the integral condition (3.1). From (3.1) we have that $\int_{I_j} \phi\,d\mu \ge \lambda\,\mu(I_j)$, for every j. Since the family $(I_j)_j$ is pairwise disjoint we have that

$$\int_{E_\lambda} \phi\,d\mu \ge \lambda\,\mu(E_\lambda), \quad \text{so} \quad \frac{1}{\mu(E_\lambda)}\int_{E_\lambda} \phi\,d\mu \ge \lambda. \tag{3.2}$$

Certainly $\int_0^{\mu(E_\lambda)} \phi^*(u)\,du \geq \int_{E_\lambda} \phi\,d\mu$, so (3.2) gives

$$\frac{1}{\mu(E_\lambda)} \int_0^{\mu(E_\lambda)} \phi^*(u)\,du \geq \lambda. \tag{3.3}$$

Now let $a(\lambda)$ be the unique element of $[0,1]$ such that $\frac{1}{a(\lambda)} \int_0^{a(\lambda)} \phi^*(u)\,du = \lambda$. Its existence is guaranteed by the fact that $\lambda > f = \int_0^1 \phi^*(u)\,du$ (in fact we can suppose without loss of generality that $g(0+) = +\infty$, otherwise we work on $\lambda \in (f, \|g\|_\infty]$. Notice that if $\|g\|_\infty = A$ and $\phi^* = g$, then $\mathcal{M}_\mathcal{T}\,\phi \leq A$ a.e. on X). Let also $A_\lambda = \{t \in (0,k] : (\mathcal{M}_\mathcal{T}\,\phi)^*(t) \geq \lambda\}$. Additionally $A_\lambda \subset \{t \in (0,1] : (\mathcal{M}_\mathcal{T}\phi)^*(t) \geq \lambda\} =: B_\lambda$, so $|A_\lambda| \leq |B_\lambda| = \mu(E_\lambda)$.

Let $\beta(\lambda)$ be the unique $\beta \in (0,1]$ for which there holds: $(0,\beta) \subset A_\lambda$ and such that for every $t > \beta$ we have either $(\mathcal{M}_\mathcal{T}\,\phi)^*(t) < \lambda$ or $t > k$. So A_λ differs from $(0,\beta)$ except possibly at the endpoint β. As a consequence $A_\lambda \subset (0,\beta(\lambda)]$ and $|A_\lambda| = \beta(\lambda)$. From (3.3) and the definition of $a(\lambda)$ we have that

$$\frac{1}{\mu(E_\lambda)} \int_0^{\mu(E_\lambda)} \phi^*(u)\,du \geq \lambda = \frac{1}{a(\lambda)} \int_0^{a(\lambda)} \phi^*(u)\,du,$$

Since $\phi^* = g$ is nonincreasing we obtain that $\mu(E_\lambda) \leq a(\lambda)$. As a result $|A_\lambda| \leq a(\lambda)$. So $\beta(\lambda) \leq a(\lambda)$ and consequently we have that $A_\lambda \subset (0, a(\lambda)]$. However, of course, $A_\lambda \subset (0,k]$. Consequently $A_\lambda \subset \{t \in (0,k] : t \in (0,a(\lambda)] = \{t \in (0,k] : \frac{1}{t}\int_0^t g(u)\,du \geq \lambda\}$ from the definition of $a(\lambda)$. Obviously then

$$III \leq \int_{\lambda=f}^{+\infty} v\Big(\Big\{t \in (0,k] : \frac{1}{t}\int_0^t g(u)\,du \geq \lambda\Big\}\Big)\,dG(\lambda). \tag{3.4}$$

From the above estimates of II and III we obtain

$$I \leq \int_{\lambda=0}^{+\infty} v\Big(\Big\{t \in (0,k] : \frac{1}{t}\int_0^t g(u)\,du \geq \lambda\Big\}\Big)\,dG(\lambda) = \int_0^k G\Big(\frac{1}{t}\int_0^t g(u)\,du\Big)\,dv(t) \tag{3.5}$$

and Lemma 3.1 is proved. □

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Because of Lemma 3.1 we only need to construct for every $a \in (0,1)$ a μ-measurable function $\phi_a : X \to \mathbb{R}^+$ such that $\phi_a^* = g$ and

$$\limsup_{a\to 0^+} \int_0^k G\big[\big(\mathcal{M}_\mathcal{T}\,\phi_a\big)^*\big]\,dv \geq \int_0^k G\Big(\frac{1}{t}\int_0^t g(u)\,du\Big)\,dv(t).$$

We proceed to this as follows. Let $a \in (0,1)$. Using Lemma 2.2 we choose for every $I \in \mathcal{T}$ a family $\mathcal{F}(I) \subseteq \mathcal{T}$ of disjoint subsets of I such that

$$\sum_{J\in\mathcal{F}(I)} \mu(I) = (1-a)\,\mu(I). \tag{3.6}$$

We define $S = S_a$ to be the smallest subset of $\mathcal{T}$ such that $X \in S$ and for every $I \in S$, $\mathcal{F}(I) \subseteq S$. For $I \in S$ we write $A_I = I \setminus \bigcup_{J\in\mathcal{F}(I)} J$. Then if $a_I = \mu(A_I)$ we have, because of (3.6), that $a_I = a\mu(I)$. It is also clear that

$$S = \bigcup_{m\geq 0} S_{(m)}, \quad \text{where} \quad S_{(0)} = \{X\}, \quad S_{(m+1)} = \bigcup_{I\in S_{(m)}} \mathcal{F}(I).$$

For $I \in S$ we define $\operatorname{rank}(I) = r(I)$ to be the unique integer m such that $I \in S_{(m)}$.

Additionally, for every $I \in S$ with $r(I) = m$ we define

$$\gamma(I) = \gamma_m = \frac{1}{a(1-a)^m} \int_{(1-a)^{m+1}}^{(1-a)^m} g(u)\, du. \tag{3.7}$$

For $I \in S$ we also set

$$b_m(I) = \sum_{\substack{S\ni J\subseteq I \\ r(J)=r(I)+m}} \mu(J).$$

Then we easily see inductively that

$$b_m(I) = (1-a)^m \mu(I). \tag{3.8}$$

It is also clear that for every $I \in S$

$$I = \bigcup_{S\ni J\subseteq I} A_J. \tag{3.9}$$

Finally, for every m we define the measurable subset of X, $S_m := \bigcup_{I\in S_{(m)}} I$. Now, for every $m \geq 0$, we choose $\tau_a^{(m)} : S_m \setminus S_{m+1} \to \mathbb{R}^+$ such that

$$[\tau_a^{(m)}]^* = \big(g/[(1-a)^{m+1}, (1-a)^m)\big)^*. \tag{3.10}$$

This is possible since $\mu(S_m \setminus S_{m+1}) = \mu(S_m) - \mu(S_{m+1}) = b_m(X) - b_{m+1}(X) = (1-a)^m - (1-a)^{m+1} = a\,(1-a)^m$ and X is nonatomic.

Then we define $\tau_a \colon X \to \mathbb{R}^+$ by $\tau_a(x) = \tau_a^{(m)}(x)$ for $x \in S_m \setminus S_{m+1}$, so, because of (3.10), $\tau_a^* = g$.

It is now obvious that $S_m \setminus S_{m+1} = \bigcup_{I\in S_{(m)}} A_I$ and that

$$\int_{S_m\setminus S_{m+1}} \tau_a^{(m)}\, d\mu = \int_{(1-a)^{m+1}}^{(1-a)^m} g(u)\, du \Longrightarrow \frac{1}{\mu(S_m\setminus S_{m+1})} \int_{S_m\setminus S_{m+1}} \tau_a\, d\mu = \gamma_m. \tag{3.11}$$

Using Lemma 2.3 we see that there exists a rearrangement of $\tau_a/S_{(m)} \setminus S_{(m+1)} = \tau_a^{(m)}$, called $\phi_a^{(m)}$, for which $\frac{1}{a_I}\int_{A_I} \phi_a^{(m)} = \gamma_m$, for every $I \in S_m$. Define $\phi_a : X \to \mathbb{R}^+$ by $\phi_a(x) = \phi_a^{(m)}(x)$, for $x \in S_{(m)} \setminus S_{(m+1)}$. Of course $\phi_a^* = g$.

Let $I \in S_{(m)}$. Then

$$\begin{aligned}
Av_I(\phi_a) &= \frac{1}{\mu(I)} \int_I \phi_a \, d\mu = \frac{1}{\mu(I)} \sum_{S \ni J \subseteq I} \int_{A_J} \phi_a \, d\mu = \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{\substack{S \ni J \subseteq I \\ r(J) = r(I) + \ell}} \int_{A_J} \phi_a \, d\mu \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{\substack{S \ni J \subseteq I \\ r(J) = m + \ell}} \gamma_{m+\ell} \, a_J \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \sum_{\substack{S \ni J \subseteq I \\ r(J) = m + \ell}} a \, \mu(J) \, \frac{1}{a(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) \, du \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) \, du \sum_{\substack{S \ni J \subseteq I \\ r(J) = m + \ell}} \mu(J) \\
&= \frac{1}{\mu(I)} \sum_{\ell \geq 0} \frac{1}{(1-a)^{m+\ell}} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) \, du \cdot b_\ell(I)
\end{aligned}$$

$$\overset{(3.6)}{=} \frac{1}{(1-a)^m} \sum_{\ell \geq 0} \int_{(1-a)^{m+\ell+1}}^{(1-a)^{m+\ell}} g(u) \, du = \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u) \, du. \tag{3.12}$$

Now, for $x \in S_m \setminus S_{m+1}$, there exists $I \in S_{(m)}$ such that $x \in I$, so

$$\mathcal{M}_{\mathcal{T}}(\phi_a)(x) \geq A\,v_I(\phi_a) = \frac{1}{(1-a)^m} \int_0^{(1-a)^m} g(u) \, du =: \vartheta_m. \tag{3.13}$$

Since $\mu(S_m) = (1-a)^m$, we see easily from the above that for every $m \geq 0$ we have

$$(\mathcal{M}_{\mathcal{T}} \phi_a)^*(t) \geq \vartheta_m, \quad \text{for every} \quad t \in \big[(1-a)^{m+1}, (1-a)^m\big).$$

For any $a \in (0,1)$ we choose $m = m_a$ such that $(1-a)^{m+1} \leq k < (1-a)^m$. Hence we have $\lim_{a \to 0^+} (1-a)^{m_a} = k$.

We consider now two cases. The first one is the following:

$$\limsup_{a \to 0^+} \int_0^k G\big[(\mathcal{M}_{\mathcal{T}} \, \phi_a)^*\big] \, dv(t) = +\infty \, .$$

In this case Theorem 1.1 is obvious, by Lemma 3.1. In the second one,

$$\limsup_{a \to 0^+} \int_0^k G\big[\big(\mathcal{M}_{\mathcal{T}} \, \phi_a\big)^*\big] \, dv(t) < +\infty \, .$$

In this case

$$\begin{aligned}
\int_0^{(1-a)^{m_a}} G\big[\big(\mathcal{M}_{\mathcal{T}} \, \phi_a\big)^*\big] \, dv &\geq \sum_{l \geq 0} \int_{(1-a)^{m_a+l+1}}^{(1-a)^{m_a+l}} G(\vartheta_m) \, dv \\
&= \sum_{l \geq 0} G\Big(\frac{1}{(1-a)^{m_a+l}} \int_0^{(1-a)^{m_a+l}} g(u) \, du\Big) \, v\big(\big[(1-a)^{m_a+l+1}, (1-a)^{m_a+l}\big)\big).
\end{aligned} \tag{3.14}$$

Since $\lim_{a\to 0^+}(1-a)^{m_a} = k$ and the right hand side of (3.14) is a Riemann sum for the integral $\int_0^{(1-a)^{m_a}} G[\frac{1}{t}\int_0^t g(u)\,du]\,dv(t)$, we conclude, because of the monotonicity of G, $\frac{1}{t}\int_0^t g(u)\,du$ and h, that it converges to $\int_0^k G(\frac{1}{t}\int_0^t g(u)\,du)\,dv(t)$. Hence

$$\limsup_{a\to 0^+}\int_0^{(1-a)^{m_a}} G\big[\big(\mathcal{M}_{\mathcal{T}}\,\phi_a\big)^*\big]\,dv \geq \int_0^k G\Big(\frac{1}{t}\int_0^t g(u)\,du\Big)\,dv(t).$$

Further

$$\int_k^{(1-a)^{m_a}} G\big[\big(\mathcal{M}_{\mathcal{T}}\,\phi_a\big)^*\big]\,dv \leq \Big(\int_k^{(1-a)^{m_a}} h(u)\,du\Big)G\big[\big(\mathcal{M}_{\mathcal{T}}\,\phi_a\big)^*(k)\big].$$

However, if

$$\limsup_{a\to 0^+} G\big[\big(\mathcal{M}_{\mathcal{T}}\,\phi_a\big)^*(k)\big] = +\infty$$

we must have that

$$\limsup_{a\to 0^+}\int_0^k G\big[\big(\mathcal{M}_{\mathcal{T}}\,\phi_a\big)^*(t)\big]\,dv(t) = +\infty$$

which is not the case. As a result,

$$\lim_{a\to 0^+}\int_k^{(1-a)^{m_a}} G\big[\big(\mathcal{M}_{\mathcal{T}}\,\phi_a\big)^*(t)\big]\,dv(t) = 0.$$

Theorem 1.1 is now proved. □

Proof of Corollary 1.2. It is obvious, since

$$\int_X (\mathcal{M}_{\mathcal{T}}\,\phi)^p\,d\mu = \int_0^1 \big[(\mathcal{M}_{\mathcal{T}}\,\phi)^*\big]^p\,dt.$$

for any $\phi : (X,\mu) \to \mathbb{R}^+$. □

4. Applications

Now we give some applications.

a) First application. We seek to calculate

$$\Delta(f,F,k) = \sup\Big\{\int_K \big(\mathcal{M}_{\mathcal{T}}\,\phi\big)^q\,d\mu : \phi \geq 0,\ \int_X \phi\,d\mu = f,\ \|\phi\|_{p,\infty} = F,\ K \text{ a measurable subset of X with } \mu(K) = k\Big\} \tag{4.1}$$

for $0 < f \leq \frac{p}{p-1}F$ and $1 < q < p$. We prove:

Theorem 4.1. *For $F = (p-1)/p$ we have*

$$\Delta(f,F,k) = \begin{cases} \dfrac{p}{p-q}\, k^{1-q/p}, & k \le f^{p/(p-1)} \\ \dfrac{q(p-1)}{(p-q)(q-1)}\, f^{(p-q)/(p-1)} - \dfrac{1}{q-1}\, k^{1-q} f^q, & f^{p/(p-1)} \le k \le 1, \end{cases} \tag{4.2}$$

for $0 < f \le 1$.

Proof. Let ϕ be as in (4.1), and let K be a measurable subset of X with $\mu(K) = k$. Using Lemma 3.1 we have that

$$\int_K \big(\mathcal{M}_{\mathcal{T}}\phi\big)^q\, d\mu \le \int_0^k \Big(\frac{1}{t}\int_0^t \phi^*(u)\, du\Big)^q\, dt.$$

Since $\|\phi\|_{p,\infty} = (p-1)/p$ we have that $\phi^*(u) \le \frac{p-1}{p}\, u^{-1/p}$, $u \in (0,1]$. Hence, for every t such that $0 < t \le k$,

$$\frac{1}{t}\int_0^t \phi^*(u)\, du \le \frac{1}{t}\int_0^t \frac{p-1}{p}\, u^{-1/p} = t^{-1/p} \quad \text{and} \quad \frac{1}{t}\int_0^t \phi^*(u)\, du \le \frac{f}{t}.$$

Thus, if we set $A(t) = (1/t)\int_0^t \phi^*(u)\, du$, we have $A(t) \le \min\big\{f/t, t^{-1/p}\big\}$, for all $t \in (0,k]$.

Thus, if $k \le f^{p/(p-1)}$,

$$\int_0^k [A(t)]^q\, dt \le \int_0^k t^{-q/p}\, dt = \frac{p}{p-q}\, k^{1-q/p},$$

and for $f^{p/(p-1)} < k \le 1$,

$$\begin{aligned}\int_0^k [A(t)]^q\, dt &\le \int_0^{f^{p/(p-1)}} t^{-q/p}\, dt + \int_{f^{p/(p-1)}}^k \frac{f^q}{t^q}\, dt \\ &= \frac{p}{p-q}\, f^{(p-q)/(p-1)} - \frac{1}{q-1}\, f^q k^{1-q} + \frac{1}{q-1}\, f^{q+p(1-q)/(p-1)} \\ &= \frac{q(p-1)}{(p-q)(q-1)}\, f^{(p-q)/(p-1)} - \frac{1}{q-1}\, f^q k^{1-q}.\end{aligned}$$

We have proved that $\Delta\big(f,(p-1)/p,k\big) \le \mathcal{T}(f,k)$, where $T(f,k)$ is the right side of (4.2). We now prove the reverse inequality. Obviously, we have that

$$\Delta\Big(f, \frac{p-1}{p}, k\Big) \ge \int_0^k \Big(\frac{1}{t}\int_0^t \psi(u)\, du\Big)^q\, dt, \tag{4.3}$$

where $\psi : (0,1] \to \mathbb{R}^+$ is defined by

$$\psi(u) = \begin{cases} \dfrac{p-1}{p}\, u^{-1/p}, & 0 < u \le f^{p/(p-1)}, \\ 0, & f^{p/(p-1)} < u \le 1. \end{cases}$$

Since $\int_0^1 \psi(u)\,du = f$ and $\|\psi\|_{p,\infty}^{[0,1]} = (p-1)/p$, (4.3) is obvious because of Theorem 1.1. However, if ψ is as above we have that

$$\frac{1}{t}\int_0^t \psi(u)\,du = \frac{f}{t}, \qquad \text{for} \quad f^{p/(p-1)} < t \leq 1 \quad \text{and}$$
$$\frac{1}{t}\int_0^t \psi(u)\,du = t^{-1/p}, \quad \text{for} \quad 0 < t \leq f^{p/(p-1)}.$$

From the above calculations we conclude

$$\varDelta\Big(f, \frac{p-1}{p}, k\Big) = T(f,k)$$

and Theorem 4.1 is proved. □

b) Second application. In [10] it is proved that

$$\sup\Big\{\|\mathcal{M}_{\mathcal{T}}\,\phi\|_{p,\infty} : \phi \geq 0,\ \int_X \phi\,d\mu = f,\ \|\phi\|_{p,\infty} = F\Big\} = \frac{p}{p-1}\,F, \tag{4.4}$$

for $0 < f \leq \frac{p}{p-1}\,F$. That is the inequality $\|\mathcal{M}_{\mathcal{T}}\phi\|_{p,\infty} \leq \frac{p}{p-1}\,\|\phi\|_{p,\infty}$ is sharp and is independent of the integral of ϕ. A related problem is to find

$$E(f,F) = \sup\Big\{|||\mathcal{M}_{\mathcal{T}}\phi|||_{p,\infty} : \phi \geq 0,\ \int_X \phi\,d\mu = f,\ \|\phi\|_{p,\infty} = F\Big\},$$

where the integral norm $|||\cdot|||_{p,\infty}$ is given by (1.14). In fact, we prove:

Theorem 4.2. *With the above notation we have*

$$E(f,F) = \Big(\frac{p}{p-1}\Big)^2 F. \tag{4.5}$$

Proof. We prove this for $F = (p-1)/p$. It is obvious that

$$|||\mathcal{M}_{\mathcal{T}}\phi|||_{p,\infty} \leq \Big(\frac{p}{p-1}\Big)^2 \|\phi\|_{p,\infty}.$$

for every $\phi \in L^{p,\infty}$. Indeed, because of (1.15) and (4.4),

$$|||\mathcal{M}_{\mathcal{T}}\phi|||_{p,\infty} \leq \frac{p}{p-1}\,\|\mathcal{M}_{\mathcal{T}}\,\phi\|_{p,\infty} \leq \Big(\frac{p}{p-1}\Big)^2 \|\phi\|_{p,\infty}, \tag{4.6}$$

for every $\phi \in L^{p,\infty}$.

We prove now that (4.6) is the best possible and is independent of the integral of ϕ. Let $0 < f \leq 1$. Choose k_0 such that $0 < k_0 \leq f^{p/(p-1)}$. Set

$$\psi(u) := \begin{cases} \dfrac{p-1}{p}\,u^{-1/p}, & 0 < u \leq f^{p/(p-1)} \\ 0, & f^{p/(p-1)} < u \leq 1. \end{cases}$$

Then, obviously,

$$
\begin{aligned}
&E\Big(f, \frac{p-1}{p}\Big)\\
&\geq \sup\Big\{k_0^{-1+1/p}\int_E (\mathcal{M}_{\mathcal{T}}\,\phi)\,d\mu : E \text{ measurable subset of } X \text{ with } \mu(E)=k_0, \phi^*=\psi\Big\}\\
&= k_0^{-1+1/p}\int_0^{k_0}\Big(\frac{1}{t}\int_0^t \psi(u)\,du\Big)\,dt = \frac{p}{p-1},
\end{aligned}
$$

and Theorem 4.2 is proved. □

c) Third application. We give the last application. We know that the Lorentz space $L^{p,q}(X,\mu)\equiv L^{p,q}$ is defined by

$$
L^{p,q} = \Big\{\phi : (X,\mu)\to\mathbb{R}^+ \quad \text{such that} \quad \int_0^1 \big[\phi^*(t)\,t^{1/p}\big]^q\,\frac{dt}{t} < +\infty\Big\}
$$

with the topology induced by the quasi-norm $\|\cdot\|_{p,q}$ given by

$$
\|\phi\|_{p,q} = \Big[\int_0^1 \big[\phi^*(t)\,t^{1/p}\big]^q\,\frac{dt}{t}\Big]^{1/p}.
$$

Now we prove the following.

Theorem 4.3. *$\mathcal{M}_{\mathcal{T}}$ maps $L^{p,q}$ to $L^{p,q}$ and $\|\mathcal{M}_{\mathcal{T}}\|_{L^{p,q}\to L^{p,q}} = p/(p-1)$, where $q<p$.*

Proof. We set $v(A) = \int_A h(t)\,dt$, where $h(t)=t^{q/p-1}$, for all Borel subsets A of $[0,1]$. Then

$$
\begin{aligned}
\|\mathcal{M}_{\mathcal{T}}\phi\|_{p,q}^q &= \int_0^1 \big[(\mathcal{M}_{\mathcal{T}}\,\phi)^* t^{1/p}\big]^q\,\frac{dt}{t} = \int_0^1 \big[(\mathcal{M}_{\mathcal{T}}\,\phi)^*\big]^q\,dv(t)\\
&\leq \int_0^1 \Big(\frac{1}{t}\int_0^t \phi^*(u)\,du\Big)^q\,dv(t). \qquad (4.7)
\end{aligned}
$$

We set $A(t) = \frac{1}{t}\int_0^t \phi^*(u)\,du$. Then $A(t) = \int_0^1 \phi^*(tu)\,du$. So by the continuous form of the Minkowski inequality we then have

$$
\begin{aligned}
\|\mathcal{M}_{\mathcal{T}}\phi\|_{p,q}^q &\leq \Big[\int_0^1\Big(\int_0^1 \big[\phi^*(tu)\big]^q\,dv(t)\Big)^{1/q}du\Big]^q\\
&= \Big[\int_0^1\Big(\int_0^1 \big[\phi^*(tu)\big]^q t^{q/p-1}\,dt\Big)^{1/q}du\Big]^q\\
&= \Big[\int_0^1\Big(\int_0^u \big[\phi^*(t)\big]^q\,\frac{t^{q/p-1}}{u^{q/p-1}}\cdot\frac{dt}{u}\Big)^{1/q}du\Big]^q\\
&= \Big[\int_0^1 u^{-1/p}\Big(\int_0^u \big[\phi^*(t)\big]^q t^{q/p-1}\,dt\Big)^{1/q}du\Big]^q\\
&\leq \|\phi\|_{p,q}^q\Big[\int_0^1 u^{-1/p}\,du\Big]^q = \Big(\frac{p}{p-1}\Big)^q\cdot\|\phi\|_{p,q}^q,
\end{aligned}
$$

and so

$$\|\mathcal{M}_{\mathcal{T}}\phi\|_{p,q} \le \frac{p}{p-1}\,\|\phi\|_{p,q}, \quad \text{for} \quad \phi \in L^{p,q}, \quad q<p. \tag{4.8}$$

Now we finish the proof of Theorem 4.3. Let $g:(0,1] \to \mathbb{R}^+$ be nonincreasing. Then, by Theorem 1.1,

$$\sup_{\phi^*=g} \|\mathcal{M}_{\mathcal{T}}\phi\|_{p,q} = \Big[\int_0^1 \Big(\frac{1}{t}\int_0^t g(u)\,du\Big)^q dv(t)\Big]^{1/q},$$

so in order to prove that (4.8) is sharp we just need to construct, for every a such that $-1/p<a<0$, a nonincreasing $g_a:(0,1] \to \mathbb{R}^+$ such that

$$\frac{I}{II} \to \Big(\frac{p}{p-1}\Big)^q \quad \text{as} \quad a \to -\frac{1^+}{p},$$

where

$$I = \int_0^1 \Big(\frac{1}{t}\int_0^t g_a(u)\,du\Big)^q t^{q/p-1}\,dt, \quad \text{and} \quad II = \int_0^1 \big[g_a(u)\big]^q t^{q/p-1}\,dt.$$

If $g_a(t)=t^a$, for a such that: $-1/p<a<0$, we have

$$I = \Big(\frac{1}{a+1}\Big)^q \frac{1}{q(a+1/p)} \quad \text{and} \quad II = \frac{1}{q(a+1/p)},$$

so that

$$\frac{I}{II} = \Big(\frac{1}{a+1}\Big)^q \xrightarrow{a\to -1^+/p} \Big(\frac{p}{p-1}\Big)^q,$$

and Theorem 4.3 is proved. □

References

[1] Burkholder, D. L.: Boundary value problems and sharp inequalities for martingale transforms. *Ann. Probab.* **12** (1984), no. 3, 647–702.

[2] Burkholder, D. L.: Martingales and Fourier analysis in Banach spaces. In *Probability and analysis (Varenna, 1985)*, 61–108. Lecture Notes in Math. 1206, Springer, Berlin, 1986.

[3] Grafakos, L.: *Classical and modern Fourier analysis*. Pearson Education, Upper Saddle River, NJ, 2004.

[4] Melas, A. D.: Sharp general local estimates for dyadic-like maximal operators and related Bellman functions. *Adv. Math.* **220** (2009), no. 2, 367–426.

[5] Melas, A. D.: The Bellman functions of dyadic-like maximal operators and related inequalities. *Adv. Math.* **192** (2005), no. 2, 310–340.

[6] Melas, A. D. and Nikolidakis, E. N.: Dyadic-like maximal operators on integrable functions and Bellman functions related to Kolmogorov's inequality. *Trans. Amer. Math. Soc.* **362** (2010), no. 3, 1571–1597.

[7] MELAS, A. D. AND NIKOLIDAKIS, E. N.: On weak type inequalities for dyadic maximal functions. *J. Math. Anal. Appl.* **348** (2008), no. 1, 404–410.

[8] NAZAROV, F. AND TREIL, S.: The hunt for a Bellman function: Applications to estimates for singular integral operators and to other classical problems of harmonic analysis. *St. Petersbg. Math. J.* **8** (1997), no. 5, 721–824.

[9] NAZAROV, F., TREIL, S. AND VOLBERG, A.: Bellman function in stochastic optimal control and harmonic asnalysis. In *Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000),* 393–423. Oper. Theory Adv. Appl. 129, Birkhäuser, Basel, 2001.

[10] NAZAROV, F., TREIL, S. AND VOLBERG, A.: The Bellman functions and two-weight inequalities for Haar multipliers. *J. Amer. Math. Soc.* **12** (1999), no. 4, 909–928.

[11] NIKOLIDAKIS, E. N.: Extremal problems related to maximal dyadic like operators. *J. Math. Anal. Appl.* **369** (2010), no. 1, 377–385.

[12] NIKOLIDAKIS, E. N.: Optimal weak type estimates for dyadic-like maximal operators. *Ann. Acad. Sci. Fenn. Math.* **38** (2013), no. 1, 229–244.

[13] NIKOLIDAKIS, E. N.: Sharp weak type inequalities for the dyadic maximal operator. *J. Fourier Anal. Appl.* **19** (2013), no. 1, 115–139.

[14] SLAVIN, L., STOKOLOS, A. AND VASYUNIN, V.: Monge–Ampère equations and Bellman functions: the dyadic maximal operator. *C. R. Math. Acad. Sci. Paris* **346** (2008), no. 9-10, 585–588.

[15] SLAVIN, L. AND VOLBERG, A.: The explicit BF for a dyadic Chang–Wilson–Wolff theorem. The s-function and the exponential integral. In *Topics in harmonic analysis and ergodic theory*, 215–228. Contemp. Math. 444, Amer. Math. Soc., Providence, RI, 2007.

[16] VASYUNIN, V.: The exact constant in the inverse Holder inequality for Muckenhoupt weights. *St. Petersburg Math. J.* **15** (2004), no. 1, 49–79.

[17] VASYUNIN, V. AND VOLBERG, A.: Monge–Ampère equation and Bellman optimization of Carleson embedding therorems. In *Linear and complex analysis*, 195–238. Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

[18] VASYUNIN, V. AND VOLBERG, A.: The Bellman functions for the simplest two-weight inequality: an investigation of a particular case. *St. Petersburg Math. J.* **18** (2007), no. 2, 201–222.

[19] WANG, G.: Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion. *Proc. Amer. Math. Soc* **112** (1991), no. 2, 579–586.

Received November 13, 2012; revised February 5, 2013.

ELEFTHERIOS NIKOLIDAKIS: Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR-157 84, Athens, Greece.
E-mail: lefteris@math.uoc.gr

This research has been cofinanced by the European Union and Greek national funds through the operational program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF), aristeia code: MAXBELLMAN 2760, research code: 70/3/11913.