Calderón commutators and the Cauchy integral on Lipschitz curves revisited III. Polydisc extensions

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Dedicated to Professor Nicolae Popa, on the occasion of his 70*th birthday*

Abstract. This article is the last in a series of three papers, whose aim is to give new proofs of the well-known theorems of Calderón, Coifman, McIntosh and Meyer ([1], [3] and [4]). Here we extend the results of the previous two papers to the polydisc setting. In particular, we solve completely a question of Coifman open since the nineteen-eighties.

1. Introduction

The present article is a continuation of [8], [9] and is the last paper in the series. Its goal is to show that the method developed in these papers to give new proofs of the L^p boundedness of the Calderón commutators and the Cauchy integral on Lipschitz curves [1], [3] and [4], can be used to extend these classical results to the *n-parameter polydisc* setting for any $n \geq 2$.

Suppose that F is an analytic function on a disc centered at the origin in the complex plane and suppose A is a complex valued function on $\mathbb{R}^n$, so that $\partial^n A / \partial x_1 \ldots \partial x_n \in L^\infty(\mathbb{R}^n)$ with an L^∞ norm strictly smaller than the radius of convergence of F. Define the linear operator $C_{n,F,A}$ by the formula

$$(1.1) \qquad C_{n,F,A}f(x) := p.v. \int_{\mathbb{R}^n} f(x+t) \, F\Big(\frac{\Delta_{t_1}^{(1)}}{t_1} \circ \cdots \circ \frac{\Delta_{t_n}^{(n)}}{t_n} A(x) \Big) \, \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$$

for n-variable functions $f(x)$ for which the principal value integral exists, where $\Delta_s^{(i)}$ denotes the finite difference operator at scale s in the direction of e_i, given by

$$\Delta_s^{(i)} B(x) := B(x + se_i) - B(x),$$

and $e_1, \ldots, e_n$ is the standard basis in $\mathbb{R}^n$.

The main theorem we are going to prove is the following.

Theorem 1.1. *The operator $C_{n,F,A}$ extends naturally as a bounded linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for every $1 < p < \infty$.*

This answers completely a question of Coifman open since the early nineteen-eighties (see [5] and [6]). The case when the L^∞ norm of $\partial^n A/\partial x_1 \ldots \partial x_n$ is small and the generic $n = 2$ case were understood earlier by Journé in [5] and [6] respectively. Our proof is quite different from the approach in [5] and [6] and works equally well in all dimensions. In fact, as we will describe in the last section of the paper, much more can be proved in the same way. Not only are the operators of (1.1) are bounded, so are (for example) those given by expressions of the form

$$(1.2) \quad f \to p.v. \int_{\mathbb{R}^4} f(x+t+s) \, F\Big(\frac{\Delta_{t_1}^{(1)}}{t_1} \circ \frac{\Delta_{t_2}^{(2)}}{t_2} \circ \frac{\Delta_{s_1}^{(1)}}{s_1} \circ \frac{\Delta_{s_2}^{(2)}}{s_2} \, A(x)\Big) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{ds_1}{s_1} \frac{ds_2}{s_2}$$

and their natural generalizations. Of course, in (1.2) one has to assume that $\partial^4 A/\partial x_1^2 \partial x_2^2 \in L^\infty(\mathbb{R}^2)$. When $F(z) = z^d$ with $d \geq 1$ the operator in (1.1) is the natural n-parameter extension of the dth Calderón commutator, whereas for $F(z) = 1/(1 + iz)$ one obtains the n-parameter generalization of the Cauchy integral on Lipschitz curves (see [1], [3] and [4]).

For simplicity, henceforth we shall denote the n-parameter dth Calderón commutator by $C_{n,d,A}$. It is easy to see that when $f(x)$ and $A(x)$ have the particular forms

$$f(x) = f_1(x_1) \cdots f_n(x_n) \quad \text{and} \quad A(x) = A_1(x_1) \cdots A_n(x_n) \, ,$$

one has

$$C_{n,d,A}f(x) = C_{1,d,A_1}f_1(x_1) \cdots C_{1,d,A_n}f_n(x_n).$$

To motivate the introduction of the operators $C_{n,F,A}$ one just has to recall the context in which the original Calderón commutators appeared [1], [2] and [3]. If one tries to extend Calderón's algebra to $\mathbb{R}^n$ and to include in it pseudodifferential operators containing partial derivatives, one is naturally led to the study of the operators in (1.1) and their natural generalizations.

It is clear and well known that to prove statements such as the one in Theorem 1.1, one needs to prove *polynomial bounds* for the corresponding Calderón commutators $C_{n,d,A}$. More specifically, Theorem 1.1 reduces to the estimate

$$(1.3) \qquad \|C_{n,d,A}f\|_p \leq C(n,d) \cdot C(p) \cdot \|f\|_p \cdot \Big\| \frac{\partial^n A}{\partial x_1 \cdots \partial x_n} \Big\|_\infty^d$$

for any $f \in L^p$, where $C(n,d)$ grows at most polynomially in d [1].

The argument in [5] for proving the *small L^∞ norm* theorem used induction on the dimension n. We work instead directly in $\mathbb{R}^n$ and since our method is

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[1]This reduction is a simple consequence of the fact that if one writes the analytic function F as a power series, the generic operator $C_{n,F,A}$ itself becomes a series involving all the commutators $C_{n,d,A}$. The polynomial bounds are necessary for this series to be absolutely convergent.

essentially similar in every dimension, to keep the technicalities to a minimum, we chose for the reader's convenience to describe the proof of the main Theorem 1.1 in the particular case of the plane $\mathbb{R}^2$. However, it will be clear that the same proof works equally well in every dimension.

So from now on $n = 2$ and the goal is to prove the corresponding (1.3). The operators $C_{2,d,A}$ that we would like to understand, are given by

$$(1.4) \qquad C_{2,d,A}f(x) = p.v. \int_{\mathbb{R}^2} f(x+t) \Big(\frac{\Delta_{t_1}^{(1)}}{t_1} \circ \frac{\Delta_{t_2}^{(2)}}{t_2} A(x) \Big)^d \frac{dt_1}{t_1} \frac{dt_2}{t_2}.$$

If $a := \partial^2 A / \partial x_1 \partial x_2$, then one observes that

$$(1.5) \qquad \frac{\Delta_{t_1}^{(1)}}{t_1} \circ \frac{\Delta_{t_2}^{(2)}}{t_2} A(x) = \int_{[0,1]^2} a\big(x_1 + \alpha t_1, x_2 + \beta t_2\big) \, d\alpha \, d\beta.$$

As in [9], using (1.5) d times, one can see that if a and f are Schwartz functions, the implicit limit in (1.4) exists and can be rewritten as

$$(1.6) \qquad \int_{\mathbb{R}^{2d+2}} m_{2,d}(\xi, \xi_1, \ldots, \xi_d, \eta, \eta_1, \ldots, \eta_d) \, \widehat{f}(\xi, \eta) \, \widehat{a}(\xi_1, \eta_1) \ldots \widehat{a}(\xi_d, \eta_d)$$
$$\cdot e^{2\pi i (x_1, x_2) \cdot [(\xi, \eta) + (\xi_1, \eta_1) + \cdots + (\xi_d, \eta_d)]} \, d\xi \, d\xi_1 \cdots d\xi_d \, d\eta \, d\eta_1 \cdots d\eta_d$$

where

$$(1.7) \quad m_{2,d}(\xi, \xi_1, \ldots, \xi_d, \eta, \eta_1, \ldots, \eta_d) := m_{1,d}(\xi, \xi_1, \ldots, \xi_d) \cdot m_{1,d}(\eta, \eta_1, \ldots, \eta_d)$$

with $m_{1,d}(\xi, \xi_1, \ldots, \xi_d)$ and $m_{1,d}(\eta, \eta_1, \ldots, \eta_d)$ given by

$$\int_{[0,1]^d} \mathrm{sgn}\big(\xi + \alpha_1 \xi_1 + \cdots + \alpha_d \xi_d \big) \, d\alpha_1 \cdots d\alpha_d$$

and

$$\int_{[0,1]^d} \mathrm{sgn}\big(\eta + \beta_1 \eta_1 + \cdots + \beta_d \eta_d \big) \, d\beta_1 \cdots d\beta_d \, ,$$

respectively. Because of the formula (1.6) $C_{2,d}$ can be seen as a $(d+1)$-linear operator. However, it is important to realize (as in [9]) that even though its symbol $m_{2,d}$ has the nice *product structure* in (1.7), it is not a classical *biparameter* symbol, since $m_{1,d}$ itself is not a classical Marcinkiewicz–Hörmander–Mihlin multiplier.[2] As a consequence of this fact, the general *polydisc Coifman–Meyer theorem* proved in [10] and [11] cannot be applied in this case. The strategy would be to combine the techniques of [10] and [11] with the new ideas of [8] and [9] and to show that (together with some other logarithmic estimates that will be proved in this paper) they are enough to obtain the polynomial bounds of (1.3). Given these remarks, it would clearly be of great help for the reader to be familiar with our earlier arguments in [8] and [9].

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[2]$m_{1,d}$ is of course the symbol of the one dimensional dth Calderón commutator [9].

We will prove the following.

Theorem 1.2. *Let $1 < p_1, \ldots, p_{d+1} \leq \infty$ and $1 \leq p < \infty$ be such that $1/p_1 + \cdots + 1/p_{d+1} = 1/p$. Denote by l the number of indices i for which $p_i \neq \infty$. The operator $C_{2,d}$ extends naturally as a bounded $(d+1)$-linear operator $L^{p_1} \times \cdots \times L^{p_{d+1}} \to L^p$ with an operator bound of type*

$$(1.8) \qquad\qquad C(d) \cdot C(l) \cdot C(p_1) \cdots C(p_{d+1}),$$

where $C(d)$ grows at most polynomially in d and $C(p_i) = 1$ as long as $p_i = \infty$ for $1 \leq i \leq d+1$.

Theorem 1.2 is the biparameter extension of the corresponding Theorem 1.1 in [9]. If we assume it, we see that (1.3) follows from it by taking $p_1 = p$ and $p_2 = \cdots = p_{d+1} = \infty$.

To show Theorem 1.2 we will prove that for every $1 \leq i \leq d + 2$ and for every $\phi_1, \ldots, \phi_{d+1}$ Schwartz functions, one has

$$(1.9) \quad \left\| C_{2,d}^{*i}(\phi_1, \ldots, \phi_{d+1}) \right\|_p \leq C(d) \cdot C(l) \cdot C(p_1) \cdots C(p_{d+1}) \cdot \|\phi_1\|_{p_1} \cdots \|\phi_{d+1}\|_{p_{d+1}}$$

where $(p_j)_{j=1}^{d+1}$ and p are as before and $(C_{2,d}^{*i})_{i=1}^{d+2}$ are the adjoints of the multilinear operator $C_{2,d}$.[3] Standard density and duality arguments, as in [9], then allow one to conclude that the estimates in (1.9) can be extended naturally to arbitrary products of L^{p_j} and L^∞ spaces.[4]

The plan of the rest of the paper is as follows. In Section 2 we describe some discrete model operators whose analysis will play an important role in understanding (1.9). In Section 3 we prove that the main estimates (1.9) can be reduced to a general theorem for the model operators. In Section 4 we prove the theorem for the discrete model operators of Section 2. In Section 5 we show logarithmic bounds for some shifted Hardy–Littlewood–Paley hybrid operators that appear naturally in the study of the discrete models. Finally, in Section 6 we describe various generalizations of Theorem 1.1.

Acknowledgements. We thank the referee for pointing out to us that logarithmic estimates for some "shifted singular integrals", related to the ones used in our paper, go back (in the martingale setting at least) to the work of Figiel in [7].

2. Discrete model operators

As mentioned earlier, the main task here is to describe some discrete model operators, whose analysis is deeply related to the analysis of (1.9). Because of the formula (1.6), we now know that

$$(2.1) \qquad\qquad C_{2,d} = C_{1,d} \otimes C_{1,d}$$

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[3]For symmetry, we aso use the notation $C_{2,d} = C_{2,d}^{*d+2}$.

[4]The reader is also referred to our earlier [9] for an explanation of why one needs the larger set of estimates in (1.9) for $C_{2,d}$ and its adjoints, even though one is interested in the more particular (1.3).

and so one should not be at all surprised, to find out that these *biparameter* model operators that will be introduced are in fact tensor products of the *one-parameter* discrete model operators of [9]. Also as in [9], these operators are *l*-linear, rather than $(d+1)$-linear, for some $1 \leq l \leq d+1$. The explanation for this is similar to the one in [9]. To to prove (1.9), one first decomposes $C_{2,d}$ into polynomially (in d) many *biparameter paraproduct-like pieces* and then estimates each piece independently in d. To be able to achieve this, one has first to realize that one can estimate most of the L^∞ functions easily by their L^∞ norms and reduce (1.9) in this way to the corresponding estimate for some *minimal l*-linear operators. To prove the desired bounds for these minimal operators, one has to interpolate between some Banach and quasi-Banach estimates, as in [9]. The Banach estimates are easy, but the quasi-Banach estimates are hard. One has to discretize the operators carefully, in order to understand them completely. This is (in a few words) how one arrives at the model operators. Their definition is as follows.

A smooth function $\Phi(x)$ of one variable is said to be a bump function adapted to a dyadic interval I if and only if one has

$$\left|\partial^\alpha \Phi(x)\right| \lesssim \frac{1}{|I|^\alpha} \frac{1}{\left(1 + \operatorname{dist}(x,I)/|I|\right)^M}$$

for all derivatives α satisfying $|\alpha| \leq 5$ and any large $M > 0$ with the implicit constants depending on it. Then, if $1 \leq q \leq \infty$, we say that $|I|^{-1/q}\Phi$ is an L^q normalized bump adapted to I. The function $\Phi(x)$ is said to be of Ψ *type* if $\int_{\mathbb{R}} \Phi(x)\,dx = 0$, otherwise is said to be of Φ *type*.

A smooth function $\Phi(x,y)$ of two variables is said to be a bump function adapted to the dyadic rectangle $R = I \times J$ if and only if it is of the form $\Phi(x,y) = \Phi_1(x) \cdot \Phi_2(y)$ with $\Phi_1(x)$ adapted to I and $\Phi_2(y)$ adapted to J. If I is a dyadic interval and $\mathbf{n}$ an integer, we denote by $I_\mathbf{n} := I + n|I|$ the dyadic interval having the same length as I but sitting $\mathbf{n}$ units of length $|I|$ away from it.

Fix $1 \leq l \leq d+1$ and arbitrary pairs of integers $\mathbf{n}_1 = (\mathbf{n}_1^1, \mathbf{n}_1^2), \ldots, \mathbf{n}_l = (\mathbf{n}_l^1, \mathbf{n}_l^2)$. Define also $\mathbf{n}_{l+1} := (0,0)$. For $1 \leq j \leq l+1$ consider families $(\Phi^j_{R_{\mathbf{n}_j}})_R$ of L^2 normalized bump functions adapted to dyadic rectangles $R_{\mathbf{n}_j} = I_{\mathbf{n}_j^1} \times J_{\mathbf{n}_j^2}$ where $R = I \times J$ runs over a given finite collection $\mathcal{R}$ of dyadic rectangles in the plane. Assume also that for $1 \leq j \leq l+1$ at least two of the families $(\Phi^j_{I_{\mathbf{n}_j^1}})_I$ are of Ψ *type* and that the same is true for the families $(\Phi^j_{J_{\mathbf{n}_j^2}})_J$ for $1 \leq j \leq l+1$.

The discrete model operator associated to these families of functions is defined by

$$(2.2) \qquad T_\mathcal{R}(f_1, \ldots, f_l) = \sum_{R \in \mathcal{R}} \frac{1}{|R|^{(l-1)/2}} \left\langle f_1, \Phi^1_{R_{\mathbf{n}_1}} \right\rangle \cdots \left\langle f_l, \Phi^l_{R_{\mathbf{n}_l}} \right\rangle \Phi^{l+1}_R.$$

The following theorem holds.

Theorem 2.1. *For any such finite family of arbitrary dyadic rectangles, the l-linear operator $T_\mathcal{R}$ maps $L^{p_1} \times \cdots \times L^{p_l} \to L^p$ boundedly, for any $1 < p_1, \ldots, p_l < \infty$*

with $1/p_1 + \cdots + 1/p_l = 1/p$ and $0 < p < \infty$, with a bound of type

$$(2.3) \qquad O\Big(\prod_{j=1}^{2} \log^2 <\mathbf{n}_1^j> \cdots \log^2 <\mathbf{n}_l^j> \Big)$$

where, in general, $<m>$ simply denotes $2 + |m|$, and the implicit constants are allowed to depend on l.

This theorem is the biparameter generalization of Theorem 3.1 in [9]. As pointed out there, standard arguments based on scale invariance and interpolation allow one to reduce Theorem 2.1 to the more precise statement that for every $f_j \in L^{p_j}$ with $\|f_j\|_{p_j} = 1$ and every measurable set $E \subseteq \mathbb{R}^2$ of measure 1, there exists a subset $E' \subseteq E$ with $|E'| \sim |E|$ so that

$$(2.4) \qquad \sum_{R \in \mathcal{R}} \frac{1}{|R|^{(l-1)/2}} \, \big| \langle f_1, \Phi^1_{R_{\mathbf{n}_1}} \rangle \big| \cdots \big| \langle f_l, \Phi^l_{R_{\mathbf{n}_l}} \rangle \big| \, \big| \langle f_{l+1}, \Phi^{l+1}_R \rangle \big|$$

$$\lesssim \prod_{j=1}^{2} \log^2 <\mathbf{n}_1^j> \cdots \log^2 <\mathbf{n}_l^j>$$

where $f_{l+1} := \chi_{E'}$. As in [9], the fact that one loses only logarithmic bounds in the above estimates, will be of a crucial importance later.

3. Reduction to the model operators

The goal of this section is to show that indeed (1.9) can be reduced to Theorem 2.1 or more precisely to its weaker but more precise variant (2.4). In particular, one can find here a description of all the ideas that are necessary to understand why it is possible to estimate the biparameter Calderón commutators $C_{2,d}$ with bounds that grow at most polynomially in d.

The reader familiar with our previous work will realize that this section is in fact a *tensor product* of the corresponding section in [9] with itself. As in [9], the first task is to decompose $C_{2,d}$ into polynomially many *biparameter paraproduct-like pieces* which will be studied later on.

Noncompact and compact Littlewood–Paley decompositions

Let $\Phi(x)$ be an even, positive Schwartz function satisfying $\int_{\mathbb{R}} \Phi(x)\, dx = 1$. Define also $\Psi(x)$ by

$$\Psi(x) = \Phi(x) - \frac{1}{2}\, \Phi\Big(\frac{x}{2}\Big)$$

and observe that $\int_{\mathbb{R}} \Psi(x)\, dx = 0$.

Then, as always, consider the functions $\Psi_k(x)$ and $\Phi_k(x)$ defined by $2^k \Psi(2^k x)$ and $2^k \Phi(2^k x)$ respectively, for every integer $k \in \mathbb{Z}$. Notice also that all the L^1

norms of Φ_k are all equal 1. Since $\Psi_k(x) = \Phi_k(x) - \Phi_{k-1}(x)$ one can see that

$$\sum_{k \leq k_0} \Psi_k = \Phi_{k_0}, \quad \text{and so} \quad \sum_{k \in \mathbb{Z}} \Psi_k = \delta_0;$$

or, equivalently,

$$(3.1) \qquad\qquad \sum_{k \in \mathbb{Z}} \widehat{\Psi}_k(\xi) = 1$$

for almost every $\xi \in \mathbb{R}$. On the other hand, as observed in [9], since $\widehat{\Psi}(0) = \widehat{\Psi}'(0) = 0$ one can write $\widehat{\Psi}(\xi)$ as

$$\widehat{\Psi}(\xi) = \xi^2 \varphi(\xi)$$

for some other smooth and rapidly decaying function φ.

These are what we called the noncompact (in frequency) Littlewood–Paley decompositions. The compact decompositions are obtained similarly, the only difference being that instead of considering the Schwartz function Φ from before, one begins with a different one having the property that supp $\widehat{\Phi} \subseteq [-1, 1]$ and $\widehat{\Phi}(0) = 1$.

As explained in [9], the advantage of the noncompact Littlewood–Paley projections is reflected in the perfect estimate

$$(3.2) \qquad\qquad |f * \Phi_k(x)| \leq \|f\|_\infty$$

which plays an important role in the argument.

The generic decomposition of $C_{2,d}$

Using (1.6), if $f, f_1, \ldots, f_{d+1}$ are all Schwartz functions, one can write the $(d+2)$-linear form associated to $C_{2,d}$ as

$$(3.3) \quad \int_{\substack{\xi+\xi_1+\cdots+\xi_{d+1}=0 \\ \eta+\eta_1+\cdots+\eta_{d+1}=0}} \left(\int_{[0,1]^d} \operatorname{sgn}\big(\xi + \alpha_1 \xi_1 + \cdots + \alpha_d \xi_d\big)\, d\alpha_1 \ldots d\alpha_d \right)$$

$$\cdot \left(\int_{[0,1]^d} \operatorname{sgn}\big(\eta + \beta_1 \eta_1 + \cdots + \beta_d \eta_d\big)\, d\beta_1 \ldots d\beta_d \right)$$

$$\cdot \widehat{f}(\xi, \eta)\, \widehat{f_1}(\xi_1, \eta_1) \cdots \widehat{f_{d+1}}(\xi_{d+1}, \eta_{d+1})\, d\xi\, d\xi_1 \ldots d\xi_{d+1}\, d\eta\, d\eta_1 \ldots d\eta_{d+1}.$$

Then, by using the Littlewood–Paley decompositions in (3.1) several times one can write

$$(3.4) \qquad 1 = \sum_{l_0, l_1, \ldots, l_{d+1} \in \mathbb{Z}} \widehat{\Psi}_{l_0}(\xi)\, \widehat{\Psi}_{l_1}(\xi_1) \cdots \widehat{\Psi}_{l_d}(\xi_d)\, \widehat{\Psi}_{l_{d+1}}(\xi_{d+1}).$$

As in [9], since for every $(d+2)$ tuple $(l_0, l_1, \ldots, l_{d+1}) \in \mathbb{Z}^{d+2}$, one has that either $l_0 \geq l_1, \ldots, l_{d+1}$ or $l_1 \geq l_0, \ldots, l_{d+1} \ldots$ or $l_{d+1} \geq l_0, \ldots, l_d$, fixing always the biggest parameter and summing over the rest, one can rewrite (3.4) as

$$(3.5) \ \sum_l \widehat{\Psi}_l(\xi)\, \widehat{\Phi}_l(\xi_1) \cdots \widehat{\Phi}_l(\xi_d)\, \widehat{\Phi}_l(\xi_{d+1}) + \cdots + \sum_l \widehat{\Phi}_l(\xi)\, \widehat{\Phi}_l(\xi_1) \cdots \widehat{\Phi}_l(\xi_d)\, \widehat{\Psi}_l(\xi_{d+1}).$$

Also as in [9], we use in (3.5) compact Littlewood–Paley decompositions for the ξ and ξ_{d+1} variables and noncompact ones for the remaining variables. Every individual term in the decomposition (3.5) contains only one Ψ type of a function and we would like to have (at least) two. To be able to produce another one, one has to recall that $\xi + \xi_1 + \cdots + \xi_{d+1} = 0$. Taking this into account, we examine the second (for instance) term in (3.5) in the particular case when $l = 0$. For simplicity, we rewrite it as

$$(3.6) \qquad\qquad \widehat{\Phi}(\xi)\,\widehat{\Psi}(\xi_1)\cdots\widehat{\Phi}(\xi_d)\,\widehat{\Phi}(\xi_{d+1}).$$

We know from before that $\widehat{\Psi}(\xi_1) = \xi_1^2\widehat{\varphi}(\xi_1)$ and so we can write

$$\widehat{\Psi}(\xi_1) = \xi_1\,\widehat{\varphi}(\xi_1)(-\xi-\xi_2-\cdots-\xi_{d+1}) = -\xi_1\,\xi\,\widehat{\varphi}(\xi_1)-\xi_1\,\xi_2\,\widehat{\varphi}(\xi_1)-\cdots-\xi_1\,\xi_{d+1}\,\widehat{\varphi}(\xi_1).$$

Using this in (3.6) allows one to decompose it as another sum of $O(d)$ terms, containing this time two functions of Ψ type, since besides $\xi_1\widehat{\varphi}(\xi_1)$ one finds now either a factor of type $\xi\widehat{\Phi}(\xi)$ or of type $\xi_j\widehat{\Phi}_j(\xi_j)$ for some $j = 2, \ldots, d+1$.

If one performs a similar decomposition for every scale $l \in \mathbb{Z}$ and each of the terms in (3.5) one obtains a splitting of the function $1_{\{\xi+\xi_1+\cdots+\xi_{d+1}=0\}}$ as a sum of $O(d^2)$ expressions whose generic inner terms contain two functions of Ψ type as desired.

Since we are in the biparameter setting, one has to decompose $1_{\{\eta+\eta_1+\cdots+\eta_{d+1}=0\}}$ in a completely similar manner. Combining these two decompositions allows us to rewrite the $(d + 2)$ linear form of $C_{2,d}$ as

$$(3.7) \quad \sum_{k_1,k_2\in\mathbb{Z}} \int_{\substack{\xi+\xi_1+\cdots+\xi_{d+1}=0\\\eta+\eta_1+\cdots+\eta_{d+1}=0}} \Big(\int_{[0,1]^d} \operatorname{sgn}\big(\xi+\alpha_1\xi_1+\cdots+\alpha_d\xi_d\big)\,d\alpha_1\ldots d\alpha_d\Big)$$

$$\cdot\Big(\int_{[0,1]^d} \operatorname{sgn}\big(\eta+\beta_1\eta_1+\cdots+\beta_d\eta_d\big)\,d\beta_1\ldots d\beta_d\Big)$$

$$\cdot\,\widehat{\Phi}_{k_1}^{1,0}(\xi)\,\widehat{\Phi}_{k_1}^{1,1}(\xi_1)\cdots\widehat{\Phi}_{k_1}^{1,d}(\xi_d)\,\widehat{\Phi}_{k_1}^{1,d+1}(\xi_{d+1})$$

$$\cdot\,\widehat{\Phi}_{k_2}^{2,0}(\eta)\,\widehat{\Phi}_{k_2}^{2,1}(\eta_1)\cdots\widehat{\Phi}_{k_2}^{2,d}(\eta_d)\,\widehat{\Phi}_{k_2}^{2,d+1}(\eta_{d+1})$$

$$\cdot\,\widehat{f}(\xi,\eta)\,\widehat{f}_1(\xi_1,\eta_1)\cdots\widehat{f}_{d+1}(\xi_{d+1},\eta_{d+1})\,d\xi\,d\xi_1\ldots d\xi_{d+1}\,d\eta\,d\eta_1\ldots d\eta_{d+1},$$

which completes our generic decomposition.

Recall that at least two of the families $\big(\widehat{\Phi}_{k_1}^{1,j}(\xi_j)\big)_{k_1}$ for $0 \le j \le d+1$ are of Ψ type and likewise at least two of the families $\big(\widehat{\Phi}_{k_2}^{2,j}(\eta_j)\big)_{k_2}$ for $0 \le j \le d+1$ are of Ψ type. We denote these indices by i_1 and i_2 and j_1 and j_2 respectively. There are several cases that one has to consider which correspond to the positions of these indices. We call an index *intermediate* if it is between 1 and d and *extremal* if it is either 0 or $d + 1$. In [8] we encountered essentially only two cases. Case 1 was when at least one of the Ψ positions corresponded to an intermediate index and Case 2 was when both of the Ψ positions were extremal. Since we now work in the biparameter setting, there are four possible cases of types Case $i \otimes$ Case j for $1 \le i,j \le 2$.

Case 1 ⊗ Case 1

Assume here that $i_1 = j_1 = 0$ and $i_2 = j_2 = 1$. As mentioned earlier, the fact that $i_1 = i_2 = 0$ is not important, and i_1 and i_2 can be anywhere else in the interval $[0, d+1]$. Also, the fact that the intermediate indices i_2 and j_2 have been chosen to be equal is not important either. We chose them so for notational simplicity. As in [9] we would like now to expand the two implicit symbols in (3.7).

As in [9] we define $\widetilde{\xi} := \xi + \alpha_2 \xi_2 + \cdots + \alpha_d \xi_d$ and $\widetilde{\eta} := \eta + \beta_2 \eta_2 + \cdots + \beta_d \eta_d$. Recall from [9] that the idea is to treat the first symbol of (3.7) as depending on the variables ξ_1 and $\widetilde{\xi}$ and similarly the second symbol of (3.7) as depending on η_1 and $\widetilde{\eta}$. Also, since most of our functions do not have compact support in frequency, we need to consider some other compact Littlewood–Paley decompositions. We first write, as in [9],

$$1 = \sum_{l_0, l_1} \widehat{\Psi}_l(\widetilde{\xi})\, \widehat{\Psi}_{l_1}(\xi_1) = \sum_{l_0 \ll l_1} \cdots + \sum_{l_0 \sim l_1} \cdots + \sum_{l_0 \gg l_1} \cdots$$

which can be rewritten as

$$(3.8) \qquad \sum_{r_1} \widehat{\Phi}_{r_1}(\widetilde{\xi})\, \widehat{\Psi}_{r_1}(\xi_1) + \sum_{r_1} \widehat{\Psi}_{r_1}(\widetilde{\xi})\, \widehat{\Psi}_{r_1}(\xi_1) + \sum_{r_1} \widehat{\Psi}_{r_1}(\widetilde{\xi})\, \widehat{\Phi}_{r_1}(\xi_1).$$

Then, we consider an identical decomposition, but for the variables $\widetilde{\eta}$ and η_1, where the summation index is r_2. If we insert (3.8) into (3.7) it becomes a sum of three distinct expressions that generate the subcases 1_a, 1_b, and 1_c respectively. If, in addition, one inserts in (3.7) the formula for the variables $\widetilde{\eta}$ and η_1 analogous to (3.8), one ends up with nine *biparameter subcases* of types Case $1_a \otimes$ Case 1_a, Case $1_a \otimes$ Case 1_b, etc.

Case $1_a \otimes$ Case 1_a

To analyze the impact that these extra decompositions have, we consider for simplicity (as in [9]) the particular term corresponding to $k_1 = k_2 = 0$. However, the argument we use is scale invariant.

For now we ignore the symbol in (3.7) and concentrate on the remaining expression which becomes

$$(3.9) \quad \left(\sum_{r_1} \left[\widehat{\Phi}_{r_1}(\widetilde{\xi})\, \widehat{\Psi}_{r_1}(\xi_1) \right] \widehat{\Phi}_0^{1,0}(\xi)\, \widehat{\Phi}_0^{1,1}(\xi_1) \cdots \widehat{\Phi}_0^{1,d}(\xi_d)\, \widehat{\Phi}_0^{1,d+1}(\xi_{d+1}) \right)$$

$$\cdot \left(\sum_{r_2} \left[\widehat{\Phi}_{r_2}(\widetilde{\eta})\, \widehat{\Psi}_{r_2}(\eta_1) \right] \widehat{\Phi}_0^{2,0}(\eta)\, \widehat{\Phi}_0^{2,1}(\eta_1) \cdots \widehat{\Phi}_0^{2,d}(\eta_d)\, \widehat{\Phi}_0^{2,d+1}(\eta_{d+1}) \right)$$

$$= \left(\sum_{r_1 \leq 0} \cdots + \sum_{r_1 > 0} \cdots \right) \cdot \left(\sum_{r_2 \leq 0} \cdots + \sum_{r_2 > 0} \cdots \right)$$

$$:= \left(1'_a + 1''_a \right) \otimes \left(1'_a + 1''_a \right),$$

which allows us to split our existing subcase into four additional subcases.

Case $1'_a \otimes$ Case $1'_a$

This corresponds to the situation when both r_1 and r_2 are negative. As in [9], using the fact that $\widehat{\Psi}_{r_1}(\xi_1)$ is compactly supported and given that $\widehat{\Phi}_0^{1,1}(\xi_1)$ is also of Ψ type (in fact it is of the form $\xi_1\widehat{\varphi}(\xi_1)$) one can rewrite *the ξ part* of (3.9) as

$$\sum_{r_1 \leq 0} 2^{r_1} \widehat{\Phi}_{r_1}(\widetilde{\xi})\,\widehat{\Phi}_0^{1,0}(\xi)\,\widehat{\Psi}_{r_1}^{1,1}(\xi_1) \cdots \widehat{\Phi}_0^{1,d}(\xi_d)\,\widehat{\Phi}_0^{1,d+1}(\xi_{d+1})$$

$$= \sum_{r_1 \leq 0} 2^{r_1} \big[\widehat{\widetilde{\Phi}}_{r_1}(\widetilde{\xi})\,\widehat{\widetilde{\Psi}}_{r_1}^{1,1}(\xi_1)\big] \cdot \widehat{\Phi}_0^{1,0}(\xi)\,\widehat{\Psi}_{r_1}^{1,1}(\xi_1) \cdots \widehat{\Phi}_0^{1,d+1}(\xi_{d+1})\,\widehat{\Phi}_{r_1}(\widetilde{\xi})$$

for naturally chosen compactly supported functions $\widehat{\widetilde{\Phi}}_{r_1}(\widetilde{\xi})$, $\widehat{\widetilde{\Psi}}_{r_1}^{1,1}(\xi_1)$ and $\widehat{\Psi}_{r_1}^{1,1}(\xi_1)$.

This allows us to split the symbol

$$\Big(\int_0^1 \mathrm{sgn}(\widetilde{\xi} + \alpha_1\xi_1)\,d\alpha_1\Big)\widehat{\widetilde{\Phi}}_{r_1}(\widetilde{\xi})\,\widehat{\widetilde{\Psi}}_{r_1}^{1,1}(\xi_1)$$

as a double Fourier series of the form

(3.10)
$$\sum_{\widetilde{n},\widetilde{n}_1 \in \mathbb{Z}} C_{\widetilde{n},\widetilde{n}_1}^{r_1}\, e^{2\pi i \frac{\widetilde{n}}{2^{r_1}}\widetilde{\xi}} \cdot e^{2\pi i \frac{\widetilde{n}_1}{2^{r_1}}\xi_1},$$

where the Fourier coefficients satisfy the quadratic estimates

(3.11)
$$\big|C_{\widetilde{n},\widetilde{n}_1}^{r_1}\big| = \big|C_{\widetilde{n},\widetilde{n}_1}\big| \lesssim \frac{1}{<\widetilde{n}>^2}\frac{1}{<\widetilde{n}_1>^{\#}}$$

for an arbitrarily large number $\# > 0$. See [9] for these important estimates.

Clearly, one can make similar calculations for *the η part* of (3.9). Using them, one can see that the contribution of $1'_a \otimes 1'_a$ in (3.7) (at scale 1) is

$$\int_{[0,1]^{d-1}} \int_{[0,1]^{d-1}} \sum_{r_1 \leq 0} 2^{r_1} \sum_{r_2 \leq 0} 2^{r_2} \sum_{\widetilde{n},\widetilde{n}_1} C_{\widetilde{n},\widetilde{n}_1}^{r_1} \sum_{\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1} C_{\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1}^{r_2}$$

$$\int_{\substack{\xi+\xi_1+\cdots+\xi_{d+1}=0\\ \eta+\eta_1+\cdots+\eta_{d+1}=0}} \Big[\widehat{\Phi}_0^{1,0}(\xi)\,e^{2\pi i \frac{\widetilde{n}}{2^{r_1}}\xi} \cdot \widehat{\Phi}_0^{2,0}(\eta)\,e^{2\pi i \frac{\widetilde{\widetilde{n}}}{2^{r_2}}\eta}\Big]$$

$$\cdot \Big[\widehat{\Psi}_{r_1}^{1,1}(\xi_1)\,e^{2\pi i \frac{\widetilde{n}_1}{2^{r_1}}\xi_1} \cdot \widehat{\Psi}_{r_2}^{2,1}(\eta_1)\,e^{2\pi i \frac{\widetilde{\widetilde{n}}_1}{2^{r_2}}\eta_1}\Big]$$

$$\cdot \Big[\widehat{\Phi}_0^{1,2}(\xi_2)\,e^{2\pi i \frac{\widetilde{n}}{2^{r_1}}\alpha_2\xi_2} \cdot \widehat{\Phi}_0^{2,2}(\eta_2)\,e^{2\pi i \frac{\widetilde{\widetilde{n}}}{2^{r_2}}\beta_2\eta_2}\Big]$$

$$\cdots$$

$$\cdot \Big[\widehat{\Phi}_0^{1,d}(\xi_d)\,e^{2\pi i \frac{\widetilde{n}}{2^{r_1}}\alpha_d\xi_d} \cdot \widehat{\Phi}_0^{2,d}(\eta_d)\,e^{2\pi i \frac{\widetilde{\widetilde{n}}}{2^{r_2}}\beta_d\eta_d}\Big]$$

$$\cdot \Big[\widehat{\Phi}_0^{1,d+1}(\xi_{d+1}) \cdot \widehat{\Phi}_0^{2,d+1}(\eta_{d+1})\Big]$$

(3.12)
$$\cdot \Big[\widehat{\Phi}_{r_1}(\widetilde{\xi}) \cdot \widehat{\Phi}_{r_2}(\widetilde{\eta})\Big]$$

$$\cdot \widehat{f}(\xi,\eta)\,\widehat{f_1}(\xi_1,\eta_1) \ldots \widehat{f_{d+1}}(\xi_{d+1},\eta_{d+1})$$

$$\cdot d\xi\,d\xi_1 \ldots d\xi_{d+1}\,d\eta\,d\eta_1 \ldots d\eta_{d+1}\,d\alpha_2 \ldots d\alpha_d\,d\beta_2 \ldots d\beta_d.$$

Now, if one fixes $\vec{\alpha}, \vec{\beta}, r_1, r_2, \widetilde{n}, \widetilde{n}_1, \widetilde{\widetilde{n}}$, and $\widetilde{\widetilde{n}}_1$, the corresponding inner expression in (3.12) becomes

$$(3.13) \quad \int_{\substack{\xi+\xi_1+\cdots+\xi_{d+1}=0 \\ \eta+\eta_1+\cdots+\eta_{d+1}=0}} \left[\widehat{f}(\xi,\eta) \cdot \widehat{\Phi}_0^{1,0}(\xi) \, e^{2\pi i \frac{\widetilde{\widetilde{n}}}{2^{r_1}}\xi} \cdot \widehat{\Phi}_0^{2,0}(\eta) \, e^{2\pi i \frac{\widetilde{\widetilde{n}}}{2^{r_2}}\eta} \right]$$

$$\cdot \left[\widehat{f}_1(\xi_1,\eta_1) \cdot \widehat{\Psi}_{r_1}^{1,1}(\xi_1) \, e^{2\pi i \frac{\widetilde{n}_1}{2^{r_1}}\xi_1} \cdot \widehat{\Psi}_{r_2}^{2,1}(\eta_1) \, e^{2\pi i \frac{\widetilde{\widetilde{n}}_1}{2^{r_2}}\eta_1} \right]$$

$$\cdots$$

$$\cdot \left[\widehat{f}_d(\xi_d,\eta_d) \cdot \widehat{\Phi}_0^{1,d}(\xi_d) \, e^{2\pi i \frac{\widetilde{n}}{2^{r_1}}\alpha_d \xi_d} \cdot \widehat{\Phi}_0^{2,d}(\eta_d) \, e^{2\pi i \frac{\widetilde{\widetilde{n}}}{2^{r_2}}\beta_d \eta_d} \right]$$

$$\cdot \left[\widehat{f}_{d+1}(\xi_{d+1},\eta_{d+1}) \cdot \widehat{\Phi}_0^{1,d+1}(\xi_{d+1}) \cdot \widehat{\Phi}_0^{2,d+1}(\eta_{d+1}) \right]$$

$$\cdot \widehat{\Phi}_{r_1}\big(\xi + \alpha_2\xi_2 + \cdots + \alpha_d\xi_d\big) \cdot \widehat{\Phi}_{r_2}\big(\eta + \beta_2\eta_2 + \cdots + \beta_d\eta_d\big)$$

$$\cdot d\xi \, d\xi_1 \ldots d\xi_{d+1} \, d\eta \, d\eta_1 \ldots d\eta_{d+1}.$$

To continue the calculations we need the following lemma.

Lemma 3.1. *If $F, F_1, \ldots, F_{d+1}, \widetilde{\Phi}$, and $\widetilde{\widetilde{\Phi}}$ are Schwartz functions, then one has*

$$\int_{\substack{\xi+\xi_1+\cdots+\xi_{d+1}=0 \\ \eta+\eta_1+\cdots+\eta_{d+1}=0}} \widehat{F}(\xi,\eta) \, \widehat{F}_1(\xi_1,\eta_1) \cdots \widehat{F}_{d+1}(\xi_{d+1},\eta_{d+1})$$

$$\cdot \widehat{\widetilde{\Phi}}\big(a\xi + a_1\xi_1 + \cdots + a_{d+1}\xi_{d+1}\big) \cdot \widehat{\widetilde{\widetilde{\Phi}}}\big(b\eta + b_1\eta_1 + \cdots + \beta_{d+1}\eta_{d+1}\big)$$

$$\cdot d\xi \, d\xi_1 \ldots d\xi_{d+1} \, d\eta \, d\eta_1 \ldots d\eta_{d+1}$$

$$= \int_{\mathbb{R}^4} F\big(x_1 - at_1, x_2 - at_2\big) F_1\big(x_1 - a_1t_1, x_2 - b_1t_2\big)$$

$$\cdots F_{d+1}\big(x_1 - a_{d+1}t_1, x_2 - b_{d+1}t_2\big) \cdot \widetilde{\Phi}(t_1) \, \widetilde{\widetilde{\Phi}}(t_2) \, dx_1 \, dx_2 \, dt_1 \, dt_2$$

for arbitrary real numbers $a, a_1, \ldots, a_{d+1}, b, b_1, \ldots,$ and b_{d+1}.

This Lemma 3.1 is the biparameter extension of Lemma 4.1 in [9] and since its proof requires no new ideas it is left to the reader. As pointed out in [9] there is also a natural generalization of it, which states that the formula works for more than two averages (so one can take an arbitrary number of $\widetilde{\Phi}$ functions and another arbitrary number of $\widetilde{\widetilde{\Phi}}$ function).

As in [9], if G is an arbitrary Schwartz function and a is a real number we denote by G^a the function defined by

$$\widehat{G^a}(\xi) = \widehat{G}(\xi) \, e^{2\pi i a\xi}.$$

Alternatively, one has $G^a(x) = G(x-a)$. Using Lemma 3.1 and this notation, the previous (3.13) can be rewritten as

$$\int_{\mathbb{R}^4} \left(f * \Phi_0^{1,0,\frac{\widetilde{n}}{2^{r_1}}} \otimes \Phi_0^{2,0,\frac{\widetilde{\widetilde{n}}}{2^{r_2}}} \right)(x_1 - t_1, x_2 - t_2) \left(f_1 * \Psi_{r_1}^{1,1,\frac{\widetilde{n}_1}{2^{r_1}}} \otimes \Psi_{r_2}^{2,1,\frac{\widetilde{\widetilde{n}}_1}{2^{r_2}}} \right)(x_1, x_2)$$

$$\cdot \prod_{j=2}^{d} \left(f_j * \Phi_0^{1,j,\frac{\tilde{n}}{2^{r_1}}\alpha_j} \otimes \Phi_0^{2,j,\frac{\tilde{\tilde{n}}}{2^{r_2}}\beta_j} \right)(x_1 - \alpha_j\, t_1, x_2 - \beta_j\, t_2)$$

$$\cdot \left(f_{d+1} * \Phi_0^{1,d+1} \otimes \Phi_0^{2,d+1} \right)(x_1, x_2)\, \Phi_{r_1}(t_1)\, \Phi_{r_2}(t_2)\, dt_1\, dt_2\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}^4} \left(f * \Phi_0^{1,0,\frac{\tilde{n}}{2^{r_1}}} \otimes \Phi_0^{2,0,\frac{\tilde{\tilde{n}}}{2^{r_2}}} \right)\left(x_1 - \frac{t_1}{2^{r_1}}, x_2 - \frac{t_2}{2^{r_2}} \right)\left(f_1 * \Psi_{r_1}^{1,1,\frac{\tilde{n}_1}{2^{r_1}}} \otimes \Psi_{r_2}^{2,1,\frac{\tilde{\tilde{n}}_1}{2^{r_2}}} \right)(x_1, x_2)$$

$$\cdot \prod_{j=2}^{d} \left(f_j * \Phi_0^{1,j,\frac{\tilde{n}}{2^{r_1}}\alpha_j} \otimes \Phi_0^{2,j,\frac{\tilde{\tilde{n}}}{2^{r_2}}\beta_j} \right)\left(x_1 - \frac{\alpha_j t_1}{2^{r_1}}, x_2 - \frac{\beta_j t_2}{2^{r_2}} \right)$$

$$\cdot \left(f_{d+1} * \Phi_0^{1,d+1} \otimes \Phi_0^{2,d+1} \right)(x_1, x_2)\, \Phi_0(t_1)\, \Phi_0(t_2)\, dt_1\, dt_2\, dx_1\, dx_2$$

$$= \int_{\mathbb{R}^4} \left(f * \Phi_0^{1,0,\frac{\tilde{n}-t_1}{2^{r_1}}} \otimes \Phi_0^{2,0,\frac{\tilde{\tilde{n}}-t_2}{2^{r_2}}} \right)(x_1, x_2)\left(f_1 * \Psi_{r_1}^{1,1,\frac{\tilde{n}_1}{2^{r_1}}} \otimes \Psi_{r_2}^{2,1,\frac{\tilde{\tilde{n}}_1}{2^{r_2}}} \right)(x_1, x_2)$$

$$\cdot \prod_{j=2}^{d} \left(f_j * \Phi_0^{1,j,\frac{\tilde{n}-t_1}{2^{r_1}}\alpha_j} \otimes \Phi_0^{2,j,\frac{\tilde{\tilde{n}}-t_2}{2^{r_2}}\beta_j} \right)(x_1, x_2)$$

(3.14)

$$\cdot \left(f_{d+1} * \Phi_0^{1,d+1} \otimes \Phi_0^{2,d+1} \right)(x_1, x_2)\, \Phi_0(t_1)\, \Phi_0(t_2)\, dt_1\, dt_2\, dx_1\, dx_2.$$

While we must remember that all the calculations so far have been made under the assumption that $k_1 = k_2 = 0$, they can clearly be performed in general and then the formula analogous to (3.14) is

$$\int_{\mathbb{R}^4} \left(f * \Phi_{k_1}^{1,0,\frac{\tilde{n}-t_1}{2^{k_1+r_1}}} \otimes \Phi_{k_2}^{2,0,\frac{\tilde{\tilde{n}}-t_2}{2^{k_2+r_2}}} \right)(x_1, x_2)\left(f_1 * \Psi_{k_1+r_1}^{1,1,\frac{\tilde{n}_1}{2^{k_1+r_1}}} \otimes \Psi_{k_2+r_2}^{2,1,\frac{\tilde{\tilde{n}}_1}{2^{k_2+r_2}}} \right)(x_1, x_2)$$

(3.15) $$\cdot \prod_{j=2}^{d} \left(f_j * \Phi_{k_1}^{1,j,\frac{\tilde{n}-t_1}{2^{k_1+r_1}}\alpha_j} \otimes \Phi_{k_2}^{2,j,\frac{\tilde{\tilde{n}}-t_2}{2^{k_2+r_2}}\beta_j} \right)(x_1, x_2)$$

$$\cdot \left(f_{d+1} * \Phi_{k_1}^{1,d+1} \otimes \Phi_{k_2}^{2,d+1} \right)(x_1, x_2)\, \Phi_0(t_1)\, \Phi_0(t_2)\, dt_1\, dt_2\, dx_1\, dx_2.$$

In conclusion, if one writes $\vec{\alpha} = (\alpha_2, \ldots, \alpha_d)$ and $\vec{\beta} = (\beta_2, \ldots, \beta_d)$ one sees that the part of (3.7) that corresponds to Case $1_a' \otimes$ Case $1_a'$ can be written as

$$\int_{[0,1]^{d-1}} \int_{[0,1]^{d-1}} \left(\sum_{r_1 \leq 0} 2^{r_1} \sum_{r_2 \leq 0} 2^{r_2} \sum_{\tilde{n},\tilde{n}_1} C_{\tilde{n},\tilde{n}_1}^{r_1} \sum_{\tilde{\tilde{n}},\tilde{\tilde{n}}_1} C_{\tilde{\tilde{n}},\tilde{\tilde{n}}_1}^{r_2} \cdot C_{2,d}^{r_1,r_2,\tilde{n},\tilde{n}_1,\tilde{\tilde{n}},\tilde{\tilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2} \right)$$

(3.16) $$\cdot \Phi_0(t_1)\, \Phi_0(t_2)\, dt_1\, dt_2\, d\vec{\alpha}\, d\vec{\beta},$$

where $C_{2,d}^{r_1,r_2,\tilde{n},\tilde{n}_1,\tilde{\tilde{n}},\tilde{\tilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$ is the operator whose $(d+2)$-linear form is given by summing the inner expressions of (3.15) over k_1 and k_2. To prove (1.9) for (3.16) one would need to prove it for the operators $C_{2,d}^{r_1,r_2,\tilde{n},\tilde{n}_1,\tilde{\tilde{n}},\tilde{\tilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$ with bounds that are summable over $r_1, r_2, \tilde{n}, \tilde{n}_1, \tilde{\tilde{n}}$, and $\tilde{\tilde{n}}_1$ and integrable over $\vec{\alpha}, \vec{\beta}, t_1$, and t_2.

It is clear that these operators are essentially biparameter paraproducts and therefore one expects that the method of [10], and [11] should be used. That will be indeed the case, but on the other hand the appearance of the parameters mentioned earlier, has the consequence of *shifting* the implicit bump functions that appear in their definitions and as a consequence one has to be very precise when evaluates the size of their boundedness constants.

Since in our case all the bump functions are of Φ type away from the indices 0 and 1, the idea is to use the perfect estimate (3.2) (or rather, its biparameter variant) to bound all the functions which are in L^∞. To be more specific, as in [9], we denote by S the set of all the indices $2 \leq j \leq d$ for which $p_j \neq \infty$. Set $l := |S| + 2$ and freeze all the L^∞ normalized Schwartz functions f_j corresponding to the indices in $\{2, \ldots, d\} \setminus S$. The resulting operator is a *minimal l-linear* operator which will be denoted by $C_{2,d}^{l,r_1,r_2,\tilde{n},\tilde{n}_1,\tilde{\tilde{n}},\tilde{\tilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$.

Shifted hybrid maximal and square functions

It is now time to recall a few basic facts about biparameter paraproducts, to be able to continue. Consider two generic families $(\Phi_{k_1}^j)_{k_1}$ and $(\Phi_{k_2}^j)_{k_2}$ of L^1 normalized bump functions for $1 \leq j \leq l+1$ so that in each of them for two indices the corresponding sequences are of Ψ type.

An l-linear biparameter paraproduct is an l-linear operator whose $(l+1)$-linear form is given by

$$(3.17) \qquad \int_{\mathbb{R}^2} \sum_{k_1,k_2 \in \mathbb{Z}} \prod_{j=1}^{l+1} \left(f_j * \Phi_{k_1}^j \otimes \Phi_{k_2}^j \right)(x_1, x_2)\, dx_1\, dx_2.$$

First assume that we are in a case similar to the one considered before and that the Ψ functions appear for the indices $j = 1, 2$. Then one can estimate the absolute value of (3.17) by

$$\int_{\mathbb{R}^2} SS(f_1)(x_1, x_2) \cdot SS(f_2)(x_1, x_2) \cdot \prod_{j=2}^{l+1} MM(f_j)(x_1, x_2)\, dx_1\, dx_2$$

where, in general, $MM(f)(x_1, x_2)$ and $SS(f)(x_1, x_2)$ are defined by

$$MM(f)(x_1, x_2) = \sup_{k_1,k_2} \left| f * \Phi_{k_1} \otimes \Phi_{k_2}(x_1, x_2) \right|$$

and

$$(3.18) \qquad SS(f)(x_1, x_2) = \left(\sum_{k_1,k_2} \left| f * \Phi_{k_1} \otimes \Phi_{k_2}(x_1, x_2) \right|^2 \right)^{1/2},$$

respectively. In order for (3.18) to make sense, we assume of course that both $(\Phi_{k_1})_{k_1}$ and $(\Phi_{k_2})_{k_2}$ are of Ψ type. Since both MM and SS are known to be bounded in every L^p space for $1 < p < \infty$, the above argument proves that our particular biparameter paraproduct in (3.17) is bounded from $L^{p_1} \times \cdots \times L^{p_l}$ into L^p as long as $1/p_1 + \cdots + 1/p_l = 1/p$ and $1 < p_1, \ldots, p_l, p < \infty$.

As one can imagine, the above $l^2 \times l^2 \times l^\infty$ argument can be *twisted*, in which case one naturally obtains *hybrid maximal and square functions* of type MS and SM defined, respectively, by

$$(3.19) \qquad MS(f)(x_1, x_2) = \sup_{k_1} \Big(\sum_{k_2} \big| f * \Phi_{k_1} \otimes \Phi_{k_2}(x_1, x_2) \big|^2 \Big)^{1/2},$$

$$(3.20) \qquad SM(f)(x_1, x_2) = \Big(\sum_{k_1} \sup_{k_2} \big| f * \Phi_{k_1} \otimes \Phi_{k_2}(x_1, x_2) \big|^2 \Big)^{1/2}.$$

One has to assume that the family $(\Phi_{k_2})_{k_2}$ is of Ψ type in (3.19) and that $(\Phi_{k_1})_{k_1}$ is of Ψ type in (3.20), for these expressions to make sense.

As observed in [11] all these hybrid operators are bounded in L^p for $1 < p < \infty$ as well and, as a consequence, one can bound every biparameter paraproduct in arbitrary products of L^p spaces, as long as all of their indices are strictly between 1 and ∞. This discussion shows that in order to understand the operator $C_{2,d}^{l,r_1,r_2,\widetilde{n},\widetilde{n}_1,\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$ (and of course, all the other possible operators) one has to understand how to bound not only the above operators, but also their *shifted* analogs of types $M_{n_1}M_{n_1}$, $S_{n_1}S_{n_2}$, $M_{n_1}S_{n_2}$ and $S_{n_1}M_{n_2}$ which are defined similarly, but with respect to the *shifted* functions $(\Phi_{k_1}^{n_1/2^{k_1}})_{k_1}$ and $(\Phi_{k_2}^{n_2/2^{k_2}})_{k_2}$.

In [8] we obtained a complete understanding of the one-parameter shifted maximal and square functions M_n and S_n and proved their boundedness on L^p spaces with operator bounds of type $O(\log <n>)$.[5]

Now the arguments of [8] and [11] show that their hybrid biparameter analogs mentioned before, will also be bounded on L^p spaces with operatorial bounds of type $O(\log^2 <n_1> \log^2 <n_2>)$, as long as one can prove logarithmic bounds for the so called Fefferman–Stein inequality, namely

$$(3.21) \qquad \Big\| \Big(\sum_{j=1}^{N} |M_n f_j|^2 \Big)^{1/2} \Big\|_p \leq C_p \log^2 <n> \cdot \Big\| \Big(\sum_{j=1}^{N} |f_j|^2 \Big)^{1/2} \Big\|_p$$

which should hold true for every $1 < p < \infty$.

This inequality will be proven in detail in a later section. Until then, we will use all these logarithmic bounds freely.

Banach estimates for $C_{2,d}^{l,r_1,r_2,\widetilde{n},\widetilde{n}_1,\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$

Given the logarithmic bounds for the shifted maximal and square functions described earlier, it is not difficult to see (as in [9]) that $C_{2,d}^{l,r_1,r_2,\widetilde{n},\widetilde{n}_1,\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$ is indeed bounded from $L^{s_1} \times \cdots \times L^{s_l}$ into L^s as long as $1/s_1 + \cdots + 1/s_l = 1/s$ and $1 < s_1, \ldots, s_l, s < \infty$ with operator bounds no greater than

$$(3.22) \quad <r_1><r_2> \log <\widetilde{n}> \log <\widetilde{\widetilde{n}}> \log <\widetilde{n}_1> \log <\widetilde{\widetilde{n}}_1> \log <[t_1]> \log <[t_2]>$$

---

[5]The logarithmic bounds for M_n can be found in chapter II of [12], while martingale analogues of some shifted singular integrals have been studied in [7], as we mentioned earlier.

raised to the power $2l$. Moreover, this contribution is perfect, given the extra factors 2^{r_1} and 2^{r_2} that appeared before (recall that both r_1 and r_2 are negative in our case) and the quadratic decay in $\widetilde{n}, \widetilde{n}_1, \widetilde{\widetilde{n}}$, and $\widetilde{\widetilde{n}}_1$.

Quasi-Banach estimates for $C_{2,d}^{l,r_1,r_2,\widetilde{n},\widetilde{n}_1,\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$

Assume now that the index s above satisfies $0 < s < \infty$ and so it can be sub-unitary. We would like to estimate the boundedness constants of

$$(3.23) \qquad C_{2,d}^{l,r_1,r_2,\widetilde{n},\widetilde{n}_1,\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2} : L^{s_1} \times \cdots \times L^{s_l} \to L^s.$$

This time one has to discretize the operators in the x_1 and x_2 variables and then take advantage of the general result in Theorem 2.1. Arguing as in [9] we see that the problem reduces to estimating expressions of the form

$$\frac{1}{2^{6r_1 l}} \frac{1}{2^{6r_2 l}} \sum_R \frac{1}{|R|^{(l-1)/2}} \left| \left\langle f, \Phi_{I_{[\widetilde{n}-t_1]}}^{1,0} \otimes \Phi_{J_{[\widetilde{\widetilde{n}}-t_2]}}^{2,0} \right\rangle \right| \cdot \left| \left\langle f_1, \Phi_{I_{\widetilde{n}_1}}^{1,1} \otimes \Phi_{J_{\widetilde{\widetilde{n}}_1}}^{2,1} \right\rangle \right|$$

$$\cdot \prod_{j \in S} \left| \left\langle f_j, \Phi_{I_{[(\widetilde{n}-t_1)\alpha_j]}}^{1,j} \otimes \Phi_{J_{[(\widetilde{\widetilde{n}}-t_2)\beta_j]}}^{2,j} \right\rangle \right| \cdot \left| \left\langle f_{d+1}, \Phi_I^{1,d+1} \otimes \Phi_J^{2,d+1} \right\rangle \right|$$

where the sum runs over dyadic rectangles of the form $R = I \times J$. By applying Theorem 2.1 we see that the operator norm of (3.23) can be majorized by

$$\left(2^{-6r_1} \, 2^{-6r_2} \, \log <\widetilde{n}> \log <\widetilde{\widetilde{n}}> \log <\widetilde{n}_1> \log <\widetilde{\widetilde{n}}_1> \log <[t_1]> \log <[t_2]> \right)^{2l},$$

and the same is true for all its adjoint operators. In the end, by using the same interpolation argument as in [9], one can see that the operator $C_{2,d}^{l,r_1,r_2,\widetilde{n},\widetilde{n}_1,\widetilde{\widetilde{n}},\widetilde{\widetilde{n}}_1,\vec{\alpha},\vec{\beta},t_1,t_2}$ satisfies the inequality (1.9) with bounds that are clearly acceptable in (3.16) as desired.

These complete the discussion of Case $1'_a \otimes$ Case $1'_a$. The rest of the cases can be treated similarly after certain adjustments. Since all of these adjustments have been described carefully in [9], the only thing that is left is to realize that they work equally well in our *tensor product* framework. The straightforward (but quite delicate) details are left to the reader.

4. Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the method developed in [10] and [11]. First, we need to recall the following lemma whose detailed proof can be found in [11].

Lemma 4.1. *Let $J \subseteq \mathbb{R}$ be an arbitrary interval. Then, every bump function ϕ_J adapted to J can be written as*

$$(4.1) \qquad \phi_J = \sum_{k \in \mathbb{N}} 2^{-1000 \, l \, k} \, \phi_J^k,$$

where for each $k \in \mathbb{N}$, ϕ_J^k is also a bump adapted to J but with the additional property that $\operatorname{supp}(\phi_J^k) \subseteq 2^k J$.[6] Moreover, if we assume $\int_R \phi_J(x)dx = 0$ then all the functions ϕ_J^k can be chosen so that $\int_{\mathbb{R}} \phi_J^k(x)dx = 0$ for every $k \in \mathbb{N}$.

Fix the normalized functions $f_1, \ldots, f_l$ and the set E as in (2.4). Using Lemma 4.1, one can estimate the $(l+1)$-linear form on the left-hand side of (2.4) by

$$
\begin{aligned}
&|\Lambda_{\mathcal{R}}(f_1, \ldots, f_{l+1})| \\
(4.2) \quad &\leq \sum_{\vec{k} \in \mathbb{N}^2} 2^{-1000\, l\, |\vec{k}|} \sum_{R \in \mathcal{R}} \frac{1}{|R|^{(l-1)/2}} \left| \langle f_1, \Phi_{R_{\mathbf{n}_1}}^1 \rangle \right| \cdots \left| \langle f_l, \Phi_{R_{\mathbf{n}_l}}^l \rangle \right| \left| \langle f_{l+1}, \Phi_R^{l+1,\vec{k}} \rangle \right|,
\end{aligned}
$$

where the new functions $\Phi_R^{l+1,\vec{k}}$ have basically the same structure as the old Φ_R^{l+1} but they have the additional property that $\operatorname{supp}(\Phi_R^{l+1,\vec{k}}) \subseteq 2^{\vec{k}}R$. Here we have written $2^{\vec{k}}R := 2^{k_1}I \times 2^{k_2}J$, $\vec{k} = (k_1, k_2)$ and $|\vec{k}| = k_1 + k_2$.

As before, the form (4.2) will be majorized by *tensoring* two separate $l^2 \times l^2 \times l^\infty \times \cdots \times l^\infty$ estimates with respect to the parameters I and J. As a consequence, for every index $1 \leq j \leq l+1$ there are *hybrid square and maximal functions* naturally attached to that position which we denote by $(M-S)_j$. More specifically $(M-S)_j$ can be the *discrete variant* of $M_{\mathbf{n}_j^1} M_{\mathbf{n}_j^2}$ or of $S_{\mathbf{n}_j^1} S_{\mathbf{n}_j^2}$ or of $M_{\mathbf{n}_j^1} S_{\mathbf{n}_j^2}$ or of $S_{\mathbf{n}_j^1} M_{\mathbf{n}_j^2}$, depending on the positions of the corresponding Ψ functions. For simplicity, we do not write explicitly the dependence of these functions $(M-S)_j$ on the *shifting* parameters $\mathbf{n}_j$. Recall also that each of them comes with a boundedness constant which is no greater than $O(\log^2 <\mathbf{n}_j^1> \log^2 <\mathbf{n}_j^2>)$.[7]

We construct now an exceptional set as follows. For each $\vec{k} \in \mathbb{N}^2$ define
(4.3)
$$
\Omega_{-5|\vec{k}|} = \bigcup_{j=1}^{l} \left\{ (x,y) \in \mathbb{R}^2 : (M-S)_j(f_j)(x,y) > C 2^{5|\vec{k}|} \log^2 <\mathbf{n}_j^1> \log^2 <\mathbf{n}_j^2> \right\}.
$$

Also, define

$$
(4.4) \qquad \widetilde{\Omega}_{-5|\vec{k}|} = \left\{ (x,y) \in \mathbb{R}^2 : MM\big(\chi_{\Omega_{-5|\vec{k}|}}\big)(x,y) > \frac{1}{2l} \right\}
$$

and

$$
(4.5) \qquad \widetilde{\widetilde{\Omega}}_{-5|\vec{k}|} = \left\{ (x,y) \in \mathbb{R}^2 : MM\big(\chi_{\widetilde{\Omega}_{-5|\vec{k}|}}\big)(x,y) > \frac{1}{2^{|\vec{k}|}} \right\}.
$$

Finally, we define

$$
\Omega = \bigcup_{\vec{k} \in \mathbb{N}^2} \widetilde{\widetilde{\Omega}}_{-5|\vec{k}|}.
$$

It is clear that $|\Omega| < 1/2$ if C is a big enough constant. Now we fix C. Then define $E' := E \setminus \Omega$ and observe that $|E'| \sim 1$.

---

[6]$2^k J$ is the interval having the same center as J and whose length is $2^k|J|$.

[7]It is a standard fact that the continuous and the discrete variants of these operators behave similarly.

Fix $\vec{k} \in \mathbb{N}^2$ and look at the corresponding inner sum in (4.2). We split it into two parts as follows. Part I comprises the sum over those rectangles R with the property that

$$(4.6) \qquad\qquad R \cap \widetilde{\Omega}^c_{-5|\vec{k}|} \neq \emptyset$$

while Part II comprises the sum over those rectangles with the property that

$$(4.7) \qquad\qquad R \cap \widetilde{\Omega}^c_{-5|\vec{k}|} = \emptyset.$$

We observe that Part II equals zero, because if $R \cap \widetilde{\Omega}^c_{-5|\vec{k}|} \neq \emptyset$ then $R \subseteq \widetilde{\Omega}_{-5|\vec{k}|}$ and in particular this implies that $2^{\vec{k}} R \subseteq \widetilde{\widetilde{\Omega}}_{-5|\vec{k}|}$ which is a set disjoint from E'. It is therefore enough to estimate Part I. For simplicity, we will still denote by $\mathcal{R}$ the collection of rectangles that occur in Part I of the sum.

Since $R \cap \widetilde{\Omega}^c_{-5|\vec{k}|} \neq \emptyset$, it follows that $|R \cap \Omega_{-5|\vec{k}|}|/|R| \leq 1/2l$ or equivalently, $|R \cap \Omega^c_{-5|\vec{k}|}| > (2l-1)/(2l)\,|R|$.

We are now going to describe $l+1$ decomposition procedures, one for each function f_j for $1 \leq j \leq l+1$. Later on, we will combine them, in order to estimate our sum.

First of all, independently, for every index $1 \leq j \leq l$, we define recursively a sequence of larger and larger sets $\{\Omega^j_{s_j}\}$, and also a sequence of disjoint collections of rectangles $\{\mathcal{R}^j_{s_j}\}$ as follows. We start by defining

$$\Omega^j_{-5|\vec{k}|+1} = \left\{ (x,y) \in \mathbb{R}^2 : (M-S)_j(f_j)(x,y) > \frac{C\,2^{5|\vec{k}|}\log^2 <\mathbf{n}^1_j> \log^2 <\mathbf{n}^2_j>}{2^1} \right\}$$

and set

$$\mathcal{R}^j_{-5|\vec{k}|+1} = \left\{ R \in \mathcal{R} : \left| R \cap \Omega^j_{-5|\vec{k}|+1} \right| > \tfrac{1}{2l}|R| \right\}.$$

Then we define

$$\Omega^j_{-5|\vec{k}|+2} = \left\{ (x,y) \in \mathbb{R}^2 : (M-S)_j(f_j)(x,y) > \frac{C\,2^{5|\vec{k}|}\log^2 <\mathbf{n}^1_j> \log^2 <\mathbf{n}^2_j>}{2^2} \right\}$$

and similarly set

$$\mathcal{R}^j_{-5|\vec{k}|+2} = \left\{ R \in \mathcal{R} \setminus \mathcal{R}^j_{-5|\vec{k}|+1} : \left| R \cap \Omega^j_{-5|\vec{k}|+2} \right| > \tfrac{1}{2l}|R| \right\},$$

and so on. The constant $C > 0$ used above is the one in the definition of the set E' from before. Since there are finitely many rectangles, these iterative procedures will eventually terminate, producing the aforementioned sets $\{\Omega^j_{s_j}\}$ and $\{\mathcal{R}^j_{s_j}\}$. In particular, for each $1 \leq j \leq l$, they allow us to decompose the collection of rectangles $\mathcal{R}$ as $\mathcal{R} = \cup_{s_j} \mathcal{R}^j_{s_j}$.

We would clearly like to have such a decomposition available for the final function f_{l+1} as well. To do this, we first need to construct the analogue for it of

the set $\Omega_{-5|\vec{k}|}$. Choose an integer $N > 0$ so large that for every $R \in \mathcal{R}$ we have $|R \cap \Omega_{-N}^{l+1c}| > (2l-1)/(2l)\,|R|$, where

$$\Omega_{-N}^{l+1} = \left\{ (x,y) \in \mathbb{R}^2 : (M-S)_{l+1}(f_{l+1})(x,y) > C\,2^N \right\}.$$

Then, similarly to the previous procedures, we define

$$\Omega_{-N+1}^{l+1} = \left\{ (x,y) \in \mathbb{R}^2 : (M-S)_{l+1}(f_{l+1})(x,y) > C\,2^N/2^1 \right\}$$

and set

$$\mathcal{R}_{-N+1}^{l+1} = \left\{ R \in \mathcal{R} : \left| R \cap \Omega_{-N+1}^{l+1} \right| > \tfrac{1}{2l}|R| \right\},$$

then define

$$\Omega_{-N+2}^{l+1} = \left\{ x \in \mathbb{R}^2 : (M-S)_{l+1}(f_{l+1})(x,y) > C\,2^N/2^2 \right\}$$

and set

$$\mathcal{R}_{-N+2}^{l+1} = \left\{ R \in \mathcal{R} \setminus \mathcal{R}_{-N+1}^{l+1} : \left| R \cap \Omega_{-N+2}^{l+1} \right| > \tfrac{1}{2l}|R| \right\},$$

and so on, constructing the sets $\{\Omega_{s_{l+1}}^{l+1}\}$ and $\{\mathcal{R}_{s_{l+1}}^{l+1}\}$ such that the same collection of rectangles which appear in Part I, splits also as $\cup_{s_{l+1}} \mathcal{R}_{s_{l+1}}^{l+1}$.

Then we write Part I as

$$(4.8) \qquad \sum_{s_1,\ldots,s_l > -5|\vec{k}|,\, s_{l+1} > -N} \ \sum_{R \in \mathcal{R}_{s_1,\ldots,s_{l+1}}} \frac{1}{|R|^{(l+1)/2}} \left| \langle f_1, \Phi_{R_{\mathbf{n}_1}}^1 \rangle \right|$$

$$\cdots \left| \langle f_l, \Phi_{R_{\mathbf{n}_l}}^l \rangle \right| \left| \langle f_{l+1}, \Phi_R^{l+1,\vec{k}} \rangle \right| |R|,$$

where $\mathcal{R}_{s_1,\ldots,s_{l+1}} := \mathcal{R}_{s_1}^1 \cap \cdots \cap \mathcal{R}_{s_{l+1}}^{l+1}$. Now, if R belongs to $\mathcal{R}_{s_1,\ldots,s_{l+1}}$ this means in particular that R has not been selected at either of the previous $s_j - 1$ steps for $1 \leq j \leq l+1$, which means that $|R \cap \Omega_{s_1-1}^1| \leq \tfrac{1}{2l}|R|$, ..., $|R \cap \Omega_{s_{l+1}-1}^{l+1}| \leq \tfrac{1}{2l}|R|$ or equivalently $|R \cap \Omega_{s_1-1}^{1c}| > \tfrac{2l-1}{2l}|R|$, ..., $|R \cap \Omega_{s_{l+1}-1}^{l+1c}| > \tfrac{2l-1}{2l}|R|$. This implies that

$$(4.9) \qquad \left| R \cap \Omega_{s_1-1}^{1c} \cap \cdots \cap \Omega_{s_{l+1}-1}^{l+1c} \right| > \frac{1}{2}|R|.$$

In particular, using (4.9), the term in (4.8) is smaller than

$$(4.10) \qquad \sum_{\substack{s_1,\ldots,s_l > -5|\vec{k}| \\ s_{l+1} > -N}} \ \sum_{R \in \mathcal{R}_{s_1,\ldots,s_{l+1}}} \frac{1}{|R|^{(l+1)/2}} \left| \langle f_1, \Phi_{R_{\mathbf{n}_1}}^1 \rangle \right| \cdots \left| \langle f_l, \Phi_{R_{\mathbf{n}_l}}^l \rangle \right|$$

$$\cdot \left| \langle f_{l+1}, \Phi_R^{l+1,\vec{k}} \rangle \right| \left| R \cap \Omega_{s_1-1}^{1c} \cap \cdots \cap \Omega_{s_{l+1}-1}^{l+1c} \right|$$

$$= \sum_{\substack{s_1,\ldots,s_l > -5|\vec{k}| \\ s_{l+1} > -N}} \int_{\Omega_{s_1-1}^{1c} \cap \cdots \cap \Omega_{s_{l+1}-1}^{l+1c}} \ \sum_{R \in \mathcal{R}_{s_1,\ldots,s_{l+1}}} \frac{1}{|R|^{(l+1)/2}}$$

$$\cdot \left| \langle f_1, \Phi_{R_{\mathbf{n}_1}}^1 \rangle \right| \cdots \left| \langle f_l, \Phi_{R_{\mathbf{n}_l}}^l \rangle \right| \left| \langle f_{l+1}, \Phi_R^{l+1,\vec{k}} \rangle \right| \chi_R(x,y)\,dx\,dy$$

$$\lesssim \sum_{\substack{s_1,\ldots,s_{l+1}>-5|\vec{k}| \\ s_{l+1}>-N}} \int_{\Omega^{1c}_{s_1-1}\cap\cdots\cap\Omega^{l+1c}_{s_{l+1}-1}\cap\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}} \prod_{j+1}^{l+1}(M-S)_j(f_j)(x,y)\,dx\,dy$$

$$\lesssim \sum_{\substack{s_1,\ldots,s_{l+1}>-5|\vec{k}| \\ s_{l+1}>-N}} 2^{5l|\vec{k}|}\prod_{j=1}^{l}\log^2<\mathbf{n}_j^1>\log^2<\mathbf{n}_j^2>2^{-s_1}\cdots 2^{-s_{l+1}}|\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}|,$$

where

$$\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}} := \bigcup_{R\in\mathcal{R}_{s_1,\ldots,s_{l+1}}} R.$$

On the other hand we can write

$$|\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}| \leq |\Omega_{\mathcal{R}^1_{s_1}}| \leq \left|\left\{(x,y)\in\mathbb{R}^2 : MM(\chi_{\Omega^1_{s_1}})(x,y)>1/(2l)\right\}\right|$$

$$\lesssim |\Omega^1_{s_1}| = \left|\left\{(x,y)\in\mathbb{R}^2 : (M-S)_1(f_1)(x,y)>\frac{C(\vec{k},\mathbf{n}_1)}{2^{s_1}}\right\}\right| \lesssim 2^{s_1 p_1}.$$

Similarly, we have

$$\left|\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}\right| \lesssim 2^{s_j p_j}$$

for every $1\leq j\leq l$ and also

$$\left|\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}\right| \lesssim 2^{s_{l+1}\alpha},$$

for every $\alpha>1$. Here we used the facts that all the operators $(M-S)_j$ are bounded on L^s as long as $1<s<\infty$ and that $|E'|\sim 1$. In particular, it follows that

(4.11) $$\left|\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}\right| \lesssim 2^{s_1 p_1\theta_1}\cdots 2^{s_l p_l\theta_l}2^{s_{l+1}\alpha\theta_{l+1}}$$

for any $0\leq\theta_1,\ldots,\theta_{l+1}<1$ such that $\theta_1+\cdots+\theta_{l+1}=1$.

Now we split the sum in (4.10) into

(4.12) $$\sum_{\substack{s_1,\ldots,s_l>-5|\vec{k}| \\ s_{l+1}>0}} 2^{5l|\vec{k}|}\prod_{j=1}^{l}\log^2<\mathbf{n}_j^1>\log^2<\mathbf{n}_j^2>2^{-s_1}\cdots 2^{-s_{l+1}}|\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}|$$

$$+ \sum_{\substack{s_1,\ldots,s_l>-5|\vec{k}| \\ 0>s_{l+1}>-N}} 2^{5l|\vec{k}|}\prod_{j=1}^{l}\log^2<\mathbf{n}_j^1>\log^2<\mathbf{n}_j^2>2^{-s_1}\cdots 2^{-s_{l+1}}|\Omega_{\mathcal{R}_{s_1,\ldots,s_{l+1}}}|.$$

To estimate the terms in (4.12) we use the inequality (4.11) as follows. First, we choose $\theta_1,\ldots,\theta_l$ small enough so that $1-p_j\theta_j>0$ for every $1\leq j\leq l$. Because of this, θ_{l+1} can get quite close to 1. To estimate the first term in (4.12) we pick α very close to 1 so that $1-\alpha\theta_{l+1}>0$, while to estimate the second term we pick α large enough so that $1-\alpha\theta_{l+1}<0$.

With these choices, the sum in (4.12) is at most $O(2^{100\,l\,|\vec{k}|}\prod_{j=1}^{l}\log^2<\mathbf{n}_j^1>$ $\log^2<\mathbf{n}_j^2>)$ and after summing over $\vec{k}\in\mathbb{N}^2$ this causes the expression in (4.2) to be $O(\prod_{j=1}^{l}\log^2<\mathbf{n}_j^1>\log^2<\mathbf{n}_j^2>)$, as desired. This ends the proof.

5. Logarithmic bounds for the shifted hybrid maximal and square functions

To complete the proof of the main theorem, we need to demonstrate the logarithmic bounds that have been used for the *shifted hybrid maximal and square functions*. As we mentioned before, the arguments of [8] and [10] show that they would follow from the following logarithmic bound for the vector valued Fefferman–Stein inequality.

Theorem 5.1. *Let $n \in \mathbb{Z}$ be a fixed integer and denote by M_n the shifted maximal operator associated to n. Then, one has, for every N and any $1 < p < \infty$,*

$$(5.1) \qquad \left\| \Big(\sum_{j=1}^{N} |M_n f_j|^2 \Big)^{1/2} \right\|_p \leq C_p \log^2 <n> \left\| \Big(\sum_{j=1}^{N} |f_j|^2 \Big)^{1/2} \right\|_p.$$

Proof. The proof is a combination of the classical argument of Fefferman and Stein from [12] with the new ideas from [8]. Workman's [13] gives a nice presentation of the Fefferman–Stein inequality and we follow that presentation closely. There are three cases. Clearly, $|n|$ is supposed to be large, otherwise there is nothing to prove. Assume also that n is positive, since the negative case is completely similar.

Case 1: $p = 2$

This case is very simple and it follows immediately from the theorem in [8] which says that M_n is bounded on L^2 (and in fact on any L^p) with an operator bound of type $O(\log <n>)$.

Case 2: $p > 2$

To understand this case one first needs to observe the following lemma.

Lemma 5.2. *The inequality*

$$(5.2) \qquad \alpha \cdot \int_{\{x : M_n f(x) > \alpha\}} |\Phi(x)| \, dx \lesssim \int_{\mathbb{R}} |f(x)| \mathcal{M}_n \, \Phi(x) \, dx$$

holds for every $\alpha > 0$ and measurable functions f and Φ, where $\mathcal{M}_n$ is defined as

$$\mathcal{M}_n \Phi(x) := \sum_{k=0}^{[\log_2 n]} M_{-2^k} \, \Phi(x).$$

Proof. To prove this we need to recall a few facts from [8]. Denote by I_n maximal dyadic intervals chosen with the property that

$$(5.3) \qquad \frac{1}{|I_n|} \int_{I_n} |f(x)| \, dx > \alpha.$$

Clearly, they are disjoint and their union equals $\{x : Mf(x) > \alpha\}$. Each I_n comes with $[\log_2 n]$ equal length dyadic intervals, denoted by $I_n^1, \ldots, I_n^{[\log_2 n]}$, *attached* to it.

More precisely, I_n^k lies 2^k steps of length $|I_n|$ to the left of I_n. It was observed in [8] that

$$\big\{x : M_n f(x) > \alpha\big\} \subseteq \bigcup_{I_n} I_n \cup I_n^1 \cup \cdots \cup I_n^{[\log_2 n]}.$$

Using these, one can majorize the left hand side of (5.2) by

$$\alpha \cdot \sum_{I_n} \sum_{k=1}^{[\log_2 n]} \int_{I_n^k} |\Phi(x)| \, dx \lesssim \sum_{I_n} \sum_{k=1}^{[\log_2 n]} \Big(\frac{1}{|I_n|} \int_{I_n} |f(y)| \, dy \Big) \cdot \Big(\int_{I_n^k} |\Phi(x)| \, dx \Big)$$

$$(5.4) \hspace{3cm} = \sum_{I_n} \sum_{k=1}^{[\log_2 n]} \Big(\int_{I_n} |f(y)| \, dy \Big) \cdot \Big(\frac{1}{|I_n|} \int_{I_n^k} |\Phi(x)| \, dx \Big).$$

Now, for every $y \in I_n$ one can see that

$$\frac{1}{|I_n|} \int_{I_n^k} |\Phi(x)| dx \lesssim M_{-2^k} \Phi(y).$$

Using this in (5.4) one immediately obtains the desired (5.2).   □

The result of the above lemma implies that

$$M_n : L^1(\mathbb{R}, \mathcal{M}_n \Phi dx) \to L^{1,\infty}(\mathbb{R}, |\Phi| dx).$$

Since it can be assumed that Φ is not identically equal to zero, we know that $\mathcal{M}_n \Phi > 0$. In particular, we also have the trivial bound

$$M_n : L^\infty(\mathbb{R}, \mathcal{M}_n \Phi dx) \to L^\infty(\mathbb{R}, |\Phi| dx)$$

and by interpolation, we obtain the following L^2 estimate:

$$(5.5) \hspace{2cm} \int_{\mathbb{R}} |M_n f(x)|^2 |\Phi(x)| \, dx \lesssim \int_{\mathbb{R}} |f(x)|^2 \mathcal{M}_n \, \Phi(x) \, dx.$$

Returning to the proof of Case 2, since $p > 2$ we know that $q := p/2 > 1$. By picking an appropriate $\|\Phi\|_{q'} = 1$ and relying on the previous (5.5), one can write

$$\Big\| \Big(\sum_{j=1}^N |M_n f_j|^2 \Big)^{1/2} \Big\|_p^2 = \Big\| \sum_{j=1}^N |M_n f_j|^2 \Big\|_q = \int_{\mathbb{R}} \Big(\sum_{j=1}^N |M_n f_j|^2 \Big) |\Phi| \, dx$$

$$\lesssim \int_{\mathbb{R}} \Big(\sum_{j=1}^N |f_j|^2 \Big) \mathcal{M}_n \, \Phi \, dx \lesssim \Big\| \sum_{j=1}^N |f_j|^2 \Big\|_q \|\mathcal{M}_n \, \Phi\|_{q'} \lesssim \Big\| \Big(\sum_{j=1}^N |f_j|^2 \Big)^{1/2} \Big\|_p^2 \|\mathcal{M}_n\|_{q' \to q'}.$$

On the other hand, from the definition of $\mathcal{M}_n$ and the result of [8], we know that

$$\|\mathcal{M}_n\|_{q' \to q'} \leq \sum_{k=1}^{[\log_2 n]} \|\mathcal{M}_{-2^k}\|_{q' \to q'} \lesssim \sum_{k=1}^{[\log_2 n]} \log 2^k \lesssim \log^2 <n>,$$

which completes Case 2.

Case 3: $1 < p < 2$

The idea here is to prove the following endpoint case

$$(5.6) \qquad \left\| \left(\sum_{j=1}^{N} |M_n f_j|^2 \right)^{1/2} \right\|_{1,\infty} \lesssim \log^2 <n> \cdot \left\| \left(\sum_{j=1}^{N} |f_j|^2 \right)^{1/2} \right\|_{1}$$

directly and then to apply standard vector-valued interpolation with the corresponding L^2 estimate.

To prove (5.6), let $\alpha > 0$ and define

$$F(x) := \left(\sum_{j=1}^{N} |f_j(x)|^2 \right)^{1/2}.$$

Select maximal dyadic intervals I_n with the property

$$\frac{1}{|I_n|} \int_{I_n} F(x)\,dx > \alpha.$$

As before, we think of each I_n as being related to the dyadic interval I, having the same length as I_n, and lying n steps of length $|I_n|$ to the left of it. If we define $\Omega := \bigcup_I I_n$ one has as usual

$$(5.7) \qquad |\Omega| = \sum_I |I_n| \leq \frac{1}{\alpha} \sum_I \int_{I_n} F(x)\,dx \leq \frac{1}{\alpha} \|F\|_1.$$

Observe that $F \leq \alpha$ on Ω^c and also that

$$\alpha < \frac{1}{|I_n|} \int_{I_n} F(x)\,dx \leq 2\alpha$$

because of the maximality of I_n.

Now split each f_k as $f_k = f'_k + f''_k$ where $f'_k := f_k \chi_{\Omega^c}$ and $f''_k := f_k \chi_{\Omega}$.

Contribution of $\{f'_k\}$

One can write

$$\left| \left\{ x : \left(\sum_j |M_n f'_j(x)|^2 \right)^{1/2} > \alpha/2 \right\} \right| \leq \frac{1}{\alpha^2} \left\| \left(\sum_j |M_n f'_j(x)|^2 \right)^{1/2} \right\|_2^2$$

$$\lesssim \frac{1}{\alpha^2} \log^2 <n> \left\| \left(\sum_j |f'_j(x)|^2 \right)^{1/2} \right\|_2^2 \lesssim \log^2 <n> \frac{1}{\alpha^2} \int_{\Omega^c} F^2(x)\,dx$$

$$\leq \log^2 <n> \frac{1}{\alpha} \|F\|_1$$

as desired.

Contribution of $\{f_k''\}$

Estimating the corresponding contribution for $\{f_k''\}$, requires a bit more care. First define the functions g_k by

$$(5.8) \qquad g_k := \sum_I \left(\frac{1}{|I_n|} \int_{I_n} |f_k(x)| \, dx \right) \cdot \chi_{I_n},$$

and then define

$$G(x) := \left(\sum_j |g_j(x)|^2 \right)^{1/2}.$$

Fix $x \in I_n$ and observe that by the Minkowski inequality one can write

$$G(x) = \left(\sum_j \left(\frac{1}{|I_n|} \int_{I_n} |f_j(y)| \, dy \right)^2 \right)^{1/2} \leq \frac{1}{|I_n|} \int_{I_n} \left(\sum_j |f_j(y)|^2 \right)^{1/2} dy$$

$$= \frac{1}{|I_n|} \int_{I_n} F(y) \, dy \leq 2\alpha.$$

Using that G is supported in Ω and arguing as before, we have

$$\left| \left\{ x : \left(\sum_j |M_n g_j(x)|^2 \right)^{1/2} > \alpha/2 \right\} \right| \lesssim \frac{1}{\alpha^2} \log^2 <n> \left\| \left(\sum_j |g_j(x)|^2 \right)^{1/2} \right\|_2^2$$

$$= \frac{1}{\alpha^2} \log^2 <n> \|G\|_2^2 \lesssim \log^2 <n> |\Omega| \leq \frac{1}{\alpha} \log^2 <n> \|F\|_1.$$

Now we would like to compare $M_n f_k''(x)$ with $M_n g_k(x)$ if possible. Denote by $\widetilde{\Omega}$ the set

$$\widetilde{\Omega} := \bigcup_I 3I_n \cup 3I_n^1 \cup \cdots \cup 3I_n^{[\log_2 n]}$$

and observe that $|\widetilde{\Omega}| \lesssim \log <n> |\Omega|$. We will prove that for every $x \in \widetilde{\Omega}^c$ one has

$$(5.9) \qquad M_n f_k''(x) \leq M_n g_k(x)$$

and this will clearly allow us to reduce the contribution of $\{f_k''\}$ to the contribution of $\{g_k\}$ which was understood earlier. Fix $x \in \widetilde{\Omega}^c$ and $x \in J$, where J is a dyadic interval such that the corresponding $\frac{1}{|J_n|} \int_{J_n} |f_k''(y)| \, dy$ is nonzero. In particular, J_n has to intersect Ω which is the support of f_k''. Suppose now that I is so that $J_n \cap I_n \neq \emptyset$. Then, one must have $I_n \subseteq J_n$ as the other alternative $J_n \subseteq I_n$ is not possible since $x \in \widetilde{\Omega}^c$. This implies that

$$\frac{1}{|J_n|} \int_{J_n} |f_k''(x)| \, dx = \frac{1}{|J_n|} \int_{J_n} |g_k(x)| \, dx$$

which is enough to guarantee (5.9) and conclude the proof. $\qquad\square$

6. Generalizations

The goal of this section is to point out that virtually all the earlier generalizations that we described in [8] and [9], have *natural* extensions in this *multiparameter setting* and can be proved by the same method. We give here just two samples and leave the rest (and the straightforward details) to the imaginative reader. Suppose for simplicity that we are in $\mathbb{R}^2$ and write $D_1 := \partial/\partial x_1$ and $D_2 := \partial/\partial x_2$. A direct computation shows that the *double commutator* $[|D_2|, [|D_1|, A]]$ can be rewritten as

$$(6.1) \qquad \big[|D_2|, [|D_1|, A]\big] f(x) = p.v. \int_{\mathbb{R}^2} f(x+t) \Big(\frac{\Delta_{t_1}^{(1)}}{t_1} \circ \frac{\Delta_{t_2}^{(2)}}{t_2} A(x) \Big) \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

which is precisely the *bidisc* extension of the first commutator of Calderón. There is of course a similar formula available in every dimension.

Theorem 6.1. *Let $a_1, \ldots, a_n$ be real numbers, all different from zero. The expression*

$$p.v. \int_{\mathbb{R}^n} f(x+t) \Big(\frac{\Delta_{a_1 t_1}^{(1)}}{t_1} \circ \cdots \circ \frac{\Delta_{a_n t_n}^{(n)}}{t_n} A(x) \Big) \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n},$$

viewed as a bilinear map in f and $\partial^n A/\partial x_1 \cdots \partial x_n$, is bounded from $L^p \times L^q$ into L^r for every $1 < p, q \leq \infty$ with $1/p + 1/q = 1/r$ and $1/2 < r < \infty$.

The particular case $q = \infty$ is in Journé's [5] but the rest of the estimates seem to be new.

Then, one can also observe by a direct calculation that

$$\Big[|D_2|, \big[|D_1|, \big[|D_2|, [|D_1|, A] \big] \big] \Big] f(x)$$
$$= p.v. \int_{\mathbb{R}^4} f(x+t+s) \Big(\frac{\Delta_{t_1}^{(1)}}{t_1} \circ \frac{\Delta_{t_2}^{(2)}}{t_2} \circ \frac{\Delta_{s_1}^{(1)}}{s_1} \circ \frac{\Delta_{s_2}^{(2)}}{s_2} A(x) \Big) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{ds_1}{s_1} \frac{ds_2}{s_2}$$

which is the *bidisc* analogue of an operator introduced in [9]. As we promised, we record now the following theorem.

Theorem 6.2. *Let F be an analytic function on a disc centered at the origin in the complex plane and let A be a complex valued function in $\mathbb{R}^2$ so that $\partial^4 A/\partial x_1^2 \partial x_2^2 \in L^\infty(\mathbb{R}^2)$ with an L^∞ norm strictly smaller than the radius of convergence of F. Then, the linear operator*

$$f \to p.v. \int_{\mathbb{R}^4} f(x+t+s) \, F\Big(\frac{\Delta_{t_1}^{(1)}}{t_1} \circ \frac{\Delta_{t_2}^{(2)}}{t_2} \circ \frac{\Delta_{s_1}^{(1)}}{s_1} \circ \frac{\Delta_{s_2}^{(2)}}{s_2} A(x) \Big) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{ds_1}{s_1} \frac{ds_2}{s_2}$$

is bounded on $L^p(\mathbb{R}^2)$ for every $1 < p < \infty$.

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