Semiclassical hypoelliptic estimates
with a loss of many derivatives

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Abstract. We study the pseudospectral properties of general pseudodifferential operators around a doubly characteristic point and provide necessary and sufficient conditions for semiclassical hypoelliptic a priori estimates with a loss of many derivatives.

1. Introduction

In recent years, there has been renewed interest in the analysis of the spectra and resolvents of non-self-adjoint operators with double characteristics. This interest partly originates in the study of the long-time behavior of evolution equations associated with non-self-adjoint operators

$$\begin{cases} (\partial_t + P)\, u(t,x) = 0 \\ \qquad u(t,\cdot)|_{t=0} = u_0. \end{cases}$$

This is for instance the case in the analysis of kinetic equations and the study of the trend to equilibrium in statistical physics.

The study of doubly characteristic operators has a long and distinguished tradition in the analysis of partial differential equations [14], [15], [29]. The simplest examples of such operators are the quadratic differential operators

$$Q(x, D_x) = \sum_{|\alpha+\beta|=2} q_{\alpha,\beta}\, x^\alpha D_x^\beta, \quad x \in \mathbb{R}^n,$$

with $q_{\alpha,\beta} \in \mathbb{C}$, $D_{x_j} = i^{-1}\partial_{x_j}$, $\alpha, \beta \in \mathbb{N}^n$. In the elliptic case, the spectra of these operators were understood and described explicitly in [29]. On the other hand, the pseudospectral study of these operators is much more recent. Studying the

pseudospectrum of an operator is studying the regions

$$\mathrm{Spec}_\varepsilon(A) = \left\{ z \in \mathbb{C};\ \|(A - z)^{-1}\| \geq \frac{1}{\varepsilon} \right\}, \quad \varepsilon > 0,$$

in the complex plane where its resolvent is large in norm, with the convention that $\|(A-z)^{-1}\| = +\infty$ if z belongs to the spectrum $\mathrm{Spec}(A)$ of A. The spectral stability of the operator under small perturbations can be analyzed using the breadth of the pseudospectrum. Indeed, the pseudospectrum may be defined in an equivalent way [28] in terms of the spectra

$$\mathrm{Spec}_\varepsilon(A) = \bigcup_{\substack{B \in \mathscr{L}(H), \\ \|B\|_{\mathscr{L}(H)} \leq \varepsilon}} \mathrm{Spec}(A + B)$$

of the operator perturbations, where $\mathscr{L}(H)$ stands for the set of bounded linear operators on H. The pseudospectral study of a variety of operators has recently received much attention in diverse contexts. For further details and motivations, we refer the reader to the overview of this topic in the book [32], and to the references therein. For now, we simply remark that the study of the pseudospectrum is nontrivial only for non-self-adjoint operators, or more precisely for nonnormal operators. In fact, the classical formula

$$(1.1) \qquad \forall z \notin \mathrm{Spec}(A), \quad \|(A - z)^{-1}\| = \frac{1}{\mathrm{dist}(z, \mathrm{Spec}(A))},$$

emphasizes that the resolvent of a normal operator cannot blow up far from its spectrum, and that the spectrum is stable under small perturbations

$$(1.2) \qquad \mathrm{Spec}_\varepsilon(A) = \left\{ z \in \mathbb{C};\ \mathrm{dist}(z, \mathrm{Spec}(A)) \leq \varepsilon \right\}.$$

However, formula (1.1) does not hold any longer for nonnormal operators and the behavior of the resolvent for such operators can be complicated, becoming very large in norm far from the spectrum. As a consequence, the spectra of these operators may be unstable under small perturbations. The rotated harmonic oscillator

$$P = -\partial_x^2 + e^{i\theta} x^2, \quad -\pi < \theta < \pi, \quad \theta \neq 0,$$

is a notable example of an elliptic quadratic operator whose spectrum is unstable under small perturbations. The seminal works [1], [4] have indeed shown that its resolvent $\|(P - z)^{-1}\|$ exhibits rapid growth in some regions of the resolvent set far from the spectrum, and that some strong spectral instabilities develop in some regions with specific geometry, which have been precisely described in the works [1], [22]. These phenomena of spectral instabilities are not peculiar to the rotated harmonic oscillator. They were shown to be the typical behavior of any nonnormal elliptic quadratic operator [23], [25], [26], with rapid resolvent growth along any ray lying inside the range of their Weyl symbols. This is linked to some properties of microlocal nonsolvability and to violations of the adjointness condition in the so-called Nirenberg–Treves condition (Ψ), which allow the construction

of quasimodes [3], [5], [15], [21], [23], [36], [37]. Similar types of spectral instabilities were shown to occur for general pseudodifferential operators around a doubly characteristic point, when the quadratic approximations of these operators at the doubly characteristic set are nonnormal [24]. Beginning with these early insights, there has been a series of recent works [5], [8], [9], [10], [12], [13], [24], [30], [33], and [34], aimed at providing a sharp description of the spectral and pseudospectral properties of general pseudodifferential operators around a doubly characteristic point. In the present work, we aim at sharpening this picture and at describing the pseudospectral behavior of a general pseudodifferential operator around a doubly characteristic point by refining the understanding of the underlying geometry ruling these phenomena.

2. Setting of the analysis

Let $m(\cdot; h) : \mathbb{R}^{2n} \longrightarrow (0, +\infty)$ be an order function (see Dimassi–Sjöstrand's book [6]), that is,

$$\exists C_0, N_0 > 0, \forall 0 < h \leq 1, \forall X, Y \in \mathbb{R}^{2n}, \quad m(X; h) \leq C_0 \langle X - Y \rangle^{N_0} m(Y; h),$$

with $\langle X \rangle = (1 + |X|^2)^{1/2}$, where $|\cdot|$ is the Euclidean norm. We consider the symbol class of h-dependent symbols whose growth is controlled by the order function m given by

$$(2.1) \qquad S(m) = \big\{ a(\cdot; h) \in C^\infty(\mathbb{R}^{2n}, \mathbb{C}); \forall \alpha \in \mathbb{N}^{2n}, \exists C_\alpha > 0,$$
$$\forall 0 < h \leq 1, \forall X \in \mathbb{R}^{2n}, \quad |\partial_X^\alpha a(X; h)| \leq C_\alpha m(X; h) \big\}.$$

In the present work, we study a semiclassical pseudodifferential operator

$$(2.2) \qquad P = p^w(x, hD_x; h) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} \, p\left(\frac{x+y}{2}, h\,\xi; h\right) u(y) \, dy \, d\xi,$$

defined by the semiclassical Weyl quantization of a symbol $p(x, \xi; h)$ admitting a semiclassical asymptotic expansion

$$(2.3) \qquad p(x, \xi; h) \sim \sum_{j=0}^{+\infty} p_j(x, \xi) \, h^j$$

in the symbol class $S(1)$. The symbols $p_j \in S(1)$ in the asymptotic expansion are assumed to be independent of the semiclassical parameter $0 < h \leq 1$. We assume that the real part of the principal symbol is nonnegative

$$(2.4) \qquad \operatorname{Re} p_0(X) \geq 0, \quad X = (x, \xi) \in \mathbb{R}^{2n},$$

and elliptic at infinity

$$(2.5) \qquad \exists C > 1, \forall |X| \geq C, \quad \operatorname{Re} p_0(X) \geq \frac{1}{C}.$$

These two assumptions imply that there exists a neighborhood of zero in the complex plane such that the analytic family of bounded operators

$$P - z : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \quad z \in \text{neigh}(0, \mathbb{C}),$$

is Fredholm of index 0, when the semiclassical parameter $0 < h \ll 1$ is sufficiently small [5]. An application of the analytic Fredholm theory shows that the spectrum of the operator P in a small neighborhood V of 0, that can be taken to have the form $V = D(0, c)$ (the open disk in $\mathbb{C}$ of radius c centered at 0) with $0 < c \le 1$, is discrete and is composed only of eigenvalues with finite algebraic multiplicity.

We assume further that the characteristic set of the real part of the principal symbol is reduced to a single point,

$$(2.6) \qquad\qquad (\text{Re } p_0)^{-1}(\{0\}) = \{0\} \subset \mathbb{R}^{2n},$$

and that this point is doubly characteristic for the principal symbol p_0,

$$(2.7) \qquad\qquad p_0(0) = \nabla p_0(0) = 0,$$

so that we may write

$$(2.8) \qquad\qquad p_0(Y) = q(Y) + \mathcal{O}(Y^3), \quad Y \to 0,$$

q being the quadratic term in the Taylor expansion of the principal symbol at 0.

We aim to study the spectral and pseudospectral properties of the operator P in a neighborhood of 0. As mentioned above, the study of this problem was started in [12], [13], where the first lines of this spectral and pseudospectral picture were sketched.

The results of [12] actually provide an initial localization of the spectrum of the operator P in any given h-ball centered at $z = 0$. More specifically, when the quadratic approximation of the principal symbol is elliptic on a particular vector subspace S of phase space defined as its singular space[1]

$$(2.9) \qquad\qquad X \in S, \quad q(X) = 0 \Longrightarrow X = 0,$$

then, for any given constant $C > 1$ and any fixed neighborhood $\Omega \subset \mathbb{C}$ of the spectrum $\text{Spec}(q^w(x, D_x))$ of the quadratic operator $q^w(x, D_x)$ described in the appendix (Section 6), there exist positive constants $0 < h_0 \le 1$, $C_0 > 0$ such that for all $0 < h \le h_0$, $|z| \le C$ satisfying

$$z - p_1(0) \notin \Omega,$$

we have

$$(2.10) \qquad\qquad h\|u\|_{L^2} \le C_0 \, \|(P - hz)u\|_{L^2}, \quad u \in \mathscr{S}(\mathbb{R}^n),$$

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[1] We refer the reader to the appendix, Section 6, for miscellaneous facts about quadratic operators and the definition of the singular space.

where $p_1(0)$ stands for the value of the subprincipal symbol at the doubly characteristic point $0 \in \mathbb{R}^{2n}$. This result indicates that the spectrum of P in any h-ball centered at $z = 0$ is localized in an h-neighborhood of the spectrum of its quadratic approximation shifted by the value of the subprincipal symbol at the doubly characteristic point

$$p_1(0) + \mathrm{Spec}\big(q^w(x, D_x)\big).$$

Under the same assumptions, this pseudospectral picture was complemented by the following result about the spectrum [13]: for any given $C > 0$, there exists $0 < h_0 \le 1$, such that for all $0 < h \le h_0$, the spectrum of the operator P in the open disk $D(0, Ch)$ is given by eigenvalues z_k having a semiclassical expansion of the form

$$(2.11) \qquad z_k \sim h\big(\lambda_k + p_1(0) + h^{1/N_k}\lambda_{k,1} + h^{2/N_k}\lambda_{k,2} + \cdots\big),$$

where λ_k is an eigenvalue of the quadratic operator $q^w(x, D_x)$ located in the fixed ball $D(0, C)$, N_k is the dimension of the corresponding generalized eigenspace, and the $\lambda_{k,j} \in \mathbb{C}$ are complex constants.

Next we consider the remainder term in the principal symbol

$$(2.12) \qquad r(X) = p_0(X) - q(X),$$

and assume further the existence of a closed angular sector Γ with vertex at 0, and a neighborhood V of the origin in $\mathbb{R}^{2n}$ such that

$$(2.13) \qquad r(V) \setminus \{0\} \subset \Gamma \setminus \{0\} \subset \big\{z \in \mathbb{C}; \mathrm{Re}\, z > 0\big\}.$$

When the quadratic approximation $q^w(x, D_x)$ enjoys some subelliptic properties, sharp resolvent estimates may be derived outside an h-ball centered at $z = 0$, in a parabolic region with a particular geometry. More specifically, when the quadratic form q has a zero singular space, i.e., $S = \{0\}$, we consider the smallest integer $0 \le k_0 \le 2n - 1$ satisfying

$$(2.14) \qquad \Big(\bigcap_{j=0}^{k_0} \mathrm{Ker}\big(\mathrm{Re}\, F(\mathrm{Im}\, F)^j\big)\Big) \bigcap \mathbb{R}^{2n} = \{0\},$$

where F is the Hamilton map of q (see the appendix, Section 6). It was shown in [13] that for any given sufficiently small constant $c_0 > 0$ there exist positive constants $0 < h_0 \le 1$, $C \ge 1$, and $C_0 > 0$, such that for all $0 < h \le h_0$, $u \in \mathscr{S}(\mathbb{R}^n)$, and $z \in \Omega_h$,

$$(2.15) \qquad h^{2k_0/(2k_0+1)} |z|^{1/(2k_0+1)} \|u\|_{L^2} \le C_0 \|Pu - zu\|_{L^2},$$

where

$$(2.16) \qquad \Omega_h = \Big\{z \in \mathbb{C}; \ \mathrm{Re}\, z \le \frac{1}{C} h^{2k_0/(2k_0+1)} |z|^{1/(2k_0+1)}, \ C\,h \le |z| \le c_0\Big\}.$$

The term $h^{2k_0/(2k_0+1)}|z|^{1/(2k_0+1)}$ increases when the spectral parameter z moves away from the origin in the region where $Ch \leq |z| \leq c_0$. When the spectral parameter has magnitude h, we recover the semiclassical estimate (2.10), and we emphasize that the resolvent estimate

$$(P-z)^{-1} = \mathcal{O}\big(h^{-2k_0/(2k_0+1)}|z|^{-1/(2k_0+1)}\big) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n),$$

and the geometry of the parabolic region where it holds, are directly related to the loss of $2k_0/(2k_0+1)$ derivatives appearing in the global subelliptic estimate satisfied by the quadratic approximation of the operator at the doubly characteristic point (see the appendix, Section 6),

$$\big\|\langle(x,D_x)\rangle^{2/(2k_0+1)}u\big\|_{L^2} \leq C\left(\|q^w(x,D_x)\,u\|_{L^2} + \|u\|_{L^2}\right).$$
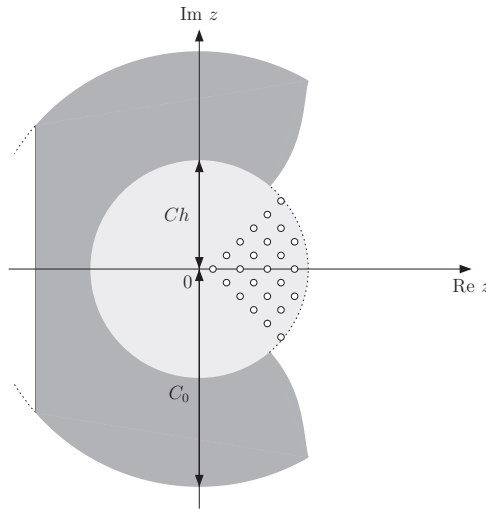

FIGURE 1. The estimate $h^{2k_0/(2k_0+1)}|z|^{1/(2k_0+1)}\|u\|_{L^2} \leq C_0\,\|Pu-zu\|_{L^2}$ is valid when z belongs to the dark grey region of the figure, whereas the estimate $h\|u\|_{L^2} \leq C_0\,\|Pu - zu\|_{L^2}$ is valid in the light grey one.

These results show that the algebraic structure of the singular space makes it possible to give a sharp description of the spectral and pseudospectral properties of pseudodifferential operators around a doubly characteristic point. The picture drawn so far has been sharpened recently by Viola [33], [34]. In these papers, Viola studies the case when the spectral parameter z enters more deeply into the numerical range and may grow slightly more rapidly than the semiclassical parameter h outside the parabolic region Ω_h. His result shows that polynomial resolvent bounds still hold in a larger $h(\log\log h^{-1})^{1/n}$-neighborhood of $z = 0$. More precisely, under the previous assumptions with a zero singular space, Viola shows that for any given $\rho > 0$, there exist positive constants C_0 and C_1 such that

the resolvent

$$(P - z)^{-1} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n),$$

exists and satisfies the bound

$$\|(P - z)^{-1}\|_{\mathscr{L}(L^2)} = \mathcal{O}(h^{-1-\rho})$$

when $0 < h \ll 1$, as long as the spectral parameter z obeys

$$|z| \le \frac{1}{C_0} h \Big(\log \log \frac{1}{h} \Big)^{1/n}, \quad \operatorname{dist}\big(z, \operatorname{Spec}(q^w(x, D_x))\big) \ge h \, e^{-\frac{1}{C_1}(\log \log \frac{1}{h})^{1/n}}.$$

Figure 2 below[2] is an illustration of a typical region in the complex plane where this resolvent estimate holds, for decreasing values of h.
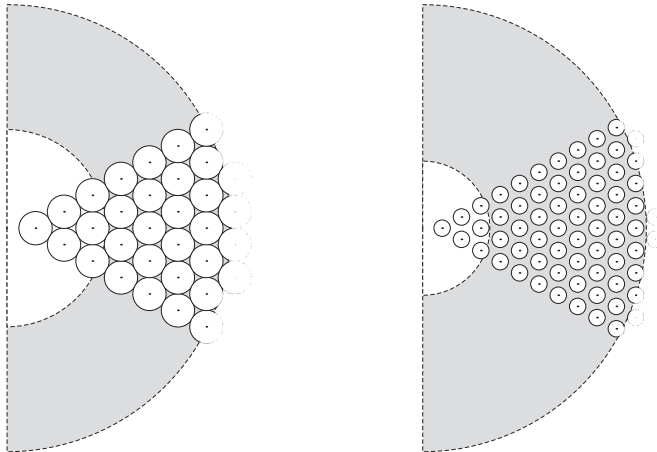

Figure 2.

The disks surrounding the spectral values of the quadratic operator $q^w(x, D_x)$ correspond to the forbidden region

$$\operatorname{dist}\big(z, \operatorname{Spec}(q^w(x, D_x))\big) < h \, e^{-\frac{1}{C_1}(\log \log \frac{1}{h})^{1/n}}.$$

Returning to the resolvent estimate (2.15), we observe that the estimate

$$(2.17) \qquad h^{2k_0/(2k_0+1)} \|u\|_{L^2} \le C_0 \|Pu - zu\|_{L^2},$$

holds true at the boundary of the parabolic set Ω_h, when

$$\operatorname{Re} z \le c_1 h^{2k_0/(2k_0+1)}, \quad \Big| |\operatorname{Im} z| - \frac{c_0}{2} \Big| \le c_1,$$

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[2]Courtesy of Joe Viola.

with $0 < c_1 \ll 1$. By using semigroup techniques, this resolvent estimate was improved by Sjöstrand [30] to

$$(2.18) \qquad |\text{Re } z| \, \|u\|_{L^2} \leq C_0 \, \|Pu - zu\|_{L^2},$$

when

$$-c_1 \leq \text{Re } z \leq -h^{2k_0/(2k_0+1)}, \quad \left| |\text{Im } z| - \frac{c_0}{2} \right| \leq c_1,$$

and to

$$(2.19) \qquad h^{2k_0/(2k_0+1)} \, \|u\|_{L^2} \leq C_0 \exp\left(\frac{C_0}{h} (\text{Re } z)_+^{(2k_0+1)/(2k_0)} \right) \|Pu - zu\|_{L^2},$$

when

$$-h^{2k_0/(2k_0+1)} \leq \text{Re } z \leq c_1 \left(h \log \frac{1}{h} \right)^{2k_0/(2k_0+1)}, \quad \left| |\text{Im } z| - \frac{c_0}{2} \right| \leq c_1.$$

For $\text{Re } z \sim h^{2k_0/(2k_0+1)}$, we recover the estimate (2.17). Furthermore, this result shows that the spectral parameter may enter more deeply in a logarithmic fashion into the numerical range outside the parabolic region Ω_h,

$$\text{Re } z \sim \left(h \log \frac{1}{h} \right)^{2k_0/(2k_0+1)},$$

while a polynomial resolvent bound

$$\|(P - z)^{-1}\|_{\mathscr{L}(L^2)} = \mathcal{O}\left(h^{-2k_0/(2k_0+1)-\rho_0} \right), \quad \rho_0 > 0,$$

continues to hold.

In the present work, we aim at completing this picture by investigating further the pseudospectral properties of the operator P inside the h-neighborhood of the set

$$\Sigma = p_1(0) + \text{Spec}(q^w(x, D_x)).$$

More specifically, we study necessary and sufficient conditions for the validity of the following a priori estimates

$$(2.20) \quad \exists \, c_0 > 0, \, \exists \, 0 < h_0 \leq 1, \, \forall \, u \in \mathscr{S}(\mathbb{R}^n), \, \forall \, 0 < h \leq h_0,$$
$$\|Pu - hzu\|_{L^2} \geq c_0 \, h^{N_0/2+1} \|u\|_{L^2},$$

where $N_0 \geq 1$ is a positive integer, when z belongs to a neighborhood of Σ. While the resolvent estimate (2.15) and the geometry of the parabolic region (2.16) were shown to be related to the subelliptic properties of the quadratic approximation $q^w(x, D_x)$, we show in this work that the resolvent estimates (2.20) are actually linked to some properties of hypoellipticity with a loss of many derivatives. The proof of the main result of this article (Theorem 3.1) is indeed based on a Grushin-reduction method following closely and adapting to the semiclassical setting the approach developed by Parenti and the first author in the study of hypoellipticity with a loss of many derivatives for operators with multiple characteristics [18].

We recall that the Grushin-reduction method has proved itself fundamental in many different problems, especially in spectral theory and in the study of hypoellipticity (and, more recently in the study of solvability and semiglobal solvability [29], [7], [18], [19], [20]) for operators with multiple characteristics. The idea of Grushin is roughly the following. Let H be a Hilbert space and let $A\colon \mathsf{H} \longrightarrow \mathsf{H}$ be a Fredholm operator (with nonzero kernel) of index 0. The vector subspaces $V_1 = \operatorname{Ker} A$ and $V_2 = \operatorname{Ker} A^*$ must then have the same finite dimension d. Consider orthonormal bases $\{w_1, \ldots, w_d\}$ of V_1 and $\{v_1, \ldots, v_d\}$ of V_2. Given a vector subspace $V \subset \mathsf{H}$ spanned by the orthonormal basis $\{e_1, \ldots, e_d\}$, we define the maps

$$h_V^+\colon \mathsf{H} \ni u \longmapsto \begin{bmatrix} (u, e_1)_{\mathsf{H}} \\ \vdots \\ (u, e_d)_{\mathsf{H}} \end{bmatrix} \in \mathbb{C}^d, \quad h_V^-\colon \mathbb{C}^d \ni v = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_d \end{bmatrix} \longmapsto \sum_{j=1}^d \zeta_j\, e_j \in V.$$

Then, the system

$$\mathbf{A} = \begin{bmatrix} A & h_{\operatorname{Ker} A^*}^- \\ h_{\operatorname{Ker} A}^+ & 0 \end{bmatrix} \colon \mathsf{H} \times \mathbb{C}^d \longrightarrow \mathsf{H} \times \mathbb{C}^d,$$

is invertible, with an inverse of the form

$$\mathbf{E} = \begin{bmatrix} \tilde{E} & h_{\operatorname{Ker} A}^- \\ h_{\operatorname{Ker} A^*}^+ & 0 \end{bmatrix} \colon \mathsf{H} \times \mathbb{C}^d \longrightarrow \mathsf{H} \times \mathbb{C}^d,$$

where

$$\tilde{E} \colon H = V_2 \oplus V_2^\perp \longrightarrow V_1^\perp$$
$$u = u_1 + u_2 \longmapsto (A|_{V_1^\perp})^{-1} u_2.$$

Next, given an operator P with multiple characteristics, we take for A a polynomial-coefficient differential operator called the localized operator [2], [29]. The latter is the Weyl-quantization, in the normal directions to the characteristic set, of the relevant piece in the Taylor expansion of the symbol at the characteristic points obtained by keeping track of the orders of vanishing of its various parts. Then, the system

$$\begin{bmatrix} P & R_- \\ R_+ & 0 \end{bmatrix},$$

which is approximated by the system $\mathbf{A}$, can be inverted in a suitable pseudodifferential calculus (in the sense of left and right parametrices) by a system

$$\begin{bmatrix} E & E_- \\ E_+ & E_\pm \end{bmatrix},$$

which is approximated by the system $\mathbf{E}$.

As already mentioned, this method proved itself successful in Sjöstrand's paper [29] in which, for the first time, the hypoellipticity with a loss of one derivative

(and solvability) for general pseudodifferential operators with multiple characteristics was studied. Later, along the same lines of problems, Helffer [7] studied the hypoellipticity with a loss of $3/2$-derivatives for operators with multiple characteristics. Pushing the machinery of localized operators to all orders (to describe the "transport terms" in the parametrix), Parenti and the first author [18] studied the hypoellipticity with a loss of many derivatives for operators with multiple symplectic characteristics. They showed in particular that the various examples of C^∞ hypoelliptic operators with multiple characteristics and loss of derivatives, such as the Stein example, the Christ flat-Kohn example and others, were manifestations of the same phenomenon [19]. More recently, they could also obtain, by the approach developed in [18], the local and semiglobal solvability of certain operators with multiple symplectic characteristics [20].

We close this section by describing the organization of the article. The next section provides the statement of the main result (Theorem 3.1). Section 4 is dedicated to some case studies, while the proof of Theorem 3.1 is given in Section 5. Finally, Section 6 is an appendix gathering miscellaneous facts and notation related to quadratic differential operators.

3. Statement of the main result

We consider the semiclassical pseudodifferential operator P given in (2.2) whose Weyl symbol $p(x, \xi; h)$ admits the semiclassical asymptotic expansion (2.3) in the symbol class $S(1)$, and we assume that the principal symbol p_0 satisfies the assumptions (2.4), (2.5), (2.6), and (2.7).

Let q be the quadratic term in the Taylor expansion of the principal symbol at the doubly characteristic point $X = 0$,

$$(3.1) \qquad p_0(X) = q(X) + \mathcal{O}(X^3), \quad X = (x, \xi) \in \mathbb{R}^{2n},$$

when $X \to 0$. The assumption $\operatorname{Re} p_0 \geq 0$ implies that the complex-valued quadratic form q has also a nonnegative real part, $\operatorname{Re} q \geq 0$.

In the present work, we do not consider the degenerate case, that is the case when the quadratic form q is only partially elliptic (i.e., it satisfies the ellipticity condition (2.9) on its singular space). Indeed, we assume that the quadratic form q is elliptic on the whole phase space:

$$(3.2) \qquad (x, \xi) \in \mathbb{R}^{2n}, \quad q(x, \xi) = 0 \implies (x, \xi) = 0.$$

Under this assumption, the spectrum of the quadratic operator

$$q^w(x, D_x)\, u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\, q\Big(\frac{x+y}{2}, \xi\Big) u(y)\, d\xi\, dy,$$

is composed only of eigenvalues with finite algebraic multiplicities (Theorem 3.5 in [29], see also [2]),

$$(3.3) \qquad \operatorname{Spec}\big(q^w(x, D_x)\big) = \Big\{ \sum_{\substack{\lambda \in \operatorname{Spec}(F), \\ -i\lambda \in \Sigma(q)}} (r_\lambda + 2k_\lambda)(-i\lambda);\ k_\lambda \in \mathbb{N} \Big\},$$

where $\Sigma(q) = \overline{q(\mathbb{R}^{2n})}$, and where r_λ is the dimension of the complex vector space spanned by the generalized eigenvectors associated with the eigenvalue $\lambda \in \mathbb{C}$ of the Hamilton map of q; see the appendix (Section 6).

Let $K \subset \mathbb{C}$ be a compact set and let $N_0 \geq 1$ be a positive integer. We consider a spectral parameter $z(h)$ with the semiclassical expansion

$$(3.4) \qquad z(h) = \sum_{k=0}^{2N_0+2} z_k \, h^{k/2},$$

with $z_k \in K$ for all $0 \leq k \leq 2N_0 + 2$, where the leading term is assumed to satisfy

$$(3.5) \qquad z_0 \in p_1(0) + \mathrm{Spec}\big(q^w(x, D_x)\big).$$

We define the symbols

$$(3.6) \qquad a_k(X) = \tilde{a}_k(X) - z_k := \sum_{\substack{j+|\alpha|/2=1+k/2 \\ 0 \leq j \leq 1+[N_0/2],\ |\alpha| \leq N_0+2}} \frac{p_j^{(\alpha)}(0)}{\alpha!} X^\alpha - z_k,$$

for $0 \leq k \leq 2N_0 + 2$, where $[x]$ stands for the integer part of x. Notice that

$$(3.7) \qquad a_0(X) = q(X) + p_1(0) - z_0,$$

with q the quadratic form defined in (3.1). The two operators

$$(3.8) \qquad Q = a_0^w(x, D_x) = q^w(x, D_x) + p_1(0) - z_0 : B \longrightarrow L^2(\mathbb{R}^n),$$

$$(3.9) \qquad Q^* = \overline{a_0}^w(x, D_x) = \overline{q}^w(x, D_x) + \overline{p_1(0)} - \overline{z_0} : B \longrightarrow L^2(\mathbb{R}^n),$$

are known to be Fredholm operators of index 0 (see Lemma 3.1 in [14], or Theorem 3.5 in [29]), where B is the Hilbert space

$$B = \big\{ u \in L^2(\mathbb{R}^n);\ x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n),\ \alpha, \beta \in \mathbb{N}^n,\ |\alpha + \beta| \leq 2 \big\},$$

equipped with the norm

$$\|u\|_B^2 = \sum_{|\alpha+\beta| \leq 2} \big\| x^\alpha D_x^\beta u \big\|_{L^2}^2.$$

Setting

$$(3.10) \qquad V_1 = \mathrm{Ker}\, Q, \quad V_2 = \mathrm{Ker}\, Q^*,$$

we can decompose $L^2(\mathbb{R}^n)$ as

$$(3.11) \qquad L^2(\mathbb{R}^n) = V_1 \oplus V_1^\perp = V_2 \oplus V_2^\perp,$$

with $V_1^\perp = \mathrm{Ran}\, Q^*$, $V_2^\perp = \mathrm{Ran}\, Q$. Since

$$0 = \mathrm{ind}\, Q = \dim \mathrm{Ker}\, Q - \mathrm{codim}\, \mathrm{Ran}\, Q = \dim V_1 - \mathrm{codim}\, V_2^\perp,$$

the kernels of the operators Q and Q^* have the same dimension

$$1 \leq d = \dim V_1 = \dim V_2 < +\infty.$$

Let $\phi_1, \ldots, \phi_d$ and $\psi_1, \ldots, \psi_d$ be orthonormal bases of V_1 and V_2, so that

$$(3.12) \qquad Q\phi_j = 0, \quad Q^*\psi_k = 0, \quad 1 \leq j, k \leq d.$$

Because of the ellipticity of the quadratic symbols q and $\overline{q}$, the eigenfunctions ϕ_j, ψ_k belong to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. We denote by π_1 and π_2 the orthogonal projections onto the vector spaces $V_1^\perp$ and $V_2^\perp$,

$$(3.13) \qquad \pi_1 u = u - \sum_{j=1}^{d} (u, \phi_j)_{L^2}\, \phi_j, \quad \pi_2 u = u - \sum_{j=1}^{d} (u, \psi_j)_{L^2}\, \psi_j.$$

The unbounded operators

$$(3.14) \qquad Q|_{V_1^\perp} : V_1^\perp \longrightarrow V_2^\perp = \operatorname{Ran} Q, \quad Q^*|_{V_2^\perp} : V_2^\perp \longrightarrow V_1^\perp = \operatorname{Ran} Q^*,$$

are isomorphisms when equipped with the domains

$$D(Q|_{V_1^\perp}) = B \cap V_1^\perp, \quad D(Q^*|_{V_2^\perp}) = B \cap V_2^\perp.$$

We define the operator

$$(3.15) \qquad S : L^2(\mathbb{R}^n) = V_2 \oplus V_2^\perp \longrightarrow L^2(\mathbb{R}^n)$$
$$u = u_1 + u_2 \longmapsto (Q|_{V_1^\perp})^{-1} u_2.$$

The main result of this article is the following theorem:

Theorem 3.1. *Let $K \subset \mathbb{C}$ be a compact subset and let $N_0 \geq 1$ be a positive integer. Let P be a semiclassical pseudodifferential operator* (2.2) *satisfying the assumptions* (2.3), (2.4), (2.5), (2.6), (2.7), *and* (3.2). *Let $z(h)$ be the spectral parameter* (3.4) *whose leading part satisfies the assumption* (3.5). *Let Ω be a compact subset of K^{2N_0+2} (the Cartesian product of K with itself $2N_0 + 2$ times). The a priori estimate*

$$(3.16) \quad \exists c_0 > 0, \exists 0 < h_0 \leq 1, \forall u \in L^2(\mathbb{R}^n), \forall 0 < h \leq h_0, \forall (z_1, \ldots, z_{2N_0+2}) \in \Omega,$$
$$\|Pu - hz(h)u\|_{L^2} \geq c_0\, h^{N_0/2+1} \|u\|_{L^2}$$

holds if and only if the a priori estimate

$$(3.17) \quad \exists c_0 > 0, \exists 0 < h_0 \leq 1, \forall u_- \in \mathbb{C}^d, \forall 0 < h \leq h_0, \forall (z_1, \ldots, z_{2N_0+2}) \in \Omega,$$
$$|E_\pm u_-| \geq c_0\, h^{N_0/2+1} |u_-|$$

holds, where $E_\pm$ stands for the $d \times d$ matrix

$$(3.18) \qquad E_\pm = \sum_{j=1}^{2N_0+2} A_j\, h^{1+j/2}, \quad A_j = \big(A_{k,l}^{(j)}\big)_{1 \leq k, l \leq d} \in M_d(\mathbb{C}),$$

and where the entries $A_{k,l}^{(j)}$ of each A_j are given by

$$(3.19) \qquad A_{k,l}^{(j)} = \sum_{i=1}^{j} (-1)^i \sum_{\substack{1 \le k_p \le 2N_0+2 \\ k_1+\cdots+k_i=j}} \left(a_{k_1}^w \, S \, a_{k_2}^w \, S \cdots a_{k_{i-1}}^w \, S \, a_{k_i}^w \phi_l, \psi_k \right)_{L^2}.$$

The operators $a_k^w(x, D_x)$ are the Weyl quantizations of the symbols defined in (3.6).

Remark 3.2. It will be shown in the proof of Theorem 3.1 that the operator S is a pseudodifferential operator. This accounts for the fact that $S : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}(\mathbb{R}^n)$ and the definition of the entries $A_{k,l}^{(j)}$.

This result indicates that the resolvent growth, up to large powers of $1/h$, is entirely determined by the Taylor series of the symbol up to a certain specified order.

4. Some case studies

Before plunging into the proof of Theorem 3.1, which will be given in Section 5, we wish to discuss in this section some case studies. We begin by studying the case of semiclassical hypoelliptic estimates with a loss of $3/2$ derivatives.

4.1. Semiclassical hypoelliptic estimates with a loss of $3/2$ derivatives

When $N_0 = 1$, Theorem 3.1 shows that the semiclassical hypoelliptic estimate with a loss of $3/2$ derivatives

$$(4.1) \quad \exists c_0 > 0, \, \exists 0 < h_0 \le 1, \, \forall u \in L^2(\mathbb{R}^n), \, \forall 0 < h \le h_0, \, \forall z_1 \in K,$$
$$\left\| Pu - hz_0 u - h^{3/2} z_1 u \right\|_{L^2} \ge c_0 \, h^{3/2} \left\| u \right\|_{L^2},$$

holds if and only if the matrix

$$A_1(z_1) = \left((a_1^w(x, D_x) \, \phi_l, \psi_k)_{L^2} \right)_{1 \le k,l \le d},$$

is invertible for all $z_1 \in K$, where $a_1^w(x, D_x)$ is the differential operator defined by the Weyl quantization of the symbol

$$a_1(X) = \sum_{|\alpha|=3} \frac{p_0^{(\alpha)}(0)}{\alpha!} \, X^\alpha + \sum_{|\alpha|=1} \frac{p_1^{(\alpha)}(0)}{\alpha!} \, X^\alpha - z_1.$$

Denoting by d_0 the rank of the matrix $A_0 = ((\phi_l, \psi_k)_{L^2})_{1 \le k,l \le d}$, we observe that its determinant $\det A_1(z_1)$ is a polynomial function in the variable z_1 of degree d_0. We distinguish two cases:

  (i) $d_0 = 0$;

  (ii) $1 \le d_0 \le d$.

When $d_0 = 0$, the matrix A_0 is zero and the invertibility of the matrix $A_1(z_1) = A_1(0)$ is independent of the parameter z_1. When $\det A_1(0) \neq 0$, the a priori estimate (4.1) holds. This indicates that there is no eigenvalue for the operator P in any $h^{3/2}$-neighborhood of the point hz_0, when $0 < h \ll 1$. On the other hand, when $\det A_1(0) = 0$, the a priori estimate (4.1) is violated for every $z_1 \in \mathbb{C}$ and the resolvent cannot be bounded in norm as $\mathcal{O}(h^{-3/2})$ in any $h^{3/2}$-neighborhood of the point hz_0.

When $1 \leq d_0 \leq d$, we consider an open neighborhood ω of the finite set

$$\Lambda = \big\{ z \in \mathbb{C};\ \det A_1(z) = 0 \big\}.$$

We deduce from Theorem 3.1 that

$$\exists c_0 > 0,\ \exists 0 < h_0 \leq 1,\ \forall u \in \mathscr{S}(\mathbb{R}^n),\ \forall 0 < h \leq h_0,\ \forall z_1 \in K \cap (\mathbb{C} \setminus \omega),$$
$$\big\| Pu - hz_0 u - h^{3/2} z_1 u \big\|_{L^2} \geq c_0\, h^{3/2}\, \|u\|_{L^2}.$$

In this case, the spectrum of the operator P in the disk $D(hz_0, Ch^{3/2})$ is localized in any $h^{3/2}$-neighborhood U of the set $hz_0 + h^{3/2}\Lambda$, and the resolvent of P is bounded in norm as $\mathcal{O}(h^{-3/2})$ on the set $D(hz_0, Ch^{3/2}) \cap (\mathbb{C} \setminus U)$.

4.2. Case when the eigenfunctions have some parity properties

When the eigenfunctions $\phi_1, \ldots, \phi_d,\ \psi_1, \ldots, \psi_d$ have some parity properties, the conclusions of Theorem 3.1 can be sharpened further as follows.

Proposition 4.1. *Under the hypotheses of Theorem 3.1, we make the additional assumptions:*

(i) *The functions $\phi_1, \ldots, \phi_d$ are all even, or all odd.*

(ii) *The functions $\psi_1, \ldots, \psi_d$ are all even, or all odd.*

(iii) *All the terms with odd indices in the semiclassical expansion of the spectral parameter (3.4) are zero*

$$z(h) = \sum_{k=0}^{N_0+1} z_{2k}\, h^k.$$

Then, the conclusions of Theorem 3.1 hold with

$$E_\pm = \sum_{j=1}^{2N_0+2} A_j\, h^{1+j/2},$$

where

$$A_{2j+1} = 0, \quad \forall j \ with \ \ 1 \leq 2j+1 \leq 2N_0 + 2,$$

when the functions $\phi_1, \ldots, \phi_d$ and $\psi_1, \ldots, \psi_d$ have the same parity, or else

$$A_{2j} = 0, \quad \forall j \ with \ \ 1 \leq 2j \leq 2N_0 + 2,$$

when the functions $\phi_1, \ldots, \phi_d$ and $\psi_1, \ldots, \psi_d$ have opposite parities.

Proof. To begin, we claim that when the functions $\psi_1, \ldots, \psi_d \in V_2$ are all even, or all odd, then Su is even (respectively odd) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is an even (respectively odd) function. To see this, observe that when all the functions $\psi_1, \ldots, \psi_d$ are even, the function

$$\pi_2 u = u - \sum_{j=1}^{d} (u, \psi_j)_{L^2} \, \psi_j,$$

is even (respectively odd) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd), because $(u, \psi_j)_{L^2} = 0$ when u is odd. On the other hand, when all the functions $\psi_1, \ldots, \psi_d$ are odd, $\pi_2 u$ is also even (respectively odd) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd) because $(u, \psi_j)_{L^2} = 0$ when u is even. Then we observe that Qu is even (respectively odd) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd). Indeed, recalling that

$$Q = q^w(x, D_x) + p_1(0) - z_0,$$

the parity property holds true for Qu since it trivially holds true in the case of the operators

$$(x^\alpha \xi^\beta)^w = \frac{1}{2} \left(x^\alpha D_x^\beta + D_x^\beta x^\alpha \right), \quad |\alpha + \beta| = 2.$$

For $u \in \mathscr{S}(\mathbb{R}^n)$, we write $Su = v_1 + v_2$ with $v_1, v_2 \in \mathscr{S}(\mathbb{R}^n)$ with v_1 even and v_2 odd (see Remark 3.2). We assume that $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd). It follows from (3.14) and (3.15) that

$$\operatorname{Ran} S \subset V_1^\perp, \quad \pi_2 = QS.$$

Since $\pi_2 u = QSu = Qv_1 + Qv_2$ is even (respectively odd), then $Qv_2 = 0$ (respectively $Qv_1 = 0$), that is, $v_2 \in V_1$ (respectively $v_1 \in V_1$). On the other hand, we have

$$0 = (Su, v_2)_{L^2} = (v_1 + v_2, v_2)_{L^2} = (v_1, v_2)_{L^2} + \|v_2\|_{L^2}^2 = \|v_2\|_{L^2}^2,$$

respectively

$$0 = (Su, v_1)_{L^2} = (v_1 + v_2, v_1)_{L^2} = \|v_1\|_{L^2}^2 + (v_2, v_1)_{L^2} = \|v_1\|_{L^2}^2,$$

because $\operatorname{Ran} S \subset V_1^\perp$, and $(v_1, v_2)_{L^2} = 0$ when v_1 and v_2 have opposite parities. It follows that $Su = v_1$ is even (respectively $Su = v_2$ is odd). This concludes the proof of the claim.

Next, we observe that

$$\left(a^w(x, D_x) u \right)(-x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(-x-y) \cdot \xi} \, a\left(\frac{-x+y}{2}, \xi \right) u(y) \, dy \, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} \, a\left(-\frac{x+y}{2}, -\xi \right) u(-y) \, dy \, d\xi.$$

It follows that the function $a^w u$ is even (respectively odd) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd) when the symbol a is even, whereas $a^w u$ is odd (respectively

even) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd) when a is odd. Therefore, when all the terms with odd indices in the semiclassical expansion of the spectral parameter (3.4) are zero, that is

$$z(h) = \sum_{k=0}^{N_0+1} z_{2k}\, h^k,$$

we have from (3.6) that

$$a_{2k}(X) = \sum_{\substack{j+|\alpha|/2=1+k \\ 0\le j \le 1+[N_0/2],\ |\alpha|\le N_0+2}} \frac{p_j^{(\alpha)}(0)}{\alpha!}\, X^\alpha - z_{2k},$$

is an even function and that

$$a_{2k+1}(X) = \sum_{\substack{j+|\alpha|/2=1+k+1/2 \\ 0\le j \le 1+[N_0/2],\ |\alpha|\le N_0+2}} \frac{p_j^{(\alpha)}(0)}{\alpha!}\, X^\alpha,$$

is an odd function. Under assumption (ii), we deduce from Remark 3.2 and the previous claim that the function

$$a_{k_1}^w\, S\, a_{k_2}^w\, S \cdots a_{k_{i-1}}^w\, S\, a_{k_i}^w\, u \in \mathscr{S}(\mathbb{R}^n), \quad \text{with}\ \ 1 \le k_1 + \cdots + k_i = 2j \le 2N_0 + 2,$$

is even (respectively odd) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd). On the other hand, the function

$$a_{k_1}^w\, S\, a_{k_2}^w\, S \cdots a_{k_{i-1}}^w\, S\, a_{k_i}^w\, u \in \mathscr{S}(\mathbb{R}^n), \quad \text{with}\ \ 1 \le k_1 + \cdots + k_i = 2j+1 \le 2N_0 + 2$$

is odd (respectively even) whenever $u \in \mathscr{S}(\mathbb{R}^n)$ is even (respectively odd). It follows that

$$\left(a_{k_1}^w\, S\, a_{k_2}^w\, S \cdots a_{k_{i-1}}^w\, S\, a_{k_i}^w\, \phi_l, \psi_k\right)_{L^2} = 0,$$

with $1 \le k_1 + \cdots + k_i = 2j+1 \le 2N_0 + 2$ (respectively $1 \le k_1 + \cdots + k_i = 2j \le 2N_0 + 2$), when the functions $\phi_1, \ldots, \phi_d$ and $\psi_1, \ldots, \psi_d$ have the same parity (respectively opposite parities). This ends the proof of Proposition 4.1. $\qquad\square$

4.3. Case d = dim V₁ = dim V₂ = 1

We now consider the case when the kernels V_1 and V_2 are one-dimensional, that is

$$V_1 = \text{Ker}\, Q = \text{Span}\ \phi_1, \quad V_2 = \text{Ker}\, Q^* = \text{Span}\ \psi_1,$$

spanned by eigenfunctions satisfying $(\phi_1, \psi_1)_{L^2} \ne 0$. In this case, the matrix (3.18) can be written as

$$E_\pm = \sum_{j=1}^{2N_0+2} h^{1+j/2} \sum_{i=1}^{j} (-1)^i \sum_{\substack{1\le k_p \le 2N_0+2 \\ k_1+\cdots+k_i=j}} \left(a_{k_1}^w\, S\, a_{k_2}^w\, S \cdots a_{k_{i-1}}^w\, S\, a_{k_i}^w\, \phi_1, \psi_1\right)_{L^2}.$$

We define successively, for every $1 \le j \le 2N_0 + 2$,

$$(4.2) \qquad \tilde{z}_1 = \frac{1}{(\phi_1, \psi_1)_{L^2}} \left(\tilde{a}_1^w(x, D_x) \phi_1, \psi_1 \right)_{L^2},$$

and

$$(4.3) \quad \tilde{z}_j = \frac{1}{(\phi_1, \psi_1)_{L^2}} \Big[(\tilde{a}_j^w \phi_1, \psi_1)_{L^2} + \sum_{i=2}^{j} (-1)^{i+1} \sum_{\substack{1 \le k_p \le 2N_0+2 \\ k_1 + \cdots + k_i = j}}$$

$$\left((\tilde{a}_{k_1}^w - \tilde{z}_{k_1}) S (\tilde{a}_{k_2}^w - \tilde{z}_{k_2}) S \cdots (\tilde{a}_{k_{i-1}}^w - \tilde{z}_{k_{i-1}}) S (\tilde{a}_{k_i}^w - \tilde{z}_{k_i}) \phi_1, \psi_1 \right)_{L^2} \Big].$$

The following result follows from Theorem 3.1.

Corollary 4.2. *With the hypotheses of Theorem 3.1, we assume further that the kernels $V_1 = Span\, \phi_1$ and $V_2 = Span\, \psi_1$ are one-dimensional, spanned by eigenfunctions satisfying $(\phi_1, \psi_1)_{L^2} \ne 0$. Let $N_0 \ge 1$ be a positive integer and let $K \subset \mathbb{C} \backslash \{\tilde{z}_{N_0}\}$ be a compact subset, where the complex numbers $\tilde{z}_j$, $1 \le j \le 2N_0 + 2$, are defined in (4.2) and (4.3). Then, there exist $c_0 > 0$ and $0 < h_0 \le 1$ such that for all $u \in L^2(\mathbb{R}^n)$, $0 < h \le h_0$, and $z \in K$ one has*

$$\left\| Pu - hz_0 u - \sum_{j=1}^{N_0-1} h^{1+j/2}\, \tilde{z}_j\, u - h^{1+N_0/2}\, z\, u \right\|_{L^2} \ge c_0\, h^{1+N_0/2}\, \|u\|_{L^2}.$$

Recalling (3.3) and (3.5), we shall now consider the specific case when

$$(4.4) \qquad z_0 - p_1(0) = \sum_{\substack{\lambda \in \mathrm{Spec}(F), \\ -i\lambda \in \Sigma(q)}} -i\lambda r_\lambda,$$

is the first eigenvalue in the bottom of the spectrum of the elliptic quadratic operator $q^w(x, D_x)$. In this case, we recall from [29] (see also [17], Theorem 2.1) that the eigenvalue $z_0 - p_1(0)$ has algebraic multiplicity 1 and that the eigenspace

$$V_1 = \mathrm{Ker}\, Q = \mathbb{C}\phi_1,$$

is spanned by a ground state of exponential type

$$\phi_1(x) = e^{-a(x)} \in \mathscr{S}(\mathbb{R}^n),$$

where a is a complex-valued quadratic form whose real part is positive definite. The quadratic form a is defined as (with $\langle \cdot, \cdot \rangle$ the Euclidean inner product in $\mathbb{R}^n$)

$$a(x) = -\frac{1}{2} i \langle x, B^+ x \rangle, \quad x \in \mathbb{R}^n, \quad \mathrm{Im}\, B^+ > 0,$$

where B^+ is the symmetric matrix with positive definite imaginary part $\mathrm{Im}\, B^+$ defining the positive Lagrangian plane (see [29], Proposition 3.3)

$$V^+ = \bigoplus_{\substack{\lambda \in \mathrm{Spec}(F) \\ -i\lambda \in \Sigma(q)}} V_\lambda = \left\{ (x, B^+ x);\ x \in \mathbb{C}^n \right\},$$

where V_λ is the space of the generalized eigenvectors associated with the eigenvalue λ of the Hamilton map of q. On the other hand, we observe that the adjoint operator is actually given by the quadratic operator

$$q^w(x, D_x)^* = \overline{q}^w(x, D_x),$$

whose symbol is the complex conjugate of the symbol q. This quadratic symbol is also elliptic. It follows that $\overline{z}_0 - \overline{p_1(0)}$ is the first eigenvalue in the bottom of the spectrum of the quadratic operator $q^w(x, D_x)^*$. Therefore this eigenvalue has algebraic multiplicity 1 and its eigenspace is also spanned by a ground state

$$\psi_1(x) = e^{-\tilde{a}(x)} \in \mathscr{S}(\mathbb{R}^n)$$

of exponential type, where $\tilde{a}$ is a complex-valued quadratic form whose real part is positive definite. Under these assumptions, we are therefore in the situation where $d = \dim V_1 = \dim V_2 = 1$, with even eigenfunctions ϕ_1 and ψ_1,

$$V_1 = \text{Span } \phi_1, \quad V_2 = \text{Span } \psi_1.$$

These two eigenspaces are equal, $V_1 = V_2$, that is $\phi_1 = \psi_1$, if and only if $V^+ = \overline{V^-}$ (see [35], Theorem 1.7), where

$$V^- = \bigoplus_{\substack{\lambda \in \text{Spec}(F) \\ i\lambda \in \Sigma(q)}} V_\lambda.$$

In any case, we notice that $(\phi_1, \psi_1)_{L^2} \neq 0$ since this scalar product is equal to the Fourier transform evaluated at zero,

$$(4.5) \qquad (\phi_1, \psi_1)_{L^2} = \widehat{f}(0) \neq 0,$$

of the Gaussian function $f(x) = e^{-b(x)}$, with $b = a + \overline{\tilde{a}}$ a complex-valued quadratic form whose real part is positive definite, see e.g. [15] (Theorem 7.6.1). We consider the case when $N_0 = 2\tilde{N}_0$, with $\tilde{N}_0 \geq 1$. Notice that (4.5) means that $\pi_{V_2}|_{V_1} : V_1 \longrightarrow V_2$ is invertible, where $\pi_{V_2} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is the orthogonal projection onto V_2 (of course, the same holds for $\pi_{V_1}|_{V_2} : V_2 \longrightarrow V_1$). Under these assumptions, we define successively for every $1 \leq j \leq 2\tilde{N}_0 + 1$,

$$(4.6) \qquad \tilde{z}_2 = \frac{1}{(\phi_1, \psi_1)_{L^2}} \left[(\tilde{a}_2^w \phi_1, \psi_1)_{L^2} - (\tilde{a}_1^w S \tilde{a}_1^w \phi_1, \psi_1)_{L^2} \right],$$

$$(4.7) \qquad \tilde{z}_{2j} = \frac{1}{(\phi_1, \psi_1)_{L^2}} \left[(\tilde{a}_{2j}^w \phi_1, \psi_1)_{L^2} + \sum_{i=2}^{2j} (-1)^{i+1} \right.$$
$$\left. \times \sum_{\substack{1 \leq k_p \leq 2N_0 + 2 \\ k_1 + \cdots + k_i = 2j}} (a_{k_1}^w S a_{k_2}^w S \cdots a_{k_{i-1}}^w S a_{k_i}^w \phi_1, \psi_1)_{L^2} \right].$$

We therefore deduce from Proposition 4.1 and Theorem 3.1 the following result.

Corollary 4.3. *Under the hypotheses of Theorem* 3.1*, we make the additional assumption* (4.4)*. Let $\tilde{N}_0 \geq 1$ be a positive integer and let $K \subset \mathbb{C} \setminus \{\tilde{z}_{2\tilde{N}_0}\}$ be a compact subset, where the complex numbers $\tilde{z}_{2j}$, $1 \leq j \leq 2\tilde{N}_0 + 1$, are defined in* (4.6) *and* (4.7)*. Then, there exist $c_0 > 0$ and $0 < h_0 \leq 1$ such that for all $u \in \mathscr{S}(\mathbb{R}^n)$, $0 < h \leq h_0$, and $z \in K$, one has*

$$\left\| Pu - hz_0 u - \sum_{j=1}^{\tilde{N}_0 - 1} h^{1+j}\, \tilde{z}_{2j}\, u - h^{\tilde{N}_0 + 1}\, z\, u \right\|_{L^2} \geq c_0\, h^{\tilde{N}_0 + 1} \|u\|_{L^2}.$$

On the other hand, we may calculate explicitly all the terms in the semiclassical expansion (2.11) of an eigenvalue of P with leading term hz_0, when the assumptions of Corollary 4.3 are satisfied. Indeed, the semiclassical expansion is given in this case by

$$z_k \sim h\big(\lambda_k + p_1(0) + h\,\lambda_{k,1} + h^2 \lambda_{k,2} + \cdots\big),$$

since $z_0 - p_1(0)$ is an eigenvalue with algebraic multiplicity 1 of the quadratic operator $q^w(x, D_x)$. Then, we directly deduce from Proposition 4.1 and Theorem 3.1 that the coefficients $\lambda_{k,j}$ must correspond to the terms $\tilde{z}_{2j}$ given in Corollary 4.3, that is

$$\lambda_{k,j} = \tilde{z}_{2j}, \quad 1 \leq j \leq \tilde{N}_0.$$

5. Proof of Theorem 3.1

Let $K \subset \mathbb{C}$ be a compact subset and let $N_0 \geq 1$ be a positive integer. Let

$$P = p^w(x, hD_x; h) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi}\, p\left(\frac{x+y}{2}, h\,\xi; h\right) u(y)\, dy\, d\xi,$$

be a semiclassical pseudodifferential operator satisfying the assumptions (2.3), (2.4), (2.5), (2.6), (2.7), and (3.2), and let

$$z(h) = \sum_{k=0}^{2N_0 + 2} z_k\, h^{k/2}, \quad z_k \in K,$$

be the spectral parameter whose leading part satisfies assumption (3.5). After conjugating by the unitary operator

$$(5.1) \qquad T_h : L^2(\mathbb{R}^n) \ni u(x) \longmapsto h^{n/4} u(h^{1/2} x) \in L^2(\mathbb{R}^n)$$

it is sufficient to prove Theorem 3.1 for the operator

$$(5.2) \qquad P = T_h\, p^w\big(x, h\, D_x; h\big)\, T_h^{-1} = p^w\big(h^{1/2} x, h^{1/2} D_x; h\big).$$

In the following, the standard notation $c = a\, \#^w b$ denotes the Weyl symbol of the operator obtained by composition

$$c^w(x, D_x) = a^w(x, D_x)\, b^w(x, D_x) = (\text{Op}^w a)(\text{Op}^w b),$$

with the standard normalization of the Weyl quantization

$$a^w(x, D_x)\, u(x) = (\mathrm{Op}^w a)\, u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\, a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi.$$

We refer the reader to the notation introduced in Section 3 and begin by noting that the orthogonal projections π_1 and π_2 onto the vector spaces $V_1^\perp$ and $V_2^\perp$ satisfy

$$\pi_1, \pi_2 : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}(\mathbb{R}^n),$$

since the eigenfunctions ϕ_j and ψ_k belong to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$. Next, we observe that the L^2-adjoint of the bounded operator $S : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined in (3.15) is given by

$$(5.3) \qquad\qquad S^* : L^2(\mathbb{R}^n) = V_1 \oplus V_1^\perp \longrightarrow L^2(\mathbb{R}^n)$$
$$u = u_1 + u_2 \longmapsto (Q^*|_{V_2^\perp})^{-1} u_2.$$

By definition, these two operators are continuous on $L^2(\mathbb{R}^n)$ and satisfy the identities

$$(5.4) \qquad SQ = Q^* S^* = 1 - (1 - \pi_1), \quad QS = S^* Q^* = 1 - (1 - \pi_2).$$

Using standard notation for a metric on phase space [15], [16], for $m \in \mathbb{R}$ we write

$$(5.5) \quad \mathbf{S}^m = S(\langle X \rangle^m, g) = \big\{ a \in C^\infty(\mathbb{R}^{2n}, \mathbb{C}); \, \forall \alpha \in \mathbb{N}^{2n}, \, \exists C_\alpha > 0, \, \forall X \in \mathbb{R}^{2n},$$
$$|\partial_X^\alpha a(X)| \le C_\alpha \langle X \rangle^{m - |\alpha|} \big\}$$

for the class of (h-independent) global pseudodifferential operators (after Shubin, see [31]), where g is the admissible, geodesically temperate metric (see, e.g., [16], Lemma 2.6.23) given by

$$g = \frac{|dX|^2}{\langle X \rangle^2}, \quad X = (x, \xi) \in \mathbb{R}^{2n},$$

and we write $\mathrm{Op}^w(\mathbf{S}^m)$ for the set of associated pseudodifferential operators with symbols in $\mathbf{S}^m$.

As global pseudodifferential operators in $\mathbb{R}^n$, the operators $1 - \pi_1$ and $1 - \pi_2$ are smoothing since their symbols in the standard quantization, given respectively by

$$\sum_{j=1}^d e^{-ix\cdot\xi}\, \phi_j(x)\, \overline{\widehat{\phi_j}(\xi)} \in \mathbf{S}^{-\infty} \quad \text{and} \quad \sum_{j=1}^d e^{-ix\cdot\xi}\, \psi_j(x)\, \overline{\widehat{\psi_j}(\xi)} \in \mathbf{S}^{-\infty},$$

belong to the Schwartz space $\mathscr{S}(\mathbb{R}^{2n})$. This also implies that

$$1 - \pi_1 \in \mathrm{Op}^w(\mathbf{S}^{-\infty}) \quad \text{and} \quad 1 - \pi_2 \in \mathrm{Op}^w(\mathbf{S}^{-\infty}).$$

Setting

$$T_+ u = \sum_{j=1}^d (u, \psi_j)_{L^2}\, \phi_j \quad \text{and} \quad T_- u = \sum_{j=1}^d (u, \phi_j)_{L^2}\, \psi_j,$$

the same arguments show that $T_\pm \in \mathrm{Op}^w(\mathbf{S}^{-\infty})$. Next, we deduce from (3.10), (3.12) and (3.14) that the mapping

$$\Phi : B \longrightarrow L^2(\mathbb{R}^n) = V_2 \oplus V_2^\perp$$

$$u \longmapsto \sum_{j=1}^{d}(u, \phi_j)_{L^2}\,\psi_j + Qu,$$

is invertible with inverse given by

$$\Phi^{-1} : L^2(\mathbb{R}^n) \longrightarrow B$$

$$u \longmapsto \sum_{j=1}^{d}(u, \psi_j)_{L^2}\,\phi_j + Su.$$

We can write $\Phi = a^w(x, D_x)$, with $a \in \mathbf{S}^2$. Referring to [16] (Section 2.6) for the definition of Sobolev spaces attached to a pseudodifferential calculus, we notice that $L^2(\mathbb{R}^n) = H(1, g)$ and $B = H(\langle X \rangle^2, g)$. Then we deduce from Corollary 2.6.28 in [16] that

$$\Phi^{-1} \in \mathrm{Op}^w(\mathbf{S}^{-2}).$$

Since $T_+ \in \mathrm{Op}^w(\mathbf{S}^{-\infty})$, this implies that $S \in \mathrm{Op}^w(\mathbf{S}^{-2})$ and justifies Remark 3.2.

Now we prove Theorem 3.1. We start by observing that for $X = (x, \xi) \in \mathbb{R}^{2n}$,

$$p\left(h^{1/2}X; h\right) - h\, z(h) = \sum_{k=0}^{1+[N_0/2]} p_k(h^{1/2}X)h^k - \sum_{k=0}^{2N_0+2} z_k\, h^{1+k/2} \mod S\big(h^{[N_0/2]+2}\big),$$

up to a symbol belonging to the class $S(h^{[N_0/2]+2})$, defined in (2.1). From the Taylor expansions

$$p_k(X) = \sum_{|\alpha| \leq N_0+2} \frac{p_k^{(\alpha)}(0)}{\alpha!}\, X^\alpha + \sum_{|\alpha|=N_0+3} \frac{N_0+3}{\alpha!}\, X^\alpha \int_0^1 (1-t)^{N_0+2}\, p_k^{(\alpha)}(tX)\, dt$$

of the symbols, we obtain that

$$p\left(h^{1/2}X; h\right) - h\, z(h)$$

$$= \sum_{\substack{k=0,\ldots,1+[N_0/2] \\ |\alpha| \leq N_0+2}} \frac{p_k^{(\alpha)}(0)}{\alpha!}\, X^\alpha\, h^{k+|\alpha|/2} - \sum_{k=0}^{2N_0+2} z_k\, h^{1+k/2} + \sum_{\substack{k=0,\ldots,1+[N_0/2] \\ |\alpha|=N_0+3}} \frac{N_0+3}{\alpha!}$$

$$\text{(5.6)} \qquad \times\, X^\alpha h^{k+|\alpha|/2} \int_0^1 (1-t)^{N_0+2} p_k^{(\alpha)}\big(t\, h^{1/2}X\big)\, dt \mod S\big(h^{[N_0/2]+2}\big).$$

By assumption (2.7), the point $0 \in \mathbb{R}^{2n}$ is doubly characteristic for the principal symbol, that is $p_0(0) = \nabla p_0(0) = 0$. Setting

$$R_\alpha(X; h) = \sum_{k=0,\ldots,1+[N_0/2]} \frac{N_0+3}{\alpha!}\, h^k \int_0^1 (1-t)^{N_0+2} p_k^{(\alpha)}\big(t\, h^{1/2}X\big)\, dt,$$

we can therefore write

$$p\left(h^{1/2}X;h\right) - h\,z(h)$$

$$(5.7) \quad = \sum_{k=0}^{2N_0+2} a_k(X)\,h^{1+k/2} + h^{N_0/2+3/2} \sum_{|\alpha|=N_0+3} X^\alpha R_\alpha(X;h) \mod S\big(h^{[N_0/2]+2}\big),$$

with the symbols a_k defined in (3.6). As $p_k \in S(1)$, we readily see that

$$(5.8) \qquad\qquad\qquad\qquad R_\alpha \in S(1).$$

Following [18], we now use a Grushin-reduction method. To this end, we define

$$(5.9) \qquad\qquad\qquad\qquad R_- : \mathbb{C}^d \longrightarrow V_2$$

$$u_- \longmapsto \sum_{j=1}^{d} u_-(j)\,\psi_j,$$

$$(5.10) \qquad\qquad\qquad R_+ : L^2(\mathbb{R}^n) \longrightarrow \mathbb{C}^d$$

$$u \longmapsto \big((u,\phi_j)_{L^2}\big)_{1\le j\le d},$$

where ϕ_j and ψ_k are the eigenfunctions defined in (3.12). Setting

$$(5.11) \qquad\qquad \phi_{0,k}^+ = \phi_k, \quad \psi_{0,k}^- = \psi_k, \quad k = 1,\dots,d,$$

we shall construct by induction functions $\phi_{j,k}^+$, $\psi_{j,k}^- \in \mathscr{S}(\mathbb{R}^n)$, for $1 \le k \le d$, $1 \le j \le 2N_0+2$, and $d \times d$ complex matrices $A_j \in M_d(\mathbb{C})$ for $j = 1,\dots,2N_0+2$, which are all independent of the semiclassical parameter and satisfy the equations

$$(5.12) \qquad\qquad\qquad R_+ E_+ = \mathrm{Id} + \mathcal{O}_{\mathscr{L}(\mathbb{C}^d)}(h^{1/2}),$$

$$(5.13) \qquad\qquad \sum_{k=0}^{2N_0+2} h^{1+k/2}\, a_k^w(x,D_x)E_+ + R_-E_\pm = \mathcal{O}_{\mathscr{L}(\mathbb{C}^d,L^2)}\big(h^{N_0+5/2}\big),$$

$$(5.14) \qquad E_-\Big(\sum_{k=0}^{2N_0+2} a_k^w(x,D_x)\,h^{1+k/2} \Big) + E_\pm R_+ = \mathcal{O}_{\mathscr{L}(L^2,\mathbb{C}^d)}\big(h^{N_0+5/2}\big),$$

$$(5.15) \qquad\qquad S\,a_0^w(x,D_x) + E_+ R_+ = \mathrm{Id} + \mathcal{O}_{\mathscr{L}(L^2)}(h^{1/2}),$$

where

$$(5.16) \qquad\qquad\qquad E_+ : \mathbb{C}^d \longrightarrow L^2(\mathbb{R}^n)$$

$$u_- \longmapsto \sum_{k=1}^{d} u_-(k)\Big(\sum_{j=0}^{2N_0+2} \phi_{j,k}^+\, h^{j/2} \Big),$$

$$(5.17) \qquad E_- : L^2(\mathbb{R}^n) \longrightarrow \mathbb{C}^d$$

$$u \longmapsto \left(\left(u, \sum_{j=0}^{2N_0+2} \psi_{j,k}^- \, h^{j/2} \right)_{L^2} \right)_{1 \le k \le d},$$

and

$$(5.18) \qquad E_\pm = \sum_{j=1}^{2N_0+2} A_j \, h^{1+j/2}.$$

The notation $\mathcal{O}_{\mathscr{L}(E,F)}(h^N)$ stands for a remainder which is a bounded operator $T : E \to F$ with a norm satisfying $\|T\|_{\mathscr{L}(E,F)} \lesssim h^N$.

We next observe that equation (5.12) is satisfied immediately since the functions $(\phi_{0,k}^+)_{1 \le k \le d} = (\phi_k)_{1 \le k \le d}$ are chosen orthonormal

$$\Big(\sum_{k=1}^d u_-(k) \Big(\sum_{j=0}^{2N_0+2} \phi_{j,k}^+ \, h^{j/2} \Big), \phi_l \Big)_{L^2} = u_-(l) + \sum_{\substack{k=1,\dots,d \\ j=1,\dots,2N_0+2}} h^{j/2} \, u_-(k) \big(\phi_{j,k}^+, \phi_l \big)_{L^2}.$$

Equation (5.13) can be written as

$$(5.19) \qquad \sum_{\substack{1 \le l \le d \\ 0 \le j,k \le 2N_0+2}} h^{1+(k+j)/2} u_-(l) \, a_k^w(x, D_x) \, \phi_{j,l}^+$$

$$+ \sum_{\substack{1 \le l \le d \\ 1 \le j \le 2N_0+2}} h^{1+j/2} (A_j \, u_-)(l) \, \psi_l = \mathcal{O}\big(h^{N_0+5/2} |u_-| \big).$$

We deduce from (3.8), (3.12), and (5.11) that the coefficient of h on the left-hand side of (5.19) is zero, that is,

$$\sum_{1 \le l \le d} u_-(l) \, a_0^w(x, D_x) \, \phi_{0,l}^+ = \sum_{1 \le l \le d} u_-(l) \, Q \, \phi_l = 0.$$

Next, we observe that the coefficient of $h^{1+j/2}$, with $1 \le j \le 2N_0 + 2$, on the left-hand side of equation (5.19) is zero if and only if we have

$$(5.20) \qquad \sum_{\substack{1 \le l \le d \\ 0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j}} u_-(l) \, a_{k_1}^w(x, D_x) \, \phi_{k_2,l}^+ + \sum_{1 \le l \le d} (A_j u_-)(l) \, \psi_l = 0.$$

By assuming that the Schwartz functions $\phi_{k,l}^+$ have already been determined for all $0 \le k \le j-1$, $1 \le l \le d$, for satisfying equation (5.20) it will be sufficient to choose the functions $(\phi_{j,l}^+)_{1 \le l \le d}$ and the matrix $A_j = (A_{k,l}^{(j)})_{1 \le k,l \le d}$ to satisfy the identities

$$(5.21) \qquad Q\phi_{j,l}^+ = - \sum_{1 \le k \le d} A_{k,l}^{(j)} \, \psi_k - \sum_{\substack{0 \le k_1,k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} a_{k_1}^w(x, D_x) \, \phi_{k_2,l}^+,$$

for every $1 \le l \le d$. Taking

$$(5.22) \qquad A_{k,l}^{(j)} = - \sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} \left(a_{k_1}^w(x, D_x) \phi_{k_2,l}^+, \psi_k \right)_{L^2},$$

yields that the right-hand side of (5.21) is fully determined and belongs to $V_2^\perp$,

$$Q\phi_{j,l}^+ = -\pi_2 \Big(\sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} a_{k_1}^w(x, D_x) \phi_{k_2,l}^+ \Big) \in V_2^\perp.$$

It follows from (3.14) and (3.15) that we can choose the functions $\phi_{j,l}^+$ as

$$(5.23) \qquad \phi_{j,l}^+ = -(Q|_{V_1^\perp})^{-1} \pi_2 \Big(\sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} a_{k_1}^w(x, D_x) \phi_{k_2,l}^+ \Big)$$

$$= -S \Big(\sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} a_{k_1}^w(x, D_x) \phi_{k_2,l}^+ \Big).$$

Since $S : \mathscr{S}(\mathbb{R}^n) \longrightarrow \mathscr{S}(\mathbb{R}^n)$ (see Remark 3.2), the functions $(\phi_{j,l}^+)_{1 \le l \le d}$ belong to $\mathscr{S}(\mathbb{R}^n)$. By iterating this process, we obtain functions $\phi_{j,l}^+$ for $1 \le l \le d$ and $0 \le j \le 2N_0 + 2$ and matrices A_j for $1 \le j \le 2N_0 + 2$, satisfying equation (5.13). Next, equation (5.14) can be written as

$$\sum_{0 \le j,k \le 2N_0+2} h^{1+\frac{j+k}{2}} \left(a_k^w(x, D_x) u, \psi_{j,l}^- \right)_{L^2} + \sum_{\substack{1 \le k \le d \\ 1 \le j \le 2N_0+2}} h^{1+j/2} A_{l,k}^{(j)}(u, \phi_k)_{L^2} = \mathcal{O}\big(h^{N_0+\frac{5}{2}} \|u\|_{L^2} \big),$$

for $1 \le l \le d$. We therefore need to satisfy the equations

$$\Big(u, \sum_{0 \le j,k \le 2N_0+2} h^{1+\frac{j+k}{2}} \overline{a}_k^w(x, D_x) \psi_{j,l}^- + \sum_{\substack{1 \le k \le d \\ 1 \le j \le 2N_0+2}} h^{1+j/2} \overline{A_{l,k}^{(j)}} \phi_k \Big)_{L^2} = \mathcal{O}\big(h^{N_0+\frac{5}{2}} \|u\|_{L^2} \big).$$

We get from (3.9), (3.12) and (5.11) that the coefficient of h in

$$\sum_{0 \le j,k \le 2N_0+2} h^{1+(j+k)/2} \overline{a}_k^w(x, D_x) \psi_{j,l}^- + \sum_{\substack{1 \le k \le d \\ 1 \le j \le 2N_0+2}} h^{1+j/2} \overline{A_{l,k}^{(j)}} \phi_k,$$

is zero. It will therefore suffice to choose the Schwartz functions $\psi_{j,l}^-$, for $1 \le l \le d$ and $1 \le j \le 2N_0 + 2$, such that

$$\sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j}} \overline{a}_{k_1}^w(x, D_x) \psi_{k_2,l}^- + \sum_{1 \le k \le d} \overline{A_{l,k}^{(j)}} \phi_k = 0,$$

that is

$$(5.24) \qquad Q^*\psi_{j,l}^- = -\sum_{1\le k\le d} \overline{A_{l,k}^{(j)}}\,\phi_k - \sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \overline{a}_{k_1}^w(x,D_x)\,\psi_{k_2,l}^-,$$

for all $1\le l\le d$ and $1\le j\le 2N_0+2$. Assuming that the Schwartz functions $\psi_{k,l}^-$ have already been determined for all $0\le k\le j-1$ and $1\le l\le d$, by using (3.14) and (5.3) we define the functions $(\psi_{j,l}^-)_{1\le l\le d}$ as

$$\psi_{j,l}^- = -(Q^*|_{V_2^\perp})^{-1}\pi_1\Big(\sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \overline{a}_{k_1}^w(x,D_x)\,\psi_{k_2,l}^-\Big)$$

$$(5.25) \qquad = -S^*\Big(\sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \overline{a}_{k_1}^w(x,D_x)\,\psi_{k_2,l}^-\Big).$$

The next lemma establishes the identity

$$\overline{A_{l,k}^{(j)}} = -\sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \big(\overline{a}_{k_1}^w(x,D_x)\,\psi_{k_2,l}^-,\phi_k\big)_{L^2},$$

which yields that equations (5.24) are satisfied and therefore that equation (5.14) holds.

Lemma 5.1. *The functions $\phi_{k,l}^+$ and $\psi_{k,l}^-$ constructed above satisfy the identities*

$$\sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \big(a_{k_1}^w(x,D_x)\,\phi_{k_2,l}^+,\psi_k\big)_{L^2} = \sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \big(\phi_l,\overline{a}_{k_1}^w(x,D_x)\,\psi_{k_2,k}^-\big)_{L^2},$$

for every $1\le j\le 2N_0+2$ and $1\le k,l\le d$. Furthermore, the entries of the matrices $A_j = (A_{k,l}^{(j)})_{1\le k,l\le d}$ are given by

$$A_{k,l}^{(j)} = \sum_{i=1}^j (-1)^i \sum_{\substack{1\le k_p\le 2N_0+2 \\ k_1+\cdots+k_i=j}} \big(a_{k_1}^w\, S\, a_{k_2}^w\, S\cdots a_{k_{i-1}}^w\, S\, a_{k_i}^w\phi_l,\psi_k\big)_{L^2},$$

for all $1\le j\le 2N_0+2$ and $1\le k,l\le d$.

Proof. For $1\le k,l\le d$, we have from (5.11) and (5.23) that

$$\sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \big(a_{k_1}^w\,\phi_{k_2,l}^+,\psi_k\big)_{L^2} = \sum_{\substack{0\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j,\ k_1\ge 1}} \big(\phi_{k_2,l}^+,\overline{a}_{k_1}^w\,\psi_k\big)_{L^2} = \big(\phi_l,\overline{a}_j^w\,\psi_k\big)_{L^2}$$

$$-\sum_{\substack{1\le k_1,k_2\le 2N_0+2 \\ k_1+k_2=j}}\ \sum_{\substack{0\le k_3,k_4\le 2N_0+2 \\ k_3+k_4=k_2,\ k_3\ge 1}} \big(S\,a_{k_3}^w\,\phi_{k_4,l}^+,\overline{a}_{k_1}^w\,\psi_k\big)_{L^2}.$$

We can write

$$\sum_{\substack{1 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j}} \sum_{\substack{0 \leq k_3, k_4 \leq 2N_0+2 \\ k_3+k_4=k_2, \ k_3 \geq 1}} \left(S \, a_{k_3}^w \, \phi_{k_4,l}^+, \overline{a}_{k_1}^w \, \psi_k\right)_{L^2}$$

$$= \sum_{\substack{1 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j}} \left(\phi_l, \overline{a}_{k_2}^w \, S^* \, \overline{a}_{k_1}^w \, \psi_k\right)_{L^2} + \sum_{\substack{1 \leq k_1, k_3, k_4 \leq 2N_0+2 \\ k_1+k_3+k_4=j}} \left(\phi_{k_4,l}^+, \overline{a}_{k_3}^w \, S^* \, \overline{a}_{k_1}^w \, \psi_k\right)_{L^2},$$

for all $1 \leq j \leq 2N_0 + 2$, which yields

$$\sum_{\substack{0 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j, \ k_1 \geq 1}} \left(a_{k_1}^w \, \phi_{k_2,l}^+, \psi_k\right)_{L^2} = \left(\phi_l, \overline{a}_j^w \, \psi_k\right)_{L^2} - \sum_{\substack{1 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j}} \left(\phi_l, \overline{a}_{k_1}^w \, S^* \, \overline{a}_{k_2}^w \, \psi_k\right)_{L^2}$$

$$- \sum_{\substack{1 \leq k_1, k_2, k_3 \leq 2N_0+2 \\ k_1+k_2+k_3=j}} \left(\phi_{k_1,l}^+, \overline{a}_{k_2}^w \, S^* \, \overline{a}_{k_3}^w \, \psi_k\right)_{L^2}.$$

By using the definition (5.23) of the functions $\phi_{k,l}^+$ and iterating this process, we obtain

$$\sum_{\substack{0 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j, \ k_1 \geq 1}} \left(a_{k_1}^w \, \phi_{k_2,l}^+, \psi_k\right)_{L^2} = \sum_{i=1}^{j} (-1)^{i+1} \sum_{\substack{1 \leq k_p \leq 2N_0+2 \\ k_1+\cdots+k_i=j}} \left(\phi_l, \overline{a}_{k_1}^w \, S^* \, \overline{a}_{k_2}^w \, S^* \cdots \overline{a}_{k_{i-1}}^w \, S^* \overline{a}_{k_i}^w \, \psi_k\right)_{L^2}.$$

On the other hand, from (5.11) and (5.25) it follows that

$$\sum_{\substack{0 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j, \ k_1 \geq 1}} \left(\phi_l, \overline{a}_{k_1}^w \, \psi_{k_2,k}^-\right)_{L^2} = \sum_{\substack{0 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j, \ k_1 \geq 1}} \left(a_{k_1}^w \, \phi_l, \psi_{k_2,k}^-\right)_{L^2} = \left(a_j^w \, \phi_l, \psi_k\right)_{L^2}$$

$$- \sum_{\substack{1 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j}} \sum_{\substack{0 \leq k_3, k_4 \leq 2N_0+2 \\ k_3+k_4=k_2, \ k_3 \geq 1}} \left(a_{k_1}^w \, \phi_l, S^* \, \overline{a}_{k_3}^w \, \psi_{k_4,k}^-\right)_{L^2}.$$

Since we can write

$$\sum_{\substack{1 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j}} \sum_{\substack{0 \leq k_3, k_4 \leq 2N_0+2 \\ k_3+k_4=k_2, \ k_3 \geq 1}} \left(a_{k_1}^w \, \phi_l, S^* \, \overline{a}_{k_3}^w \, \psi_{k_4,k}^-\right)_{L^2}$$

$$= \sum_{\substack{1 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j}} \left(a_{k_2}^w \, S \, a_{k_1}^w \, \phi_l, \psi_k\right)_{L^2} + \sum_{\substack{1 \leq k_1, k_3, k_4 \leq 2N_0+2 \\ k_1+k_3+k_4=j}} \left(a_{k_3}^w \, S \, a_{k_1}^w \, \phi_l, \psi_{k_4,k}^-\right)_{L^2},$$

we get

$$\sum_{\substack{0 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j, \ k_1 \geq 1}} \left(\phi_l, \overline{a}_{k_1}^w \, \psi_{k_2,k}^-\right)_{L^2} = \left(a_j^w \, \phi_l, \psi_k\right)_{L^2} - \sum_{\substack{1 \leq k_1, k_2 \leq 2N_0+2 \\ k_1+k_2=j}} \left(a_{k_1}^w \, S \, a_{k_2}^w \, \phi_l, \psi_k\right)_{L^2}$$

$$- \sum_{\substack{1 \leq k_1, k_2, k_3 \leq 2N_0+2 \\ k_1+k_2+k_3=j}} \left(a_{k_1}^w \, S \, a_{k_2}^w \, \phi_l, \psi_{k_3,k}^-\right)_{L^2}.$$

By using the definition (5.25) of the functions $\psi_{k,l}^-$ and iterating this process, we obtain that

$$\sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} \left(\phi_l, \overline{a}_{k_1}^w \psi_{k_2,k}^-\right)_{L^2} = \sum_{i=1}^{j}(-1)^{i+1}$$

$$\times \sum_{\substack{1 \le k_p \le 2N_0+2 \\ k_1+\cdots+k_i=j}} \left(a_{k_1}^w \, S \, a_{k_2}^w \, S \cdots a_{k_{i-1}}^w \, S \, a_{k_i}^w \, \phi_l, \psi_k\right)_{L^2}.$$

As

$$\left(\phi_l, \overline{a}_{k_1}^w \, S^* \, \overline{a}_{k_2}^w \, S^* \cdots \overline{a}_{k_{i-1}}^w \, S^* \, \overline{a}_{k_i}^w \, \psi_k\right)_{L^2} = \left(a_{k_i}^w \, S \, a_{k_{i-1}}^w \, S \cdots a_{k_2}^w \, S \, a_{k_1}^w \, \phi_l, \psi_k\right)_{L^2},$$

we conclude from (5.22) that

$$A_{k,l}^{(j)} = - \sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} \left(a_{k_1}^w \, \phi_{k_2,l}^+, \psi_k\right)_{L^2} = - \sum_{\substack{0 \le k_1, k_2 \le 2N_0+2 \\ k_1+k_2=j, \ k_1 \ge 1}} \left(\phi_l, \overline{a}_{k_1}^w \, \psi_{k_2,k}^-\right)_{L^2},$$

for all $1 \le j \le 2N_0 + 2$. □

Writing $u = u_1 + u_2 \in L^2(\mathbb{R}^n)$, with $(u_1, u_2) \in V_1 \times V_1^\perp$, we finally obtain from (3.8), (5.4) and (5.11) that equation (5.15) holds:

$$SQu + \sum_{k=1}^{d}(u, \phi_k)_{L^2}\left(\sum_{j=0}^{2N_0+2} \phi_{j,k}^+ \, h^{j/2}\right) = u_2 + \sum_{k=1}^{d}(u, \phi_k)_{L^2}\phi_k + \mathcal{O}(h^{1/2}\|u\|_{L^2})$$

$$= u_2 + u_1 + \mathcal{O}(h^{1/2}\|u\|_{L^2})$$

$$= u + \mathcal{O}(h^{1/2}\|u\|_{L^2}).$$

We shall now use the Grushin-reduction, (5.12), (5.13), (5.14), and (5.15), in order to prove Theorem 3.1.

Let Ω be a compact subset of K^{2N_0+2}. We first assume that there exist $c_0 > 0$ and $0 < h_0 \le 1$ such that for all $u \in L^2(\mathbb{R}^n)$, $0 < h \le h_0$, and $(z_1, \ldots, z_{2N_0+2}) \in \Omega$,

$$(5.26) \qquad \|Pu - hz(h)\,u\|_{L^2} \ge c_0\,h^{N_0/2+1}\|u\|_{L^2}.$$

From (5.7) and (5.26) we get that, for any given $u_- \in \mathbb{C}^d$,

$$(5.27) \quad c_0\,h^{N_0/2+1}\|E_+u_-\|_{L^2} \le \|(P - hz(h))E_+u_-\|_{L^2}$$

$$\le \left\| \sum_{k=0}^{2N_0+2} h^{1+k/2} a_k^w(x, D_x)\,E_+u_- + R_-E_\pm u_- \right\|_{L^2} + \|R_-E_\pm u_-\|_{L^2}$$

$$+ h^{N_0/2+3/2}\left\| \sum_{|\alpha|=N_0+3} \mathrm{Op}^w\big(X^\alpha R_\alpha(X;h)\big)E_+u_- \right\|_{L^2} + \mathcal{O}\big(h^{[N_0/2]+2}\big)\|E_+u_-\|_{L^2}.$$

By using the symbolic calculus in the Weyl quantization, we readily obtain from (5.8) and the exact formula ([15], Theorem 18.5.4)

$$(5.28) \qquad X^\alpha \#^w R_\alpha = \sum_{p=0}^{|\alpha|} \frac{1}{p!} \left(\frac{1}{2i}\, \sigma(\partial_{X_1}, \partial_{X_2}) \right)^p X_1^\alpha R_\alpha(X_2; h) \Big|_{X_1 = X_2 = X},$$

that the operator $\mathrm{Op}^w(X^\alpha R_\alpha(X; h))$ can be written as

$$(5.29) \qquad \mathrm{Op}^w\big(X^\alpha R_\alpha(X; h)\big) = \sum_{\beta \leq \alpha} \mathrm{Op}^w\big(\tilde{R}_\beta(X; h)\big)\, \mathrm{Op}^w(X^\beta),$$

for some symbols $\tilde{R}_\beta$ belonging to the class $S(1)$. It follows from (5.16) and (5.29) that

$$\big\| \mathrm{Op}^w\big(X^\alpha R_\alpha(X; h)\big) E_+ u_- \big\|_{L^2} \leq \sum_{\substack{1 \leq k \leq d \\ 0 \leq j \leq 2N_0+2}} |u_-(k)|\, h^{j/2}\, \big\| \mathrm{Op}^w\big(X^\alpha R_\alpha(X; h)\big) \phi_{j,k}^+ \big\|_{L^2}$$

$$(5.30) \qquad\qquad \lesssim |u_-| \sum_{\substack{1 \leq k \leq d \\ 0 \leq j \leq 2N_0+2 \\ \beta \leq \alpha}} \big\| \mathrm{Op}^w(X^\beta) \phi_{j,k}^+ \big\|_{L^2} = \mathcal{O}(1)|u_-|,$$

since the functions $\phi_{j,k}^+$ belong to $\mathscr{S}(\mathbb{R}^n)$. As $\|R_-\|_{\mathscr{L}(\mathbb{C}^d, L^2)} = \mathcal{O}(1)$, it follows from (5.13), (5.27) and (5.30) that

$$(5.31) \qquad h^{N_0/2+1} \|E_+ u_-\|_{L^2} \lesssim |E_\pm u_-| + \mathcal{O}\big(h^{N_0/2+3/2}\big)|u_-|.$$

Since $\|R_+\|_{\mathscr{L}(L^2, \mathbb{C}^d)} = \mathcal{O}(1)$, we get from (5.12) that

$$(5.32) \qquad |u_-| \leq |R_+ E_+ u_-| + \mathcal{O}(h^{1/2})|u_-| \leq \|E_+ u_-\|_{L^2} + \mathcal{O}(h^{1/2})|u_-|,$$

whence from (5.31) and (5.32) we obtain

$$h^{N_0/2+1}|u_-| \lesssim |E_\pm u_-| + \mathcal{O}(h^{N_0/2+3/2})|u_-|,$$

that is, there exist constants $c_0 > 0$ and $0 < h_0 \leq 1$, such that
$$(5.33)$$
$$\forall u_- \in \mathbb{C}^d,\ \forall 0 < h \leq h_0,\ \forall (z_1, \ldots, z_{2N_0+2}) \in \Omega, \quad |E_\pm u_-| \geq c_0\, h^{N_0/2+1}\, |u_-|.$$

This concludes the proof of the first implication.

We shall now prove the converse implication. We therefore assume that the estimate (5.33) holds. It follows from (2.3), (2.7), (3.1), (3.4), and (3.7) that

$$p\,(h^{1/2}X; h) - hz(h) = p_0(h^{1/2}X) + h\, p_1(h^{1/2}X) - h\, z_0 \mod S(h^{3/2})$$
$$= h\big(q(X) + p_1(0) - z_0\big) + r_{0,h}(X) + r_{1,h}(X) \mod S(h^{3/2})$$
$$(5.34) \qquad\qquad = h\, a_0(X) + r_{0,h}(X) + r_{1,h}(X) \mod S(h^{3/2}),$$

with

$$(5.35) \qquad r_{0,h}(X) = \sum_{|\alpha|=3} \frac{3}{\alpha!} X^\alpha h^{3/2} \int_0^1 (1-t)^2 p_0^{(\alpha)}(t\,h^{1/2}X)\,dt,$$

$$(5.36) \qquad r_{1,h}(X) = \sum_{|\alpha|=1} X^\alpha h^{3/2} \int_0^1 p_1^{(\alpha)}(t\,h^{1/2}X)\,dt.$$

Let $\chi_0 \in C_0^\infty(\mathbb{R}^{2n})$ be a cutoff function satisfying $0 \le \chi_0 \le 1$ and

$$(5.37) \quad \operatorname{supp}\chi_0 \subset \big\{X \in \mathbb{R}^{2n};\ |X| \le 2\big\}, \quad \chi_0 = 1 \ \text{ on } \ \big\{X \in \mathbb{R}^{2n};\ |X| \le 1\big\},$$

and let $A \gg 1$ be a large positive constant to be chosen later. Setting

$$(5.38) \qquad M_0 = \chi_0^w\big(A\,h^{1/2}x, A\,h^{1/2}D_x\big),$$

it follows from (3.8) and (5.15) that for all $u \in \mathscr{S}(\mathbb{R}^n)$,

$$
\begin{aligned}
h\|u\|_{L^2} &\le h\,\|E_+ R_+ u\|_{L^2} + h\,\|SQ\,u\|_{L^2} + \mathcal{O}(h^{3/2})\,\|u\|_{L^2} \\
&\lesssim h\,|R_+u| + h\,\|SQM_0\,u\|_{L^2} + h\,\|SQ(1-M_0)\,u\|_{L^2} + \mathcal{O}(h^{3/2})\,\|u\|_{L^2},
\end{aligned}
$$

since $\|E_+\|_{\mathscr{L}(\mathbb{C}^d,L^2)} = \mathcal{O}(1)$. From (5.4) one has

$$\|SQ(1-M_0)\,u\|_{L^2} = \|\pi_1(1-M_0)\,u\|_{L^2} \le \|(1-M_0)\,u\|_{L^2},$$

whence

$$(5.39) \qquad h\|u\|_{L^2} \lesssim h|R_+u| + h\|SQM_0\,u\|_{L^2} + h\|(1-M_0)\,u\|_{L^2}.$$

Observing that $\|S\|_{\mathscr{L}(L^2)} = \mathcal{O}(1)$ and $\|M_0\|_{\mathscr{L}(L^2)} = \mathcal{O}(1)$, when $0 < h \le A^{-2} \le 1$, we then deduce from (3.8) and (5.34) that

$$
\begin{aligned}
(5.40) \qquad h\,\big\|SQM_0\,u\big\|_{L^2} &\le \big\|S(P - hz(h))M_0\,u\big\|_{L^2} + \big\|Sr_{0,h}^w M_0\,u\big\|_{L^2} \\
&\quad + \big\|Sr_{1,h}^w M_0\,u\big\|_{L^2} + \mathcal{O}(h^{3/2})\|u\|_{L^2},
\end{aligned}
$$

that in turn yields

$$
\begin{aligned}
(5.41) \qquad \big\|S(P - hz(h))M_0\,u\big\|_{L^2} &\lesssim \big\|(P - hz(h))M_0\,u\big\|_{L^2} \\
&\lesssim \big\|M_0(P - hz(h))\,u\big\|_{L^2} + \big\|[P, M_0]\,u\big\|_{L^2} \\
&\lesssim \big\|Pu - hz(h)\,u\big\|_{L^2} + \big\|[P, M_0]\,u\big\|_{L^2},
\end{aligned}
$$

which, along with (5.40), gives

$$
\begin{aligned}
(5.42) \quad h\,\big\|SQM_0\,u\big\|_{L^2} &\lesssim \big\|Pu - hz(h)\,u\big\|_{L^2} + \big\|[P, M_0]\,u\big\|_{L^2} \\
&\quad + \big\|Sr_{0,h}^w M_0\,u\big\|_{L^2} + \big\|Sr_{1,h}^w M_0\,u\big\|_{L^2} + \mathcal{O}(h^{3/2})\|u\|_{L^2}.
\end{aligned}
$$

We shall need the following technical lemma.

Lemma 5.2. *We have*

$$\left\| S r_{1,h}^w M_0\, u \right\|_{L^2} = \left\| S r_{1,h}^w\, \chi_0^w(A\, h^{1/2} X)\, u \right\|_{L^2} = \mathcal{O}\!\left(\frac{h}{A}\right)\|u\|_{L^2} + \mathcal{O}_A(h^2)\|u\|_{L^2},$$

when $0 < h \le A^{-2} \le 1$.

Proof. Since $\|S\|_{\mathscr{L}(L^2)} = \mathcal{O}(1)$, we first observe that

$$\left\| S r_{1,h}^w M_0\, u \right\|_{L^2} \le \left\| r_{1,h}^w M_0\, u \right\|_{L^2}.$$

We refer the reader to Section 2 for the definitions of the symbol classes and we recall from [6] (Proposition 7.7) that for any given $a(\cdot;h) \in S(m_1)$, $b(\cdot;h) \in S(m_2)$, there exists a symbol $c(\cdot;h) \in S(m_1 m_2)$ such that

$$a\big(x, h\,\xi; h\big)\#^w b\big(x, h\,\xi; h\big) = a\big(x, h\,\xi; h\big)\, b\big(x, h\,\xi; h\big) + h\, c\big(x, h\,\xi; h\big).$$

By using the symplectic change of coordinates $(x,\xi) \mapsto (h^{1/2}x, h^{-1/2}\xi)$ corresponding, at the operator level as in (5.1) and (5.2), to the conjugation by the unitary operator T_h, we obtain that

$$(5.43) \quad a\big(h^{1/2}X; h\big)\#^w b\big(h^{1/2}X; h\big) = a\big(h^{1/2}X; h\big)\, b\big(h^{1/2}X; h\big) + h\, c\big(h^{1/2}X; h\big).$$

We deduce from (5.43) that if $R_1(\cdot;h) \in S(1)$ and $R_2(\cdot;h) \in S(\langle X\rangle^{-1})$, then there exists $R_3(\cdot;h) \in S(h)$ such that

$$(5.44) \quad \big(h^{1/2}X_j\, R_1(h^{1/2}X; h)\big)\#^w R_2(h^{1/2}X; h)$$
$$= h^{1/2}X_j\, R_1(h^{1/2}X; h)\, R_2(h^{1/2}X; h) + R_3(h^{1/2}X; h),$$

since $X_j\, R_1(\cdot;h) \in S(\langle X\rangle)$. We next observe that (2.3) and (5.37) yield that

$$(5.45) \qquad \int_0^1 p_1^{(\alpha)}(tX)\, dt \in S(1), \quad \chi_0(AX) \in S\big(\mathcal{O}_A(\langle X\rangle^{-1})\big),$$

where the notation $S(\mathcal{O}_A(m))$ stands for the following class of symbols possibly depending on the parameter $A \ge 1$ with seminorms also possibly depending on this parameter:

$$(5.46) \quad S\big(\mathcal{O}_A(m)\big) = \big\{ a(\cdot;h,A) \in C^\infty(\mathbb{R}^{2n}, \mathbb{C});\ \forall \alpha \in \mathbb{N}^{2n},\ \forall A \ge 1,\ \exists C_{\alpha,A} > 0,$$
$$\forall 0 < h \le 1,\ \forall X \in \mathbb{R}^{2n},\ \big|\partial_X^\alpha a(X;h,A)\big| \le C_{\alpha,A} m(X;h)\big\}.$$

It therefore follows from (5.36), (5.44), and (5.45) that

$$\frac{A}{h}\, r_{1,h}\#^w \chi_0(A\, h^{1/2}X) = \Big(\sum_{|\alpha|=1} (A\, h^{1/2}X)^\alpha \int_0^1 p_1^{(\alpha)}(t\, h^{1/2}X)\, dt \Big)\#^w \chi_0(A\, h^{1/2}X)$$

$$= \sum_{|\alpha|=1} (A\, h^{1/2}X)^\alpha \chi_0(A\, h^{1/2}X) \int_0^1 p_1^{(\alpha)}(t\, h^{1/2}X)\, dt + R_8(h^{1/2}X; h, A),$$

with $R_8(\cdot;h,A) \in S(\mathcal{O}_A(h))$. Since the symbol

$$\sum_{|\alpha|=1} (A\,h^{1/2}X)^\alpha \chi_0(A\,h^{1/2}X) \int_0^1 p_1^{(\alpha)}(t\,h^{1/2}X)\,dt$$

belongs to the class $S(1)$ uniformly with respect to the parameters when $0 < h \le A^{-2} \le 1$, we thus get

$$\big\|r_{1,h}^w M_0\,u\big\|_{L^2} = \big\|r_{1,h}^w \chi_0^w(A\,h^{1/2}X)\,u\big\|_{L^2} = \mathcal{O}\Big(\frac{h}{A}\Big)\|u\|_{L^2} + \mathcal{O}_A(h^2)\|u\|_{L^2},$$

when $0 < h \le A^{-2} \le 1$, and this concludes the proof of the lemma. $\qquad\square$

We shall also need the following technical result.

Lemma 5.3. *We have*

$$\big\|Sr_{0,h}^w M_0\,u\big\|_{L^2} = \mathcal{O}\Big(\frac{h}{A}\Big)\|u\|_{L^2} + \mathcal{O}_A(h^2)\|u\|_{L^2},$$

when $0 < h \le A^{-2} \le 1$.

Proof. From (5.35) we have

$$(5.47)\quad \frac{A}{h}\,r_{0,h}(X)\#^w\chi_0(A\,h^{1/2}X)$$
$$= \Big(\sum_{|\alpha|=3}\frac{3}{\alpha!}X^\alpha A\,h^{1/2}\int_0^1 (1-t)^2 p_0^{(\alpha)}(t\,h^{1/2}X)\,dt\Big)\#^w\chi_0(A\,h^{1/2}X).$$

Observing from (5.28) that if $R(\cdot;h) \in S(1)$ then for each $\alpha \in \mathbb{N}^n$ there exist symbols $R_\beta(\cdot;h) \in S(1)$, with $\beta \le \alpha$, $|\beta| < |\alpha|$, $\beta \in \mathbb{N}^n$, such that

$$(5.48)\quad X^\alpha R(h^{1/2}X;h) = X^\alpha \#^w R(h^{1/2}X;h) + \sum_{\substack{\beta \le \alpha \\ |\beta| < |\alpha|}} h^{(|\alpha|-|\beta|)/2} X^\beta R_\beta(h^{1/2}X;h),$$

by induction we readily have that if $R(\cdot;h) \in S(1)$, then for each $\alpha \in \mathbb{N}^n$ there exist symbols $R_\beta(\cdot;h) \in S(1)$, with $\beta \le \alpha$, $|\beta| < |\alpha|$, $\beta \in \mathbb{N}^n$, such that

$$(5.49)\quad X^\alpha R(h^{1/2}X;h) = X^\alpha \#^w R(h^{1/2}X;h) + \sum_{\substack{\beta \le \alpha \\ |\beta| < |\alpha|}} h^{(|\alpha|-|\beta|)/2} X^\beta \#^w R_\beta(h^{1/2}X;h).$$

We deduce from (2.3), (5.47) and (5.49) that there exist symbols $R_\beta(\cdot;h,A) \in S(\mathcal{O}_A(1))$, for $|\beta| \le 2$, such that

$$\frac{A}{h}\,r_{0,h}(X)\#^w\chi_0(A\,h^{1/2}X)$$
$$= \sum_{|\alpha|=3}\frac{3}{\alpha!}X^\alpha A\,h^{1/2}\#^w\Big(\int_0^1 (1-t)^2 p_0^{(\alpha)}(t\,h^{1/2}X)\,dt\Big)\#^w\chi_0(A\,h^{1/2}X)$$
$$+ h\sum_{|\beta|\le 2} h^{(2-|\beta|)/2} X^\beta \#^w R_\beta(h^{1/2}X;h,A)\#^w\chi_0(A\,h^{1/2}X).$$

The symbolic calculus shows that

$$r_1(X; h, A)$$

$$(5.50) \quad = \sum_{|\beta| \leq 2} h^{(2-|\beta|)/2} X^\beta \#^w R_\beta(h^{1/2}X; h, A) \#^w \chi_0(A h^{1/2}X) \in S(\mathcal{O}_A(\langle X \rangle^2)),$$

since $X^\beta \in S(\langle X \rangle^2)$ when $|\beta| \leq 2$, $R_\beta(h^{1/2}X; h, A) \in S(\mathcal{O}_A(1))$, and $\chi_0(Ah^{1/2}X) \in S(1)$ uniformly with respect to the parameters h and A, when $0 < h \leq A^{-2} \leq 1$. On the other hand, we get from (2.3), (5.37) and another use of the symbolic calculus that there exists a symbol $r_2(\cdot; h, A) \in S(\mathcal{O}_A(\langle X \rangle^{-\infty}))$ such that

$$\left(\int_0^1 (1-t)^2 p_0^{(\alpha)}(t h^{1/2}X) \, dt \right) \#^w \chi_0(A h^{1/2}X)$$

$$(5.51) \qquad = \chi_0(A h^{1/2}X) \int_0^1 (1-t)^2 p_0^{(\alpha)}(t h^{1/2}X) \, dt + h \, r_2(h^{1/2}X; h, A).$$

It follows that

$$(5.52) \quad \frac{A}{h} r_{0,h}(X) \#^w \chi_0(A h^{1/2}X) = h \sum_{|\alpha|=3} \frac{3}{\alpha!} X^\alpha A h^{1/2} \#^w r_2(h^{1/2}X; h, A)$$

$$+ \sum_{|\alpha|=3} \frac{3}{\alpha!} X^\alpha A h^{1/2} \#^w \left(\chi_0(A h^{1/2}X) \int_0^1 (1-t)^2 p_0^{(\alpha)}(t h^{1/2}X) \, dt \right)$$

$$+ h \, r_1(X; h, A).$$

Using (5.28) shows that there exist symbols

$$r_3(\cdot; h, A), \quad r_4(\cdot; h, A) \in S\big(\mathcal{O}_A(\langle X \rangle^{-\infty})\big), \quad r_5(\cdot; h, A) \in S\big(\mathcal{O}_A(\langle X \rangle^2)\big),$$

such that

$$(h^{1/2} X_{j_1} X_{j_2} X_{j_3}) \#^w r_2(h^{1/2}X; h, A) = (X_{j_1} X_{j_2}) \#^w (h^{1/2} X_{j_3}) \#^w r_2(h^{1/2}X; h, A)$$

$$+ \frac{i}{2} \big(h^{1/2} \{ X_{j_1} X_{j_2}, X_{j_3} \} \big) \#^w r_2(h^{1/2}X; h, A)$$

$$= \big((X_{j_1} X_{j_2}) \#^w r_3(h^{1/2}X; h, A) \big) + r_4(h^{1/2}X; h, A) = r_5(X; h, A),$$

when $0 < h \leq A^{-2} \leq 1$. It follows from (5.52) and the preceding identity that there exists a symbol $r_6(\cdot; h, A) \in S(\mathcal{O}_A(\langle X \rangle^2))$ such that

$$(5.53) \quad \frac{A}{h} r_{0,h}(X) \#^w \chi_0(A h^{1/2}X)$$

$$= \sum_{|\alpha|=3} \frac{3}{\alpha!} X^\alpha A h^{1/2} \#^w \left(\chi_0(A h^{1/2}X) \int_0^1 (1-t)^2 p_0^{(\alpha)}(t h^{1/2}X) \, dt \right) + h \, r_6(X; h, A),$$

when $0 < h \leq A^{-2} \leq 1$. When $|\alpha| = 3$, formula (5.28) once more gives that there exist symbols $R_\beta(\cdot; h, A) \in S(\mathcal{O}_A(1))$, with $|\beta| \leq 2$, such that

$$X^\alpha \#^w \Big(\chi_0(A\, h^{1/2} X) \int_0^1 (1 - t)^2 p_0^{(\alpha)}(t\, h^{1/2} X)\, dt \Big)$$

$$= X^\alpha \chi_0(A\, h^{1/2} X) \int_0^1 (1-t)^2 p_0^{(\alpha)}(t\, h^{1/2} X)\, dt$$

$$+ h^{1/2} \sum_{|\beta| \leq 2} h^{(2-|\beta|)/2} X^\beta R_\beta(h^{1/2} X; h, A).$$

It follows from (5.53) and the previous identity that there exists a symbol

$$r_7(\cdot; h, A) \in S\big(\mathcal{O}_A(\langle X \rangle^2)\big),$$

such that

$$\frac{A}{h} r_{0,h}(X) \#^w \chi_0(A\, h^{1/2} X)$$

$$(5.54) \quad = \sum_{|\alpha|=3} \frac{3}{\alpha!} X^\alpha A\, h^{1/2} \chi_0(A\, h^{1/2} X) \int_0^1 (1-t)^2 p_0^{(\alpha)}(t\, h^{1/2} X)\, dt + h\, r_7(X; h, A),$$

because

$$h^{(2-|\beta|)/2} X^\beta R_\beta(h^{1/2} X; h, A) \in S\big(\mathcal{O}_A(\langle X \rangle^2)\big),$$

when $|\beta| \leq 2$ and $0 < h \leq A^{-2} \leq 1$. Consider then the symbol

$$(5.55) \quad r_{2,h}(X) = \sum_{|\alpha|=3} \frac{3}{\alpha!} X^\alpha A\, h^{1/2} \chi_0(A\, h^{1/2} X) \int_0^1 (1-t)^2 p_0^{(\alpha)}(t\, h^{1/2} X)\, dt,$$

which can be written as

$$r_{2,h}(X) = \sum_{|\alpha_1|=2, |\alpha_2|=1} X^{\alpha_1} (A\, h^{1/2} X)^{\alpha_2} \chi_0(A\, h^{1/2} X) p_{\alpha_1, \alpha_2}(h^{1/2} X),$$

for some symbols p_{α_1, α_2} belonging to the class $S(1)$, since $p_0 \in S(1)$. We therefore deduce from (5.37) that the symbol $r_{2,h}$ belongs to the class $S(\langle X \rangle^2)$ uniformly with respect to the parameters when $0 < h \leq A^{-2} \leq 1$. By using the fact that the symbol of the operator S belongs to the class $\mathbf{S}^{-2}$, we obtain from (5.54) and (5.55) that

$$S r_{0,h}^w M_0 = S r_{0,h}^w \chi_0^w(A\, h^{1/2} X) = \frac{h}{A} r_{3,h}^w + h^2 r_{4,h}^w,$$

for some symbols $r_{3,h} \in S(1)$ and $r_{4,h} \in S(\mathcal{O}_A(1))$, uniformly with respect to the parameters when $0 < h \leq A^{-2} \leq 1$. It follows that

$$\big\| S r_{0,h}^w M_0\, u \big\|_{L^2} \lesssim \frac{h}{A} \|u\|_{L^2} + \mathcal{O}_A(h^2) \|u\|_{L^2},$$

when $0 < h \leq A^{-2} \leq 1$. This concludes the proof of Lemma 5.3. $\qquad \square$

We now resume the proof of Theorem 3.1 and deduce from (5.42) and Lemmas 5.2 and 5.3 that

$$
h\big\| SQM_0\, u\big\|_{L^2} \lesssim \big\| Pu - hz(h)\, u\big\|_{L^2} + \big\| [P, M_0]\, u\big\|_{L^2}
$$

(5.56)
$$
+ \mathcal{O}\Big(\frac{h}{A}\Big)\|u\|_{L^2} + \mathcal{O}_A(h^2)\|u\|_{L^2} + \mathcal{O}(h^{3/2})\|u\|_{L^2},
$$

when $0 < h \le A^{-2} \le 1$. Then, from (5.39) and (5.56), we get

$$
h\|u\|_{L^2} \lesssim \big\| Pu - hz(h)u\big\|_{L^2} + \big\| [P, M_0]\, u\big\|_{L^2} + h\, |R_+u|
$$
$$
+ h\big\|(1 - M_0)\, u\big\|_{L^2} + \mathcal{O}\Big(\frac{h}{A}\Big)\|u\|_{L^2} + \mathcal{O}_A(h^2)\|u\|_{L^2} + \mathcal{O}(h^{3/2})\|u\|_{L^2},
$$

when $0 < h \le A^{-2} \le 1$. We next choose the large parameter $A \gg 1$ to control the term $\mathcal{O}(h/A)\|u\|_{L^2}$ by the left-hand side of the preceding estimate. With this definitive choice fixing the parameters $A_0 \ge 1$ and $0 < h_0 \ll 1$, we obtain

$$
h\|u\|_{L^2} \lesssim \big\| Pu - hz(h)\, u\big\|_{L^2} + \big\| [P, M_0]\, u\big\|_{L^2} + h\, |R_+u|
$$

(5.57)
$$
+ h\big\|(1 - M_0)\, u\big\|_{L^2} + \mathcal{O}(h^{3/2})\|u\|_{L^2},
$$

when $0 < h \le h_0$. Using (2.3) and (5.37) observe that the Weyl symbol of the operator

$$
[P, M_0] = \big[P, \chi_0^w(A_0 h^{1/2}X)\big],
$$

is given by

$$
\frac{1}{i}\big\{p\,(h^{1/2}X; h), \chi_0(A_0 h^{1/2}X)\big\} \bmod S(h^2) = \frac{h}{i}\big\{p_0, \chi_0(A_0\cdot)\big\}(h^{1/2}X) \bmod S(h^2),
$$

whence it follows that

(5.58) $\big\| [P, M_0]\, u\big\|_{L^2} \lesssim h\big\| \mathrm{Op}^w\big(\{p_0, \chi_0(A_0\cdot)\}\, (h^{1/2}X)\big)\, u\big\|_{L^2} + h^2\|u\|_{L^2}.$

From (2.5), (2.6) and (5.37) we have that the principal symbol is elliptic near the supports of the functions

$$
(1 - \chi_0)(A_0\cdot) \quad \text{and} \quad \{p_0, \chi_0(A_0\cdot)\}.
$$

This yields the estimate

$$
\big\|(1 - M_0)\, u\big\|_{L^2} + \big\| \mathrm{Op}^w\big(\{p_0, \chi_0(A_0\cdot)\}\, (h^{1/2}X)\big)\, u\big\|_{L^2}
$$
$$
= \big\|(1 - \chi_0)^w(A_0 h^{1/2}X)\, u\big\|_{L^2} + \big\| \mathrm{Op}^w\big(\{p_0, \chi_0(A_0\cdot)\}\, (h^{1/2}X)\big)u\big\|_{L^2}
$$
(5.59) $\lesssim \big\| Pu - hz(h)\, u\big\|_{L^2} + \mathcal{O}(h)\|u\|_{L^2}.$

From (5.57), (5.58) and (5.59) we then have that

$$
h\|u\|_{L^2} \lesssim \big\| Pu - hz(h)\, u\big\|_{L^2} + h\, |R_+u|,
$$

when $0 < h \ll 1$. Since $N_0 \geq 1$, this implies

$$h^{N_0/2+1}\|u\|_{L^2} \lesssim h^{N_0/2}\big\|Pu - hz(h)\,u\big\|_{L^2} + h^{N_0/2+1}|R_+u|$$
$$(5.60) \qquad\qquad \lesssim \big\|Pu - hz(h)\,u\big\|_{L^2} + h^{N_0/2+1}|R_+u|,$$

when $0 < h \ll 1$. On the other hand, from (5.7) and (5.14) we have that

$$|E_\pm R_+u| \leq \Big| E_-\Big(\sum_{k=0}^{2N_0+2} a_k^w(x, D_x)\, h^{1+k/2} \Big) u \Big| + \mathcal{O}(h^{N_0+5/2})\|u\|_{L^2}$$

$$\leq |E_-(P - hz(h))\,u| + \mathcal{O}(h^{[N_0/2]+2})\|u\|_{L^2}$$
$$+ h^{N_0/2+3/2} \sum_{|\alpha|=N_0+3} \big| E_-\mathrm{Op}^w\big(X^\alpha R_\alpha(X;h)\big)\,u\big|$$

$$(5.61) \qquad\qquad \lesssim \big\|(P - hz(h))\,u\big\|_{L^2} + \mathcal{O}\big(h^{[N_0/2]+2}\big)\,\|u\|_{L^2}$$
$$+ h^{N_0/2+3/2} \sum_{|\alpha|=N_0+3} \big| E_-\mathrm{Op}^w\big(X^\alpha R_\alpha(X;h)\big)\,u\big|,$$

because $\|E_-\|_{\mathscr{L}(L^2,\mathbb{C}^d)} = \mathcal{O}(1)$. It follows from (5.8) and (5.17) that

$$\big(E_-\mathrm{Op}^w\big(X^\alpha R_\alpha(X;h)\big)\,u\big)_{k\text{-th component}}$$

$$= \Big(\mathrm{Op}^w\big(X^\alpha R_\alpha(X;h)\big)\,u, \sum_{j=0}^{2N_0+2} \psi_{j,k}^-\, h^{j/2}\Big)_{L^2}$$

$$(5.62) \qquad\qquad = \sum_{j=0}^{2N_0+2} h^{j/2}\big(u, \mathrm{Op}^w\big(X^\alpha \overline{R_\alpha(X;h)}\big)\,\psi_{j,k}^-\big)_{L^2} = \mathcal{O}(1)\|u\|_{L^2},$$

since $\psi_{j,k}^- \in \mathscr{S}(\mathbb{R}^n)$. We therefore get from (5.61) and (5.62) that

$$|E_\pm R_+u| \lesssim \big\|Pu - hz(h)\,u\big\|_{L^2} + \mathcal{O}\big(h^{N_0/2+3/2}\big)\,\|u\|_{L^2} + \mathcal{O}\big(h^{[N_0/2]+2}\big)\,\|u\|_{L^2}.$$

If the estimate (5.33) holds, we thus have

$$c_0\, h^{N_0/2+1}|R_+u| \leq |E_\pm R_+u|$$
$$\lesssim \big\|Pu - hz(h)\,u\big\|_{L^2} + \mathcal{O}\big(h^{N_0/2+3/2}\big)\,\|u\|_{L^2} + \mathcal{O}\big(h^{[N_0/2]+2}\big)\,\|u\|_{L^2},$$

and deduce from (5.60) that

$$h^{N_0/2+1}\|u\|_{L^2} \lesssim \big\|Pu - hz(h)\,u\big\|_{L^2} + \mathcal{O}\big(h^{N_0/2+3/2}\big)\,\|u\|_{L^2} + \mathcal{O}\big(h^{[N_0/2]+2}\big)\,\|u\|_{L^2}.$$

This shows that

$$h^{N_0/2+1}\|u\|_{L^2} \lesssim \big\|Pu - hz(h)\,u\big\|_{L^2},$$

when $0 < h \ll 1$. Hence the estimate (5.26) holds true for any given Schwartz function and by density it also holds true for all $u \in L^2(\mathbb{R}^n)$. This finally proves the second implication and concludes the proof of Theorem 3.1. $\qquad\square$

6. Appendix

This appendix gathers miscellaneous facts and notation related to quadratic differential operators used in the previous sections. We refer the reader to [11], [13], and [27] for the results recalled in this section.

Associated with a complex-valued quadratic form

$$q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \longrightarrow \mathbb{C}$$
$$(x, \xi) \longmapsto q(x, \xi),$$

with $n \geq 1$, one has the Hamilton map $F \in M_{2n}(\mathbb{C})$, uniquely defined by the identity

$$(6.1) \qquad q\big((x, \xi); (y, \eta)\big) = \sigma\big((x, \xi), F(y, \eta)\big), \quad (x, \xi) \in \mathbb{R}^{2n}, \ (y, \eta) \in \mathbb{R}^{2n},$$

where $q(\cdot; \cdot)$ stands for the polarized form associated with the quadratic form q and where σ is the standard symplectic form on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$,

$$(6.2) \qquad \sigma\big((x, \xi), (y, \eta)\big) = \xi \cdot y - x \cdot \eta, \quad (x, \xi) \in \mathbb{R}^{2n}, \ (y, \eta) \in \mathbb{R}^{2n}.$$

It readily follows from the definition that the real and imaginary parts of the Hamilton map

$$\operatorname{Re} F = \frac{1}{2}(F + \overline{F}), \quad \operatorname{Im} F = \frac{1}{2i}(F - \overline{F}),$$

$\overline{F}$ being the complex conjugate of F, are the Hamilton maps associated with the quadratic forms $\operatorname{Re} q$ and $\operatorname{Im} q$. The singular space S associated with the quadratic symbol q was introduced in [11] and is defined by

$$(6.3) \qquad S = \Big(\bigcap_{j=0}^{2n-1} \operatorname{Ker}\big(\operatorname{Re} F (\operatorname{Im} F)^j\big)\Big) \bigcap \mathbb{R}^{2n}.$$

This linear subspace of the phase space plays a basic role in the understanding of the properties of the quadratic operator

$$q^w(x, D_x)\, u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}\, q\Big(\frac{x+y}{2}, \xi\Big) u(y)\, d\xi\, dy,$$

when its symbol may fail to satisfy the ellipticity condition

$$(x, \xi) \in \mathbb{R}^{2n}, \quad q(x, \xi) = 0 \implies (x, \xi) = 0.$$

In particular, the known description of the spectrum of elliptic quadratic operators [29] extends to certain classes of "partially elliptic" quadratic operators. More specifically, when q is a quadratic symbol with a nonnegative real part $\operatorname{Re} q \geq 0$, satisfying the ellipticity condition on its singular space S (partial ellipticity)

$$(6.4) \qquad (x, \xi) \in S, \quad q(x, \xi) = 0 \implies (x, \xi) = 0,$$

then the spectrum $\mathrm{Spec}(q^w(x, D_x))$ of the quadratic operator $q^w(x, D_x)$ is composed only of eigenvalues with finite algebraic multiplicities [11] (Theorem 1.2.2) and is given explicitly by

$$(6.5) \qquad \mathrm{Spec}\big(q^w(x, D_x)\big) = \Big\{ \sum_{\substack{\lambda \in \mathrm{Spec}(F), \\ -i\lambda \in \mathbb{C}_+ \cup (\Sigma(q|_S) \setminus \{0\})}} (r_\lambda + 2k_\lambda)(-i\lambda); \quad k_\lambda \in \mathbb{N} \Big\},$$

where r_λ is the dimension of the space of generalized eigenvectors of F in $\mathbb{C}^{2n}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$, and where

$$\Sigma(q|_S) = \overline{q(S)} \subset i\mathbb{R}, \quad \mathbb{C}_+ = \big\{ z \in \mathbb{C}; \ \mathrm{Re}\, z > 0 \big\}.$$

Equivalently, the singular space can be defined as the subset of phase space where all the Poisson brackets $H_{\mathrm{Im}\, q}^k \mathrm{Re}\, q$, with $k \geq 0$, vanish

$$S = \big\{ X \in \mathbb{R}^{2n}; \ H_{\mathrm{Im}\, q}^k \mathrm{Re}\, q(X) = 0, \ k \geq 0 \big\}.$$

This shows that the singular space corresponds exactly to the set of points X_0 in the phase space where the real part of the symbol q composed with the flow generated by the Hamilton vector field

$$t \longmapsto \mathrm{Re}\, q(e^{tH_{\mathrm{Im}\, q}} X_0)$$

associated with its imaginary part $\mathrm{Im}\, q$, vanishes to infinite order at $t = 0$. Furthermore, quadratic operators with zero singular space were shown to enjoy notable subelliptic properties [27]. Namely, when q is a complex-valued quadratic form with a nonnegative real part $\mathrm{Re}\, q \geq 0$, and a zero singular space $S = \{0\}$, then the quadratic operator $q^w(x, D_x)$ fulfills the subelliptic estimate with a loss of $2k_0/(2k_0 + 1)$ derivatives

$$(6.6) \quad \big\| \langle (x, D_x) \rangle^{2/(2k_0+1)} u \big\|_{L^2} \leq C \big(\| q^w(x, D_x)\, u \|_{L^2} + \| u \|_{L^2} \big), \quad u \in \mathscr{S}(\mathbb{R}^n),$$

where $\langle (x, D_x) \rangle^2 = 1 + |x|^2 + |D_x|^2$, and where $0 \leq k_0 \leq 2n - 1$ stands for the smallest integer satisfying

$$\Big(\bigcap_{j=0}^{k_0} \mathrm{Ker}\big(\mathrm{Re}\, F (\mathrm{Im}\, F)^j \big) \Big) \bigcap \mathbb{R}^{2n} = \{0\}.$$

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