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A Nullstellensatz for Łojasiewicz ideals

Francesca Acquistapace, Fabrizio Broglia and Andreea Nicoara

Abstract. For an ideal of smooth functions $\mathfrak{a}$ that is either Łojasiewicz or weakly Łojasiewicz, we give a complete characterization of the ideal of functions vanishing on its variety $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ in terms of the global Łojasiewicz radical and Whitney closure. We also prove that the Łojasiewicz radical of such an ideal is *analytic-like* in the sense that its saturation equals its Whitney closure. This allows us to revisit Nullstellensatz results due to Bochnak and Adkins–Leahy and to resolve positively a modification of the Nullstellensatz conjecture due to Bochnak.

1. Introduction

In this paper we characterize a class of ideals $\mathfrak{a}$ having the zero property in the algebra $\mathcal{E}(M)$ of real-valued smooth functions on a smooth manifold M. Recall that an ideal $\mathfrak{a}$ has the *zero property* if it coincides with the ideal $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ of all functions vanishing on its zero set.

The investigation of such a Nullstellensatz for the class of C^∞ functions was initiated by Bochnak in 1973 in [4] and was continued by Risler in [10]. Interesting contributions by Adkins and Leahy can be found in [2] and [3].

In particular, Bochnak formulated the following conjecture:

Conjecture. *Let $\mathfrak{a}$ be a finitely generated ideal in $\mathcal{E}(M)$. Then the following are equivalent:*

(1) *$\mathfrak{a}$ has the zero property;*

(2) *$\mathfrak{a}$ is closed and real.*

Here the closure is taken in the compact-open topology.

He proved his conjecture when $\mathfrak{a}$ is generated by finitely many analytic functions. Then Risler in [10] completely resolved the conjecture in dimension 2 and for principal ideals in dimension 3. Finally, for an ideal generated by analytic func-

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tions (not necessarily finitely many), Adkins and Leahy prove in [2] that $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ is the closure of the real radical of $\mathfrak{a}$.

Note that a closed finitely generated ideal is Łojasiewicz (see the definition below), but the converse is not true ([12], p. 104, example 4.8).

In Theorem 2.7, we give a complete characterization of $\mathcal{I}(\mathcal{Z}(\mathfrak{a}))$ for the case when $\mathfrak{a}$ is a Łojasiewicz ideal in terms of a particular notion of radical called the *Łojasiewicz radical*. This radical certainly contains the real radical, but as far as we know it is not known whether they are equal. An answer to this question probably involves the solution of Hilbert's 17th Problem without denominators in the smooth setting. We define this radical below in Definition 2.6.

Łojasiewicz ideals were considered by several authors including Malgrange in Section 6 of [7], Thom in [11], and Tougeron in [12], page 104. The Łojasiewicz radical appears in work by Kohn ([6], Theorem 1.21), and Nowak [8], though mainly as a notion applied to ideals of germs.

As a consequence of Theorem 2.7, we solve the Bochnak conjecture in terms of convexity (defined at the beginning of section 4.1):

Theorem 1.1. *Let $\mathfrak{a}$ be a Łojasiewicz ideal in $\mathcal{E}(M)$. Then the following are equivalent:*

(1) *$\mathfrak{a}$ has the zero property;*

(2) *$\mathfrak{a}$ is closed, convex, and radical.*

In fact, we obtain our result for ideals $\mathfrak{a}$ with countably many generators but still satisfying condition (2) of Definition 2.3. See Theorem 3.4.

Note that a convex radical ideal is a real ideal. If we had a good representation of positive semidefinite functions as sums of squares, the converse would also be true. This converse happens to be true when $\mathfrak{a}$ is generated by analytic functions (see Theorem 4.2). We thus recover the results of Bochnak and Adkins–Leahy.

2. Łojasiewicz ideals

Let M be a smooth manifold, and let $\mathcal{E}(M)$ be its algebra of smooth real-valued functions endowed with the compact open topology.

The *saturation* of an ideal $\mathfrak{a}$ in $\mathcal{E}(M)$ is the ideal

$$\tilde{\mathfrak{a}} = \{g \in \mathcal{E}(M) \,|\, \forall x \in M \; g_x \in \mathfrak{a}\mathcal{E}_x\}.$$

An ideal $\mathfrak{a}$ is *saturated* if $\mathfrak{a} = \tilde{\mathfrak{a}}$.

Lemma 2.1. *The following inclusions hold:*

$$\mathfrak{a} \subset \tilde{\mathfrak{a}} \subset \overline{\mathfrak{a}}.$$

Proof. Consider the ideal

$$\mathfrak{a}^* = \{g \in \mathcal{E}(M) \,|\, \forall x \in M \; T_x g \in T_x \mathfrak{a}\}.$$

The Whitney spectral theorem gives $\mathfrak{a}^* = \overline{\mathfrak{a}}$ (see for instance chapter II of [7]), and the proof follows since $\tilde{\mathfrak{a}} \subset \mathfrak{a}^*$. □

Remarks 2.2. (1) Both inclusions $\mathfrak{a} \subset \widetilde{\mathfrak{a}}$ and $\tilde{\mathfrak{a}} \subset \overline{\mathfrak{a}}$ are strict in general; consult Adkins–Leahy [2], p. 708, for an example.

(2) We turn to the analytic setting. If M is analytic and g belongs to the ring $\mathcal{O}(M)$ of analytic functions on M, then g_x can be identified with $T_x g$. A consequence of this fact is that for any ideal $\mathfrak{a} \subset \mathcal{O}(M)$, $\mathfrak{a}^* = \tilde{\mathfrak{a}}$, where the two operations on $\mathfrak{a}$ are performed in the ring $\mathcal{O}(M)$ only. Unlike what happens in the smooth case, in the analytic setting $\mathfrak{a}^*$ is not the closure of $\mathfrak{a}$ in the compact-open topology; see [5] for details. Henceforth we will call *analytic-like* any ideal in $\mathcal{E}(M)$ satisfying $\mathfrak{a}^* = \tilde{\mathfrak{a}}$.

(3) Note that

$$\begin{aligned}\tilde{\mathfrak{a}} &= \{g \in \mathcal{E}(M) \mid \forall \text{ compact } K \subset M\ \exists\, h \in \mathcal{E}(M) \text{ s.t. } \mathcal{Z}(h) \cap K = \varnothing \text{ and } hg \in \mathfrak{a}\} \\ &= \{g \in \mathcal{E}(M) \mid \forall\, x \in M\ \exists\, h \in \mathcal{E}(M) \text{ s.t. } h(x) \neq 0 \text{ and } hg \in \mathfrak{a}\}.\end{aligned}$$

Proof. It is clear that both the second and third sets are subsets of $\tilde{\mathfrak{a}}$. It is also clear that the second set is a subset of the third. Therefore, the three-way equality reduces to proving that $\tilde{\mathfrak{a}}$ is a subset of the second set. Let $g \in \tilde{\mathfrak{a}}$. Given any compact subset K of M, let $x \in K$. It follows that $g_x \in \mathfrak{a}\mathcal{E}_x$, and in a suitable neighborhood U_x of x,

$$g = \alpha_1 f_1 + \cdots + \alpha_k f_k.$$

Take a positive semidefinite bump function φ such that $x \in V_x = \{\varphi > 0\} \subset U_x$. Then

$$\varphi g = (\varphi\alpha_1) f_1 + \cdots + (\varphi\alpha_k) f_k \in \mathfrak{a}.$$

The family $\{V_x\}_{x\in K}$ is an open cover of the compact set K. Take a finite subcover $V_{x_1}, \ldots, V_{x_j}$ with corresponding bump functions $\varphi_1, \ldots, \varphi_j$. Summing the expressions for $\varphi_1 g, \ldots, \varphi_j g$, we obtain that $(\varphi_1 + \cdots + \varphi_j)g$ is a finite sum of elements of $\mathfrak{a}$ with coefficients in $\mathcal{E}(M)$. As $(\varphi_1 + \cdots + \varphi_j)(y) \neq 0$ for all $y \in K$ by construction, we set $h = \varphi_1 + \cdots + \varphi_j$ and conclude that g is an element of

$$\{g \in \mathcal{E}(M) \mid \forall \text{ compact } K \subset M\ \exists\, h \in \mathcal{E}(M) \text{ s.t. } \mathcal{Z}(h) \cap K = \varnothing \text{ and } hg \in \mathfrak{a}\}$$

as needed. □

Definition 2.3. An ideal $\mathfrak{a} \subset \mathcal{E}(M)$ is a *Łojasiewicz ideal* if

(1) $\mathfrak{a}$ is generated by finitely many smooth functions $f_1, \ldots, f_l$;

(2) $\mathfrak{a}$ contains an element f with the property that for any compact $K \subset M$, there exist a constant c and an integer m depending on K such that $|f(x)| \geq c\, d(x, \mathcal{Z}(\mathfrak{a}))^m$ on an open neighborhood of K, i.e., f satisfies a *Łojasiewicz inequality* on each compact set.

Remark 2.4. It is well known that in the definition above one can take f to be the sum of squares of the generators $f_1^2 + \cdots + f_l^2$. This can be seen as follows: $f_1, \ldots, f_l$ cannot be simultaneously flat at any point in M; otherwise, f would be flat at some point of its zero set, hence it could not satisfy the inequality in the definition. So $f_1^2 + \cdots + f_l^2$ is nowhere flat and dominates $C\,|f|^2$ on every compact set of M for an appropriately chosen constant $C > 0$. It thus satisfies the required inequality with exponent $2m$. In what follows, we will replace condition (2) by:

(2') Let $\{f_1, \dots, f_l\}$ be generators of $\mathfrak{a}$. For any compact $K \subset M$ there is a constant C and a positive integer m such that $|f(x)| \geq c\, d(x, \mathcal{Z}(\mathfrak{a}))^m$ on an open neighborhood of K, where $f = f_1^2 + \cdots + f_l^2$.

Lemma 2.5. *Let $\mathfrak{a}$ be a Łojasiewicz ideal generated by $f_1, \dots, f_l$ and $f = f_1^2 + \cdots + f_l^2$. Let $g \in \mathcal{E}(M)$ be such that $\mathcal{Z}(g) \supset \mathcal{Z}(f) = \mathcal{Z}(\mathfrak{a})$. Then for any compact set $K \subset M$, there exist a constant c and a positive integer m such that $g^{2m} \leq cf$ on an open neighborhood of K. In particular, there exist an integer m and an element $a \in \mathfrak{a}$ such that $g^{2m} \leq |a|$ on an open neighborhood of K.*

Proof. Let $X = \mathcal{Z}(\mathfrak{a})$, and fix a compact set $K \subset M$. Now let U be an open set such that $U \supset K$, $\overline{U}$ is compact, and $\overline{U} \subset M$. Then for $x, y \in U$ close enough to each other, $|g(x) - g(y)| \leq c_1\, d(x, y)$ holds by the mean value theorem, where c_1 is a suitable positive constant related to maxima of norms of first order partial derivatives of g on $\overline{U}$. Now $d(x, X) = \inf_{y \in X} d(x, y)$, so since g vanishes on X, we obtain $|g(x)| \leq c_1\, d(x, X)$ for $x \in U$ close enough to X. Since K is compact, we can find finitely many open sets $V_i \subset U$, where the previous inequality holds and such that $X \cap K \subset V = \cup V_i$. Therefore, we have $|g(x)| \leq c_1\, d(x, X)$ on V. Let W be an open neighborhood of $X \cap K$ such that its closure $\overline{W}$ satisfies $\overline{W} \subset V$. Set $H = \overline{U} \setminus W$, which is compact. Let $\min_{x \in H} d(x, X) = \alpha > 0$, $\sup_{\overline{U}} |g| = A$, and $c_2 = A/\alpha$. Hence for $x \in H$, one gets

$$|g(x)| \leq c_2\, d(x, X).$$

Let $c_3 = \max\{c_1, c_2\}$. We now have $|g(x)| \leq c_3\, d(x, X)$ on $\overline{U}$ and hence $g(x)^{2m} \leq c_3^{2m} d(x, X)^{2m}$ for any m. Since $\mathfrak{a}$ is Łojasiewicz, in a neighborhood of the compact set $\overline{U}$, one has the inequality $c\, d(x, X)^{2m} \leq f(x)$. Combining these inequalities, one gets $g^{2m} \leq c_4 f$ in an open neighborhood of K, where $c_4 = c_3^{2m}/c$. □

Lemma 2.5 motivates us to globalize the definition of the Łojasiewicz radical.

Definition 2.6. The *Łojasiewicz radical* of an ideal $\mathfrak{a} \in \mathcal{E}(M)$ is given by

$$\sqrt[Ł]{\mathfrak{a}} := \{g \in \mathcal{E}(M) \mid \exists\, f \in \mathfrak{a} \text{ and } m \geq 1 \text{ such that } f - g^{2m} \geq 0\}.$$

It is not hard to verify that $\sqrt[Ł]{\mathfrak{a}}$ is a radical real ideal for any ideal $\mathfrak{a}$.

We can now prove our main result:

Theorem 2.7. *Let $\mathfrak{a} \subset \mathcal{E}(M)$ be a Łojasiewicz ideal. Then*

- *$\sqrt[Ł]{\mathfrak{a}}$ is analytic-like, i.e., $\widetilde{\sqrt[Ł]{\mathfrak{a}}} = \overline{\sqrt[Ł]{\mathfrak{a}}}$.*
- *$\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \overline{\sqrt[Ł]{\mathfrak{a}}}$.*

Proof. Note that for any ideal $\mathfrak{b}$ we have:

- $\overline{\mathfrak{b}} \subset \mathcal{I}(\mathcal{Z}(\mathfrak{b}))$. Indeed, $g \in \overline{\mathfrak{b}}$ implies $T_x(g) \in T_x(\mathfrak{b})$ for all $x \in M$. Hence if $x \in \mathcal{Z}(\mathfrak{b})$, then $T_x(g)$ has order at least 1 because $T_x(\mathfrak{b})$ is contained in the maximal ideal of the ring of formal power series at x. Therefore, $g(x) = 0$.
- $\mathcal{Z}(\sqrt[Ł]{\mathfrak{b}}) = \mathcal{Z}(\mathfrak{b})$.

We thus have $\widetilde{\sqrt[\text{Ł}]{\mathfrak{a}}} \subset \overline{\sqrt[\text{Ł}]{\mathfrak{a}}} \subset \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$. Hence both assertions in the statement will be proved if we show $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) \subset \widetilde{\sqrt[\text{Ł}]{\mathfrak{a}}}$.

Take $g \in \mathcal{I}(\mathcal{Z}(\mathfrak{a}))$, and let $f = f_1^2 + \cdots + f_l^2$, where $f_1, \ldots, f_l$ are generators of $\mathfrak{a}$. Let K be a compact set in M. By Lemma 2.5, $g^{2m} \leq cf$ on a neighborhood of K. Let $\varphi_K \in \mathcal{E}(M)$ be a nonnegative function taking the value 1 on K and the value 0 outside the neighborhood where the inequality $g^{2m} \leq cf$ holds. Hence $(\varphi_K g)^{2m} \leq cf$ on the whole of M, which means $\varphi_K g \in \sqrt[\text{Ł}]{\mathfrak{a}}$. By Remark 2.2 (3), we are done. □

3. Weakly Łojasiewicz ideals

Examining the proofs above, we see that the main ingredient is the existence of a function $f \in \mathfrak{a}$ that is the sum of the squares of the generators and has the same zero set as $\mathfrak{a}$, making $\mathfrak{a}$ Łojasiewicz. In Lemma 3.2 below, we construct a function with this property for a more general class of ideals.

Definition 3.1. An ideal $\mathfrak{a} \subset \mathcal{E}(M)$ is *weakly Łojasiewicz* if

(1) $\mathfrak{a}$ is locally finitely generated, i.e., for any $x \in M$ there exist finitely many elements in $\mathfrak{a}$ generating $\mathfrak{a}\,\mathcal{E}(U)$, where U is a suitable neighborhood of x;

(2) There exists an element $f \in \tilde{\mathfrak{a}}$ such that for any compact $K \subset M$, there exist a constant c and an exponent m such that $|f(x)| \geq c\, d(x, \mathcal{Z}(\mathfrak{a}))^m$.

Lemma 3.2. *Let $\mathfrak{a}$ be a weakly Łojasiewicz ideal. Then there exists $f \in \tilde{\mathfrak{a}}$ satisfying property (2) in Definition 3.1 such that $\mathcal{Z}(f) = \mathcal{Z}(\mathfrak{a})$. Moreover, for any compact set $K \subset M$, there exists a neighborhood U of K such that the restriction of f to U belongs to $\mathfrak{a}\,\mathcal{E}(U)$.*

Proof. Since $\mathfrak{a}$ is locally finitely generated, we can assume it is globally generated by countably many smooth functions $\{f_j\}_{j>0}$. Let h be a smooth function satisfying property (2) of Definition 3.1. Since $h_x \in \mathfrak{a}\mathcal{E}_x$, for any $x \in M$, there is l_x such that $h_x = \sum_{j=1}^{l_x} a_{jx} f_j$ and this equality holds in a neighborhood U_x of x. Hence, if $K \subset M$ is a compact set, there exist finitely many points $x_1, \ldots, x_s$ such that $K \subset U_{x_1} \cup \cdots \cup U_{x_s}$. Take $l = \max_i \{l_{x_i}\}$, and let $\{\varphi_i\}$ be a smooth partition of unity subordinate to the covering $U = U_{x_1} \cup \cdots \cup U_{x_s}$. Then

$$h = \Big(\sum_i \varphi_i\Big) h = \sum_i \varphi_i \Big(\sum_{j=1}^{l} a_{jx_i} f_j\Big) = \sum_{j=1}^{l} \Big(\sum_i \varphi_i a_{jx_i}\Big) f_j.$$

This shows that h belongs to $\mathfrak{a}\,\mathcal{E}(U)$ and that the latter is a Łojasiewicz ideal.

Next, take an exhaustion of M by compact sets $\{K_j\}_{j>0}$ such that $K_j \subset \operatorname{Int} K_{j+1}$ for every $j \geq 1$. We can assume that $\mathfrak{a}$ is generated on a neighborhood of K_j by $f_i, \ldots, f_{i_j}$. Consider the open locally finite covering of M given by $\{U_j = \operatorname{Int} K_{j+1} \setminus K_{j-2}\}_{j\geq 1}$, where $K_{-1} = K_0 = \emptyset$. Let $\{\alpha_j\}_{j\geq 1}$ be a collection of smooth functions $\alpha_j : M \to [0,1]$ satisfying that $\alpha_j = 1$ on $K_j \setminus \operatorname{Int} K_{j-1}$ and $\operatorname{supp}(\alpha_j) \subset U_j$ for all $j \geq 1$. Note that $\alpha_j f_1, \ldots, \alpha_j f_{i_j}$ still generate $\mathfrak{a}$ in a neighborhood $V_j \subset K_{j+1}$ of $K_j \setminus \operatorname{Int} K_{j-1}$ and that $\mathfrak{a}$ is a Łojasiewicz ideal on V_j.

Now put

$$f = \sum_{j=1}^{\infty} \alpha_j \Big(\sum_{i=1}^{i_j} f_i^2 \Big).$$

We get:

1. $f \in \mathcal{E}(M)$. Indeed, for any $x \in K_j \setminus \operatorname{Int} K_{j-1} \subset M$, f is the sum of three summands of finitely many functions,
$$f = \alpha_{j-1} \Big(\sum_{i=1}^{i_{j-1}} f_i^2 \Big) + \alpha_j \Big(\sum_{i=1}^{i_j} f_i^2 \Big) + \alpha_{j+1} \Big(\sum_{i=1}^{i_{j+1}} f_i^2 \Big).$$
2. $f \in \tilde{\mathfrak{a}}$. Indeed, for $x \in K_j \setminus K_{j-1}$, the germ f_x belongs to the ideal generated by $f_1, \ldots, f_{i_{j+1}}$, which generate the ideal $\mathfrak{a}$ on V_{j+1}.
3. $f \geq 0$ and $\mathcal{Z}(f) = \mathcal{Z}(\mathfrak{a})$ since this is true locally.
4. f satisfies the inequality of Definition 3.1. Indeed, if $K \subset M$ is a compact set, then $K \subset K_j$ for some j. Hence f belongs to the restriction of $\mathfrak{a}$ to $V_1 \cup \cdots \cup V_j$, which is a Łojasiewicz ideal, and $f|_{V_1 \cup \cdots \cup V_j}$ is a finite combination with positive coefficients of the squares of a family of generators of $\mathfrak{a}\mathcal{E}(V_1 \cup \cdots \cup V_j)$.

□

Remark 3.3. Statement 4 at the end of the previous proof implies that a weakly Łojasiewicz ideal is *locally Łojasiewicz* in the sense that for any point $x \in M$, there is an open neighborhood V of x such that the restriction of $\mathfrak{a}$ to V is a Łojasiewicz ideal. It is not hard to prove the converse.

Next, note that similar statements to those of Lemma 2.5 and Theorem 2.7 hold true for a weakly Łojasiewicz ideal $\mathfrak{a}$ with an analogous proof, simply replacing $f_1^2 + \cdots + f_l^2$ by the function f constructed in Lemma 3.2 above, provided the ideal $\mathfrak{a}$ is saturated as $f \in \tilde{\mathfrak{a}}$. Hence, we obtain:

Theorem 3.4. *Let $\mathfrak{a} \subset \mathcal{E}(M)$ be a weakly Łojasiewicz ideal. Then $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt[L]{\tilde{\mathfrak{a}}}}$. In particular, if $\mathfrak{a}$ is saturated, then $\mathcal{I}(\mathcal{Z}(\mathfrak{a})) = \widetilde{\sqrt[L]{\mathfrak{a}}}$.*

4. Consequences

4.1. Resolving a modification of the Bochnak conjecture

We now want to relate the notion of being Łojasiewicz with convexity.

We say that an ideal $\mathfrak{a}$ of $\mathcal{E}(M)$ is *convex* if each $g \in \mathcal{E}(M)$ satisfying $|g| \leq f$ for some $f \in \mathfrak{a}$ belongs to $\mathfrak{a}$. In particular, the Łojasiewicz radical $\sqrt[L]{\mathfrak{a}}$ of an ideal $\mathfrak{a}$ of $\mathcal{E}(M)$ is a radical convex ideal.

Moreover, we define the *convex hull* $\mathfrak{g}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of $\mathcal{E}(M)$ by

$$\mathfrak{g}(\mathfrak{a}) := \{g \in \mathcal{E}(M) \mid \exists f \in \mathfrak{a} \text{ such that } |g| \leq f\}.$$

Note that $\mathfrak{g}(\mathfrak{a})$ is the smallest convex ideal of $\mathcal{E}(M)$ that contains $\mathfrak{a}$ and

$$\sqrt[\text{Ł}]{\mathfrak{a}} = \sqrt{\mathfrak{g}(\mathfrak{a})}.$$

Hence if $\mathfrak{a}$ is convex and radical, it coincides with its Łojasiewicz radical, and we immediately get:

Corollary 4.1. *If the ideal $\mathfrak{a} \subset \mathcal{E}(M)$ is a (weakly) Łojasiewicz ideal, the following are equivalent:*

(1) *$\mathfrak{a}$ has the zero property;*

(2) *$\mathfrak{a}$ is closed, convex, and radical.*

4.2. Recovering the Bochnak and Adkins–Leahy Nullstellensatz results

To compare our results with those of Bochnak and Adkins–Leahy, we have to relate Łojasiewicz radicals with real radicals of ideals generated by analytic functions. So assume M is an analytic manifold and the ideal $\mathfrak{a} \subset \mathcal{E}(M)$ is generated by analytic functions. It follows that the zero set of $\mathfrak{a}$ is a global analytic set X and $\mathfrak{a}$ is locally finitely generated. Furthermore, there exists an analytic function $f \in \tilde{\mathfrak{a}}$ whose zero set is X, and so $\mathfrak{a}$ is a weakly Łojasiewicz ideal (see [1]).

Theorem 4.2. *Let M be an analytic manifold, and let $\mathfrak{a} \in \mathcal{O}(M)$ be an ideal of real analytic functions. Then*

$$\widetilde{\sqrt[\text{Ł}]{\widetilde{\mathfrak{a}\mathcal{E}(M)}}} = \overline{\sqrt[r]{\mathfrak{a}\mathcal{E}(M)}}.$$

Proof. Let $X = \mathcal{Z}(\mathfrak{a})$, and consider the ideal $\sqrt[\text{Ł}]{\mathfrak{a}} \subset \mathcal{O}(M)$. We have the following:

- $(\sqrt[\text{Ł}]{\mathfrak{a}})\mathcal{O}_x \subset \sqrt[\text{Ł}]{\mathfrak{a}\mathcal{O}_x} = \sqrt[\text{r}]{\mathfrak{a}\mathcal{O}_x}$. Indeed, if $g \in (\sqrt[\text{Ł}]{\mathfrak{a}})\mathcal{O}_x$, then $g = \sum_i h_i a_i$, where $h_i \in \mathcal{O}_x$ and $a_i \in \sqrt[\text{Ł}]{\mathfrak{a}}$. Hence $a_i^{2m_i} \leq c_i f$, for some $f \in \mathfrak{a}$, and so $(h_i a_i)^{2m_i} \leq c_i' f$, for $f \in \mathfrak{a}$, which means $h_i a_i \in \sqrt[\text{Ł}]{\mathfrak{a}\mathcal{O}_x}$. So $g \in \sqrt[\text{Ł}]{\mathfrak{a}\mathcal{O}_x}$. Since the Łojasiewicz radical contains the real radical and $\mathcal{Z}(\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{O}_x}) = \mathcal{Z}(\sqrt[\text{r}]{\mathfrak{a}\mathcal{O}_x})$ the last equality is the Risler Nullstellensatz in the ring of germs of analytic functions; see [9].

- $(\sqrt[\text{Ł}]{\mathfrak{a}})\mathcal{E}_x \subset \sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}_x}$ by the same argument as before.

- $\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}_x} = (\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}(M)})_x$. Only the inclusion $\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}_x} \subset (\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}(M)})_x$ requires some justification. Indeed, if $\varphi \in \mathfrak{a}\mathcal{E}_x$, then $\varphi = \sum \varphi_i a_i$, for $\varphi_i \in \mathcal{E}_x$ and $a_i \in \mathfrak{a}$. This holds true in an open neighborhood U of x. Take a smooth bump function ψ such that $\psi = 1$ in a smaller neighborhood and its support is contained in U. Then $\psi\varphi = \sum(\psi\varphi_i)a_i \in \mathfrak{a}\,\mathcal{E}(M)$, and its germ at x is precisely φ. Thus, taking Łojasiewicz radicals and localizing at x, we have $\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}_x} \subset (\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}(M)})_x$ as needed.

We have obtained

$$(\sqrt[\text{Ł}]{\mathfrak{a}})\mathcal{E}_x \subset \sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}_x} = (\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}(M)})_x.$$

Now apply the Taylor homomorphism at x to obtain

$$T_x(\sqrt[\text{Ł}]{\mathfrak{a}}\mathcal{E}_x) = (\sqrt[\text{Ł}]{\mathfrak{a}})_x\mathcal{F}_x \subset T_x(\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}(M)})_x \subset \sqrt[\text{r}]{\mathfrak{a}\mathcal{O}_x}\mathcal{F}_x.$$

It is worth noting here that we identify the Taylor series of analytic functions with the corresponding germs. The last inclusion holds because the elements of $(\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}(M)})_x$ vanish on X_x. Therefore, their Taylor series belong to $\mathcal{I}^{\mathcal{F}_x}(X_x) = \sqrt[\text{r}]{\mathfrak{a}\mathcal{O}_x}\,\mathcal{F}_x$ by Malgrange's theorem (see Theorem 3.5 in page 90 of [7]). Arguing as in [3],

$$\sqrt[\text{r}]{\mathfrak{a}\mathcal{O}_x} \subset \sqrt[\text{r}]{\mathfrak{a}\mathcal{E}_x} = (\sqrt[\text{r}]{\mathfrak{a}\mathcal{E}(M)})_x.$$

As a result, the last inclusion is an equality.

We have now finished making preliminary observations and are ready to prove the statement. Consider

$$g \in \widetilde{\sqrt[\text{Ł}]{\widetilde{\mathfrak{a}\mathcal{E}(M)}}}.$$

For any compact set $K \subset M$, there is an open neighborhood U of K such that g belongs to $\sqrt[\text{Ł}]{\widetilde{\mathfrak{a}\mathcal{E}(U)}}$. It follows that $g^{2m} \leq cf$ for some $f \in \widetilde{\mathfrak{a}\mathcal{E}(U)}$, $c > 0$, and $m \geq 1$. In turn, on a smaller neighborhood $V \subset U$ of K, we get $f \in \mathfrak{a}\,\mathcal{E}(V)$. This means that for any $x \in M$, the germ g_x belongs to $\sqrt[\text{Ł}]{\mathfrak{a}\,\mathcal{E}_x}$, which equals $(\sqrt[\text{Ł}]{\mathfrak{a}\,\mathcal{E}(M)})_x$. Applying what was stated before, we see that

$$T_x g \in T_x(\sqrt[\text{Ł}]{\mathfrak{a}\mathcal{E}(M)}) \subset (\sqrt[\text{r}]{\mathfrak{a}\mathcal{E}(M)})_x,$$

which implies $g_x \in (\sqrt[\text{r}]{\mathfrak{a}\mathcal{E}(M)})_x$ for all $x \in M$ and so $g \in \overline{\sqrt[\text{r}]{\mathfrak{a}\mathcal{E}(M)}}$. The converse inclusion comes from the fact that the Łojasiewicz radical is analytic-like and bigger than the real one by Theorem 2.7. □

As a consequence of Theorems 2.7 and 4.2, we recover the result of Adkins and Leahy in [3]. Concerning Bochnak's result, note that a finitely generated analytic ideal $\mathfrak{a} \subset \mathcal{E}(M)$ is closed and if it is real, it coincides with its real radical. Hence it has the zero property.

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Francesca Acquistapace: Dipartimento di Matematica, Università degli Studi di Pisa, Largo Bruno Pontecorvo, 5, 56127 Pisa, Italy.
E-mail: acquistf@dm.unipi.it

Fabrizio Broglia: Dipartimento di Matematica, Università degli Studi di Pisa, Largo Bruno Pontecorvo, 5, 56127 Pisa, Italy.
E-mail: broglia@dm.unipi.it

Andreea Nicoara: Department of Mathematics, University of Pennsylvania, 209 South 33$^{\text{rd}}$ St., Philadelphia, PA 19104, USA.
E-mail: anicoara@math.upenn.edu