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On the product of two π-decomposable groups

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Abstract. The aim of this paper is to prove the following result: let π be a set of odd primes. If the finite group $G = AB$ is a product of two π-decomposable subgroups $A = \mathrm{O}_\pi(A) \times \mathrm{O}_{\pi'}(A)$ and $B = \mathrm{O}_\pi(B) \times \mathrm{O}_{\pi'}(B)$, then $\mathrm{O}_\pi(A)\mathrm{O}_\pi(B) = \mathrm{O}_\pi(B)\mathrm{O}_\pi(A)$ and this is a Hall π-subgroup of G.

1. Introduction and statement of the main result

All groups considered here are finite. Within the framework of factorized groups, a well known theorem by Kegel and Wielandt yields the solubility of a group which is the product of two nilpotent subgroups. This theorem has been the starting point for a number of results on factorized groups, in particular, by considering the case when one of the factors is π-decomposable for a set π of primes. A group X is said to be π-*decomposable* if $X = X_\pi \times X_{\pi'}$ is the direct product of a π-subgroup X_π and a π'-subgroup $X_{\pi'}$, where π' stands for the complement of π in the set of prime numbers. For any set σ of primes, X_σ will denote a Hall σ-subgroup of a group X. For instance, different extensions of the Kegel and Wielandt theorem for products of a 2-decomposable group and a group of odd order, with coprime orders, were obtained by Berkovich [5], Arad and Chillag [3], Rowley [20] and Kazarin [13].

The present paper contributes to this investigation. More precisely we complete the study of products of π-decomposable groups carried out in [14] and [15] (see also [17]) and prove the following general result:

Main Theorem. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.*

This result was stated as a conjecture in [15], [16], and [17], and also was mentioned in [4]. In [14] and [15] we proved several particular cases, namely, when one of the factors is a π-group (Theorem 1 and Lemma 1 in [14]), the factors are soluble

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groups (Theorem 2 in [15]), or when the factors have coprime orders (Proposition 1 in [15]). These results substantially extend the above-mentioned previously known results on products of 2-decomposable groups. Moreover, in [14] and [15] we also obtained some π-separability criteria for products of π-decomposable groups.

The next example, which appears in [14], shows that analogous results do not hold in general if the set π of primes contains the prime 2, although some related positive results were obtained in this case in [15]. Other examples in [14] and [15] give insight into occurring phenomena.

Example. Let G be a group isomorphic to $\mathrm{L}_2(2^n)$ where n is a positive integer such that 2^n+1 is divisible by two different primes (this happens if $n \neq 3$ and 2^n+1 is not a Fermat prime). Let $q = 2^n$. Then $G = AB$ where $A \cong C_{q+1}$ is a cyclic group of order $q+1$ and $B = N_G(G_2)$, with G_2 a Sylow 2-subgroup of G. Let r be a prime dividing $q+1$ and take $\pi = \pi(N_G(G_2)) \cup \{r\}$. Clearly, $2 \in \pi$. Then $A = A_\pi \times A_{\pi'}$ and B is a π-group, but $A_\pi B$ is not a subgroup.

On the other hand, the paper [17] describes completely a minimal counterexample of our main theorem. In particular, it is shown that such a minimal counterexample has to be an almost simple group. Hence, after providing in Section 2 some necessary preliminaries, mainly related to finite simple groups, we will prove in Section 3 the main theorem by carrying out a case-by-case analysis of the simple groups occurring as the socle of the minimal counterexample, leading to a final contradiction.

If n is an integer and p a prime, n_p will denote the largest power of p dividing n and $\pi(n)$ the set of prime divisors of n. In particular, for the order $|G|$ of a group G we set $\pi(G) = \pi(|G|)$. Also, $\mathrm{Syl}_p(G)$ will denote the set of Sylow p-subgroups of G.

2. Preliminaries

The following result on factorized groups will be used throughout the paper, usually without further reference.

Lemma 1 (Lemma 1.3.1 in [2]). *Let the group $G = AB$ be the product of the subgroups A and B. If x and y are elements of G, then $G = A^x B^y$. Moreover, there exists an element z of G such that $A^x = A^z$ and $B^y = B^z$.*

The following basic lemma will also be used.

Lemma 2. *If G is a soluble group with an abelian Sylow r-subgroup R, for a prime r, then $G = \mathrm{O}_{r'}(G) N_G(R)$.*

Proof. It is well known (see, for example, Theorem 6.3.2 in [9]) that in a soluble group $C_G(R \cap \mathrm{O}_{r',r}(G)) \leq \mathrm{O}_{r',r}(G)$, where $\mathrm{O}_{r',r}(G)$ is the r-nilpotent radical of G. Then, when R is abelian, $R\mathrm{O}_{r'}(G)$ is a normal subgroup of G and the result follows by applying the Frattini argument. □

Next we record some arithmetical lemmas, that will be applied later in the paper.

Lemma 3 (Zsigmondy, [21]). *Let q and n be integers, $q, n \geq 2$. A prime number r is called primitive with respect to the pair (q, n) (or a primitive prime divisor of $q^n - 1$) if r divides $q^n - 1$ but r does not divide $q^i - 1$ for $i < n$. Then:*

(1) *There exists a primitive prime divisor of $q^n - 1$ unless $n = 2$ and q is a Mersenne prime or $(q, n) = (2, 6)$.*

(2) *If the prime r is a primitive prime divisor of $q^n - 1$, then $r - 1 \equiv 0 \pmod{n}$. In particular, $r \geq n + 1$.*

Lemma 4. *Let q and n be integers, $q, n \geq 2$. If an odd prime t divides $q^n + 1$ and is not primitive with respect to the pair $(q, 2n)$, then there exists j dividing n, $j \neq n$, such that t divides $q^j + 1$.*

Proof. Assume that t divides $q^n + 1$ and is not a primitive prime divisor of $q^{2n} - 1$. Then there exists $j < 2n$ such that t is a primitive prime divisor of $q^j - 1$. Since $(q^{2n} - 1, q^j - 1) = q^{(2n,j)} - 1$, it is clear that j divides $2n$. Assume first that j is odd. Since j divides $2n$, it follows that j divides n. Since $q^j \equiv 1 \pmod{t}$, this implies that $q^n \equiv 1 \pmod{t}$. But then t divides $(q^n - 1, q^n + 1)$ and so $t = 2$, a contradiction. So we may assume that j is even. Then $j = 2j_0$, for some j_0 such that j_0 divides n, $j_0 \neq n$. By the choice of j it follows that t divides $q^{j_0} + 1$ and we are done. □

2.1. Preliminaries on finite simple groups

According to the classification theorem, the finite nonabelian simple groups occur in the following families: the alternating groups A_n, with $n \geq 5$; the finite simple groups of Lie type (classical and exceptional); and the 26 sporadic groups. The book [11], by Gorenstein, Lyons and Solomon, is a general reference containing the background on finite simple groups necessary for the paper. In particular, we will make extensive use throughout the paper of the detailed knowledge on the orders of the finite simple groups and their automorphisms groups. This can be found in [11] and also in the Atlas of Finite Groups [6].

On the other hand, we will use information about the maximal factorizations of the finite simple groups and their automorphism groups from [19]. Also in this reference the orders of such groups are collected nicely in Table 2.1.

In this paper we will treat mainly the classical simple groups of Lie type (for the exceptional groups we will use a different strategy (see Lemma 11 below)). The definition and basic properties of such groups can be found in Carter's book [7] and also in Chapters 2, 3, and 4 of [11]. Moreover, the survey [18] is a good source for this topic. We collect next the notation and fundamental facts that will be used later to prove our main theorem.

Let $L = \mathrm{G}(q)$ be a classical finite group of Lie type over a finite field of characteristic p, where q is a power of p. The base field will in most cases be $\mathrm{GF}(q)$, the finite field of q elements, except for some twisted groups (see Chapter 3 of [6], or Section 14.1 in [7]).

Denote by Φ the root system corresponding to the group L, let $\Pi = \{r_1, \ldots, r_l\}$ be the set of all fundamental roots, and let $\Phi^+ \supseteq \Pi$ be the set of all positive roots. The integer l is called the *Lie rank* of L. Denote by X_r the root subgroup corresponding to the root r. In the case where L is a group of untwisted type ($A_l(q), B_l(q), C_l(q)$, or $D_l(q)$), there holds $X_r = \{x_r(t) | t \in \mathrm{GF}(q)\}$. In the remaining cases (twisted groups of types ${}^2A_l(q)$ and ${}^2D_l(q)$) we use the description of the root subgroups of the corresponding groups in Chapter 13 of [7]. The structure of such subgroups can also be found in Theorem 2.4.1 of [11].

Let U be the unipotent subgroup $\langle X_r \mid r \in \Phi^+\rangle$ (a Sylow p-subgroup) of L. Let B be the *Borel subgroup* containing U, that is, the normalizer of U in L. Then we have that $B = UH$, where $U \cap H = 1$ and H is a *Cartan subgroup* of L. The normalizer of H in L contains a subgroup N such that $N/H \cong W$, the *Weyl group* of L (associated with Φ). The subgroups B and N form a so-called (B, N) *pair* with Weyl group W (see Sections 8.3 and 13.5 in [7] or Theorem 2.3.1 in [11]). Any subgroup which contains some conjugate of the Borel subgroup B is called a *parabolic subgroup*. A subgroup X of L is called p-*local* if $X = N_L(Q)$ for some nontrivial p-subgroup Q of L. For each $w \in W$, we choose a coset representative $n_w \in N$. We recall in addition the following properties:

- **(P1)** Each element $g \in L$ can be expressed in the form $g = bn_w u$ with $b \in B = UH$, $n_w \in N$ and $u \in U$ (see Theorem 8.4.3 and Proposition 13.5.3 in [7], or Theorem 2.3.5 in [11]).
- **(P2)** $U = \prod_{r\in\Phi^+} X_r$, where the product is taken over all positive roots in an arbitrary ordering (see [7], Theorems 5.3.3 and 13.6.1, or [11], Theorem 2.3.7).
- **(P3)** Any p-local subgroup of L is contained in some parabolic subgroup of L. Moreover, if Q is a nontrivial p-subgroup of L, then there exists a parabolic subgroup P such that $Q \le \mathrm{O}_p(P)$ and $N_L(Q) \le P$ (see Theorem 3.1.3 in [11]).
- **(P4)** The main properties of parabolic subgroups can be found in Section 2.6 of [11] and Section 8.3 of [7]. In particular, if P is a parabolic subgroup of L, then $C_L(\mathrm{O}_p(P)) \le \mathrm{O}_p(P)$ and $|\mathrm{O}_p(P)|$ is a power of q. Moreover, P has a *Levi decomposition* $P = \mathrm{O}_p(P)H\langle X_r, X_{-r} | r \in J\rangle$ for some set of fundamental roots $J \subseteq \Pi$ (see 2.6.5, 2.6.6 in [11] or Section 8.5 in [7]).
- **(P5)** The order of a Sylow p-subgroup of the centralizer of a p'-element in the simple group L of Lie rank at least 2 is at most $|U|q^{-2}$ (see Propositions 7-12 in [8]).

The structure of the automorphism groups of the groups of Lie type is described thoroughly in Section 2.5 of [11] and Chapter 12 of [7]. Moreover, we will also use the information about the centralizers of outer automorphisms of prime order of such groups which can be found in 9.1 of [10] (see also Chapter 4 of [11]).

The following results on simple groups of Lie type will be essential for the proof of our main theorem.

Lemma 5. *Let $L = \mathrm{G}(q)$ be a classical simple group of Lie type over the field $\mathrm{GF}(q)$ of characteristic p. Then $|\mathrm{Out}(L)|_p \le q$ and equality holds only when $q \in \{2, 3, 4\}$. Moreover, if $q = 3$, the only case in which $|\mathrm{Out}(L)|_p = q$ is possible is $L \cong \mathrm{P}\Omega_8^+(q)$. In particular, $|\mathrm{Out}(L)|_p < q^2$ for any classical simple group of Lie type.*

Proof. In Table 5 of [6] (see also [19], Table 2.1.A), we find the order of $\mathrm{Out}(L)$ when L is a classical group of Lie type. From this table it follows that $|\mathrm{Out}(L)|_p \leq 2\log_p(q)$ if $p = 2$ and $|\mathrm{Out}(L)|_p \leq \log_p(q)$ if $p \neq 2$ and $L \ncong \mathrm{P}\Omega_8^+(q)$ with $q = p = 3$. Hence $|\mathrm{Out}(L)|_p \leq q$ and equality holds only in the asserted cases. □

Lemma 6. *Let $L = \mathrm{G}(q)$ be a classical simple group of Lie type over the field $\mathrm{GF}(q)$ of characteristic p of Lie rank at least 2. Let U be a Sylow p-subgroup of L and let $S \neq 1$ be a subgroup of U such that $|U : S| < q^2$. Then $C_L(S)$ is a p-group.*

Moreover, if $L \leq G \leq \mathrm{Aut}(L)$, then $C_G(S)$ is a p-group.

Proof. We will use the notation and properties (P1)–(P5) of the simple groups of Lie type described above.

Take a subgroup $S \neq 1$ of U such that $|U : S| < q^2$ and assume that $C_L(S)$ is not a p-group. Then there exists a p'-element g of prime order r in $C_L(S)$. We claim first that r divides $q^2 - 1$.

By (P3), there exists a parabolic subgroup P of L such that $S \leq \mathrm{O}_p(P)$ and $N_L(S) \leq P$. Without loss of generality we may assume that $B \leq P$. Let $D = \mathrm{O}_p(P)$. By [7], $\pi(H) \subseteq \pi(q^2 - 1)$. Hence, by (P1), $g = bn_w u \in P$, where $b \in B$, $1 \neq w \in W$, and $u \in U$. Now the fact that $|U : S| < q^2$ and (P4) imply that $P = B \cup Bn_wB = DH\langle X_\gamma, X_{-\gamma}\rangle$ for some fundamental root $\gamma \in \Pi$, $w = w_\gamma$ and $|X_\gamma| = q$. Then we obtain that the subgroup $\langle X_\gamma, X_{-\gamma}\rangle$ is isomorphic to $\mathrm{SL}_2(q)$ or $\mathrm{L}_2(q)$ and hence r divides $|\mathrm{SL}_2(q)|$. Therefore, r divides $q^2 - 1$. Applying (P5) we get a contradiction which allows us to deduce that $C_L(S)$ is a p-group.

Now assume that $L \leq G \leq \mathrm{Aut}(L)$. Using the information about the centralizers of outer automorphisms of prime order of groups of Lie type in 9.1 of [10] (see also Chapter 4 of [11]), it can be deduced that $C_G(S)$ is also a p-group. □

We will need later the following lemma on sporadic simple groups.

Lemma 7. *Assume that N is a sporadic simple group which is isomorphic to one in the set $\{\mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{HS}, \mathrm{He}, \mathrm{Ru}, \mathrm{Suz}, \mathrm{Fi}_{22}, \mathrm{Co}_1\}$. If s is the largest prime dividing $|N|$, then $C_{\mathrm{Aut}(N)}(S)$ is an s-group, for any $S \in \mathrm{Syl}_s(N)$.*

Proof. The result follows from a case-by-case analysis of the orders of the centralizers of Sylow s-subgroups (see [6] or [11] for the details). □

3. Proof of the main theorem

In this section we assume that G is a counterexample of minimal order to our main theorem. The main result in [17] gives a precise description of the structure of such a group:

Theorem 1 (Theorem 3 in [17]). *Let π be a set of odd primes. Assume that the group $G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ and G is a counterexample of minimal order to the assertion $A_\pi B_\pi = B_\pi A_\pi$.*

Then G has a unique minimal normal subgroup N, which is a nonabelian simple group, so that $N \trianglelefteq G \leq \mathrm{Aut}(N)$.

Moreover, the following properties hold:

(i) $G = AN = BN = AB$*; in particular,* $|N||A \cap B| = |G/N||N \cap A||N \cap B|$.

(ii) $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$*,* $A_{\pi'} \cap B_{\pi'} = 1$*, and* $A \cap B$ *is a* π*-group.*

(iii) *Neither* A *nor* B *is a* π*-group or a* π'*-group.*

(iv) $\pi(G) = \pi(N) \geq 5$.

(v) *If, in addition,* N *is a simple group of Lie type of characteristic* p *and* $p \notin \pi$*, then* $A \cap B = 1$.

For such a group G and its unique minimal normal subgroup N we have the following results.

Lemma 8. *Assume that* $S \leq X$ *and* S *is an* s*-group for* $X \in \{A, B\}$ *and a prime number* $s \in \sigma$*, with* $\sigma \in \{\pi, \pi'\}$*. Then* $\pi(|X : C_X(S)|) \subseteq \sigma$*. In particular,* $C_X(S)$ *is not an* s*-group.*

Proof. The first part is clear since $X_{\sigma'} \leq C_X(S)$. Consequently, if $C_X(S)$ were an s-group, X would be a σ-group, a contradiction. □

Lemma 9. *N is not a sporadic simple group.*

Proof. By Theorem C in [19], if N is a sporadic simple group, $N \trianglelefteq G \leq \mathrm{Aut}(N)$ and G is factorized, we have that

$$N \in \{\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{J}_2, \mathrm{HS}, \mathrm{He}, \mathrm{Ru}, \mathrm{Suz}, \mathrm{Fi}_{22}, \mathrm{Co}_1\}.$$

Note also that by Theorem 1(iv) the cases $N \cong \mathrm{M}_{11}$, $N \cong \mathrm{M}_{12}$, and $N \cong \mathrm{J}_2$ are impossible. Then Lemmas 7 and 8 provide the contradiction. □

Lemma 10. *N is not an alternating group of degree $n \geq 5$.*

Proof. First note that, by Theorem 1(iv), we may assume that $N \cong A_n$ with $n \geq 11$. By Theorem D in [19], if $N \trianglelefteq G \leq \mathrm{Aut}(N)$, the only factorizations $G = AB$ where A and B are subgroups of G not containing N satisfy that $A_{n-k} \triangleleft A \leq S_{n-k} \times S_k$ for some $1 \leq k \leq 5$. Since A_{n-k} is a simple group, because $n - k \geq 5$, and $2 \in \pi(A_{n-k})$, it follows that A_{n-k} is a π'-group. Then $A \leq S_{n-k} \times S_k$ is also a π'-group, which is a contradiction. □

Lemma 11. *N is not an exceptional group of Lie type.*

Proof. By Theorem B in [19], if N is an exceptional group of Lie type, $N \trianglelefteq G \leq \mathrm{Aut}(N)$ and G is factorized, then

$$N \in \{G_2(q), q = 3^c;\ F_4(q), q = 2^c; G_2(4)\}.$$

We check next that each of the possibilities for the group N leads to a contradiction. Recall that $\pi(G) = \pi(N)$.

Case $N \cong G_2(4)$. In this case $|N| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ and $|\mathrm{Out}(N)| = 2$. Since a Sylow 13-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$ (see [6]), we get a contradiction by Lemma 8.

Case $N \cong G_2(q)$, $q = 3^c$. In this case all possible factorizations $G = AB$ (not only the maximal ones) with subgroups A and B not containing N satisfy $A \cap N \in \{\mathrm{SL}_3(q), \mathrm{SL}_3(q).2\}$, either $B \cap N \in \{\mathrm{SU}_3(q), \mathrm{SU}_3(q).2\}$ or $B \cap N = {}^2G_2(q)$ in the case when c is odd, and $N = (A \cap N)(B \cap N)$. Since 2 divides $(|A \cap N|, |B \cap N|)$ and each of the subgroups has a Sylow 3-subgroup containing its centralizer in the corresponding subgroup, we deduce that all these are π'-groups and hence N is a π'-group, a contradiction.

Case $N \cong F_4(q)$, $q = 2^c$. In this case all possible factorizations $G = AB$ (not only the maximal ones) with subgroups A and B not containing N are as follows: $A \cap N = \mathrm{Sp}_8(q)$, $B \cap N \in \{{}^3D_4(q), {}^3D_4(q).3\}$, and $N = (A \cap N)(B \cap N)$. Since 2 divides $(|A \cap N|, |B \cap N|)$ and each of these subgroups has a Sylow 2-subgroup containing its centralizer in the corresponding subgroup, it follows that N is a π'-group, which is a contradiction. □

Henceforth we assume that $N = \mathrm{G}(q)$ *is a classical simple group of Lie type over a field* $\mathrm{GF}(q)$ *of prime characteristic* p*, with* $q = p^e$.

Lemma 12. *Assume that N is of Lie rank $l > 1$. Then*

$$(|A \cap N|, |B \cap N|) \equiv 0 \pmod{p}.$$

Proof. From the fact that $G = AN = BN = AB$, it follows that

$$\frac{|N|_p |A \cap B|_p}{|N \cap A|_p |N \cap B|_p} = |G/N|_p,$$

which divides $|\mathrm{Out}(N)|_p$. Suppose that $|B \cap N|$ is not divisible by p. It follows that $|N|_p / |N \cap A|_p$ divides $|G/N|_p$ and, in particular, $|N|_p/|N \cap A|_p \le |\mathrm{Out}(N)_p| \le q$, by Lemma 5. For $S \in \mathrm{Syl}_p(N \cap A)$ we deduce from Lemma 6 that $C_{\mathrm{G}}(S)$ is a p-group, and so we have a contradiction, by Lemma 8. □

Lemma 13. *Let $a \in X$ and $b \in Y$, for any $X, Y \in \{A, B\}$, be elements of prime orders $r = o(a)$ and $s = o(b)$, respectively (eventually $a = b$). Assume that $C_N(a)$ and $C_N(b)$ are p'-groups. Then:*

(i) *If $(|A \cap N|, |B \cap N|) \equiv 0 \pmod{p}$, then $\{p, s, r\} \subseteq \sigma$.*

(ii) *If, in addition, $a \in A$, $b \in B$, and $C_N(a)$ and $C_N(b)$ are soluble, then $\{p, s, r\} \subseteq \pi'$. In particular, $A \cap B = 1$.*

Proof. (i) Observe that here $p \in \pi(|X : C_X(a)|) \cap \pi(|X : C_X(b)|)$ and the conclusion follows from Lemma 8.

(ii) Assume now that $a \in A$, $b \in B$, and $C_N(a)$ and $C_N(b)$ are soluble. If $\{p, s, r\} \subseteq \pi$, then $A_{\pi'} \cap N \le C_N(a)$ and $B_{\pi'} \cap N \le C_N(B)$,and so $A_{\pi'} \cap N$ and

$B_{\pi'} \cap N$ are soluble groups. Since G/N is soluble, this means that $A_{\pi'}$ and $B_{\pi'}$ are soluble. Since $2 \notin \pi$, A_π and B_π are also soluble groups, and we conclude that both A and B are soluble, which contradicts Theorem 2 in [15]. Hence we have proved that $\{p, s, r\} \subseteq \pi'$. The assertion $A \cap B = 1$ follows from Theorem 1(v). □

Recall that q is a prime power, $q = p^e$, p a prime and e a positive integer. Also let $n \geq 3$ and $(q, n) \notin \{(2, 6), (4, 3)\}$. In the sequel we will denote by q_n any primitive prime divisor of $p^{en} - 1$, i.e. primitive with respect to the pair (p, ne) (so that $q_n \mid p^{en} - 1$ but $q_n \nmid p^i - 1$ for $i < en$). Note that if r is a primitive prime divisor of $q^{2k} - 1$ for some $k \geq 2$, then r divides $q^k + 1$.

Lemma 14. *For $N = \mathrm{G}(q)$ a classical group of Lie type of characteristic p and $q = p^e$, there exist primes r, $s \in \pi(N) \setminus \pi(G/N)$ and maximal tori T_1 and T_2 of N as stated in Table 1.*

Moreover, except for the case denoted $(\star)$ in Table 1, for any element $a \in N$ of order r and any element $b \in N$ of order s we may assume that $C_N(a) \leq T_1$ and $C_N(b) \leq T_2$, and these are abelian p'-groups.

On the other hand, there is neither a field automorphism nor a graph-field automorphism of N centralizing elements of N of order r or s (except for the triality automorphism in the case $\mathrm{P}\Omega_8^+(q)$).

Proof. This can be derived from the known facts about the maximal tori in these groups (see, for instance, [8]). Information about the centralizers of outer automorphisms of prime order of groups of Lie type can be found in [10], 9.1. □

Whenever Lemma 14 is applied, we will use the same notation for the primes r and s and for the elements $a \in N$ and $b \in N$. Since $|N||A \cap B| = |G/N||N \cap A|$ $|N \cap B|$ and $r, s \notin \pi(G/N)$, we note that r, and also s, divides either $|N \cap A|$ or $|N \cap B|$. In particular, we can consider either $a \in A \cap N$ or $a \in B \cap N$, and the same for $b \in N$.

In the sequel we will use the notation and the main results of [19], where the maximal factorizations of the almost simple groups are described. More exactly, factorizations $G = XY$ where X and Y are maximal subgroups of the group G with $N \trianglelefteq G \leq \mathrm{Aut}(N)$, not containing N, are described in Tables 1-5 of [19].

Lemma 15. *N is not isomorphic to $\mathrm{L}_n(q)$, $n \leq 3$.*

Proof. If $N \cong \mathrm{L}_2(q)$, apart from some exceptional cases that we will consider next, from [19] we know that possible factorizations $G = AB$ satisfy that A and B are soluble, so the result follows from Theorem 2 in [15]. The remaining cases are excluded by Theorem 1 (iv), except for $N \cong \mathrm{L}_2(q)$ when either $q = 29$ or $q = 59$. Since in both cases a Sylow q-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$ and $|G|_q = |N|_q = q$, we get a contradiction from Lemma 8.

Assume now that $N \cong \mathrm{L}_3(q)$, so $|\mathrm{Out}(N)| = 2(q-1, 3)\log_p(q)$. Observe first that the cases $q \leq 8$ are excluded by Theorem 1(iv). From [19] we know that all factorizations $G = AB$ satisfy that for one of the factors, say A, $|N \cap A|$ divides $\frac{q^3-1}{q-1} \cdot 3$, which is not divisible by $p \neq 3$, a contradiction by Lemma 12.

N	r	s	$\lvert T_1\rvert$	$\lvert T_2\rvert$	Remarks
$\mathrm{L}_n(q)$ $n \geq 4$	q_n	q_{n-1}	$\frac{q^n-1}{(n,q-1)(q-1)}$	$\frac{q^{n-1}-1}{(n,q-1)}$	$(n,q) \neq (6,2)$ $(n-1,q) \neq (6,2)$
$\mathrm{PSp}_{2n}(q)$ $\mathrm{P}\Omega_{2n+1}(q)$	q_{2n}	$q_{2(n-1)}$	$\frac{q^n+1}{(2,q-1)}$	$\frac{(q^{n-1}+1)(q+1)}{(2,q-1)}$	n even $(\star)$ $(n,q) \neq (4,2)$
$n \geq 3$	q_{2n}	q_n	$\frac{q^n+1}{(2,q-1)}$	$\frac{(q^n-1)}{(2,q-1)}$	n odd $(n,q) \neq (3,2)$
$\mathrm{P}\Omega^-_{2n}(q)$ $n \geq 4$	q_{2n}	$q_{2(n-1)}$	$\frac{q^n+1}{(4,q^n+1)}$	$\frac{(q^{n-1}+1)(q-1)}{(4,q^n+1)}$	$(n,q) \neq (4,2)$
$\mathrm{P}\Omega^+_{2n}(q)$	$q_{2(n-1)}$	q_{n-1}	$\frac{(q^{n-1}+1)(q+1)}{(4,q^n-1)}$	$\frac{(q^{n-1}-1)(q-1)}{(4,q^n-1)}$	n even $(n,q) \neq (4,2)$
$n \geq 4$	$q_{2(n-1)}$	q_n	$\frac{(q^{n-1}+1)(q+1)}{(4,q^n-1)}$	$\frac{q^n-1}{(4,q^n-1)}$	n odd

TABLE 1.

For the case $p = 3$, we get that $|N : N \cap B|_3 \leq q/3 < q^2$, so $C_G(N \cap B)$ is a p-group because of Lemma 6, and this is a contradiction by Lemma 8. □

Lemma 16. *N is not isomorphic to $\mathrm{L}_n(q)$, $n \geq 4$.*

Proof. Recall that $|\pi(N)| > 4$ because of Theorem 1(iv).

Assume first that either $N \cong \mathrm{L}_6(2)$ or $N \cong \mathrm{L}_7(2)$. In both cases, if s is the largest prime number dividing $|N|$, then $|G|_s = |N|_s = s$ and a Sylow s-subgroup of G is self-centralizing in G, a contradiction by Lemma 8.

Hence we may assume that $N \cong \mathrm{L}_n(q)$, with $n \geq 4$, $(n,q) \neq (6,2)$, and $(n,q) \neq (7,2)$. Then, by Lemma 14, there exist tori T_1 and T_2 in N with orders:

$$|T_1| = \frac{q^n - 1}{(n, q-1)(q-1)}, \quad |T_2| = \frac{q^{n-1} - 1}{(n, q-1)}.$$

With the notation of Lemma 14, let $r = q_n$ and $s = q_{n-1}$. Take an element $a \in N$ of order r, and an element $b \in N$ of order s. Then $C_N(a) \leq T_1$, $C_N(b) \leq T_2$ and both subgroups are abelian p'-groups. Since $(|A \cap N|, |B \cap N|) \equiv 0 \pmod p$ by Lemma 12, it follows from Lemma 13 (i) that $\{p, s, r\} \subseteq \sigma$.

Recall that r does not divide $|G/N|$. We may assume without loss of generality that $r \in \pi(A)$ and $a \in A \cap N$.

Assume first that $p \in \pi$ and so $\{p, s, r\} \subseteq \pi$. In this case $A_{\pi'} \cap N$ and hence A are soluble groups. Assume in addition that s does not divide $|B \cap N|$. Then, both r and s must divide the order of the soluble group $N \cap A$. By the proof of Lemma 3.1 in [1] (see also Lemmas 2.5 and 2.6 in [1]), this can only happen if $n = s \geq 5$, $p = q$, and $|N \cap A|$ divides $s(q^s - 1)$. Therefore, applying the formula for the order of N, we get that s must also divide $|B \cap N|$, a contradiction. Hence, if $p \in \pi$, we deduce that s divides $|B \cap N|$ and we can assume that $b \in B \cap N$, but this contradicts Lemma 13 (ii). Hence we conclude that $\{p, s, r\} \subseteq \pi'$.

Recall now that the field automorphisms of N do not centralize elements of order r or s. Moreover, there is no diagonal automorphism of N centralizing an element of order r. This implies that G/N is a π'-group.

If $s \in \pi(A)$, since $r, s \in \pi'$ we get $\pi \cap \pi(N \cap A) \subseteq \pi(C_N(a)) \cap \pi(C_N(b))$. Since

$$(|T_1|, |T_2|) = \Big(\frac{q^n - 1}{(q-1)(n, q-1)}, \frac{q^{n-1} - 1}{(n, q-1)}\Big) = 1,$$

this means that $A \cap N$ and hence A are π'-groups, a contradiction.

Therefore we may assume that $\{p, s, r\} \subseteq \pi'$, $a \in A \cap N$ and $b \in B \cap N$. Then $\pi \cap \pi(N \cap A) \subseteq \pi(T_1)$ and $\pi \cap \pi(N \cap B) \subseteq \pi(T_2)$, where $(|T_1|, |T_2|) = 1$. Therefore $A_\pi = A_\pi \cap N \leq T_1$, $B_\pi = B_\pi \cap N \leq T_2$ and both are Hall subgroups of N.

Assume first that there exists a prime divisor t of $|A_\pi|$ such that t is not primitive with respect to the pair (q, n). Since t divides $q^n - 1$ but is not a primitive prime divisor, t divides $q^j - 1$ with $j \neq n$ a divisor of n (recall that $(q^n - 1, q^j - 1) = q^{(n,j)} - 1$). If $n = jk$, with $k > 1$ an integer, then N contains a subgroup of order $((q^j - 1)_t)^k$. However, then, by the formula for the order of N, we deduce that t must divide $|B|$, a contradiction since $(|A_\pi|, |B_\pi|) = 1$.

Hence we may assume that any prime divisor of $|A_\pi|$ is primitive with respect to the pair (q, n). Then, if we consider any element $x \in A_\pi \leq T_1$ of prime order, we have also that $C_N(x) \leq T_1$, but this means that $A \cap N \leq T_1$, which is the final contradiction since $p \in \pi(A \cap N)$ by Lemma 12. □

Lemma 17. *N is not isomorphic to* $\mathrm{U}_n(q)$, $n \geq 3$.

Proof. Assume that $N \cong \mathrm{U}_n(q)$, $n \geq 3$. Suppose first that n is odd. From Theorem A in [19], the only groups G such that $N \leq G \leq \mathrm{Aut}(N)$ and N is a unitary group of odd dimension that are factorizable occur for $N \cong \mathrm{U}_3(3)$, $\mathrm{U}_3(5)$, $\mathrm{U}_3(8)$ or $\mathrm{U}_9(2)$. Since in our case $|\pi(N)| \geq 5$, the only possible case would be $N \cong \mathrm{U}_9(2)$. Note that in this case $\pi(N) = \{2, 3, 5, 7, 11, 17, 19, 43\}$ and $\mathrm{Out}(N) \cong S_3$. By Lemma 12 we may assume that $p = 2$ divides $(|A \cap N|, |B \cap N|)$. This group N has maximal tori of orders $19 \cdot 3$ and $17 \cdot 5$. We may let $r = 17 \in \pi(A)$. Since the centralizer of an element of order 17 in N has odd order $17 \cdot 5$ and $2 \in \pi'$, we deduce that $r = 17 \in \pi'$, $5 \in \pi$ and $|A_\pi \cap N| = 5$, so $|A_\pi|$ divides $5 \cdot 3$. On the other hand, an element of N of order $s = 19$ has a centralizer in N of order $19 \cdot 3$. Since $r \in \pi(A)$, we have that $s \notin \pi(A)$ and $s \in \pi' \cap \pi(B)$. This means that $|B_\pi|$

divides 3^2. Since the order of a 5-Sylow subgroup of N is at least 25, this gives a contradiction.

Assume now that $n = 2m$ is even, $m \geq 2$. It follows from Tables 1 and 3 in [19] (and with the same notation) that one of the maximal subgroups in the factorization of G with $N \leq G \leq \mathrm{Aut}(N)$, say X, has the property $X \cap N = N_1 \cong \mathrm{U}_{2m-1}(q)$, unless $N \cong \mathrm{U}_4(2)$ or $\mathrm{U}_4(3)$. Since $|\pi(\mathrm{U}_4(2))| < 5$ and $|\pi(\mathrm{U}_4(3))| < 5$, these possibilities are excluded.

Apart from some exceptional cases that we will check later, any group H such that $N_1 \leq H \leq \mathrm{Aut}(N_1)$ has no proper factorizations (in the sense that the factors do not contain N_1). Assume that $A \leq X$ and so $X = A(X \cap B)$. Now note that $X = N_G(N_1)$ and so $X/C_G(N_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(N_1)$ and then it has no proper factorizations. If $N_1 \cong N_1C_G(N_1)/C_G(N_1)$ were contained either in $AC_G(N_1)/C_G(N_1)$ or in $(X \cap B)C_G(N_1)/C_G(N_1)$, which are π-decomposable groups, it would follow that $N_1 = X \cap N$ would be a π-group, a contradiction. This means that either $X = AC_G(N_1)$ or $X = (X \cap B)C_G(N_1)$. In the latter case we would have $G = AB = C_G(N_1)B$. But from the structure of $\mathrm{Out}(N)$, it follows that $|C_G(N_1)|$ divides $q+1$ and such a factorization is impossible by order arguments. Now assume $X = AC_G(N_1)$. Since $A = A_\pi \times A_{\pi'}$, applying again that $X/C_G(N_1)$ has no proper factorizations, we get that either $X = A_\pi C_G(N_1)$ or $X = A_{\pi'}C_G(N_1)$. Since A_π is a soluble group and $X/C_G(N_1)$ contains a subgroup isomorphic to N_1, the case $X = A_\pi C_G(N_1)$ cannot occur. Then $X = A_{\pi'}C_G(N_1)$, and $A_\pi \leq C_G(N_1)$ is of order dividing $q+1$. Then $|X|$ divides $|A_{\pi'}|(q+1)$. But, if $n = 2m > 4$, then $(q+1)^3$ divides $|N_1| = |\mathrm{U}_{2m-1}(q)|$, and so $q+1$ divides $|A_{\pi'}|$, which means that A is a π'-group, a contradiction. Finally, if $n = 2m = 4$, then $(q+1)^2/(3, q+1)$ divides $|N_1| = |\mathrm{U}_{2m-1}(q)|$, and so $\pi(X) \subseteq \pi(A_{\pi'}) \cup \{3\}$, but $|N_1|_3 > (q+1)_3$, so $3 \in \pi'$ and A is again a π'-group, a contradiction.

The exceptional cases when $N \cong \mathrm{U}_{2m}(q)$ and $X \cap N = N_1 \cong \mathrm{U}_{2m-1}(q)$ is factorized, occur when $N_1 \cong \mathrm{U}_3(3)$, $\mathrm{U}_3(5)$, $\mathrm{U}_3(8)$ or $\mathrm{U}_9(2)$, by Table 3 in [19]. The case $N \cong \mathrm{U}_4(3)$ corresponding to the first possibility is excluded since $|\pi(N)| \leq 4$. Hence we must study the cases $N \cong \mathrm{U}_4(5)$, $\mathrm{U}_4(8)$ and $\mathrm{U}_{10}(2)$. In all these three cases there exist maximal tori T_1 and T_2 of orders

$$|T_1| = \frac{q^n - 1}{(n, q+1)(q+1)} \quad \text{and} \quad |T_2| = \frac{q^{n-1}+1}{(n, q+1)}.$$

Take $r = q_n$ and $s = q_{2(n-1)}$, so s divides $q^{n-1}+1$. It can be seen that:

$(r, s) = (13, 7)$, $|T_1| = 13 \cdot 2^2$ and $|T_2| = 7 \cdot 3^2$, for $\mathrm{U}_4(5)$;

$(r, s) = (17, 19)$, $|T_1| = 5 \cdot 7 \cdot 13$ and $|T_2| = 3^3 \cdot 19$, for $\mathrm{U}_4(8)$;

$(r, s) = (31, 19)$, $|T_1| = 11 \cdot 31$ and $|T_2| = 19 \cdot 3^3$, for $\mathrm{U}_{10}(2)$.

Note also that p divides $(|A \cap N|, |B \cap N|)$ by Lemma 12. Moreover, if a and b are elements of orders r and s, respectively, we have here that $C_N(a) = T_1$ and $C_N(b) = T_2$. Since T_1 and T_2 are soluble p'-groups, we deduce that $\{p, s, r\} \subseteq \pi'$. Moreover, from Table 1 in [19] we know that, for one of the factors, say B, $|B \cap N|$ divides $|N_1| = |\mathrm{U}_{n-1}(q)|$. By order arguments, we see in each case that r divides

$|N \cap A|$ and s divides $|N \cap B|$, and in all cases the primes 2 and 3 divide both $|A \cap N|$ and $|B \cap N|$. On the other hand, $C_N(a) = T_1$ is a $3'$-group, so $3 \in \pi'$ and this implies that G/N is a π'-group in all cases (recall that $2 \in \pi'$). Then $B_\pi = B_\pi \cap N \le C_N(b)$ and this is a π'-group, which means that B is a π'-group, a contradiction. □

Lemma 18. *N is not isomorphic to* $\mathrm{PSp}_4(q)$*, $q = p^e$.*

Proof. Assume that $N \cong \mathrm{PSp}_4(q)$ Then

$$|N| = \frac{1}{(2, q-1)}\, q^4\, (q^4 - 1)(q^2 - 1)$$

and $|\mathrm{Out}(N)| = (2, q-1)(2, p)e$. Moreover, the cases $q \le 7$ can be excluded by Theorem 1(iv).

There is a torus T in N of order $\frac{q^2+1}{(2,q-1)}$. Since $q^2 + 1$ is not divisible by 4, $|T|$ is odd. Let $r \in \pi(T)$. Since

$$\Big(\frac{q^2+1}{(2, q-1)}, q^2 - 1\Big) = 1,$$

we deduce that r is a primitive prime divisor of $q^4 - 1$ and any element of prime order in T acts irreducibly on the natural module of $\mathrm{Sp}_4(q)$. Hence we have that $C_N(a) \le T$ for any element $1 \ne a \in T$. Since T is a p'-group, applying Lemmas 12 and 13, we deduce that $\{p\} \cup \pi(T) \subseteq \sigma$, for some $\sigma \in \{\pi, \pi'\}$. Moreover, there is no field automorphism of N centralizing any element of T. Without loss of generality assume that $\pi(A) \cap \pi(T) \ne \emptyset$. Then it is easy to deduce that either A is a σ-group or $A = A_\pi \times A_2$ and A is soluble. In the latter case, looking at the orders of maximal soluble subgroups of N divisible by a primitive prime divisor of $q^4 - 1$ (see Lemma 2.8 in [1]), we get that $|A \cap N| = |A_\pi \cap N|$ divides $q^2 + 1$. This contradicts Lemma 12 and concludes the proof, since A is not a σ-group. □

Lemma 19. *N is neither isomorphic to* $\mathrm{PSp}_{2n}(q)$ *nor to* $\mathrm{P\Omega}_{2n+1}(q)$*, for $q = p^e$ and $n \ge 3$.*

Proof. Assume that N is isomorphic either to $\mathrm{PSp}_{2n}(q)$ or to $\mathrm{P\Omega}_{2n+1}(q)$, with $n \ge 3$. Then

$$|N| = \frac{1}{(2, q-1)}\, q^{n^2}\, (q^{2n} - 1)(q^{2n-2} - 1) \cdots (q^2 - 1)$$

and $|\mathrm{Out}(N)| = (2, q-1)e$.

We deal first with the cases (*) not considered in Lemma 14. If $n = 3$ and $q = 2$, then $N \cong \mathrm{PSp}_6(2) \cong \Omega_7(2)$ and, in this case, $|\pi(N)| = 4$, which contradicts Theorem 1(iv). If $n = 4$ and $q = 2$, then $N \cong \mathrm{PSp}_8(2) \cong \Omega_9(2)$ and this group has a self-centralizing Sylow subgroup of order 17, which is contained either in A or in B, a contradiction by Lemma 8.

For the cases $(n, q) \ne (3, 2)$ and $(n, q) \ne (4, 2)$, as stated in Lemma 14, N has tori T_1 and T_2 of the following orders:

(a) If n is even,

$$|T_1| = \frac{q^n+1}{(2,q-1)}\,, \quad |T_2| = \frac{(q^{n-1}+1)(q+1)}{(2,q-1)}.$$

In this case let $r = q_{2n}$ and $s = q_{2n-2}$.

(b) If n is odd,

$$|T_1| = \frac{q^n+1}{(2,q-1)}\,, \quad |T_2| = \frac{(q^n-1)}{(2,q-1)}.$$

In this case let $r = q_{2n}$ and $s = q_n$.

In both cases we will denote by $a \in N$ an element of order r and by $b \in N$ an element of order s. We study these cases separately.

Case (a): n even.

Without loss of generality we may assume that $r \in \pi(A)$ and $a \in A \cap N$.

In this case $C_N(a) \leq T_1$ (and T_1 is abelian), and $C_N(b)/Z(C_N(b)') \cong C \times L$, with $C \leq C_{q^{n-1}+1}$ and $L' \cong \mathrm{L}_2(q)$. (Recall that $\mathrm{L}_2(q) \cong \mathrm{PSp}_2(q) \cong \Omega_3(q)$.)

Suppose first that $r \in \pi$. Since $C_N(a)$ is a p'-group, and p divides $(|N \cap A|, |N \cap B|)$ by Lemma 12, we deduce by Lemma 13 that $\{p, r\} \subseteq \pi \cap \pi(A)$ (recall also that r does not divide $|G/N|$). In this case $A_{\pi'} \cap N$ is a soluble group and hence A is a soluble group. By Lemma 2.8 in [1], the order of $A \cap N$ divides either $2n(q^n+1)$ or $16n^2(q-1)r\log_2(2n)$. In the latter case we have $q = p$, $r = 2n+1$, and n is a power of 2. Since s is a primitive prime divisor of $q^{2n-2}-1$, we have that $s \geq 2n-1$. Hence we deduce that $s \notin \pi(A \cap N)$ and so $s \in \pi(N \cap B)$. If $s \in \pi'$, from the order of $C_N(b)$ we deduce that a Sylow p-subgroup of $B \cap N$ has order at most q. Since $|N|_p \leq |G/N|_p |N \cap A|_p |N \cap B|_p$ we deduce that $q^{n^2} \leq \max\{(\log_p(q) \cdot \log_p(n))_p \cdot q,\ (\log_2(2n))_p \cdot q\}$ (recall that $p \neq 2$, since we are in the case $\{p, r\} \subseteq \pi$). This gives a contradiction since $n \geq 4$. Therefore we have $s \in \pi$, so that $\{p, r, s\} \subseteq \pi$.

Now note that the only nonsoluble composition factors of $C_N(b)$ are isomorphic to $\mathrm{L}_2(q)$. Since $B_{\pi'}$ is not soluble because of Theorem 2 in [15] and its order is coprime with $p \in \pi$, by Dickson's theorem (see II, 8.27 in [12]) we deduce that the order of a nonsoluble subgroup of $N \cap B$ divides $|A_5|$ or $|S_5|$ and there holds $q \equiv \pm 1 (\mathrm{mod}\ 5)$. In this case $5 \in \pi'$, $p \neq 5$ and $q^n + 1 \equiv 2 (\mathrm{mod}\ 5)$ (recall that n is even). In particular $|A \cap N|_5$ is either n_5 or $\log_2(2n)_5$. On the other hand, $|N \cap B|_5$ does not exceed $((q^{n-1}+1)(q^2-1))_5$. Moreover, since there are no field automorphisms centralizing elements of order r, it follows that $\log_p(q)_5 = 1$. Hence $|N|_5 \leq \max\{n_5((q^{n-1}+1)(q^2-1))_5,\ \log_2(2n)_5((q^{n-1}+1)(q^2-1))_5\}$, which is a contradiction (recall that $n \geq 3$).

Therefore, we may assume $\{p, r\} \subseteq \pi'$. Suppose that $s \in \pi(A)$. Since $(|C_N(a)|, s) = 1$, this means that $s \in \pi'$. It follows that $\pi \cap \pi(A \cap N) \subseteq \pi(C_N(a)) \cap \pi(C_N(b)) \cap \pi$. However, since $\pi((q^n+1, (q^{n-1}+1)(q^2-1))) \subseteq \{2\}$ it follows that $\pi \cap \pi(A \cap N) = \emptyset$. This means that $A \cap N$ and so A are π'-groups, a contradiction (recall that there is no field automorphism centralizing an element of order r or s).

Thus we conclude that $s \in \pi(B \cap N)$. Assume first that $s \in \pi$. Since field automorphisms do not centralize elements of order $s \in \pi$, we may assume that $p \in \pi'$ does not divide $|G/N|$ (note that for $p = 2$, each outer automorphism of N is a field automorphism). Note also that $|N \cap B|_p \leq q$. Hence it follows from the order formula $|N|_p = |G/N|_p |N \cap A|_p |N \cap B|_p$, that $|N \cap A|_p \leq q^{n^2-1}$, and so $|N_p : (N \cap A)_p| \leq q$ (recall that $A \cap B = 1$, since $p \in \pi'$ by Theorem 1 (v)). By Lemma 6, this means that $C_G((N \cap A)_p)$ is a p-group, so A is a π'-group, a contradiction.

Therefore we have that $\{p, r, s\} \subseteq \pi'$. Hence $\pi \cap \pi(N \cap A) \subseteq \pi(q^n + 1)$ and $\pi \cap \pi(N \cap B) \subseteq \pi((q^{n-1} + 1)(q^2 - 1))$, and then $\pi \cap \pi(N \cap A) \cap \pi(N \cap B) = \emptyset$. On the other hand, since the field automorphisms of N do not centralize elements of order r or s, and $2 \in \pi'$, we deduce that $A_\pi \leq N$, $B_\pi \leq N$, and both are Hall subgroups of N.

Assume that there exists $t \in \pi \cap \pi(A)$ which is not a primitive prime divisor of $q^{2n} - 1$. It follows from Lemma 4 that t divides $q^j + 1$, for some $j \neq 1$ dividing n. We claim that $n = lj$, with l odd and $l \geq 3$. Indeed, if l is even, since $q^j \equiv (-1)(\mathrm{mod}\ t)$, we get $q^n = (q^j)^l \equiv 1(\mathrm{mod}\ t)$, a contradiction since t divides $q^n + 1$. Now, since N has a torus of order $(q^j + 1)^l$ which is not contained in $A_\pi = A_\pi \cap N \leq T_1$ and G/N is a π'-group, we get a contradiction with the fact that $(t, |N \cap B|) = 1$ (recall $n \geq 3$).

Hence we may assume that each prime in $\pi \cap \pi(A)$ is a primitive prime divisor of $q^{2n} - 1$. Then if we consider any element $x \in A_\pi \leq T_1$ of prime order we have also that $C_N(x) \leq T_1$, but this means that $A \cap N \leq T_1$, which is the final contradiction since $p \in \pi(A \cap N)$.

Case (b)**:** *n odd.*

Without loss of generality we may assume that $r \in \pi(A)$. In this case $C_N(a) \leq T_1$, $C_N(b) \leq T_2$ and both centralizers are abelian. If $r \in \pi$, we have also $p \in \pi$, by Lemmas 12 and 13. In this case A is soluble and we deduce that $s = q_n \notin \pi(A)$ as in case (a). Hence $s \in \pi(B \cap N)$ and since p divides $|N \cap B|$ and $|C_N(b)|$ divides $q^n - 1$, we deduce that $s \in \pi$. In this case both subgroups $A \cap N$ and $B \cap N$ are soluble, so A and B are soluble and this contradicts Theorem 2 in [15].

Thus we can assume that $r \in \pi'$, so that $p \in \pi'$ and $\pi \cap \pi(N \cap A) \subseteq \pi(C_N(a)) \subseteq \pi(q^n + 1)$. If $s \in \pi(A)$, we get $s \in \pi'$ by Lemma 13, and hence $\pi \cap \pi(N \cap A) \subseteq \pi(C_N(b)) \subseteq \pi(q^n - 1)$. Since $(q^n + 1, q^n - 1)_{2'} = 1$, this means that $A \cap N$ and hence A are π'-groups, a contradiction.

Now we may assume $s \in \pi(B \cap N) \cap \pi'$, because $p \in \pi'$. Again we have $\pi \cap \pi(N \cap A) \subseteq \pi(q^n + 1)$ and since the field automorphisms of N do not centralize an element of order r, it follows that $|G/N|$ is a π'-group and $A_\pi = A_\pi \cap N$. On the other hand, we deduce also that $\pi \cap \pi(B \cap N) \subseteq \pi(q^n - 1)$ and $B_\pi = B_\pi \cap N$. Since $(q^n + 1, q^n - 1)_{2'} = 1$, it turns out that A_π and B_π are Hall subgroups of N, and also of G. As in case (a) we deduce that for some prime divisor of $q^n + 1$, $t \in \pi$, we have $n = lj$ with $l \geq 3$ odd and $q^j + 1 \equiv 0(\mathrm{mod}\ t)$. We get a contradiction, as in case (a), since $(q^j + 1)^l$ divides $|N|$. □

Lemma 20. *N is not isomorphic to $\mathrm{P}\Omega^+_{2n}(q)$, for $q = p^e$ and $n \geq 4$.*

Proof. Note that $\mathrm{P}\Omega_6^+(q) \cong \mathrm{L}_4(q)$ and this case has been studied in Lemma 16. Assume that $N \cong \mathrm{P}\Omega_{2n}^+(q)$, $n \geq 4$. Then

$$|N| = \frac{1}{d}\, q^{n(n-1)}\, (q^{2n-2} - 1) \cdots (q^2 - 1)(q^n - 1),$$

$d = (4, q^n - 1)$, and $|\mathrm{Out}(N)| = 2de$ if $n \geq 5$ and $|\mathrm{Out}(N)| = 6de$ if $n = 4$.

As stated in Lemma 14, N has tori T_1 and T_2 of the following orders:

(a) If n is even,

$$|T_1| = \frac{1}{d}(q^{n-1} + 1)(q + 1)\,, \quad |T_2| = \frac{1}{d}(q^{n-1} - 1)(q - 1).$$

With the notation of Lemma 14, let $r = q_{2n-2}$ and $s = q_{n-1}$.

(b) If n is odd,

$$|T_1| = \frac{1}{d}(q^{n-1} + 1)(q + 1)\,, \quad |T_2| = \frac{1}{d}(q^n - 1).$$

In this case let $r = q_{2n-2}$ and $s = q_n$.

If $n = 4$ and $q = 2$, $|\pi(\mathrm{P}\Omega_8^+(2))| = 4$, hence $(n, q) \neq (4, 2)$ and, in particular, all such primitive prime divisors exist.

Let $a, b \in N$ be elements of orders r and s, respectively, and let $C_N(a) \leq T_1$ and $C_N(b) \leq T_2$, and recall that these subgroups are abelian p'-groups. Since p divides $(|N \cap A|, |N \cap B|)$ we deduce that $\{p, r, s\} \subseteq \sigma$, for $\{\sigma, \sigma'\} = \{\pi, \pi'\}$.

Now note that, for $n > 4$, since a field or a graph-field automorphism centralizes no element of order r or s, it follows that $\pi(G/N) \setminus \{2\} \subseteq \sigma$ if $r, s \in \sigma$. In the case $n = 4$ there exist graph automorphisms of order 3 and $|G/N|_3 \leq 3 \cdot \log_p(q)_3$. We claim that in this case $\{r, s, 3\} \subseteq \sigma$ and so the previous conclusion for $\pi(G/N)$ remains valid when $n = 4$. Assume that $3 \in \sigma'$. If $r \in \pi(A)$ and $s \in \pi(B)$, then $|N \cap A|_3 |N \cap B|_3$ divides $((q^3 + 1)(q + 1))_3((q^3 - 1)(q - 1))_3$, which is not the case by comparison with $|N|_3$. Without loss of generality if $r, s \in \pi(A)$, then $\pi(A) \cap \sigma' \subseteq \pi(((q^3 + 1)(q + 1), (q^3 - 1)(q - 1))) \subseteq \{2\}$ and so $3 \notin \sigma'$, a contradiction which proves the claim.

Without loss of generality assume that $r \in \pi(A \cap N)$. Observe that in both cases (a) and (b), $|N_N(\langle a\rangle)/C_N(\langle a\rangle)|$ divides $2(n-1)$ and $r \equiv 1 \pmod{2n-2}$. Moreover, in case (a) there holds that $|N_N(\langle b\rangle)/C_N(\langle b\rangle)|$ divides $2(n-1)$ and $s \geq n$. On the other hand, in case (b) we have that $|N_N(\langle b\rangle)/C_N(\langle b\rangle)|$ divides $2n$ and $s \geq n + 1$.

Assume that $\{p, r, s\} \subseteq \pi$. Since $A_{\pi'} \cap N \leq C_N(a) \leq T_1$, we deduce that $A_{\pi'} \cap N$ and hence A are soluble groups. Since a Sylow r-subgroup of A is cyclic, we have that $A \cap N = \mathrm{O}_{r'}(A \cap N) N_{A\cap N}(\langle a\rangle)$ by Lemma 2. Moreover, $|N_{N\cap A}(\langle a\rangle)/C_{N\cap A}(\langle a\rangle)|$ divides $2n - 2$.

Suppose first that $s \in \pi(A)$ and $b \in N \cap A$. Since s divides neither $|T_1|$, nor $|C_{N\cap A}(a)|$, it follows that either s divides $2(n - 1)$ or $s \in \pi(\mathrm{O}_{r'}(A \cap N))$. Since $s \geq n$, the first case cannot occur. Hence $s \in \pi(\mathrm{O}_{r'}(A \cap N))$. Since Sylow s-subgroups of A are also cyclic, we have that $A \cap N = \mathrm{O}_{s'}(A \cap N) N_{A\cap N}(\langle b\rangle)$.

Observe that elements of order sr do not exist in N. Consequently, r divides $2n-2$ or $2n$, which is not the case as $r \geq 2n-1$. Hence $s \in \pi(B \cap N)$ and we can assume that $b \in N \cap B$. This contradicts that $\{p, r, s\} \subseteq \pi$ by Lemma 13 (ii).

Hence we have $\{p, r, s\} \subseteq \pi'$ and so $\pi(G/N) \subseteq \pi'$. Suppose that $s \in \pi(A)$. Then we deduce that $\pi \cap \pi(A) \subseteq \pi(C_N(a)) \cap \pi(C_N(b))$. However, $\pi(|T_1|, |T_2|) \subseteq \{2\}$, in both cases (a) and (b), and so $\pi \cap \pi(A) = \emptyset$, which means that A is a π'-group, a contradiction.

Now we have that $s \in \pi(B \cap N)$. It follows that $A_\pi = A_\pi \cap N \leq T_1$ and $B_\pi = B_\pi \cap N \leq T_2$ are Hall subgroups of N, and also of G. Arguing as in cases $\mathrm{L}_n(q)$ or $\mathrm{PSp}_{2n}(q)$, by using the formula for the order of N, we get the final contradiction. □

Lemma 21. *N is not isomorphic to $\mathrm{P}\Omega^-_{2n}(q)$, for $q = p^e$ and $n \geq 4$.*

Proof. If $N \cong \mathrm{P}\Omega^-_8(2))$, we can take $r = 17$ and there exists a self-centralizing Sylow subgroup of this order, so we get a contradiction by Lemma 8.

Assume that $N \cong \mathrm{P}\Omega^-_{2n}(q)$, $n > 4$. By Lemma 14 we can consider tori T_1 and T_2 of N of the orders

$$|T_1| = \frac{q^n + 1}{(4, q^n + 1)}, \quad |T_2| = \frac{(q^{n-1} + 1)(q - 1)}{(4, q^n + 1)},$$

primitive divisors $r = q_{2n}$ and $s = q_{2n-2}$, and elements a and b of orders r and s, respectively, such that $C_N(a) \leq T_1$, $C_N(b) \leq T_2$, and these subgroups are abelian p'-groups. In particular, $\{p, r, s\} \subseteq \sigma$, for $\{\sigma, \sigma'\} = \{\pi, \pi'\}$, since $(|N \cap A|, |N \cap B|) \equiv 0 \pmod{p}$. Moreover, $\pi(\log_p(q)) \subseteq \sigma$ because field automorphisms of N do not centralize elements of order r or s.

Without loss of generality assume that $r \in \pi(A)$. Suppose first that $r \in \pi$. Then $A_{\pi'} \cap N$ and hence A are soluble groups. Moreover, by Lemma 13 (ii) we deduce that $s \in \pi(A)$. Since Sylow r-subgroups of N and Sylow s-subgroups of N are cyclic, we can consider $A \cap N = \mathrm{O}_{r'}(A \cap N) N_{A \cap N}(\langle a \rangle) = \mathrm{O}_{s'}(A \cap N) N_{A \cap N}(\langle b \rangle)$. Observe that $|N_N(\langle a \rangle)/C_N(\langle a \rangle)|$ divides $2n$ and $|N_N(\langle b \rangle)/C_N(\langle b \rangle)|$ divides $2(n-1)$, where $r \geq 2n+1$ and $s \geq 2n-1$. Since there are no elements of order rs in N we deduce that $s \notin \pi(A)$, a contradiction.

Hence we may assume $\{p, r, s\} \subseteq \pi'$ and then $|G/N|$ is a π'-group. If $r, s \in \pi(A)$, the order of A_π would divide $(|T_1|, |T_2|)_{2'} = 1$ and so A would be a π'-group, a contradiction.

Therefore we have that $r \in \pi(A \cap N)$, $s \in \pi(B \cap N)$, and so $A_\pi = A_\pi \cap N \leq T_1$ and $B_\pi = B_\pi \cap N \leq T_2$ are Hall subgroups of N and G. Arguing as in the previous cases, using the formula for the order of N, we get the final contradiction. □

The main theorem is proved.

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