



Regularity and geometric estimates for minima of discontinuous functionals

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Abstract. In this paper we study nonnegative minimizers of general degenerate elliptic functionals, $\int F(X, u, \nabla u) dX \rightarrow \min$, for variational kernels F that are discontinuous in u with discontinuity of order $\sim \chi_{\{u>0\}}$. The Euler–Lagrange equation is therefore governed by a nonhomogeneous, degenerate elliptic equation with free boundary between the positive and the zero phases of the minimizer. We show optimal gradient estimate as well as nondegeneracy of minima. We also address weak and strong regularity properties of the free boundary. We show the set $\{u > 0\}$ has locally finite perimeter and that the reduced free boundary, $\partial_{\text{red}}\{u > 0\}$, has \mathcal{H}^{n-1} -total measure. For more specific problems that arise in jet flows, we show the reduced free boundary is locally the graph of a $C^{1,\gamma}$ function.

1. Introduction

Given a bounded smooth domain $\Omega \subset \mathbb{R}^n$ and a nonnegative function $\phi \in W^{1,p}(\Omega) \cap L^\infty(\overline{\Omega})$, $2 \leq p < n$, we study regularity and fine geometric properties of solutions to the minimization problem

$$(1.1) \quad \min \left\{ \int_{\Omega} F(X, u, \nabla u) dX : u \in W_{\phi}^{1,p}(\Omega) \right\},$$

where $W_{\phi}^{1,p}(\Omega)$ denotes the Sobolev space of all functions in L^p with distributional derivatives in L^p and trace value ϕ . The variational kernel $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, will be written as $F(X, u, \xi) = G(X, \xi) + g(X, u)$ and it will satisfy the following structural conditions:

(G1) For all $\xi \in \mathbb{R}^n$, the mapping $X \mapsto G(X, \xi)$ is continuous.

(G2) There exists a positive constant $0 < \lambda$ such that

$$\lambda |\xi|^p \leq G(X, \xi) \leq \lambda^{-1} |\xi|^p.$$

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(G3) For almost all $X \in \Omega$, the mapping $\xi \mapsto G(X, \xi)$ is strictly convex, differentiable and satisfies

$$G(X, t\xi) = |t|^p G(X, \xi), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n.$$

(G4) There exist constants $0 < \delta < 1$ and $C_{\mathcal{A}} > 0$ such that

$$X \mapsto \mathcal{A}(X, \xi) \in C^\delta(\Omega'), \quad \sup_{\xi \in \mathbb{R}^n} \|\mathcal{A}(X, \xi)\|_{C^\delta} \leq C_{\mathcal{A}},$$

for all subdomain $\Omega' \subset \Omega$, where $\mathcal{A}(X, \xi) := \nabla_\xi G(X, \xi)$.

(g1) The function g is defined by

$$g(X, u) = f(X)(u^+)^m + Q \chi_{\{u>0\}}, \quad 1 \leq m < p,$$

where f is measurable, $-K \leq f \leq K$, for some $K > 0$ and Q is $C^{0,\beta}$ -continuous, $0 < \kappa < Q < \kappa^{-1}$ for some $\kappa > 0$.

A prototypical kernel to keep in mind is

$$(1.2) \quad F(X, u, \xi) = |\xi|^{p-2} A(X) \xi \cdot \xi + f(X)(u^+)^m + Q \chi_{\{u>0\}},$$

for a positive definite matrix A with continuous coefficients. Motivations come from the study of jet flow and cavity problems, among many other applications. For notational convenience, we label the functional appearing in the minimization problem (1.1) by $\mathfrak{F}: W_\varphi^{1,p}(\Omega) \rightarrow \mathbb{R}$, i.e., hereafter

$$\mathfrak{F}(u) := \int_{\Omega} F(X, u, \nabla u) dX.$$

Also, any positive constant $C = C(n, p, m, \lambda, \phi, \kappa, K, \delta, C_{\mathcal{A}}, \Omega)$ that depends only on dimension and the parameter constants of the problem will henceforth be called a *universal* constant. We call a solution of the homogeneous problem

$$(1.3) \quad \operatorname{div}(\mathcal{A}(X, \nabla h)) = 0 \quad \text{in } B_r(X_0) \subset \Omega.$$

an \mathcal{A} -harmonic functions. The key feature of the functional \mathfrak{F} is that it is discontinuous with respect to u , thus the well established classical theory of the calculus of variations is not suitable for treating such problems. In fact, at a minimum u , the functional \mathfrak{F} presents discontinuity for small perturbations near points on the, in principle unknown, set $\partial\{u > 0\}$. Such a discontinuity reflects a lack of smoothness of u across the boundary of its zero level surface.

The study of the variational problem (1.1) goes back to the fundamental work of Alt and Caffarelli, [1], which provides a thorough analysis of this problem for $p = 2$, $f \equiv 0$, and $A(X) = \operatorname{Id}$ in (1.2). Danielli and Petrosyan in [6] developed the corresponding Alt and Caffarelli theory for the p -Laplacian, i.e., $f \equiv 0$ and $A(X) = \operatorname{Id}$.

In this paper we study the variational problem (1.1) in full generality, providing existence, regularity, and geometric properties of certain heterogeneous free boundary problems governed by degenerate elliptic equations. The results from this work are new even for the Poisson type equation $m = 1$. They also yield new results even in the linear setting $p = 2$.

In Section 3 we show there is a minimum for the functional \mathfrak{F} and that this minimum is nonnegative and continuous in Ω . We further show that within the set of positivity, u satisfies the desired Euler–Lagrange equation

$$\operatorname{div}(\nabla_{\xi}G(X, \nabla u)) = mf(X)u^{m-1}, \quad \text{in } \{u > 0\},$$

in the sense of distributions. Under natural structural assumptions on the kernel G , u is $C^{1,\alpha}$ in its set of positivity. Nevertheless, due to the discontinuity of \mathfrak{F} near free boundary points, ∇u jumps from positive values to zero through $\partial\{u > 0\}$. Therefore, the optimal regularity possible for a minimum is Lipschitz continuity. Such a result is established in Section 4. By Lipschitz regularity, we conclude that u grows linearly away from the free boundary. However, from energy considerations, we actually show that u grows precisely linearly on $\partial\{u > 0\}$. This is important geometric information that provides access to finer geometric-measure features of the free boundary. In fact, in Section 5 we show that

$$\Lambda := \operatorname{div}(\nabla_{\xi}G(X, \nabla u)) - mf(X)u^{m-1},$$

defines a nonnegative measure supported along the free boundary. We further show that the set of positivity of u , $\{u > 0\}$, is locally a set of finite perimeter. A finer property is actually shown: we verify that

$$\mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r(Z)) \sim r^{n-1},$$

for any ball $B_r(Z)$ centered at a free boundary point. In particular we conclude the reduced free boundary, $\partial_{\text{red}}\{u > 0\}$ has total \mathcal{H}^{n-1} -Hausdorff measure.

In the last Section we address smoothness of the (reduced) free boundary for the heterogeneous, quasilinear cavity problem

$$\begin{cases} \operatorname{div}(A(X)\nabla u) = mf(X)u^{m-1}, & \text{in } \{u > 0\}, \\ \langle A\nabla u, \nabla u \rangle = Q, & \text{on } \partial\{u > 0\} \cap \Omega, \end{cases}$$

where the matrix A is Lipschitz and positive definite. We show the free boundary is a $C^{1,\gamma}$ smooth hypersurface, up to a possible \mathcal{H}^{n-1} negligible set, providing therefore a classical solution to the corresponding quasilinear Bernoulli type problem.

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2. Preliminaries

In this section we gather some preliminary results that we will repeatedly throughout the article. We begin with the classical criterion of Ladyzhenskaya and Ural’tseva to estimate the L^{∞} -norm of a function $u \in W_{\phi}^{1,p}(\Omega)$ with boundary values $\phi \in L^{\infty}(\partial\Omega)$ for $1 < p \leq n$.

Lemma 2.1. *Let $u \in W^{1,p}(\Omega)$, $1 < p \leq n$, be such that $\|u\|_{L^\infty(\partial\Omega)} < \infty$. For $j \geq j_0 \geq \|u\|_{L^\infty(\partial\Omega)}$, let $A_j := \{u > j\}$. If u satisfies*

$$\int_{A_j} |\nabla u|^p \leq \gamma j^\alpha \mathcal{L}^n(A_j)^{1-p/n+\varepsilon}$$

where $\gamma > 0$, $\varepsilon > 0$ and $0 \leq \alpha \leq \varepsilon + p$, then there is a constant $C > 0$ depending only on $\gamma, \alpha, p, n, \varepsilon, j_0$, and $\|u\|_{L^1(A_{j_0})}$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

See [13], Chapter 2, Page 71, Lemma 5.2, for a proof of Lemma 2.1. Clearly, the case $p > n$ is a direct consequence of the Sobolev inequality.

The following is a basic fact: if h is a solution in $W_v^{1,p}(B_r(X_0))$ of equation (1.3), then h is the unique minimizer of the functional

$$(2.1) \quad \mathfrak{F}(w) := \int_{B_r(X_0)} \langle \mathcal{A}(X, \nabla w), \nabla w \rangle dX,$$

among functions $w \in W_0^{1,p}(B_1) + v$. Thus, we have

$$(2.2) \quad \begin{aligned} \lambda \int_{B_r(X_0)} |\nabla h|^p dX &\leq \int_{B_r(X_0)} \langle \mathcal{A}(X, \nabla h), \nabla h \rangle dX \\ &\leq \int_{B_r(X_0)} \langle \mathcal{A}(X, \nabla v), \nabla v \rangle dX \leq \lambda^{-1} \int_{B_r(X_0)} |\nabla v|^p dX. \end{aligned}$$

A fundamental tool in our article is the nonhomogeneous Harnack inequality.

Theorem 2.2 (Harnack inequality; see e.g. [15]). *Let $u \in W^{1,p}(B_R)$ be such that $u \geq 0$ a.e., and satisfy*

$$\operatorname{div} \mathcal{A}(X, \nabla u) = g(X), \quad \text{in } B_R$$

in the distributional sense, with $g(X) \in L^\infty(B_R)$. Then, there exists a constant $C > 0$ such that, for all $0 < r \leq R$,

$$\sup_{B_r} u \leq C \left\{ \inf_{B_r} u + (r^p \|g\|_{L^\infty(B_R)})^{1/(p-1)} \right\}.$$

Remark 2.3. Let $\Omega' \subset \mathbb{R}^n$ be a bounded domain such that $B_R \subset \Omega'$. In this article we will often regard the nonlinear term merely as a bounded source term, i.e.,

$$m f(X) u^m(X) =: g(X)$$

where $\|g\|_{L^\infty(B_R)} \leq C$ for a universal constant $C > 0$. In particular, for $0 \leq r \leq 1$ we have

$$\sup_{B_r} u \leq C \left\{ \inf_{B_r} u + r^{p/(p-1)} (\|g\|_{L^\infty(B_R)})^{1/(p-1)} \right\} \leq C \inf_{B_r} u + C r,$$

where $C = C(n, p, \lambda, \Lambda, \|g\|_{L^\infty(\Omega')}) > 0$.

To address Lipschitz regularity for a minimizer u we will need the $C^{1,\alpha}$ regularity of \mathcal{A} -harmonic functions. Precisely, the result that will be useful is the following (see [18] for superior estimates at degenerate points):

Theorem 2.4 ($C^{1,\alpha}$ regularity; see e.g. [19]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $g \in L^\infty(\Omega)$ satisfying*

$$\operatorname{div}(\mathcal{A}(X, \nabla u)) = g(X), \quad \text{in } \Omega$$

in the distributional sense. Then, there exist constants $C > 0$ and $0 < \alpha < 1$ that depend only on Ω' , $\|g\|_{L^\infty(\Omega')}$, and universal constants, such that

$$\|\nabla u\|_{L^\infty(\Omega')} \leq C \quad \text{and} \quad \|u\|_{C^{1,\alpha}(\Omega')} \leq C,$$

where $\Omega' \Subset \Omega$ is a subdomain.

Remark 2.5. We point out that Theorem 2.4 for $\Omega' = B_{2r}(X_0)$ gives

$$(2.3) \quad \|\nabla u\|_{L^\infty(B_r(X_0))} \leq C r^{\gamma-1} \quad \text{and} \quad \|\nabla u\|_{C^\alpha(B_r(X_0))} \leq C r^{\gamma-1-\alpha}$$

for some universal constant $0 < \gamma < 1$. Also we shall make use of the gradient estimate:

$$(2.4) \quad \|\nabla h\|_{L^\infty(B_{r/2}(X_0))} \leq \frac{C}{r^{n/p}} \|\nabla h\|_{L^p(B_r(X_0))},$$

for \mathcal{A} -harmonic functions, where $C > 0$ is universal constant; see for instance [8], Proposition 3.3. Sharp estimates below Lipschitz can be found, for instance, in [17].

We shall also make use of the following technical lemma regarding convergence stability.

Lemma 2.6. *Let $h_j: B_1(0) \rightarrow (0, 2)$ be a sequence of function in $W^{1,p}(B_1(0))$ such that*

$$\operatorname{div}(\mathcal{A}(W_j + c_j X, \nabla h_j)) = 0 \quad \text{in } B_1(0),$$

where $c_j \in \mathbb{R}^n$ and $W_j \in \Omega$ satisfy

$$c_j \rightarrow 0, \quad W_j \rightarrow W_0 \in \Omega \quad \text{and} \quad W_j + c_j X \in \Omega,$$

for all $X \in B_1(0)$. Suppose

$$h_j \rightarrow h_0 \quad \text{a.e. in } B',$$

where $B' \Subset B_1(0)$ is a ball. Then, we have, for all $B \Subset B'$,

$$\operatorname{div}(\mathcal{A}(W_0, \nabla h_0)) = 0 \quad \text{in } B.$$

Proof. Since each function h_j satisfies $0 \leq h_j \leq 2$ we conclude from the Caccioppoli inequality ([11], Chapter 3, Lemma 3.27) that there exists a constant $C > 0$ that depends only on B' and universal constants such that

$$(2.5) \quad \int_{B'} |\nabla h_j|^p dX \leq C.$$

Thus (see for instance [11], Theorem 1.32), $h_0 \in W^{1,p}(B')$ and

$$\nabla h_j \rightarrow \nabla h_0 \text{ weakly in } L^p(B').$$

Consider $\zeta \in C_0^\infty(B')$ such that $\zeta = 1$ in B and $0 \leq \zeta \leq 1$. Take $\zeta(h_0 - h_j)$ as a test function for the solution h_j . Then we obtain, for $Q_j := W_j + c_j X$,

$$\begin{aligned} \left| \int_{B'} \zeta \langle \mathcal{A}(Q_j, \nabla h_j), \nabla h_j - \nabla h_0 \rangle dX \right| &= \left| \int_{B'} (h_j - h_0) \langle \mathcal{A}(Q_j, \nabla h_j), \nabla \zeta \rangle dX \right| \\ &\leq C \left(\int_{B'} |h_j - h_0|^p |\nabla \zeta|^p \right)^{1/p} \cdot \left(\int_{B'} |\nabla h_j|^p \right)^{(p-1)/p} \leq C \left(\int_{B'} |h_j - h_0|^p \right)^{1/p}, \end{aligned}$$

where we use that (see [11], Chapter 5, Lemma 5.9)

$$(2.6) \quad |\mathcal{A}(X, \xi)| \leq C(p, \lambda) |\xi|^{p-1}, \quad \text{for a.e. } X \in \Omega \text{ and all } \xi \in \mathbb{R}^n,$$

and (2.5) in the second inequality.

Thus, since $h_j \rightarrow h_0$ a.e in B' , we conclude from the dominated convergence theorem that

$$\lim_{j \rightarrow \infty} \int_{B'} \zeta \langle \mathcal{A}(W_j + c_j X, \nabla h_j), \nabla h_j - \nabla h_0 \rangle dX = 0.$$

Using (2.6) we obtain

$$\left| \int_{B'} \zeta \langle \mathcal{A}(Q_j, \nabla h_0), \nabla h_j - \nabla h_0 \rangle dX \right| \leq C(p, \lambda) \int_{B'} |\nabla h_0|^{p-1} |\nabla h_j - \nabla h_0| dX.$$

Thus, since $\nabla h_0 \in L^{p/(p-1)}(B')$, we find

$$\lim_{j \rightarrow \infty} \int_{B'} \zeta \langle \mathcal{A}(W_j + c_j X, \nabla h_0), \nabla h_j - \nabla h_0 \rangle dX = 0.$$

By monotonicity of \mathcal{A} , we have

$$\int_{B'} \zeta \langle \mathcal{A}(W_j + c_j X, \nabla h_j) - \mathcal{A}(W_j + c_j X, \nabla h_0), \nabla h_j - \nabla h_0 \rangle dX \geq 0.$$

Then, we find

$$\lim_{j \rightarrow \infty} \int_B \langle \mathcal{A}(W_j + c_j X, \nabla h_j) - \mathcal{A}(W_0 + c_j X, \nabla h_0), \nabla h_j - \nabla h_0 \rangle dX = 0.$$

Moreover, it follows from (G4) that

$$\left| \int_B \langle \mathcal{A}(Q_j, \nabla h_j) - \mathcal{A}(W_0, \nabla h_j), \nabla h_j - \nabla h_0 \rangle dX \right| \leq \frac{C_{\mathcal{A}}}{2} \int_B |\nabla h_j - \nabla h_0| dX,$$

for j so large that $(|W_j - W_0| + |c_j|)^\delta \leq 1/2$. Taking into account that $\chi_B \in L^{p/(p-1)}(B)$ we obtain

$$\lim_{j \rightarrow \infty} \int_B \langle \mathcal{A}(W_j + c_j X, \nabla h_j) - \mathcal{A}(W_0, \nabla h_j), \nabla h_j - \nabla h_0 \rangle dX = 0.$$

Analogously, we have

$$\lim_{j \rightarrow \infty} \int_B \langle \mathcal{A}(W_j + c_j X, \nabla h_0) - \mathcal{A}(W_0, \nabla h_0), \nabla h_j - \nabla h_0 \rangle dX = 0.$$

Hence, we obtain

$$\lim_{j \rightarrow \infty} \int_B \langle \mathcal{A}(W_0, \nabla h_j) - \mathcal{A}(W_0, \nabla h_j), \nabla h_j - \nabla h_0 \rangle dX = 0$$

and by Lemma 3.73 in [11] we find

$$\mathcal{A}(W_0, \nabla h_j) \rightharpoonup \mathcal{A}(W_0, \nabla h_0),$$

weakly in $L^{p/(p-1)}(B)$. Finally, if $\eta \in C_0^\infty(B)$ we have

$$\begin{aligned} 0 &= \int_B \langle \mathcal{A}(Q_j, \nabla h_j), \nabla \eta \rangle dX \\ &= \int_B \langle \mathcal{A}(Q_j, \nabla h_j) - \mathcal{A}(W_0, \nabla h_j), \nabla \eta \rangle dX + \int_B \langle \mathcal{A}(W_0, \nabla h_j), \nabla \eta \rangle dX. \end{aligned}$$

Taking $j \rightarrow \infty$ we obtain

$$\int_B \langle \mathcal{A}(W_0, \nabla h_0), \nabla \eta \rangle dX = 0,$$

and the proof is complete. \square

3. Existence and continuity of minimizers

In this section we show the *discontinuous* optimization problem (1.1) has at least one minimizer. Uniqueness is known to fail even in simpler models. In the sequel we obtain a universal modulus of continuity for such a minimum.

Theorem 3.1. *There exists a minimizer $u \in W_\phi^{1,p}$ of the functional (1.1). Furthermore $u \geq 0$ in Ω .*

Proof. Let us label

$$I_0 := \min \left\{ \int_{\Omega} F(X, v, \nabla v) dX : v \in W_{\phi}^{1,p}(\Omega) \right\}.$$

To begin, we show that $I_0 > -\infty$. Indeed, for any $v \in W_{\phi}^{1,p}(\Omega)$, by the Poincaré, Young ($1 < p/m$) and Hölder inequalities, there exist universal constants $c, C > 0$ such that

$$(3.1) \quad c \|v\|_{L^p}^p - c \|\phi\|_{L^p}^p - \lambda \|\nabla \phi\|_{L^p}^p \leq \lambda \|\nabla v\|_{L^p}^p,$$

and

$$(3.2) \quad -c \|v\|_{L^p}^p - C \leq -C \|v\|_{L^p}^m \leq -K \|v\|_{L^m}^m.$$

Combining (3.1) and (3.2) we obtain

$$(3.3) \quad -C - c \|\phi\|_{L^p}^p - \lambda \|\nabla \phi\|_{L^p}^p \leq \lambda \|\nabla v\|_{L^p}^p - K \|v\|_{L^m}^m,$$

which yields

$$-C - c \|\phi\|_{L^p}^p - \lambda \|\nabla \phi\|_{L^p}^p \leq \int_{\Omega} (\lambda |\nabla v|^p - K |v|^m + \kappa \chi_{\{v>0\}}) dX.$$

Finally, from (G2) and (g1) we find

$$(3.4) \quad \int_{\Omega} (\lambda |\nabla v|^p - K |v|^m + \kappa \chi_{\{v>0\}}) dX \leq \int_{\Omega} F(X, v, \nabla v) dX.$$

Let $v_j \in W_{\phi}^{1,p}(\Omega)$ be a minimizing sequence. We can suppose for $j \gg 1$, that

$$\int_{\Omega} F(X, v_j, \nabla v_j) dX \leq I_0 + 1.$$

From (3.4) and the Hölder inequality we obtain

$$(3.5) \quad \int_{\Omega} |\nabla v_j|^p dX \leq \frac{K}{\lambda} \|v_j\|_{L^m}^m + \frac{I_0}{\lambda} + \frac{1}{\lambda} \leq C \|v_j\|_{L^p}^m + \frac{I_0}{\lambda} + C.$$

By the Poincaré inequality we have

$$(3.6) \quad c_1 \|v_j\|_{L^p}^m \leq C_2 (\|\nabla v_j\|_{L^p}^m + \|\nabla \phi\|_{L^p}^m + \|\phi\|_{L^p}^m).$$

Also we have

$$(3.7) \quad c_3 \|\nabla v_j\|_{L^p}^m \leq C_4 + \frac{1}{2} \|\nabla v_j\|_{L^p}^p.$$

Combining (3.5), (3.6) and (3.7) we obtain

$$(3.8) \quad \int_{\Omega} |\nabla v_j|^p dX \leq C_5 (\|\nabla \phi\|_{L^p}^m + \|\phi\|_{L^p}^m) + \frac{I_0}{\lambda} + C_6.$$

Thus, using the Poincaré inequality once more, we conclude that $\{v_j - \phi\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. By weak compactness, there is a function $u \in W_0^{1,p}(\Omega)$ such that, passing to a subsequence, if necessary,

$$v_j \rightarrow u \text{ weakly in } W^{1,p}(\Omega), \quad v_j \rightarrow u \text{ in } L^p(\Omega) \quad v_j \rightarrow u \text{ a.e. in } \Omega.$$

It now follows from the lower semicontinuity of G (see, for instance, [11], Chapter 5) that

$$\int_{\Omega} G(X, \nabla u) dX \leq \liminf_{j \rightarrow \infty} \int_{\Omega} G(X, \nabla v_j) dX.$$

Condition (g1) and pointwise convergence give

$$\int_{\Omega} g(X, u) dX \leq \liminf_{j \rightarrow \infty} \int_{\Omega} g(X, v_j) dX.$$

In conclusion,

$$\int_{\Omega} F(X, u, \nabla u) dX \leq \liminf_{j \rightarrow \infty} \int_{\Omega} F(X, v_j, \nabla v_j) dX,$$

which proves the existence of a minimizer.

We turn our attention to proving the nonnegativity of u . To begin, we note that

$$\chi_{\{\max(u,0)>0\}} \leq \chi_{\{u>0\}}.$$

Thus,

$$\begin{aligned} & \int_{\Omega} g(X, \max(u,0)) - g(X, u) dX \\ (3.9) \quad & = \int_{\Omega} f((u^+)^m - (u^+)^m) dX + \int_{\Omega} Q(\chi_{\{\max(u,0)>0\}} - \chi_{\{u>0\}}) dX \leq 0. \end{aligned}$$

Then, by minimality of u and (3.9) we obtain

$$\begin{aligned} 0 & \leq \int_{\Omega} F(X, \max(u,0), \nabla(\max(u,0))) - F(X, u, \nabla u) dX \\ & = \int_{\Omega} G(X, \nabla(\max(u,0))) - G(X, \nabla u) dX + \int_{\Omega} g(X, \max(u,0)) - g(X, u) dX \\ & \leq - \int_{\{u<0\}} G(X, \nabla u) dX. \end{aligned}$$

From (G2) we can write

$$0 \geq \int_{\{u \leq 0\}} G(X, \nabla u) dX \geq \lambda \int_{\{u < 0\}} |\nabla u|^p dX = \lambda \int_{\Omega} |\nabla(\min(u,0))|^p dX,$$

and the nonnegativity of u follows since the boundary data ϕ is nonnegative. \square

Remark 3.2. As previously stated, for condition (g1), throughout the whole paper we shall work under the range $1 \leq m < p$. Such a constraint is needed merely for the existence of minima of the functional \mathfrak{F} . Also, by inequalities (3.1), (3.5) and (3.6), it is possible to show existence of minimizer provided K is small enough. In addition, we can obtain critical points of the functional \mathfrak{F} , in the range $p < m \leq p^*$ (see [14]) where $p^* := np/(n-p)$.

Lemma 3.3. *Let u be a minimizer of (1.1). There exists a universal constant $M > 0$ such that $\|u\|_{L^\infty(\Omega)} \leq M$.*

Proof. Define

$$j_0 := \lceil \sup_{\partial\Omega} \phi \rceil,$$

that is, the smallest natural number above $\sup_{\partial\Omega} \phi$. For each $j \geq j_0$ we define the truncated function $u_j: \Omega \rightarrow \mathbb{R}$ by

$$u_j = \begin{cases} j, & \text{if } u > j, \\ u, & \text{if } u \leq j. \end{cases}$$

Clearly, by the choice of j_0 , $u_j \in W_\phi^{1,p}(\Omega)$ and

$$\{u_j > 0\} = \{u > 0\}.$$

If we define $A_j := \{u > j\}$, we have, for each $j > j_0$

$$u = u_j \text{ in } A_j^c \text{ and } u_j = j \text{ in } A_j.$$

Thus, by minimality of u and (G2), there holds

$$\begin{aligned} (3.10) \quad \lambda \int_{A_j} |\nabla u|^p dX &\leq \int_{A_j} G(X, \nabla u) = \int_{\Omega} G(X, \nabla u) - G(X, \nabla u_j) dX \\ &\leq \int_{\Omega} f(u_j^m - u^m) dX = \int_{A_j} f(u^m - j^m) dX. \end{aligned}$$

Taking into account the elementary inequality

$$u^m = (u - j + j)^m \leq 2^m [(u - j)^m + j^m],$$

we obtain

$$\begin{aligned} (3.11) \quad \int_{A_j} |f|(u^m - j^m) dX &\leq C \int_{A_j} |f|(u - j)^m dX + Cj^m \mathcal{L}^n(A_j) \\ &= C \int_{A_j} |f|[(u - j)^+]^m dX + Cj^m \mathcal{L}^n(A_j). \end{aligned}$$

From the range of truncation we consider, it follows that $(u - j)^+ \in W_0^{1,p}(\Omega)$. Hence, applying the Hölder inequality and Gagliardo–Nirenberg inequalities (see Chapter 7 of [10]), we find

$$\int_{A_j} |f|[(u - j)^+]^m dX \leq K [\mathcal{L}^n(A_j)]^{1-m/p^*} \|\nabla u\|_{L^p(A_j)}^m.$$

The Young inequality then gives

$$(3.12) \quad K[\mathcal{L}^n(A_j)]^{1-m/p^*} \|\nabla u\|_{L^p(A_j)}^m \leq C[\mathcal{L}^n(A_j)]^{\frac{p}{p-m} - \frac{pm}{p^*(p-m)}} + \frac{\lambda}{2C} \|\nabla u\|_{L^p(A_j)}^p.$$

Since

$$1 - \left[\frac{p}{p-m} - \frac{pm}{p^*(p-m)} \right] = -\frac{p^2 m}{np(p-m)}$$

we find

$$[\mathcal{L}^n(A_j)]^{\frac{p}{p-m} - \frac{pm}{p^*(p-m)}} \leq C \mathcal{L}^n(A_j).$$

Then, combining (3.10), (3.11), and (3.12), we obtain

$$\int_{A_j} |\nabla u|^p dX \leq C j^m \mathcal{L}^n(A_j) = C j^m [\mathcal{L}^n(A_j)]^{1-p/n+p/n}.$$

Moreover, for each $j > j_0$ we have

$$\begin{aligned} \|(u-j)^+\|_{L^1(A_j)} &= \int_{A_j} (u-j) dX \leq \int_{A_j} (u-j_0) dX \leq \int_{A_{j_0}} (u-j_0) dX \\ &= \|(u-j_0)^+\|_{L^1(A_{j_0})} \leq \|u\|_{L^1(A_{j_0})} + C(\phi, \mathcal{L}^n(\Omega)). \end{aligned}$$

We also have (see (3.1) and (3.8) replacing I_0 by $\int_{\Omega} F(X, \phi, \nabla \phi) dX$),

$$\|u\|_{L^1(A_{j_0})} \leq [\mathcal{L}^n(A_{j_0})]^{(p-1)/p} \|u\|_{L^p(A_{j_0})} \leq [\mathcal{L}^n(\Omega)]^{(p-1)/p} \|u\|_{L^p(\Omega)} \leq C.$$

Taking $\gamma = C$, $\varepsilon = p/n$ and $\alpha = m$ in Lemma 2.1 we obtain the boundedness of u . \square

Remark 3.4. A consequence of the L^∞ estimates for a minimizer u of the functional \mathfrak{F} is universal control on u in $W^{1,p}(\Omega)$. In fact, we have

$$\lambda^{-1} \int_{\Omega} |\nabla u|^p dX \leq \mathfrak{F}(\phi) - \int_{\Omega} f(X)(u^+)^m + Q \chi_{\{u>0\}} dX \leq \mathfrak{F}(\phi) + C \leq C,$$

where $C > 0$ is a universal constant. In conclusion,

$$(3.13) \quad \|u\|_{W^{1,p}(\Omega)} \leq C.$$

Lemma 3.5 (Caccioppoli type inequality). *Let u be a minimizer of the functional (1.1). If $B_r(X_0) \subset \Omega$ is a ball such that $0 < 2r \leq \text{dist}(X_0, \partial\Omega)$ then we have the estimate*

$$(3.14) \quad \int_{B_r(X_0)} |\nabla u|^p dX \leq C r^{n-p},$$

where $C > 0$ is a universal constant.

Proof. Let $\zeta \in C_0^\infty(\Omega)$ be a nonnegative function. Given $\varepsilon > 0$, by the minimality and nonnegativity of u , we have

$$\begin{aligned} 0 &\leq \frac{1}{\varepsilon} \int_{\Omega} G(X, \nabla(u - \varepsilon\zeta)) - G(X, \nabla u) dX \\ &\quad + \frac{1}{\varepsilon} \int_{\Omega} f(X) [((u - \varepsilon\zeta)^+)^m - u^m] dX + \frac{1}{\varepsilon} \int_{\Omega} Q(\chi_{\{u - \varepsilon\zeta > 0\}} - \chi_{\{u > 0\}}) dX \\ &\leq - \int_{\Omega} \langle \mathcal{A}(X, \nabla(u - \varepsilon\zeta)), \nabla\zeta \rangle dX + \frac{1}{\varepsilon} \int_{\Omega} f(X) [((u - \varepsilon\zeta)^+)^m - u^m] dX. \end{aligned}$$

Also, for $\phi := (u - \varepsilon\zeta)^+$, we have

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{\Omega} f(X) [\phi^m - u^m] dX \\ &= \frac{1}{\varepsilon} \int_{\{u > 0\}} f(X) [\phi^m - u^m] dX + \frac{1}{\varepsilon} \int_{\{u=0\}} f(X) [(-\varepsilon\zeta)^+]^m dX \\ &= \int_{\{u > 0\}} f(X) \frac{[\phi^m - u^m]}{\varepsilon} dX. \end{aligned}$$

Taking $\varepsilon \rightarrow 0$ we obtain

$$(3.15) \quad - \int_{\Omega} \langle \mathcal{A}(X, \nabla u), \nabla\zeta \rangle dX - \int_{\Omega} (m f(X) u^{m-1} \chi_{\{u > 0\}}) \zeta dX \geq 0.$$

If $B_r := B_r(X_0)$ let $\eta \in C_0^\infty(B_r)$ be a nonnegative function such that

$$(3.16) \quad \eta = \begin{cases} 1, & \text{in } B_r, \\ 0, & \text{in } B_{2r}^c, \end{cases}$$

and

$$(3.17) \quad \|\nabla\eta\|_{L^\infty(\Omega)} \leq \frac{1}{2r}.$$

From (G3) we obtain

$$(3.18) \quad \langle \mathcal{A}(X, \xi), \xi \rangle = p G(X, \xi),$$

for almost every $X \in \Omega$ and all $\xi \in \mathbb{R}^n$. Moreover, we have (see [11], Chapter 5, Lemma 5.9)

$$(3.19) \quad |\mathcal{A}(X, \xi)| \leq C(p, \lambda) |\xi|^{p-1},$$

for a.e. $X \in \Omega$ and all $\xi \in \mathbb{R}^n$.

Then, taking $\zeta = \eta^p u$ in inequality (3.15) and using (3.18) and (3.19) we find

$$\begin{aligned} 0 &\leq \int_{\Omega} \langle \mathcal{A}(X, \nabla u), -\eta^p \nabla u - p\eta^{p-1} u \nabla\eta \rangle dX - \int_{\Omega} (m f(X) u^{m-1} \chi_{\{u > 0\}}) \eta^p u dX \\ &\leq -C(p, \lambda) \int_{\Omega} |\nabla u|^p \eta^p dX + C(p, \lambda) \int_{\Omega} u |\nabla\eta| |\nabla u|^{p-1} \eta^{p-1} dX \\ &\quad + m M^m K \int_{\Omega} \eta^p dX. \end{aligned}$$

Thus, we find

$$\begin{aligned} C(p, \lambda) \int_{\Omega} |\nabla u|^p \eta^p dX &\leq C(p, \lambda) \int_{\Omega} u^p |\nabla \eta|^p dX + \frac{C(p, \lambda)}{2} \int_{\Omega} |\nabla u|^p \eta^p dX \\ &\quad + C(m, M, K) \int_{\Omega} \eta^p dX, \end{aligned}$$

where we use the Young inequality. Hence, we obtain

$$\int_{\Omega} |\nabla u|^p \eta^p dX \leq C(p, \lambda) \int_{\Omega} u^p |\nabla \eta|^p dX + C(m, M, K, p, \lambda) \int_{\Omega} \eta^p dX,$$

and taking into account the conditions (3.16), (3.17) and the boundedness of u (Lemma 3.3) we obtain

$$\int_{B_r} |\nabla u|^p dX \leq C \left[\frac{C}{r^p} \mathcal{L}^n(B_{2r}) + C \mathcal{L}^n(B_{2r}) \right] \leq C (r^{n-p} + r^n) \leq C r^{n-p},$$

and the lemma is proved. \square

Even though the functional is discontinuous, it is possible to prove that minimizers are universally continuous. The delicate question of optimal regularity will be addressed in the next section.

Theorem 3.6. *Let u be a minimizer of (1.1). There exists a universal constant $\beta \in (0, 1)$ such that $u \in C_{\text{loc}}^{0, \beta}(\Omega)$.*

Proof. Let h be the solution of boundary value problem

$$\begin{cases} \operatorname{div}(\mathcal{A}(X, \nabla h)) = 0 & \text{in } B, \\ h = u & \text{on } \partial B, \end{cases}$$

where $B \Subset \Omega$ is a fixed ball. By the minimality of u , (g1) and the mean value theorem we have

$$\begin{aligned} (3.20) \quad \int_B G(X, \nabla u) - G(X, \nabla h) dX &\leq \int_B g(X, h) - g(X, u) dX \\ &= \int_B f(h^m - u^m) dX + \int_B Q(\chi_{\{h>0\}} - \chi_{\{u>0\}}) dX \\ &\leq mKM^{m-1} \int_B |u - h| dX + \kappa^{-1} \mathcal{L}^n(\{u = 0\} \cap B). \end{aligned}$$

We have used that an \mathcal{A} -harmonic function with nonnegative boundary values is positive and $0 \leq u, h \leq M$ (see [11], Chapter 3, Proposition 3.24). Thus, using (3.18) and monotonicity, see for instance, Lemma 3.2 in [16], we obtain

$$(3.21) \quad \int_B G(X, \nabla u) - G(X, \nabla h) dX \geq c \int_B |\nabla(u - h)|^p dX,$$

where $c = c(n, p, G)$ is a positive constant. The Young and Poincaré inequalities together yield

$$(3.22) \quad mM^{m-1} \int_B |u - h| dX \leq \frac{c}{2} \int_B |\nabla(u - h)|^p dX + C\mathcal{L}^n(B).$$

Thus, if B is a ball of radius $r > 0$, it follows from (3.20), (3.21) and (3.22) that

$$(3.23) \quad \|\nabla(u - h)\|_{L^p(B)} \leq Cr^{n/p}.$$

Now we will estimate $\|\nabla h\|_{L^p(B)}$. If $B = B_r(X_0)$ for some $X_0 \in \Omega$ and $r > 0$ we have (see (2.4) and (2.2) in section 2)

$$(3.24) \quad \|\nabla h\|_{L^\infty(B_{r/2}(X_0))} \leq \frac{C}{r^{n/p}} \|\nabla h\|_{L^p(B)} \leq \frac{C}{r},$$

where we use Lemma 3.5 in the last inequality.

Let $0 < r \leq r_0(\varepsilon)$ with $r_0^\varepsilon \leq 1/2$ for a constant $\varepsilon > 0$ to be chosen. By the inequalities (3.23) and (3.24) we estimate

$$(3.25) \quad \begin{aligned} \|\nabla u\|_{L^p(B_{r,1+\varepsilon})} &\leq \|\nabla(u - h)\|_{L^p(B_{r,1+\varepsilon})} + \|\nabla h\|_{L^p(B_{r,1+\varepsilon})} \\ &\leq \|\nabla(u - h)\|_{L^p(B_r)} + Cr^{\frac{(1+\varepsilon)n}{p}} \|\nabla h\|_{L^\infty(B_{r/2})} \leq C\left(r^{\frac{n}{p}} + r^{\frac{(1+\varepsilon)n}{p}-1}\right). \end{aligned}$$

Taking $\tau = r^{1+\varepsilon}$ we obtain

$$(3.26) \quad \|\nabla u\|_{L^p(B_\delta)} \leq C\tau^{n/p-(1-\theta)},$$

with $0 < \theta = \theta(\varepsilon) < 1$ if $\varepsilon = \varepsilon(n, p) > 0$ is small.

Hence, from Morrey's theorem there is a constant $\beta = \beta(n, p) > 0$ such that $u \in C_{\text{loc}}^{0,\beta}(\Omega)$. \square

Remark 3.7. In Theorem 3.6 we can consider the cases $p = n$ and $p > n$ (the later is trivial). Moreover, If h is as in Theorem 3.6 we have $\|h\|_{L^\infty(B_{r/2}(X_0))} \leq M$. Thus, if G satisfies the assumption of Theorem 2.4 (see Remark 2.4), we obtain

$$\|\nabla h\|_{L^\infty(B_{r/2}(X_0))} \leq \frac{C}{r^{1-\gamma}} \leq \frac{C\mathcal{L}^n(\Omega)^\gamma}{r} \leq \frac{C_1}{r}$$

and the arguments of Theorem 3.6 remain valid. Here we do not use (2.2) and Lemma 3.5 in the proof of Theorem 3.6.

At this stage of the program, an important consequence of Theorem 3.6 is the fact that the positivity set, $\{u > 0\}$, of u is open. Next theorem gives the Euler–Lagrange equations satisfied in this set.

Theorem 3.8. *Let u be a minimizer of (1.1). Within the open set $\{u > 0\}$, u satisfies*

$$\operatorname{div}(\mathcal{A}(X, \nabla u)) = m f(X) u^{m-1}$$

in the distributional sense.

Proof. For fixed $\zeta \in C_0^\infty(\{u > 0\})$, there is $0 < \varepsilon_0 \ll 1$, so small that

$$\{u + \varepsilon\zeta > 0\} = \{u > 0\},$$

for all $0 < \varepsilon \leq \varepsilon_0$. We can write, for $v_\varepsilon := u + \varepsilon\zeta$,

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{\{u>0\}} (F(X, v_\varepsilon, \nabla v_\varepsilon) - F(X, u, \nabla u)) dX \\ &= \frac{1}{\varepsilon} \int_{\{u>0\}} G(X, \nabla v_\varepsilon) - G(X, \nabla u) + \int_{\{u>0\}} f(X) \frac{v_\varepsilon^m - u^m}{\varepsilon}. \end{aligned}$$

Taking $\varepsilon \rightarrow 0$, and using the minimality of u , we obtain

$$0 = \int_{\{u>0\}} \langle \mathcal{A}(X, \nabla u), \nabla \zeta \rangle dX + \int_{\{u>0\}} m f(X) u^{m-1} \zeta dX$$

and the result follows. \square

4. Upper and lower gradient bounds

In the previous section we have shown that minimizers are $C^{0,\beta}$ continuous in Ω , for some unknown $\beta < 1$. From the discontinuity of the functional \mathfrak{F} along the free-surface it is also possible to check that minimizers are not C^1 -regular across the zero level surface $\partial\{u > 0\}$. Thus the optimal regularity one can expect for u is Lipschitz continuity. This is the content of the next theorem.

Theorem 4.1. *Given a subdomain $\Omega' \Subset \Omega$, there exists a constant $C > 0$ that depends only on Ω' and universal constants, such that*

$$\|\nabla u\|_{L^\infty(\Omega')} \leq C.$$

Proof. Suppose, with the aim of obtaining a contradiction, that there exists a sequence of points $X_j \in \Omega' \cap \{u > 0\}$ such that

$$(4.1) \quad X_j \rightarrow \partial\{u > 0\} \quad \text{and} \quad \frac{u(X_j)}{\text{dist}(X_j, \partial\{u > 0\})} \nearrow \infty.$$

We write

$$U_j := u(X_j) \quad \text{and} \quad d_j := \text{dist}(X_j, \partial\{u > 0\}).$$

For each j , let $Y_j \in \partial\{u > 0\}$ be such that

$$d_j = |X_j - Y_j|.$$

Recall we have proven in Theorem 3.8 that

$$\text{div}(\mathcal{A}(X, \nabla u)) = m f(X) u^{m-1} \quad \text{in } \{u > 0\}.$$

Thus, by the Harnack inequality (Theorem 2.2), universal boundedness of u , and (g1), there exists a constant $c > 0$ (see Remark 2.3) that depends only on Ω' and universal constants, such that

$$d_j + \inf_{\overline{B}_{3d_j/4}(X_j)} u \geq cU_j.$$

In turn, we have

$$(4.2) \quad \sup_{\overline{B}_{d_j/4}(Y_j)} u \geq cU_j - d_j.$$

Consider the set

$$(4.3) \quad A_j := \left\{ Z \in B_{d_j}(Y_j) : \text{dist}(Z, \partial\{u > 0\}) \leq \frac{1}{3} \text{dist}(Z, \partial B_{d_j}(Y_j)) \right\}.$$

First we claim that $B_{d_j/4}(Y_j) \subset A_j$. In fact, if $|Z - Y_j| \leq d_j/4$, then

$$\frac{1}{3} \text{dist}(Z, \partial B_{d_j}(Y_j)) \geq \frac{1}{3} \frac{3d_j}{4} = \frac{d_j}{4} \geq \text{dist}(Z, \partial\{u > 0\}).$$

Thus,

$$M_j := \sup_{Z \in A_j} \text{dist}(Z, \partial B_{d_j}(Y_j)) u(Z) = \text{dist}(Z_j, \partial B_{d_j}(Y_j)) u(Z_j) \geq \frac{3d_j}{4} \sup_{\overline{B}_{d_j/4}(Y_j)} u.$$

Therefore,

$$u(Z_j) \geq \frac{d_j}{\text{dist}(Z_j, \partial B_{d_j}(Y_j))} \frac{3}{4} \sup_{\overline{B}_{d_j/4}(Y_j)} u \geq \frac{3}{4} \sup_{\overline{B}_{d_j/4}(Y_j)} u.$$

Hence, using (4.2) we have

$$(4.4) \quad u(Z_j) \geq \frac{3}{4}(cU_j - d_j).$$

For each j , let $W_j \in \partial\{u > 0\}$ be such that

$$(4.5) \quad r_j := |Z_j - W_j| = \text{dist}(Z_j, \partial\{u > 0\}) \leq \frac{1}{3} \text{dist}(Z_j, \partial B_{d_j}(Y_j)).$$

Using (4.5) we conclude that

$$(4.6) \quad r_j \leq \frac{1}{3}(d_j - |Z_j - Y_j|) \leq \frac{1}{3}(d_j - r_j).$$

That is,

$$(4.7) \quad \frac{d_j}{r_j} \geq 4.$$

From (4.4) and (4.7) we have, for j so large that

$$\frac{U_j}{d_j} \geq \frac{1}{c}, \quad \left(c \frac{U_j}{d_j} - 1 \geq 0 \right)$$

the lower estimate

$$(4.8) \quad \frac{u(Z_j)}{r_j} \geq \frac{1}{r_j} \frac{3}{4} (cU_j - d_j) = \frac{3d_j}{4r_j} \left(c \frac{U_j}{d_j} - 1 \right) \geq 4 \frac{3}{4} \left(c \frac{U_j}{d_j} - 1 \right).$$

We have proven that

$$(4.9) \quad \frac{u(Z_j)}{r_j} \rightarrow \infty.$$

If $X \in B_{2r_j}(W_j)$ we obtain (see (4.5))

$$|X - Y_j| \leq |X - W_j| + |W_j - Z_j| + |Z_j - Y_j| \leq 2r_j + r_j + |Z_j - Y_j| \leq d_j.$$

Thus, $B_{2r_j}(W_j) \subset B_{d_j}(Y_j)$. Also we have

$$\text{dist}(X, \partial\{u > 0\}) \leq \frac{r_j}{2},$$

for all $X \in B_{r_j/2}(W_j)$. The triangular inequality and (4.5) then yield

$$\begin{aligned} \text{dist}(X, \partial B_{d_j}(Y_j)) &\geq \text{dist}(Z_j, \partial B_{d_j}(Y_j)) - |Z_j - X| \geq \text{dist}(Z_j, \partial B_{d_j}(Y_j)) - \frac{3r_j}{2} \\ &\geq \text{dist}(Z_j, \partial B_{d_j}(Y_j)) - \frac{1}{2} \text{dist}(Z_j, \partial B_{d_j}(Y_j)) = \frac{1}{2} \text{dist}(Z_j, \partial B_{d_j}(Y_j)). \end{aligned}$$

We conclude that $B_{r_j/2}(W_j) \subset A_j$ and

$$u(Z_j) \geq \frac{M_j}{\text{dist}(Z_j, \partial B_{d_j}(Y_j))} \geq \frac{\text{dist}(X, \partial B_{d_j}(Y_j)) u(X)}{\text{dist}(Z_j, \partial B_{d_j}(Y_j))} \geq \frac{1}{2} u(X),$$

for all $X \in B_{r_j/2}(W_j)$. From the preceding inequality we obtain

$$(4.10) \quad \sup_{B_{r_j/2}(W_j)} u \leq 2u(Z_j).$$

Since $B_{r_j}(Z_j) \subset \{u > 0\}$, by the Harnack inequality there exists a universal constant $c' > 0$ that depends only on Ω' and universal constants such that

$$\inf_{\overline{B}_{3r_j/4}(Z_j)} u \geq c' u(Z_j) - r_j.$$

Therefore, we conclude,

$$\sup_{\overline{B}_{r_j/4}(W_j)} u \geq c' u(Z_j) - r_j$$

and using (4.9) we find (for j sufficiently large)

$$(4.11) \quad \sup_{\overline{B}_{r_j/4}(W_j)} \frac{u}{u(Z_j)} \geq \frac{c'}{2}.$$

For each j , consider the normalized function $u_j: B_1(0) \rightarrow (0, 2)$, defined by

$$(4.12) \quad u_j(X) := \frac{u(W_j + \frac{1}{2} r_j X)}{u(Z_j)}.$$

Notice that from (4.10), and (4.11), we have (for j sufficiently large)

$$(4.13) \quad \max_{B_1(0)} u_j \leq 2, \quad \max_{B_1(0)} u_j \geq \frac{c'}{2}, \quad u_j(0) = 0.$$

Let h be the \mathcal{A} -harmonic function in $B_{r_j/2}(W_j)$ equal to u on the boundary. By (3.20), as in Theorem 3.6, we have

$$(4.14) \quad \int_{B_{r_j/2}(W_j)} \langle \mathcal{A}(X, \nabla u), \nabla u \rangle - \langle \mathcal{A}(X, \nabla h), \nabla h \rangle dX \leq K \int_{B_{r_j/2}(W_j)} |u^m - h^m| dX + C r_j^n.$$

Analogously, for each j sufficiently large, consider the normalized function $h_j: B_1(0) \rightarrow (0, 2)$, defined by

$$(4.15) \quad h_j(X) := \frac{h(W_j + \frac{1}{2} r_j X)}{u(Z_j)}.$$

It is easy to see that

$$(4.16) \quad \begin{cases} \operatorname{div}(\mathcal{A}(W_j + \frac{1}{2} r_j X, \nabla h_j)) = 0 & \text{in } B_1(0) \\ h_j = u_j & \text{on } \partial B_1(0). \end{cases}$$

Also, from the normalization,

$$(4.17) \quad \nabla u_j(X) = \frac{r_j}{2u(Z_j)} \nabla u\left(W_j + \frac{1}{2} r_j X\right), \quad \nabla h_j(X) = \frac{r_j}{2u(Z_j)} \nabla h\left(W_j + \frac{1}{2} r_j X\right),$$

for all $X \in B_1(0)$. By a change of variables and (G3) we obtain

$$\begin{aligned} \int_{B_{r_j/2}(W_j)} |u^m - h^m| dX &\leq u^m(Z_j) C(m) \left(\frac{r_j}{2}\right)^n \int_{B_1(0)} |u_j - h_j| dX \\ &\leq C(n, m) u^m(Z_j) \left(\frac{r_j}{2}\right)^n. \end{aligned}$$

Similarly,

$$\int_{B_{r_j/2}(W_j)} \langle \mathcal{A}(X, \nabla u), \nabla u \rangle dX = \varsigma_j \cdot \int_{B_1(0)} \langle \mathcal{A}(W_j + \frac{r_j}{2} X, \nabla u_j), \nabla u_j \rangle dX,$$

where

$$\varsigma_j := \left(\frac{r_j}{2u(Z_j)}\right)^{-p} \left(\frac{r_j}{2}\right)^n.$$

We conclude that

$$(4.18) \quad \begin{aligned} c \int_{B_1(0)} |\nabla(u_j - h_j)|^p dX &\leq \int_{B_1(0)} \langle \mathcal{A}(W_j + \frac{r_j}{2}X, \nabla u_j), \nabla u_j \rangle dX \\ &\quad - \int_{B_1(0)} \langle \mathcal{A}(W_j + \frac{r_j}{2}X, \nabla h_j), \nabla h_j \rangle dX \\ &\leq l_j \rightarrow 0. \end{aligned}$$

where

$$l_j := C r_j^m \left(\frac{r_j}{u(Z_j)} \right)^{p-m} = C u^m(Z_j) \left(\frac{r_j}{u(Z_j)} \right)^p \leq C \left(\frac{r_j}{u(Z_j)} \right)^p.$$

Moreover, u_j is a minimizer of the functional

$$\mathcal{F}_j(v) := \int_{B_1} G_j(X, \nabla v) + f_j(X)(v^+)^m + Q_j(X)\chi_{\{v>0\}} dX,$$

where

$$\begin{aligned} G_j(X, \xi) &:= G\left(W_j + \frac{r_j}{2}X, \xi\right), \quad \text{for all } X \in B_1, \xi \in \mathbb{R}^n, \\ f_j(X) &:= 2^m u^m(Z_j) \left(\frac{r_j}{2u(Z_j)} \right)^p f\left(W_j + \frac{r_j}{2}X\right), \quad \text{for all } X \in B_1 \end{aligned}$$

and

$$Q_j(X) := \left(\frac{r_j}{2u(Z_j)} \right)^p Q\left(W_j + \frac{r_j}{2}X\right), \quad \text{for all } X \in B_1.$$

Since the functional \mathcal{F}_j satisfies the structural conditions **(G1)**, **(G2)**, **(G3)**, **(G4)**, and **(g1)** and $0 \leq u_j \leq 2$ in B_1 we conclude, as in the proof of Theorem 3.6, that u_j and h_j are uniform Hölder continuous in $B_{8/9}(0)$. Thus, up to a subsequence,

$$u_j \rightarrow u_0 \quad \text{and} \quad h_j \rightarrow h_0,$$

uniformly in $\overline{B_{4/9}(0)}$, and (see Lemma 2.6 and Remark 3.4)

$$(4.19) \quad \nabla u_j \rightarrow \nabla u_0 \quad \text{and} \quad \nabla h_j \rightarrow \nabla h_0,$$

weakly in $L^p(B_{4/9}(0))$. Passing to the limit in (4.16), we find (see Lemma 2.6)

$$(4.20) \quad \operatorname{div}(\mathcal{A}(W_0, \nabla h_0)) = 0 \quad \text{in } B_{2/9}(0),$$

where, up to a subsequence, $W_j \rightarrow W_0 \in \partial\{u > 0\}$. From (4.18) we find

$$\nabla(u_j - h_j) \rightarrow 0 \quad \text{weakly in } L^p(B_{2/9}(0)).$$

Then, we obtain $\nabla(u_0 - h_0) = 0$ in $L^p(B_{4/9}(0))$. Thus, $u_0 = h_0 + c$, for some $c \in \mathbb{R}$, and it solves the elliptic PDE

$$\operatorname{div}(\mathcal{A}(W_0, \nabla u_0)) = 0 \quad \text{in } B_{2/9}(0).$$

Therefore, since $u_0(0) = 0$ and $u_0 \geq 0$, we obtain, by the strong maximum principle, that $u_0 \equiv 0$ in $B_{2/9}(0)$, which contradicts (4.13). Theorem 4.1 is proven. \square

The optimal regularity estimate on u established in Theorem 4.1 implies that u grows at most linearly away from the free surface $\partial\{u > 0\}$. From energy considerations, we will show next that minimizers grow precisely linearly.

Theorem 4.2. *Given a subdomain $\Omega' \Subset \Omega$, there exist constants $c_1 > 0$ and $r_1 > 0$ that depend only on Ω' and universal constants, such that if $X_0 \in \partial\{u > 0\} \cap \Omega'$, $0 < r \leq r_1$, then*

$$\sup_{X \in B_r(X_0)} u(X) \geq c_1 r.$$

Proof. Given a point $X_0 \in \partial\{u > 0\} \cap \Omega'$, define

$$v(X) = \frac{u(X_0 + rX)}{r}, \quad \forall X \in B_1(0).$$

Since u is Lipschitz continuous we have

$$(4.21) \quad \|v\|_{L^\infty(B_1(0))} \leq C.$$

Let h_r be the universal barrier given by

$$\begin{cases} \operatorname{div}(\mathcal{A}(X_0 + rX, \nabla h_r)) = 0 & \text{in } B_1(0) \setminus B_{1/2}(0) \\ h_r = 1 & \text{on } \partial B_1(0), \\ h_r = 0 & \text{in } \overline{B_{1/2}(0)}. \end{cases}$$

Define the test function ξ in $B_1(0)$ by

$$\xi(X) = \min \left\{ v(X), h_r(X) \sup_{B_1(0)} v \right\}.$$

Using (4.21) we have

$$0 \leq \xi \leq v \leq C.$$

By the minimality of v in $B_1(0)$, if we define $\Pi := \{h_r \sup_{B_1(X_0)} u < v\}$ and

$$\Phi := \int_{\Pi} G(X_0 + rX, \nabla \xi) - G(X_0 + rX, \nabla v) dX,$$

we can estimate

$$\begin{aligned} \Phi &= \int_{B_1(0)} G(X_0 + rX, \nabla \xi) - G(X_0 + rX, \nabla v) dX \\ &\geq \int_{B_1(0)} g(X_0 + rX, rv) - g(X_0 + rX, r\xi) dX \\ (4.22) \quad &= r^m \int_{\Pi} f(X_0 + rX)(v^m - \xi^m) dX \\ &\quad + \int_{B_1(0)} Q(X_0 + rX)(\chi_{\{v>0\}} - \chi_{\{\xi>0\}}) dX. \end{aligned}$$

Also we have

$$\begin{aligned}
 (4.23) \quad r^m \int_{\Pi} |f(X_0 + rX)(v^m - \xi^m)| dX &\leq r^m Km C^{m-1} \int_{\Pi} |v - \xi| dX \\
 &\leq r^m C \int_{\Pi} |v - \xi|^p dX + r^m C \leq r^m C \int_{\Pi} |\nabla(v - \xi)|^p dX + r^m C \\
 &\leq r^m C \int_{\Pi} |\nabla v|^p dX + r^m C \int_{\Pi} |\nabla \xi|^p dX + r^m C.
 \end{aligned}$$

Similarly we estimate

$$\begin{aligned}
 (4.24) \quad &\int_{B_1(0)} Q(X_0 + rX)(\chi_{\{v>0\}} - \chi_{\{\xi>0\}}) dX \\
 &= \int_{B_1(0)} Q(X_0 + rX)(1 - \chi_{\{\xi>0\}}) dX \\
 &= \int_{B_1(0)} Q(X_0 + rX)\chi_{\{\xi=0\}} dX \geq \epsilon \mathcal{L}^n(\{\xi = 0\}) \geq \epsilon \mathcal{L}^n(B_{1/2}(0)),
 \end{aligned}$$

where $C > 0$ is a universal constant. Taking into account (G2) and (G3), we obtain

$$\begin{aligned}
 (4.25) \quad &\int_{\Pi} G(X_0 + rX, \nabla \xi) - G(X_0 + rX, \nabla v) dX \\
 &\leq \lambda^{-1} \int_{\Pi} |\sup_{B_1(0)} v \nabla h_r|^p dX - \lambda \int_{\Pi} |\nabla v|^p dX \\
 &\leq C \left(\sup_{B_1(0)} v \right)^p \int_{B_1(0)} |\nabla h_r|^p dX - \lambda \int_{\Pi} |\nabla v|^p dX,
 \end{aligned}$$

for a universal $C > 0$. Also by the $C^{1,\alpha}$ estimates for h_r , we have

$$C \left(\sup_{B_1(0)} v \right)^p \int_{B_1(0)} |\nabla h_r|^p dX \leq C \left(\sup_{B_1(0)} v \right)^p$$

for a universal constant $C > 0$. Substituting inequalities (4.23), (4.24) and (4.25) into (4.22) and taking r sufficiently small, we obtain

$$C \left(\sup_{B_1(0)} v \right)^p \geq (\lambda - r^m C) \int_{\Pi} |\nabla v|^p dX + c \mathcal{L}^n(B_{1/2}(0)) \geq c \mathcal{L}^n(B_{1/2}(0)),$$

for universal constants $c > 0$ and $C > 0$. Therefore, $\sup_{B_1(0)} v \geq c$, and strong nondegeneracy is proven. \square

Corollary 4.3. *Given a subdomain $\Omega' \Subset \Omega$, there exist constants $c, d_0 > 0$ that depend only on Ω' and universal constants, such that if $X \in \{u > 0\} \cap \Omega'$ and $\text{dist}(X, \partial\{u > 0\}) \leq d_0$, then*

$$u(X) \geq c \cdot \text{dist}(X, \partial\{u > 0\}).$$

We emphasize that the technique employed in proof of Theorem 4.2 is a generalization of the ones used in [1].

5. Hausdorff estimates on the free boundary

In this section we turn our attention to fine Hausdorff estimates on the free surface $\partial\{u > 0\}$.

Theorem 5.1. *Given a subdomain $\Omega' \Subset \Omega$ and $Z \in \partial\{u > 0\} \cap \Omega'$, there exist constants $r_0 > 0$ and $0 < \varsigma < 1$ that depend only on Ω' and universal constants, such that,*

$$(5.1) \quad \varsigma \omega_n r^n \leq \mathcal{L}^n(B_r(Z) \cap \{u > 0\}) \leq (1 - \varsigma) \omega_n r^n,$$

for all $0 \leq r \leq r_0$.

Proof. Let $X_0 \in \bar{B}_{r/4}(Z)$ be a maximum point of u , i.e.,

$$u(X_0) = \sup_{X \in B_{r/4}(Z)} u(X).$$

By the strong nondegeneracy property ($0 < r \leq r_0$ with $r_0 = r_1$ and $r_1 > 0$ as in Theorem 4.2) we obtain

$$u(X_0) \geq cr > 0,$$

where $c > 0$ is a universal constant. By the Lipschitz continuity of u there exists a universal constant $C > 0$ such that

$$u(X) \geq u(X_0) - C|X - X_0| \geq \frac{cr}{2} > 0, \quad \forall X \in B_{\frac{c}{2C}r}(X_0),$$

with $C \geq c$. Hence,

$$B_{\frac{c}{2C}r}(X_0) \subset B_r(Z) \cap \{u > 0\},$$

and the lower bound in (5.1) follows.

Now we prove the upper bound. We argue by contradiction, i.e., let us assume that there exists a sequence of positive real numbers r_j with $r_j \searrow 0$ as $j \rightarrow \infty$ and

$$(5.2) \quad \frac{\mathcal{L}^n(B_{r_j}(Z) \cap \{u = 0\})}{r_j^n} \rightarrow 0.$$

We define the sequence $u_j: B_1(0) \rightarrow \mathbb{R}$ by

$$(5.3) \quad u_j(X) := \frac{u(Z + r_j X)}{r_j}.$$

Let h_j be the solution to

$$(5.4) \quad \begin{cases} \operatorname{div}(\mathcal{A}(Z + r_j X, \nabla h_j)) = 0 & \text{in } B_1(0) \\ h_j = u_j & \text{on } \partial B_1(0). \end{cases}$$

Notice that by Lipschitz continuity of u , both u_j and h_j are bounded. Estimates (3.20) and (3.21) from the proof of Theorem 3.6 give after renormalization,

$$\begin{aligned} \int_{B_1(0)} |\nabla(h_j - u_j)|^p dX &\leq r_j^m \int_{B_1(0)} |h_j^m - u_j^m| dX + C \frac{\mathcal{L}^n(B_{r_j}(Z) \cap \{u = 0\})}{r_j^n} \\ &\leq r_j^m C \int_{B_1(0)} |h_j - u_j| dX + C \frac{\mathcal{L}^n(B_{r_j}(Z) \cap \{u = 0\})}{r_j^n} \\ &\leq r_j^m C \int_{B_1(0)} |h_j - u_j|^p dX + r_j^m C + C \frac{\mathcal{L}^n(B_{r_j}(Z) \cap \{u = 0\})}{r_j^n} \\ &\leq r_j^m C \int_{B_1(0)} |\nabla(h_j - u_j)|^p dX + r_j^m C + C \frac{\mathcal{L}^n(B_{r_j}(Z) \cap \{u = 0\})}{r_j^n}, \end{aligned}$$

where $C > 0$ is a universal constant. Hence, for j sufficiently large we have

$$(5.5) \quad \int_{B_1(0)} |\nabla(h_j - u_j)(X)|^p dX \leq r_j^m C + C \frac{\mathcal{L}^n(B_{r_j}(Z) \cap \{u = 0\})}{r_j^n}.$$

Moreover, by the Lipschitz regularity of u and the $C^{1,\alpha}$ elliptic estimate we can assume that

$$u_j \rightarrow u_0 \quad \text{and} \quad h_j \rightarrow h_0$$

uniformly in $B_{4/5}(0)$. Since h_j is the solution to problem (5.4) we obtain

$$\operatorname{div}(\mathcal{A}(Z, \nabla h_0(Y))) = 0, \quad \text{in } B_{1/2}(0).$$

It follows from (5.2) and (5.5) that

$$u_0 = h_0 + c, \quad \text{in } B_{1/2}(0).$$

where c is a constant. Then,

$$\operatorname{div}(\mathcal{A}(Z, \nabla u_0(Y))) = 0, \quad \text{in } B_{1/2}(0).$$

Since $u_0 \geq 0$ and $u_0(0) = 0$, by the strong maximum principle, we obtain $u_0 \equiv 0$ in $B_{1/2}(0)$. But this contradicts the strong nondegeneracy property, shown in Theorem 4.2. \square

Theorem 3.8 gives the Euler–Lagrange equation u satisfies within its set of positivity. To further investigate the behavior of u along the free boundary, we need to obtain the equation u satisfies across the free surface of discontinuity of the functional, $\partial\{u > 0\}$.

Lemma 5.2. *Let u be a minimizer of the functional (1.1), then*

$$\operatorname{div}(\mathcal{A}(X, \nabla u)) - m f(X) u^{m-1} \chi_{\{u > 0\}} \geq 0, \quad \text{in } \Omega$$

in the sense of distribution. In particular it defines a Radon measure

$$\Lambda := \operatorname{div}(\mathcal{A}(X, \nabla u)) - m f(X) u^{m-1} \chi_{\{u > 0\}}$$

Furthermore, the support of Λ is contained in $\partial\{u > 0\}$.

Proof. By Lemma 3.5 we have

$$-\int_{\Omega} \langle \mathcal{A}(X, \nabla u), \nabla \zeta \rangle dX - \int_{\Omega} (m f(X) u^{m-1} \chi_{\{u>0\}}) \zeta dX \geq 0,$$

for all nonnegative functions $\zeta \in C_0^\infty(\Omega)$. Moreover, as in Theorem 3.8,

$$(5.6) \quad \operatorname{div} \mathcal{A}(X, \nabla u) = m f(X) u^{m-1} \quad \text{in } \{u > 0\}.$$

Hence, the measure Λ defined by

$$\int_{\Omega} \zeta d\Lambda := -\int_{\Omega} \langle \mathcal{A}(X, \nabla u), \nabla \zeta \rangle dX - \int_{\Omega} (m f(X) u^{m-1} \chi_{\{u>0\}}) \zeta dX,$$

is a nonnegative Radon measure with support on $\Omega \cap \partial\{u > 0\}$. \square

With the aid of the measure Λ , we can establish fine upper and lower control on the \mathcal{H}^{n-1} Hausdorff measure of the free boundary, which ultimately yields important geometric measure theoretic information about $\partial\{u > 0\}$.

Theorem 5.3. *The set $\{u > 0\}$ has locally finite perimeter and for fixed $\Omega' \Subset \Omega$ there exist constants \underline{c} and \overline{C} that depend only on Ω' and universal constants such that*

$$(5.7) \quad \underline{c} r^{n-1} \leq \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r(Z)) \leq \overline{C} r^{n-1}$$

for any ball $B_r(Z) \subset \Omega'$ centered at a free boundary point, $Z \in \partial\{u > 0\}$. In particular,

$$\mathcal{H}^{n-1}(\partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0.$$

Proof. Through a suitable approximation scheme, using test function $0 \leq \zeta_k \leq 1$, ζ_k with $\zeta_k \rightarrow \chi_{B_r(Z)}$, we have (for almost $r > 0$)

$$(5.8) \quad \begin{aligned} \int_{B_r(Z)} d\Lambda &= \int_{\partial B_r(Z)} \langle \mathcal{A}(X, \nabla u), \nu \rangle dS_X - \int_{B_r(Z)} m f(X) u^{m-1} \chi_{\{u>0\}} dX \\ &\leq C \|\nabla u\|_{L^\infty(\Omega')}^{p-1} r^{n-1} + C r^{n-1} \leq C r^{n-1}, \end{aligned}$$

where $C > 0$ is a constant that depend only on Ω' and universal constants. This proves the upper bound in (5.7).

To check the lower bound, let us assume, aiming at a contradiction, that there exists a sequence of positive real numbers r_j such that $r_j \searrow 0$ as $j \rightarrow \infty$ and

$$(5.9) \quad \frac{\mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_{r_j}(Z))}{r_j^{n-1}} \rightarrow 0.$$

With the notation used in Theorem 5.1 and Lemma 5.2, we obtain the sequence of nonnegative measures Λ_j in $B_{4/5}(0)$, defined by

$$(5.10) \quad \Lambda_j := [\operatorname{div}(\mathcal{A}(Z + r_j X, \nabla u_j)) - r_j^m m f(Z + r_j X) \chi_{\{u_j>0\}} u_j^{m-1}] dX.$$

By compactness we can assume that $\Lambda_j \rightharpoonup \Lambda_0$ in the sense of measures. Moreover, using (5.9) we have

$$(5.11) \quad \Lambda_j \rightharpoonup 0.$$

In the sequel we will show that

$$(5.12) \quad \Lambda_0 := \operatorname{div}(\mathcal{A}(Z, \nabla u_0)).$$

From the uniform positive density property we know $\mathcal{L}^n(\partial\{u_0 > 0\}) = 0$. Thus, we only need to verify (5.12) for balls B contained entirely in $\{u_0 = 0\}$ and in $\{u_0 > 0\}$. Let $B \subset \{u_0 > 0\}$. Define

$$A_j(X) := \mathcal{A}(Z + r_j X, \nabla u_j), \quad \forall X \in B_{4/5}(0).$$

By the Lipschitz regularity of u and (G2) we have

$$|A_j| \leq C(n, \Lambda) |\nabla u_j|^{p-1} \leq C.$$

Thus, we may extract a subsequence (that we denote by A_j) such that

$$A_j \rightarrow A_0 \quad \text{weak-} \star \text{ in } L^\infty(B_{4/5}).$$

Furthermore, u_j converges in the $C^{1,\alpha}$ topology to u_0 in B (see Section 2, Theorem 2.4 and Remark 5.5 below). Hence,

$$A_j \rightarrow \mathcal{A}q(Z, \nabla u_0) \quad \text{weak-} \star \text{ in } L^\infty(B).$$

Also we have

$$(5.13) \quad \left| \int_B r_j^m m f(Z + r_j X) \chi_{\{u_j > 0\}} u_j^{m-1} dX \right| \leq r_j^m C \mathcal{L}^n(B) \rightarrow 0.$$

Hence, for $B \subset \{u_0 > 0\}$, indeed (5.12) does hold. Now suppose $B \subset \{u_0 = 0\}$. Clearly,

$$[\operatorname{div}(\mathcal{A}q(Z, \nabla u_0))](B) = 0.$$

On the other hand, if B_k is a sequence of balls such that $B_k \nearrow B$ then for some $j_k \in \mathbb{N}$ we have

$$(5.14) \quad u_j \equiv 0 \quad \text{in } B_k \text{ for all } j > j_k.$$

Indeed, let $\tilde{B} \subset B$. If there were a subsequence u_{j_k} satisfying $u_{j_k} \neq 0$ in \tilde{B} then, by the strong nondegeneracy property (Theorem 4.2), there must exist points $P_{k_j} \in \tilde{B}$ such that

$$u_{j_k}(P_{k_j}) \geq c > 0.$$

Passing to another subsequence we can assume $P_{k_j} \rightarrow P \in \tilde{B}$. Since $u_{j_k} \rightarrow u_0$ uniformly we obtain $u_0(P) > 0$ which is a contradiction.

Thus, from (5.14) we obtain

$$(5.15) \quad \Lambda_j(B) \rightarrow 0.$$

Therefore, (5.12) holds for any $B \subset \{u_0 = 0\}$, and combining (5.11) and (5.12) we find

$$\operatorname{div}(\mathcal{A}(Z, \nabla u_0)) = 0 \quad \text{in } B_{4/5}(0).$$

However, as before, this contradicts the nondegeneracy of u_0 established in Theorem 4.2. \square

Theorem 5.4 (Representation). *Let u be a minimizer of (1.1). Then*

$$\operatorname{div}(\mathcal{A}(X, \nabla u)) - m f(X) u^{m-1} \chi_{\{u>0\}} = \tilde{Q} \mathcal{H}^{n-1} \llcorner \partial_{\text{red}}\{u > 0\},$$

in the sense of measures, for some Borel function \tilde{Q} . Moreover, \tilde{Q} is bounded away from zero and infinity. That is for a universal constant $C > 0$ there holds

$$C^{-1} \leq \tilde{Q} \leq C.$$

In particular, the free boundary $\partial\{u > 0\}$ is a set of finite perimeter.

Proof. This follows from Theorem 5.3 and standard arguments (see Theorem 4.5 in [1]). \square

Remark 5.5. Let u be a minimizer of (1.1) in Ω and let $B_{r_j}(X_j) \subset \Omega$ be a sequence of balls with $r_j \rightarrow 0$, $X_j \rightarrow X_0 \in \Omega$, and $u(X_j) = 0$. Consider the blow-up sequence

$$u_j(X) = \frac{1}{r_j} u(X_j + r_j X).$$

Since the u_j are uniformly Lipschitz continuous, for a subsequence,

$$(5.16) \quad u_j \rightarrow u_0 \quad \text{in } C_{\text{loc}}^\alpha(\mathbb{R}^n) \text{ for every } 0 < \alpha < 1,$$

$$(5.17) \quad \nabla u_j \rightarrow \nabla u_0 \quad \text{weak-} \star \text{ in } L_{\text{loc}}^\infty(\mathbb{R}^n),$$

$$(5.18) \quad \partial\{u_j > 0\} \rightarrow \partial\{u_0 > 0\} \quad \text{locally in the Hausdorff distance,}$$

$$(5.19) \quad \chi_{\{u_j > 0\}} \rightarrow \chi_{\{u_0 > 0\}} \quad \text{in } L_{\text{loc}}^1(\mathbb{R}^n).$$

Moreover, by the classical truncation argument (see for instance, [2]),

$$(5.20) \quad \nabla u_j \rightarrow \nabla u_0 \quad \text{a.e.}$$

Also, by Theorem 3.8, assertions (5.16)–(5.20) and $C^{1,\alpha}$ convergence within the positive set, we obtain

$$\begin{aligned} - \int_{\{u_0 > 0\}} \langle \mathcal{A}(X_0, \nabla u_0), \nabla \zeta \rangle dX &= - \lim_{j \rightarrow \infty} \int_{\{u_0 > 0\}} \langle \mathcal{A}(X_j + r_j X, \nabla u_j), \nabla \zeta \rangle dX \\ &= \lim_{j \rightarrow \infty} r_j^m \int_{\{u_j > 0\}} m f(X_j + r_j X) u_j^{m-1} \zeta dX = 0, \end{aligned}$$

for all $\zeta \in C_0^\infty(\{u_0 > 0\})$. In fact,

$$\left| r_j^m \int_{\{u_j > 0\}} m f(X_j + r_j X) u_j^m \zeta dX \right| \leq r_j^m C,$$

for all $\zeta \in C_0^\infty(\{u_0 > 0\})$. Hence,

$$\operatorname{div}(\mathcal{A}(X_0, \nabla u_0)) = 0 \quad \text{in } \{u_0 > 0\}.$$

We are in position to obtain the blown-up minimization problem.

Lemma 5.6. *If $u(X_j) = 0$, $X_j \rightarrow X_0 \in \Omega$, then any blow-up limit u_0 with respect to $B_{r_j}(X_j)$ is minimizer of the functional*

$$(5.21) \quad \mathfrak{F}_0(v) := \int_{B_1(0)} G(X_0, \nabla v) + Q(X_0) \chi_{\{v > 0\}} dX.$$

Proof. Set $D = B_1(0)$. Take any v , $v - u_0 \in H_0^1(D)$, $\eta \in C_0^\infty(D)$, $0 \leq \eta \leq 1$. Consider

$$v_j := v + (1 - \eta)(u_j - u_0) \quad \text{and} \quad Q_j(X) := Q(X_j + r_j X), \quad \forall X \in D.$$

We will write

$$\mathfrak{F}_j(v) := \int_{B_1(0)} G(X_j + r_j X, \nabla v) + r_j^m f(X_j + r_j X)(v^+)^m + Q_j(X) \chi_{\{v > 0\}} dX.$$

Since $v_j = u_j$ in ∂D and u_j is a local minimum we have, for large j ,

$$(5.22) \quad \mathfrak{F}_j(u_j) \leq \mathfrak{F}_j(v_j).$$

From the fact that $|\nabla u_j| \leq C$ and $\nabla u_j \rightarrow \nabla u_0$ a.e., we conclude

$$\int_D G(X_j + r_j X, \nabla u_j) dX \rightarrow \int_D G(X_0, \nabla u_0) dX.$$

Similarly

$$\int_D G(X_j + r_j X, \nabla v_j) dX \rightarrow \int_D G(X_0, \nabla v_0) dX.$$

Moreover, we have

$$\left| \int_D r_j^m f(X_j + r_j X) (u_j^m - (v_j^+)^m) dX \right| \leq r_j^m C \int_D (|u_j|^{p^*} + |v_j|^{p^*}) dX,$$

where, as before, $p^* := np/(n-p)$. Since there exists a universal constant $C > 0$ such that $\|u_j\|_{L^\infty} \leq C$ (and Ω is bounded) we have

$$\|u_j\|_{L^{p^*}} \leq C.$$

By definition of v_j

$$\|v_j\|_{L^{p^*}} \leq C \|v\|_{L^{p^*}} + C(\|u_j\|_{L^{p^*}} + \|u_0\|_{L^{p^*}}).$$

Therefore,

$$\left| \int_D r_j^m f(X_j + r_j X) u_j^m - r_j^m f(X_j + r_j X) (v_j^+)^m dX \right| \rightarrow 0.$$

Also, we have

$$\left| \int_D (Q_j - Q(X_0)) (\chi_{\{u_j > 0\}} - \chi_{\{v_j > 0\}}) dX \right| \leq 2 \int_D |Q_j - Q(X_0)| dX,$$

and using the continuity of the function Q we obtain

$$\int_D (Q_j - Q(X_0)) (\chi_{\{u_j > 0\}} - \chi_{\{v_j > 0\}}) dX \rightarrow 0.$$

Finally,

$$\chi_{\{v_j > 0\}} \leq \chi_{\{v > 0\}} + \chi_{\{\eta < 1\}}$$

and (see (5.19))

$$\int_D \chi_{\{u_j > 0\}} dX \rightarrow \int_D \chi_{\{u_0 > 0\}} dX.$$

Then it follows from (5.22) that

$$\begin{aligned} \int_D G(X_0, \nabla u_0) + Q(X_0) \chi_{\{u_0 > 0\}} dX &\leq \int_D G(X_0, \nabla v) \\ &\quad + (\chi_{\{v > 0\}} + \chi_{\{\eta < 1\}}) Q(X_0) dX. \end{aligned}$$

Taking $\eta \rightarrow 1$ finishes up the proof. \square

As a consequence, we can classify blow-ups at points in the reduced free boundary.

Theorem 5.7. *For almost every $X_0 \in \partial_{\text{red}}\{u > 0\}$ we have*

$$u(X) = \alpha(X_0) \langle X - X_0, \nu(X_0) \rangle^+ + o(|X - X_0|),$$

for any $X \in \{u > 0\}$ near X_0 , where $\nu(X_0)$ is the measure theoretic normal vector to $\partial_{\text{red}}\{u > 0\}$ at X_0 and

$$(5.23) \quad \alpha(X_0) = \left(\frac{\tilde{Q}(X_0)}{\langle \mathcal{A}(X_0), \nu(X_0) \rangle, \nu(X_0) \rangle} \right)^{1/(p-1)}.$$

Proof. From 4.5.6 (2), 2.9.8, and 2.9.9 of [9], applied to \mathcal{H}^{n-1} on $\partial\{u > 0\}$ and the Vitali relation

$$\{(X, B_r(X)) : X \in \partial\{u > 0\}, B_r(X) \Subset \Omega\},$$

for almost every $X_0 \in \partial_{\text{red}}\{u > 0\}$ we have the condition

$$(5.24) \quad \theta^{*n-1}(\mathcal{H}^{n-1} \llcorner \partial\{u > 0\}, X_0) \leq 1$$

and

$$(5.25) \quad \int_{B_r(X_0) \cap \partial\{u > 0\}} |\tilde{Q} - \tilde{Q}(X_0)| d\mathcal{H}^{n-1} = o(r^{n-1})$$

for $r \rightarrow 0$, where the symbol in (5.25) represents the average. Recalling the notation used in Remark 5.5 we have, from standard geometric measure theory arguments together with nondegeneracy and assertions (5.16)–(5.20),

$$\begin{aligned} u_0 &\equiv 0 \quad \text{in } \{X \in \mathbb{R}^n : \langle X, \nu(X_0) \rangle < 0\} \\ \text{and } \{u_0 > 0\} &= \{X \in \mathbb{R}^n : \langle X, \nu(X_0) \rangle > 0\}. \end{aligned}$$

Also we have

$$\text{div}(\mathcal{A}(X_0, \nabla u_0)) = 0 \quad \text{in } \{u_0 > 0\}.$$

Since $\partial\{u_0 > 0\}$ is the smooth surface $\{X \in \mathbb{R}^n : \langle X, \nu(X_0) \rangle = 0\}$ we obtain

$$\partial\{u_0 > 0\} = \partial_{\text{red}}\{u_0 > 0\}.$$

By Theorem 5.4 we find

$$\begin{aligned} - \int_{\Omega'} \langle \mathcal{A}(X_0 + r_j X, \nabla u_j), \nabla \zeta \rangle dX &= r_j^m \int_{\Omega'} m f(X_0 + r_j X) u_j^{m-1} \chi_{\{u_j > 0\}} \zeta dX \\ &\quad + \int_{\partial\{u_j > 0\} \cap \Omega'} \tilde{Q}(X_0 + r_j X) \zeta d\mathcal{H}^{n-1}, \end{aligned}$$

for $\zeta \in C_0^\infty(\Omega')$, where $\Omega' \Subset \{\langle X, \nu(X_0) \rangle < 0\}$. Now let $\eta \in C_0^\infty(B'_r)$ be a nonnegative function, where B'_r is a ball in $\mathbb{R}^{n-1} := \{\langle X, \nu(X_0) \rangle < 0\}$. By standard arguments (see [1], Theorem 4.8) we have

$$\int_{\partial\{u_j > 0\}} \zeta d\mathcal{H}^{n-1} \rightarrow \int_{\mathbb{R}^{n-1}} \eta d\mathcal{H}^{n-1}.$$

Furthermore, for some universal constant $C > 0$ we have

$$|r_j^m \int_{\Omega'} m f(X_0 + r_j X) u_j^{m-1} \chi_{\{u_j > 0\}} \zeta dX| \leq r^m C,$$

where we use that the sequence u_j is universally bounded. Thus, using (5.25) we obtain

$$\text{div}(\mathcal{A}(X_0, \nabla u_0)) = \tilde{Q}(X_0) \mathcal{H}^{n-1} \llcorner \{X \in \mathbb{R}^n : \langle X, \nu(X_0) \rangle = 0\}.$$

Moreover, since $\partial_{\text{red}}\{u_0 > 0\}$ is C^1 we have

$$(5.26) \quad \langle \mathcal{A}(X_0, \nu(X_0)), \nu(X_0) \rangle = \tilde{Q}(X_0),$$

for X_0 at $\partial_{\text{red}}\{u_0 > 0\}$. However,

$$\nu(X_0) = \frac{\nabla u(X_0)}{|\nabla u(X_0)|},$$

and using (G3) and the identity (5.26) we obtain

$$\alpha(X_0) = \left(\frac{\tilde{Q}(X_0)}{\langle \mathcal{A}(X_0, \nu(X_0)), \nu(X_0) \rangle} \right)^{1/(p-1)}, \quad \forall X \in \{\langle X, \nu(X_0) \rangle = 0\}.$$

Hence, we conclude

$$(5.27) \quad \nabla u_0(X) \cdot \nu(X_0) = \alpha(X_0), \quad \forall X \in \{\langle X, \nu(X_0) \rangle = 0\}.$$

Define the function v_0 by

$$(5.28) \quad v_0(X) = \begin{cases} u_0(X), & \text{if } X \in \{\langle X, \nu(X_0) \rangle < 0\} \\ -u_0(X^*), & \text{if } X \in \{\langle X, \nu(X_0) \rangle \geq 0\}, \end{cases}$$

where X^* is the reflection of X with respect to the hyperplane $\{\langle X, \nu(X_0) \rangle = 0\}$. Using standard arguments we verify that v_0 is Lipschitz continuous in \mathbb{R}^n (u_0 is Lipschitz continuous by Theorem 4.1) and

$$\text{div}(\mathcal{A}(X_0, \nabla v_0)) = 0 \quad \text{in } \mathbb{R}^n.$$

By the $C^{1,\beta}$ regularity of v_0 , we can apply the blow-up argument from [12] to conclude that v_0 is an affine function. Then, using (5.27) we find

$$u_0(X) = \alpha(X_0) \langle X - X_0, \nu(X_0) \rangle^+$$

and the result is proved. \square

Remark 5.8. If

$$\mathcal{A}(X, \xi) = |\xi|^{p-2} \xi,$$

that is, in the case of the p -Laplacian, we conclude from Lemma 5.6 that the blow-up u_0 is a minimizer of the functional

$$\int_{B_1(0)} |\nabla v|^p + Q(X_0) \chi_{\{v>0\}} dX.$$

Through a standard Hadamard domain variation argument (see [1], Theorems 2.5 and 5.5 and [6], Theorems 2.1 and 5.6), we obtain the free boundary condition

$$\alpha(X_0) = \left(\frac{Q(X_0)}{p-1} \right)^{1/p},$$

for $X_0 \in \partial_{\text{red}}\{u > 0\}$.

Using (5.23) we have the relation

$$\tilde{Q}(X_0) = \left(\frac{Q(X_0)}{p-1} \right)^{(p-1)/p}.$$

between \tilde{Q} and Q , for almost every $X_0 \in \partial_{\text{red}}\{u > 0\}$. Taking $p = 2$ we find the relation that appears in [1].

6. Jet flow problems and smoothness of the free boundary

In this section we address the question of the smoothness of the free boundary.

Because our primary motivations come from the theory of heterogeneous jet flows, in this section we shall only treat the nondegenerate problem, i.e., we will work under the assumptions

$$(6.1) \quad F(X, \xi) = \frac{1}{2} A(X) |\xi|^2 + f(X) (u^+)^m + Q \chi_{\{u > 0\}},$$

for $X \in \Omega$ and $\xi \in \mathbb{R}^n$, $1 \leq m < 2$, $f \in C(\Omega)$ and $Q \in C^{0,\beta}$, $0 < \kappa < Q < \kappa^{-1}$. The matrix A is assumed to be Lipschitz and positive definite.

The proof we will present for smoothness of the reduced free boundary is based on a flatness improvement coming from Harnack type estimates and it follows closely the recent work of [7]. There are a few subtle differences though. For instance the equation we consider is naturally in divergence form, thus it has drift terms in non-divergence form. Also the free boundary condition obtained in (5.23) is a bit more involved than the one treated in [7]. For sake of completeness and the readers' convenience, we shall carry out all the details.

We shall use Caffarelli's viscosity solution setting to address the free boundary regularity theory. We recall some terminologies. Let $u, \phi \in C(\Omega)$. If $u(X_0) = \phi(X_0)$ and there exists a neighborhood V of X_0 such that

$$(6.2) \quad u(X) \geq \phi(X) \quad (\text{respectively } u(X) \leq \phi(X)) \quad \text{in } V,$$

we say that ϕ touches u from below (respectively above) at $X_0 \in \Omega$. Moreover, if the inequality in (6.2) is strict in $V \setminus \{X_0\}$, we say that ϕ touches u strictly from below (resp. above) at $X_0 \in \Omega$.

The next proposition is standard in Caffarelli's theory ([3], [4], [5]), so we omit its proof.

Proposition 6.1. *Assume (6.1). A minimizer u of (1.1) is a viscosity solution of*

$$(6.3) \quad \begin{cases} \operatorname{div}(A(X)\nabla u) = g(X) & \text{in } \Omega_+(u) := \{u > 0\}, \\ \langle A\nabla u, \nabla u \rangle = Q & \text{on } \mathcal{F}(u) := \partial\{u > 0\} \cap \Omega, \end{cases}$$

where $g \in L^\infty(\Omega) \cap C(\Omega)$. The free boundary condition above is understood in the Caffarelli viscosity sense: if $\phi \in C^2(\Omega)$ and ϕ^+ touches u from below (respectively from above) at $X_0 \in \mathcal{F}(u)$ with $|\nabla\phi|(X_0) \neq 0$ then

$$(6.4) \quad \langle A(X_0)\nabla\phi(X_0), \nabla\phi(X_0) \rangle \leq Q(X_0) \quad (\text{resp. } \geq Q(X_0)).$$

The free boundary regularity result we will prove in this section is the following:

Theorem 6.2. *Let u be a viscosity solution to (6.3) in the ball $B_1(0)$. Suppose that $0 \in \mathcal{F}(u)$, $Q(0) = 1$, and $a_{ij}(0) = \delta_{ij}$. There exists a universal constant $\tilde{\varepsilon} > 0$ such that, if the graph of u is $\tilde{\varepsilon}$ -flat in $B_1(0)$, i.e.*

$$(6.5) \quad (X_n - \tilde{\varepsilon})^+ \leq u(X) \leq (X_n + \tilde{\varepsilon})^+ \quad \text{for } X \in B_1(0),$$

and

$$(6.6) \quad [a_{ij}]_{C^{0,1}(B_1(0))} \leq \tilde{\varepsilon}, \quad \|g\|_{L^\infty(B_1(0))} \leq \tilde{\varepsilon}, \quad [Q]_{C^{0,\beta}(B_1(0))} \leq \tilde{\varepsilon},$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}(0)$.

Corollary 6.3. *Assume (6.1). The reduced free boundary of a minimizer u of (1.1) is locally a $C^{1,\gamma}$ surface. In particular,*

$$\langle A(Z)\nabla u(Z), \nabla u(Z) \rangle = Q(Z),$$

in the classical sense for \mathcal{H}^{n-1} almost all free boundary points $Z \in \partial\{u > 0\}$.

Thus, taking $g(X) = mf(X)u^{m-1}$ we obtain the free boundary regularity theory for our problem. The approach is based fundamentally on comparison criterion.

Definition 6.4. Let $v \in C^2(\Omega)$. We say v is a strict (comparison) subsolution (respectively supersolution) to (6.3) in Ω , if the following conditions are satisfied:

1. $\operatorname{div}(A(X)\nabla v) > g(X)$ (resp. $<$) in $\Omega_+(v)$;
2. If $X_0 \in \mathcal{F}(v)$ then

$$\langle A\nabla\phi, \nabla\phi \rangle(X_0) > Q(X_0) \quad (\text{resp. } 0 < \langle A\nabla\phi, \nabla\phi \rangle(X_0) < Q(X_0)).$$

The next lemma provides a basic comparison principle for solutions to the free boundary problem (6.3).

Lemma 6.5. *Let u be a viscosity solution to (6.3) in Ω . If v is a strict subsolution to (6.3) in Ω , such that $u \geq v^+$ in Ω , then, in $\Omega^+(v) \cup \mathcal{F}(v)$, the strict inequality, $u > v^+$ holds.*

Lemma 6.5 yields the crucial tool for the proof of Theorem 6.2. More precisely, using the comparison principle established in Lemma 6.5, we prove a Harnack inequality for solution u . For $0 < \varepsilon < 1$ to be chosen later, we can assume, by normalization and dilating variables, the following conditions:

$$(6.7) \quad \|a_{ij} - \delta_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon^2,$$

$$(6.8) \quad \|g(X)\|_{L^\infty(\Omega)} \leq \varepsilon^2,$$

$$(6.9) \quad \|\nabla a_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon^2,$$

$$(6.10) \quad \|Q - 1\|_{L^\infty(\Omega)} \leq \varepsilon^2.$$

We need the following lemma.

Lemma 6.6. *Let u be a viscosity solution to (6.3) in Ω , satisfying (6.7)–(6.10). There exists a universal constant $\tilde{\varepsilon} > 0$ such that if $0 < \varepsilon \leq \tilde{\varepsilon}$ and u satisfies*

$$p^+(X) \leq u(X) \leq (p(X) + \sigma)^+, \quad |\sigma| < \frac{1}{20} \quad \text{in } B_1(0), \quad p(X) = X_n + \sigma,$$

then if at $X_0 = \frac{1}{10}e_n$

$$(6.11) \quad u(X_0) \geq \left(p(X_0) + \frac{\varepsilon}{2}\right)^+,$$

then

$$(6.12) \quad u \geq (p + c\varepsilon)^+ \quad \text{in } \overline{B}_{1/2}(0),$$

for some $0 < c < 1$. Analogously, if

$$(6.13) \quad u(X_0) \leq \left(p(X_0) + \frac{\varepsilon}{2}\right)^+,$$

then

$$(6.14) \quad u \leq (p + (1 - c)\varepsilon)^+ \quad \text{in } \overline{B}_{1/2}(0).$$

Proof. The proof is as in [7]. We will only check the first claim, as the proof of the second is analogous. Let $w: \overline{D} \rightarrow \mathbb{R}$ be defined by

$$(6.15) \quad w(X) = c \left(|X - X_0|^{-\gamma} - \left(\frac{4}{5}\right)^{-\gamma}\right),$$

where $D := B_{4/5}(X_0) \setminus \overline{B}_{1/40}(X_0)$. We choose $c > 0$ such that

$$(6.16) \quad w = \begin{cases} 0, & \text{on } \partial B_{4/5}(X_0), \\ 1, & \text{on } \partial B_{1/40}(X_0). \end{cases}$$

We compute directly

$$\partial_i w = -\gamma(X_i - X_0^i) |X - X_0|^{-\gamma-2}$$

and

$$\partial_{ij} w = \gamma |X - X_0|^{-\gamma-2} \{(\gamma + 2)(X_i - X_0^i)(X_j - X_0^j) |X - X_0|^{-2} - \delta_{ij}\}.$$

If we write $b_i := \sum_{j=1}^n \partial a_{ij} / \partial x_i$, from $\|a_{ij} - \delta_{ij}\|_{L^\infty(\Omega)} \leq \varepsilon^2$ and $\|b_i\|_{L^\infty(\Omega)} \leq C_0 \varepsilon^2$, we obtain, in D , by choosing $\gamma > 0$ large,

$$(6.17) \quad \begin{aligned} \operatorname{div}(A(X)\nabla v) &= \gamma |X - X_0|^{-\gamma-2} \left\{ (\gamma + 2) |X - X_0|^{-2} \sum_{i,j=1}^n a_{ij}(X)(X_i - X_0^i) \right. \\ &\quad \left. \cdot (X_j - X_0^j) - \sum_{i,j=1}^n a_{ij}(X)\delta_{ij} - \sum_{i=1}^n b_i(X)(X_i - X_0^i) \right\} \\ &\geq \gamma |X - X_0|^{-\gamma-2} \{(\gamma + 2) - C(n)\} \geq \delta_0, \end{aligned}$$

where $\delta_0 > 0$ is a universal constant. From (6.11) we have $u \geq p$ in $B_1(0)$. Thus,

$$(6.18) \quad B_{1/20}(X_0) \subset B_1^+(u).$$

Moreover,

$$\operatorname{div}(A(X)\nabla(u - p)) = \operatorname{div}(A(X)\nabla u) - b_n = g(X) - b_n, \quad \text{in } B_{1/20}(X_0),$$

with

$$\|g(X) - b_n\|_{L^\infty(B_{1/20}(X_0))} \leq C\varepsilon^2.$$

Hence, by the Harnack inequality, we obtain

$$\begin{aligned} u(X) - p(X) &\geq u(X_0) - p(X_0) - C \|g(X) - b_n\|_{L^\infty(B_{1/20}(X_0))} \\ &\geq u(X_0) - p(X_0) - C\varepsilon^2, \end{aligned}$$

for all $X \in B_{1/40}(X_0)$. Using (6.11) for ε sufficiently small,

$$(6.19) \quad u(X) - p(X) \geq c\varepsilon - C\varepsilon^2 \geq c_0\varepsilon, \quad \text{in } B_{1/40}(X_0).$$

Define

$$(6.20) \quad v(X) = p(X) + c_0\varepsilon(w(X) - 1), \quad X \in \overline{B}_{4/5}(X_0),$$

and for $t \geq 0$,

$$(6.21) \quad v_t(X) = v(X) + t, \quad X \in \overline{B}_{4/5}(X_0).$$

By the maximum principle (see (6.16) and (6.17)) we have $w \leq 1$ in D . Then, extending w to 1 in $B_{1/40}(X_0)$ we find

$$(6.22) \quad v_0(X) = v(X) \leq p(X) \leq u(X), \quad X \in \overline{B}_{4/5}(X_0).$$

Consider

$$t_0 = \sup \{t \geq 0 : v_t \leq u \text{ in } \overline{B}_{4/5}(X_0)\}.$$

Assume, for the moment, that we have already verified $t_0 \geq c_0\varepsilon$. From the definition of v we have

$$u(X) \geq v(X) + t_0 \geq p(X) + c_0\varepsilon w(X), \quad \forall X \in B_{4/5}(X_0).$$

Notice that $B_{1/2}(0) \subset B_{3/5}(X_0)$ and

$$w(X) \geq \begin{cases} (\frac{3}{5})^{-\gamma} - (\frac{4}{5})^{-\gamma}, & \text{in } B_{3/5}(X_0) \setminus B_{1/40}(X_0), \\ 1, & \text{on } B_{1/40}(X_0). \end{cases}$$

Hence, we conclude (ε small) that

$$u(X) - p(X) \geq c\varepsilon, \quad \text{in } B_{1/2}(0),$$

and the result is proved.

Now we prove that indeed $t_0 \geq c_0\varepsilon$. For this, we suppose for the sake of contradiction, that $t_0 < c_0\varepsilon$. Then there would exist $Y_0 \in \overline{B}_{4/5}(X_0)$ such that

$$v_t(Y_0) = u(Y_0).$$

In the sequel, we show that $Y_0 \in B_{1/40}(X_0)$. From definition of v_t and by the fact that w has zero boundary values on $\partial B_{4/5}(X_0)$ we have

$$v_t = p - c_0\varepsilon + t_0 < u \quad \text{in } \partial B_{4/5}(X_0),$$

where we have used that $u \geq p$ and $t_0 < c_0\varepsilon$. Moreover,

$$\operatorname{div}(A(X)\nabla v_t) \geq (c_0\delta_0 - \varepsilon)\varepsilon > \varepsilon^2 \quad \text{in } D$$

and

$$(6.23) \quad |\nabla v_{t_0}| \geq |\partial_n v| = |1 + c_0 \varepsilon \partial_n w|, \quad \text{in } D.$$

By the radial symmetry of w , we have

$$(6.24) \quad \partial_n w(X) = |\nabla w(X)| \langle \nu_X, e_n \rangle, \quad X \in D,$$

where ν_X is the unit vector in the direction of $X - X_0$. From (6.17) we have

$$|\nabla w|^2 = \gamma^2 |X - X_0|^{-2(\gamma+2)} |X - X_0|^2 = \gamma^2 |X - X_0|^{-2(\gamma+1)} \geq c > 0, \quad \text{in } D.$$

Also we have $\langle \nu_X, e_n \rangle \geq c$ in $\{v_{t_0} \leq 0\} \cap D$ (for ε small enough). In fact, if ε is small enough

$$\{v_{t_0} \leq 0\} \cap D \subset \{p \leq c_0 \varepsilon\} = \{X_n \leq c_0 \varepsilon - \sigma\} \subset \{X_n < 1/20\}.$$

We therefore conclude that

$$\begin{aligned} \langle \nu_X, e_n \rangle &= \frac{1}{|X_0 - X|} \langle X - X_0, e_n \rangle \geq \frac{5}{4} \langle X - X_0, e_n \rangle \\ &= \frac{5}{4} \left(-X_n + \frac{1}{20} - \frac{1}{20} + \frac{1}{10} \right) > \frac{1}{16}, \quad \text{in } \{v_{t_0} \leq 0\} \cap D. \end{aligned}$$

Moreover, from $\|a_{ij} - \delta_{ij}\| \leq \varepsilon^2$ we have

$$(6.25) \quad \langle A(X)\xi, \xi \rangle \geq |\xi|^2 (1 - \varepsilon^2), \quad \forall X \in \Omega, \forall \xi \in \mathbb{R}^n.$$

Therefore, from (6.23), (6.24) and (6.25) we obtain

$$\langle A\nabla v_{t_0}, \nabla v_{t_0} \rangle \geq |\nabla v_{t_0}|^2 - C\varepsilon^2 \geq 1 + c_1 \varepsilon + \varepsilon(c_1 - C\varepsilon) + c_1^2 \varepsilon^2 > 1 + \varepsilon^2 > Q,$$

in $\{v_{t_0} \leq 0\} \cap D$. In particular, we have

$$\langle A\nabla v_{t_0}, \nabla v_{t_0} \rangle > Q \quad \text{in } D \cap \mathcal{F}(v_{t_0}).$$

Thus, v_{t_0} is a strict subsolution in D and by Lemma 6.5 (u is a viscosity solution of problem (6.3) in $B_1(0)$) we conclude that $Y_0 \in B_{1/40}(X_0)$. This is a contradiction. In fact, we would get

$$u(Y_0) = v_{t_0}(Y_0) = v(Y_0) + t_0 \leq p(Y_0) + t_0 < p(Y_0) + c_0 \varepsilon.$$

which leads to a contradiction with (6.19). This concludes the proof of the lemma. \square

We can now establish the main tool needed for the proof of Theorem 6.2.

Theorem 6.7. *Let u be a viscosity solution to (6.3) in Ω satisfying (6.7)–(6.10). There exists a universal constant $\tilde{\varepsilon} > 0$ such that if u satisfies*

$$(6.26) \quad (X_n + a_0)^+ \leq u(X) \leq (X_n + d_0)^+ \quad \text{in } B_r(X_0) \subset \Omega,$$

at some $X_0 \in \Omega^+(u) \cup F(u)$, with

$$(6.27) \quad d_0 - a_0 \leq \varepsilon r, \quad \varepsilon \leq \tilde{\varepsilon},$$

then

$$(6.28) \quad (X_n + a_1)^+ \leq u(X) \leq (X_n + d_1)^+ \quad \text{in } B_{r/40}(X_0),$$

with

$$(6.29) \quad a_0 \leq a_1 \leq d_1 \leq d_0, \quad d_1 - a_1 \leq (1 - c) \varepsilon r,$$

and $0 < c < 1$ universal.

Proof. With no loss of generality, we assume $X_0 = 0$ and $r = 1$. We analyze two distinct cases.

1) $|a_0| < 1/20$: We put $p(X) = X_n + a_0$ and by (6.26)

$$(6.30) \quad p^+(X) \leq u(X) \leq (p(X) + a_0)^+ \quad (d_0 \leq a_0 + \varepsilon).$$

Thus, we can apply Lemma 6.6 to obtain the result.

2) $|a_0| \geq 1/20$: Assume that $a_0 < -1/20$. If we take $\varepsilon < 1/20$, it follows that $(p + \varepsilon)^+ = 0$. Thus 0 belongs to the interior of the zero phase of u , which yields a contradiction.

If $a_0 > 1/20$, then $B_{1/20}(0) \subset B_1^+(u)$ and the result follows from the Harnack inequality. In fact, if $p = x_n + d_0$

$$\operatorname{div}(A(X)\nabla(p - u)) = b_n - g(X),$$

with $\|b_n - g(X)\|_{L^\infty(B_{1/20}(0))} \leq C\varepsilon^2$. Hence, since $(p - u)(0) = a_0$ we have

$$p(X) - u(X) \geq c_0 d_0 - C\varepsilon^2, \quad (0 < c_1 < 1)$$

which implies

$$u(X) \leq p(X) - d_0 c_0 + C\varepsilon^2 = X_n + d_0(1 - c_0) + C\varepsilon^2, \quad \forall X \in B_{1/40}(X_0).$$

Let $c_1 = d_0 - a_0$. Then,

$$\begin{aligned} u(X) &\leq X_n + (c_1 + a_0)(1 - c_0) + C\varepsilon^2 = X_n + a_0 + c_1(1 - c_0) - a_0 c_0 + C\varepsilon^2 \\ &\leq X_n + a_0 + c_1(1 - c_0) - \frac{c_0}{20} + C\varepsilon^2 \leq X_n + a_0 + c_1(1 - c_0), \end{aligned}$$

if ε is small enough. Hence, if we put $a_1 := a_0$ and $d_1 = a_0 + c_1(1 - c_0)$ we obtain the result. \square

From the Harnack inequality, Theorem 6.7, we obtain precisely as in [7] the following key estimate for flatness improvement.

Corollary 6.8. *Let u be a viscosity solution to (6.3) in Ω satisfying (6.7)–(6.10). If u satisfies (6.26) then in $B_1(X_0)$ the function $\tilde{u}_\varepsilon := (u - X_n)/\varepsilon$ has a Hölder modulus of continuity at X_0 outside of a ball of radius $\varepsilon/\tilde{\varepsilon}$, i.e., for all $X \in (\Omega^+(u) \cup F(u)) \cap B_1(X_0)$ with $|X - X_0| \geq \varepsilon/\tilde{\varepsilon}$,*

$$|\tilde{u}_\varepsilon(X) - \tilde{u}_\varepsilon(X_0)| \leq C |X - X_0|^\gamma.$$

We are ready to establish improvement of flatness.

Theorem 6.9 (Flatness improvement). *Let u be a viscosity solution to (6.3) in Ω satisfying (6.7)–(6.10). Assume that u satisfies*

$$(6.31) \quad (X_n - \varepsilon)^+ \leq u \leq (X_n + \varepsilon)^+ \quad \text{for } X \in B_1(0),$$

with $0 \in \mathcal{F}(u)$. If $0 < r \leq r_0$ for a universal constant r_0 and $0 < \varepsilon \leq \varepsilon_0(r)$, then

$$(6.32) \quad \left(X \cdot \nu - \frac{r}{2} \varepsilon\right)^+ \leq u \leq \left(X \cdot \nu + \frac{r}{2} \varepsilon\right)^+ \quad \text{for } X \in B_r(0),$$

with $|\nu| = 1$, and $|\nu - e_n| \leq C \varepsilon^2$ for a universal constant C .

Proof. Again the proof follows the lines of [7]. Fix $0 < r \leq r_0$ with r_0 a constant to be chosen. Aiming at contradiction, suppose that there exist a sequence $\varepsilon_j \rightarrow 0$ and a sequence u_j of solutions to (6.3) in $B_1(0)$ with right-hand side g_j and free boundary condition Q_j such that

$$(6.33) \quad (X_n - \varepsilon_j)^+ \leq u_j \leq (X_n + \varepsilon_j)^+ \quad \text{for } X \in B_1(0), 0 \in \mathcal{F}(u_j),$$

but u_j does not satisfy the conclusion (6.32). Define

$$\tilde{u}_j(X) = \frac{u_j(X) - X_n}{\varepsilon_j}, \quad X \in \Omega_1(u_j),$$

where $\Omega_\rho := (B_1^+(u) \cup F(u)) \cap B_\rho(0)$, for $0 < \rho < 1$. Then (by Corollary 5.8) the graphs of the \tilde{u}_j over $\Omega_{\frac{1}{2}}(u_j)$ converge (possibly passing to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function \tilde{u} over $B_{1/2}(0) \cap \{X_n \geq 0\}$. We claim that \tilde{u} is a solution of the problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2}(0) \cap \{X_n > 0\}, \\ \partial_n \tilde{u} = 0 & \text{on } B_{1/2}(0) \cap \{X_n = 0\}, \end{cases}$$

in the viscosity sense (see Definition 2.5 and the subsequent remark in [7]). Given a quadratic polynomial $P(X)$ touching \tilde{u} at $X_0 \in B_{1/2}(0) \cap \{X_n \geq 0\}$ strictly from below we need to prove that

- (i) if $X_0 \in B_{1/2}(0) \cap \{X_n > 0\}$ then $\Delta P \leq 0$;
- (ii) if $X_0 \in B_{1/2}(0) \cap \{X_n = 0\}$ then $\partial_n P(X_0) \leq 0$.

As in [7], there exist points $X_j \in \Omega_{1/2}(u_j)$, $X_j \rightarrow X_0$, and constants $c_j \rightarrow 0$ such that

$$u_j(X_j) = \tilde{P}(X_j) \quad \text{and} \quad u_j(X) \geq \tilde{P}(X) \quad \text{in a neighborhood of } X_j$$

where

$$\tilde{P}(X) = \varepsilon_j(P(X) + c_j) + X_n.$$

We have two possibilities:

(a) If $X_0 \in B_{1/2}(0) \cap \{X_n > 0\}$ then, since P touches u_j from below at X_j , we obtain

$$\begin{aligned} C_1 \varepsilon_j^2 \geq g_j(X_j) &\geq \sum_{i,l=1}^n a_{il}^j(X) \partial_{il} \tilde{P} + \sum_{i=1}^n b_i^j(X_j) \partial_i \tilde{P} \\ &= \varepsilon_j \sum_{i,l=1}^n a_{il}^j(X_j) \partial_{il} P + \varepsilon_j \sum_{i=1}^n b_i^j(X_j) \partial_i P + b_n^j(X_j), \end{aligned}$$

where $\|b_i^j\|_{L^\infty} \leq C_0 \varepsilon_j^2$ and $\|\partial_i P\|_{L^\infty} \leq C$. Therefore,

$$\sum_{i,l=1}^n a_{il}^j(X_j) \partial_{il} P \leq C \varepsilon_j,$$

Thus, we have

$$\Delta P = \sum_{i,l=1}^n (\delta_{il} - a_{il}^j(X_k)) \partial_{il} P + \sum_{i,l=1}^n a_{il}^j(X_k) \partial_{il} P \leq C \varepsilon_j.$$

Hence, taking $j \rightarrow \infty$ we obtain $\Delta P \leq 0$.

(b) If $X_0 \in B_{1/2}(0) \cap \{X_n = 0\}$ we can assume (see [7]) that

$$(6.34) \quad \Delta P > 0.$$

Notice that for j sufficiently large we have $X_j \in F(u_j)$. In fact, aiming at a contradiction, suppose that there exists a subsequence $X_{j_n} \in B_1^+(u_{j_n})$ such that $X_{j_n} \rightarrow X_0$. Then, arguing as in (a), we obtain

$$\Delta P \leq C \varepsilon_j,$$

which contradicts (6.34) as $j_n \rightarrow \infty$. Therefore, there exists $j_0 \in \mathbb{N}$ such that $X_j \in F(u_j)$ for $j \geq j_0$. Moreover,

$$|\nabla \tilde{P}| \geq 1 - \varepsilon_j |\nabla P| > 0,$$

for j sufficiently large (we can assume that $j \geq j_0$). Since that \tilde{P}^+ touches u_j from below we have

$$\langle A^j(X_j) \nabla \tilde{P}(X_j), \nabla \tilde{P}(X_j) \rangle \leq Q_j(X_j) \leq (1 + \varepsilon_j^2).$$

Moreover,

$$\begin{aligned} \langle A^j(X_j) \nabla \tilde{P}(X_j), \nabla \tilde{P}(X_j) \rangle &\geq |\nabla \tilde{P}(X_j)|^2 - C \varepsilon_j^2 \\ &= \varepsilon_j^2 |\nabla P(X_j)|^2 + 1 + 2\varepsilon_j \partial_n P(X_j) - C \varepsilon_j^2, \end{aligned}$$

where we have used $|\nabla \tilde{P}|^2 \leq C$. In conclusion, we obtain

$$(6.35) \quad \varepsilon_j^2 |\nabla P(X_j)|^2 + 1 + 2\varepsilon_j \partial_n P(X_j) - C \varepsilon_j^2 \leq 1 + \varepsilon_j^2.$$

Hence, dividing (6.35) by ε_j and taking $j \rightarrow \infty$ we obtain $\partial_n P(X_0) \leq 0$.

The choice of r_0 and the conclusion of the theorem follows from the regularity of \tilde{u} . \square

We can finally conclude the proof of Theorem 6.2.

Proof of Theorem 6.2. Let

$$u_j(X) = \frac{u(\rho_j X)}{\rho_j}, \quad X \in B_1(0),$$

be the rescaling of u with $\rho_j = \tilde{r}^j$ for a fixed \tilde{r} such that

$$\tilde{r}^\beta \leq \frac{1}{4} \quad \text{and} \quad \tilde{r} \leq r_0,$$

where r_0 is the universal constant as in Theorem 6.9. Note that u_j is solution of (6.3) with

$$\begin{aligned} a_{il}^j(X) &:= a_{il}^j(\rho_j X), \\ g_j(X) &:= g(\rho_j X), \\ Q_j(X) &:= Q(\rho_j X). \end{aligned}$$

Moreover, if $\tilde{\varepsilon} := \varepsilon_0^2(\tilde{r})$ and $\varepsilon_j := 2^{-j} \varepsilon_0(\tilde{r})$ we obtain

$$\begin{aligned} |a_{il}^j(X) - \delta_{il}| &= |a_{il}(\rho_j X) - a_{il}(0)| \leq [a_{il}]_{C^{0,1}} \rho_j \leq \tilde{\varepsilon} \tilde{r}^j \leq \varepsilon_j^2, \\ \|b_i^j\|_{L^\infty} &\leq [a_{ij}]_{C^{0,1}} \rho_j \leq C_0 \tilde{\varepsilon} \tilde{r}^j \leq \varepsilon_j^2, \\ \|g_j\|_{L^\infty} &\leq \|g_j\|_{L^\infty} \rho_j \leq \tilde{\varepsilon} \tilde{r}^j \leq \varepsilon_j^2, \\ |Q_j(X) - 1| &= |Q(\rho_j X) - Q(0)| \leq [Q]_{C^{0,\beta}} \rho_j^\beta \leq \tilde{\varepsilon} \tilde{r}^{j\beta} \leq \varepsilon_j^2. \end{aligned}$$

The proof now follows as in [7]. □

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