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Blaschke-type conditions on unbounded domains, generalized convexity, and applications in perturbation theory

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Abstract. We introduce a new geometric characteristic of compact sets in the plane called r-convexity, which fits nicely into the concept of generalized convexity and extends standard convexity in an essential way. We obtain a Blaschke-type condition for the Riesz measures of certain subharmonic functions on unbounded domains with r-convex complements, having growth governed by the distance to the boundary. The result is applied to the study of the convergence of the discrete spectrum for the Schatten–von Neumann perturbations of bounded linear operators in Hilbert space.

1. Introduction

In 1915 Blaschke [3] proved his celebrated result concerning zero sets of bounded analytic functions in the unit disk, which is a gem of function theory. A vast literature with various refinements and far reaching extensions of the Blaschke condition has appeared since; see [9], [18], [22], [35], and [39], and references therein.

We focus on a series of recent papers ([4], [12], [13], [17]), where the authors study the zero sets of functions analytic in the unit disk, which grow in the direction of a prescribed subset of the unit circle. The result in [12] for analytic functions is as follows.

Theorem A. *Let $E \subset \partial\mathbb{D}$ be a closed subset of the unit circle, and let f be an analytic function in the unit disk $\mathbb{D}$ with zero set $Z_f = \{z_n\}$ (each zero z_n enters with its multiplicity) so that $|f(0)| = 1$, and*

$$\log|f(z)| \le \frac{K_f}{\operatorname{dist}^q(z,E)}, \quad z \in \mathbb{D}, \quad q > 0,$$

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where $\operatorname{dist}(E_1, E_2)$ *is the Euclidian distance between closed sets* E_1 *and* E_2*. Then for each* $\varepsilon > 0$*,*

$$\sum_n (1-|z_n|)\operatorname{dist}^p(z_n, E) \le C(q, E, \varepsilon)\, K_f, \quad p = \max(q + \kappa(E) - 1 + \varepsilon, 0),$$

where $\kappa(E)$ *is the upper Minkowski dimension of* E*.*

This result can be used in perturbation theory of bounded linear operators, although the situation there is somewhat different. The point is that the basic objects, the resolvent and the perturbation determinant, are analytic functions on the resolvent set (including infinity) of the corresponding operator, which is an *unbounded* open set of the plane with compact complement E, the spectrum of the operator. To handle this problem, attempts were made to transfer the problem to the unit disk, using conformal mapping [8], [21], or the uniformization theorem [17], apply Theorem A and then return to the initial setting by means of certain distortion results. Such attempts were by and large successful only in the cases when E is a single segment [8], [21] or a finite union of disjoint segments [17], and it is absolutely unclear whether it is possible to make such an argument work for an arbitrary compact subset of the line.

The reasoning in [12] reveals the potential theoretic character of the problem, so the natural setting is subharmonic functions v and their Riesz measures (generalized Laplacians) $\mu = 1/(2\pi)\Delta v$ rather than analytic functions and their zero sets. In the case $v = \log|f|$ with an analytic function f, the Riesz measure is a discrete and integer-valued measure supported on Z_f, and $\mu\{z\}$ equals the multiplicity of the zero at z.

In this paper we develop a straightforward approach to the study of subharmonic functions on unbounded domains with growth governed by the distance to the boundary. Let E be a compact set in the complex plane $\mathbb{C}$, which does not split the plane (its complement

$$\Omega = \overline{\mathbb{C}}\backslash E$$

in the extended plane $\overline{\mathbb{C}}$ is a domain, that is, a connected open set $\overline{\mathbb{C}}$). Consider a class of subharmonic functions on Ω subject to the growth and normalization conditions

$$v(z) \le \psi(d(z)), \quad v(\infty) = 0,$$

where

$$d(z) := \operatorname{dist}(z, E)$$

is the distance from z to E and ψ is a positive and monotone decreasing function on $\mathbb{R}_+ = [0, \infty)$.

In the study of the Riesz measures of such functions one is faced with at least two obstacles. First, the set E may be so small (polar or finite), that to apply standard techniques from potential theory we must work in the "outer neighborhood"

$$\Omega_t := \{z \in \mathbb{C} :\ d(z) > t\}, \quad t > 0, \quad \Omega = \Omega_0.$$

Its boundary $\partial\Omega_t = \{z : d(z) = t\}$ is nonpolar, since it splits the plane (see Theorem 3.6.3 in [33]), so the Green's function G_t for Ω_t exists and is unique, whenever Ω_t is a domain.

Second, it is not hard to construct a compact set E so that Ω is a domain, but Ω_t is not for $t > 0$. To cope with this problem we introduce a new geometric characteristic, r-convexity, which fits nicely in the context of generalized convexity; see [7], [37]. It can be defined in an arbitrary metric space. No linear structure is needed. Precisely, it is well known that a closed set in $\mathbb{C}$ is convex if and only if it is the intersection of all closed half-planes containing it. For an arbitrary closed set E this intersection agrees with the convex hull of E. As usual, we denote by

$$B(x,r) = \{z : |z-x| < r\}, \quad B^c(x,r) = \{z : |z-x| \ge r\},$$
$$\partial B(x,r) = \{z : |z-x| = r\}$$

an open disk of radius r centered at x, its complement, and its boundary, respectively. By replacing half-planes with exteriors of open disks B^c, we come to the following extension of the usual notion of convexity. We start with the obvious inclusion

$$E \subset \mathrm{conv}_r(E) := \bigcap \{B^c(z,r) : E \subset B^c(z,r)\}, \quad r > 0. \tag{1.1}$$

Definition 1.1. We say that a closed set E is r-convex, if $E = \mathrm{conv}_r(E)$. The set $\mathrm{conv}_r(E)$ is called the r-convex hull of E.

In other words, E is r-convex if

$$\mathbb{C}\backslash E = \bigcup \{B(z,r) : B(z,r) \subset \mathbb{C}\backslash E\}, \tag{1.2}$$

that is, the complement of E can be covered by open disks of a *fixed* radius $r > 0$ which belong to this complement. As for the usual notion of convexity, the intersection of any family of r-convex sets is r-convex. On the other hand, in contrast to the usual convexity, a finite union of disjoint r-convex sets is r'-convex for some $r' \le r$. It is also clear that $E_1 \subset E_2$ implies $\mathrm{conv}_r(E_1) \subset \mathrm{conv}_r(E_2)$.

It follows from (1.2), that each r-convex set is also r'-convex for any $r' < r$. So it is natural to consider the number $r_0(E) := \sup\{r : E = \mathrm{conv}_r(E)\}$, called the radius of convexity of E. For instance, each closed convex set E is r-convex with $r_0(E) = \infty$, and it is easy to see that the same holds for each closed subset of a line. Indeed, any open subset of a line complementary to a closed set can be covered with a disk of arbitrarily large radius. A complete characterization of compact sets E with $r_0(E) = \infty$ is given in Proposition 1. We show that $r_0(E) = R$ for each compact subset of a circle $\partial B(x,R)$, which contains more than two points (see Proposition 2.2). The sets with "interior angles", like $\{z \in \overline{\mathbb{D}} : \pi/4 \le \arg z \le 7\pi/4\}$, are not r-convex for any $r > 0$.

It turns out (see Theorem 2.9) that if an r-convex compact set E does not split the plane, then there exists $t_0 = t_0(E) > 0$ such that Ω_t is a domain for all $0 \le t \le t_0$. So for such t the Green's function G_t for Ω_t exists and is unique. A key potential theoretic result (Lemma 3.3) provides the lower bound

$$G_t(z,\infty) \ge C(E)\min\{1, d(z)\}, \quad z \in \Omega_{3t}, \quad 0 < t \le t_0,$$

for the Green's function with its pole at infinity. When E is a finite set, the result can be improved to $G_t(z,\infty) \ge C > 0$, $z \in \Omega_{kt}$, with some $k = k(E) > 1$. For various estimates of the Green's functions and harmonic measures see, for instance, [29], [14], and [6].

Here is the main result of the paper.

Theorem 1.2. *Let E be an r-convex compact set with connected complement $\Omega = \overline{\mathbb{C}}\backslash E$, and let v be a subharmonic function satisfying*[1]

$$v(z) \le K_v\, \psi(d(z)), \quad v(\infty) = 0. \tag{1.3}$$

Let φ be a positive, monotone increasing, and absolutely continuous function on $\mathbb{R}_+$, such that $\varphi_1(t) := t^{-1}\varphi(t)$ is monotone increasing in some neighborhood of the origin, $\varphi_1(0) = 0$, and

$$\int_0 \varphi_1'(t)\,\psi\Big(\frac{t}{3}\Big)\,dt + \int^{\infty} \varphi'(t)\,\psi\Big(\frac{t}{3}\Big)\,dt < \infty, \tag{1.4}$$

(*the integrals converge near the origin and infinity, respectively*). *Then there holds the Blaschke-type condition*

$$\int_\Omega \varphi(d(\zeta))\,\mu(d\zeta) \le C(E,\psi,\varphi)\,K_v \tag{1.5}$$

for the Riesz measure μ of v. If v is bounded from above in Ω, $v \le K_v$, then

$$\int_\Omega \min\{1,\, d(\zeta)\}\,\mu(d\zeta) \le C(E)\,K_v. \tag{1.6}$$

Actually, the second statement is just the Blaschke condition for subharmonic functions on unbounded domains, which might be known to experts. Note that if E is a polar set, then E is removable for the class of bounded from above subharmonic functions. Hence in this case v is constant.

Remark 1.3. Let E be an r-convex compact set, and let $\widetilde{\Omega}$ be its outer domain, that is, the unbounded component of Ω. Then the set $\mathrm{Pc}(E) = \overline{\mathbb{C}}\backslash\widetilde{\Omega}$, known as the polynomial convex hull of E, is r-convex, and $d(z) = \mathrm{dist}(z, \mathrm{Pc}(E))$ for $z \in \tilde{\Omega}$. Given a subharmonic function v satisfying (1.3), its restriction $\tilde{v}$ to $\widetilde{\Omega}$ satisfies the conditions of Theorem 1.2, so the Blaschke-type condition (1.5) holds with Ω replaced with $\widetilde{\Omega}$.

If $\psi(x) = x^{-q}$, $q > 0$, which is typical in the application of Theorem 1.2, we can take

$$\varphi(x) = x^{q+1/2}\,(\min\{x, 1/x\})^{\varepsilon+1/2} = \begin{cases} x^{q+1+\varepsilon}, & x \le 1, \\ x^{q-\varepsilon}, & x > 1, \end{cases} \qquad 0 < \varepsilon < q.$$

[1] We single out the constant K_v on purpose, in view of applications to perturbation theory in Section 5.

A special case of Theorem 1.2 with $v = \log|f|$, f an analytic function, occurs in perturbation theory in the study of discrete spectra for the Schatten–von Neumann perturbations of certain bounded linear operators. Given a bounded linear operator A_0 on the Hilbert space $\mathcal{H}$, and a compact operator B, the fundamental theorem of Weyl states that the essential spectra of A_0 and $A = A_0 + B$ agree, so the discrete spectrum of A (the set of isolated eigenvalues of finite algebraic multiplicity) can accumulate only at the joint essential spectrum. We want to gather some information on the rate of accumulation under the stronger assumption that B belongs to some Schatten–von Neumann operator ideal $\mathcal{S}_q$, $1 \le q < \infty$, that is, if $\|B\|_{\mathcal{S}_q}^q := \sum_n s_n^q(B) < \infty$, $s_n(B)$ are the singular values of B.

A number of results of the form

$$\sum_{\lambda \in \sigma_d(A)} d^p(\lambda) \le C\,\|B\|_{\mathcal{S}_q}^q, \quad d(\lambda) := \operatorname{dist}(\lambda, \sigma(A_0)) \tag{1.7}$$

with some $p = p(q)$ and $\sigma(T)$ ($\sigma_d(T)$) the spectrum (discrete spectrum) of an operator T, are known. Kato in [23] proved (1.7) for self-adjoint A_0 and $p = q \ge 1$, $C = 1$. Recently Hansmann in [20] obtained the same result for a self-adjoint A_0 and an arbitrary $B \in \mathcal{S}_q$ with $p = q > 1$ and the explicit (in a sense) constant $C = C_q > 1$.

For more general classes of operators (1.7) is known to be true for both A_0 and B normal with $p = q \ge 2$, $C = 1$, for all three A_0, B and A normal with $p = q \ge 1$, $C = 1$, and for A_0 normal, an arbitrary $B \in \mathcal{S}_q$ with $p = q \ge 1$ and $C = 1$, under the additional assumption that $\sigma(A_0)$ is a *convex* set, see [5], [2], and [19], respectively.

We apply Theorem 1.2 for the study of the rate of accumulation of the eigenvalues for an arbitrary $B \in \mathcal{S}_q$ and operators A_0 (in general, nonnormal) with r-convex spectrum and growth of the resolvent norm of A_0 governed by the distance to the spectrum (see the conditions (i)-(iii) in Section 5). The corresponding bound is as follows:

$$\sum_{\lambda \in \sigma_d(A)} \Phi\left(d(\lambda)\right) \le C\,\|B\|_{\mathcal{S}_q}^q,$$

where Φ is a continuous function on $\mathbb{R}_+$ and $\Phi(0) = 0$. This result is illustrated by several examples at the end of the paper.

Although our result on the rate of accumulation does not seem to be optimal, it enables one to extend considerably the class of unperturbed operators with the norm of resolvent growing fast near the spectrum (see, e.g., (5.3) and (5.4)).

We proceed as follows. Section 2 concerns the notion of r-convexity and its properties. In Section 3 we obtain a lower bound for the Green's function of an unbounded domain with a compact r-convex compliment. Our main result (Theorem 1.2) is proved in Section 4. Applications to perturbation theory are given in Section 5.

2. r-convexity

We begin with a characterization of r-convex compact sets E with $r_0(E) = \infty$.

Proposition 2.1. *A compact set E with $r_0(E) = \infty$ is either a convex set or a compact subset of a line. In particular, if an r-convex compact set E with $r_0(E) = \infty$ is connected, then it is convex.*

Proof. Assume that E is not convex. Then there is a pair of points $A, B \in E$ so that the open interval $(A, B) \subset \Omega = \overline{\mathbb{C}} \backslash E$. With no loss of generality we set $A = 1$ and $B = -1$. Then the vertical interval $[-i\varepsilon, i\varepsilon]$ is contained in Ω for sufficiently small ε.

We call a disk $B(z, r)$ ε-admissible if $B(z, r) \subset \Omega$ and $i\varepsilon \in B(z, r)$. By assumption there are ε-admissible disks of arbitrarily large radius. It is easy to see that the centers of such disks lie in the sector

$$\Gamma = \left\{ y > \frac{|x|}{\varepsilon} - \frac{1 - \varepsilon^2}{2\varepsilon} \right\},$$

or, otherwise, the center is closer to A or B than to $i\varepsilon$. So there are ε-admissible disks $B(z, r)$ with arbitrarily large $y = \mathrm{Im} z > 0$.

Fix $z_0 = x_0 + iy_0$ with $y_0 > 0$. We show that z_0 belongs to some ε-admissible disk. Take $0 < \varepsilon < y_0(1 + |x_0|)^{-1}$ and consider an ε-admissible disk $B(z, r)$ with $z = x + iy \in \Gamma$ and sufficiently large $y > 0$. Then

$$|z - i\varepsilon|^2 - |z - z_0|^2 = 2y(y_0 - \varepsilon) + 2xx_0 + \varepsilon^2 - x_0^2 - y_0^2.$$

Since

$$\varepsilon y + \frac{1 - \varepsilon^2}{2} > |x|, \quad -2|x| > -2\varepsilon y - (1 - \varepsilon^2), \quad (x, y) \in \Gamma,$$

we obtain

$$\begin{aligned} |z - i\varepsilon|^2 - |z - z_0|^2 &\geq 2y(y_0 - \varepsilon) - 2|x||x_0| + \varepsilon^2 - x_0^2 - y_0^2 \\ &> 2y(y_0 - \varepsilon - \varepsilon|x_0|) + \varepsilon^2 - x_0^2 - y_0^2 > 0 \end{aligned}$$

for sufficiently large y due to the choice of ε. Hence $z_0 \in B(z, r) \subset \Omega$, as claimed.

The reasoning for $y_0 < 0$ is exactly the same, so once E is not convex, then $E \subset \mathbb{R}$. The proof is complete. □

Given an r-convex set E, it is in general hard to compute its radius of convexity. In some simple instances we can solve this problem.

Proposition 2.2. *Let $E = \{a, b, c\}$ be a 3-point set in a general position and let $R = R(abc)$ be the circumradius of the triangle $\Delta = \Delta(abc)$. Then $r_0(E) = R(abc)$. Let E be a compact subset of the circle $\partial B(y, \rho)$ such that $|E| \geq 3$. Then $r_0(E) = \rho$.*

Proof. We recall some facts from elementary planar geometry. A triangle $\Delta(abc)$ is always viewed as an open planar set.

1. Given a triangle $\Delta(abc)$ and $r > R$, there exists exactly one disk $B(x, r)$ such that two vertices (say, a and b) lie on its boundary, and $c \notin B(x, r)$. We call this disk the r-disk and denote it by $(ab)_r$. If $r = R$ and Δ is acute, the R-disks $(ab)_R$, $(ac)_R$, and $(bc)_R$ are by definition the reflections of the circumdisk B_Δ through the sides of Δ. If Δ is not acute, and c is the vertex at the largest angle, then $(ac)_R$ and $(bc)_R$ are defined as above, and $(ab)_R = B_\Delta$.

2. If Δ is acute, then the circles $\partial(ab)_R$, $\partial(ac)_R$, and $\partial(bc)_R$ meet at one point in Δ, precisely at the orthocenter of Δ (see, e.g., Problem 5.9 in [31]). If Δ is not acute, and c is the vertex at the largest angle, the circles $\partial(ab)_R$, $\partial(ac)_R$, and $\partial(bc)_R$ meet at c, and there is a circular triangle with one vertex at c, which lies in $\Delta\backslash\big(\overline{(ac)_R} \cup \overline{(bc)_R}\big)$.

3. For $r \ge R$ let $[ab]_r$ be the segment of the disk $(ab)_r$ with vertices a and b, which intersects Δ. Then for $R \le r_1 < r_2$ we have $[ab]_{r_2} \subset [ab]_{r_1}$, and the inclusion is proper.

It is clear that $r_0(E) \ge R$, so we wish to show that the complement to E cannot be covered with open disks of radius $r > R$ which lie in this complement. Since Δ is convex, we can restrict our attention to the points of Δ. Assume that each point $x \in \Delta$ belongs to such disks. Then x belongs to one of the three segments from 3, so $\Delta \subset ([ab]_r \cup [ac]_r \cup [bc]_r)$. But by 2 and 3, the latter union can not cover all Δ. This contradiction completes the proof of the first statement.

As far as the second statement goes, the set E is clearly r-convex for $r \le \rho$. For $r > \rho$, as has just been proved, any 3-point set $E_1 = \{a, b, c\} \subset E$ is not r-convex, and $\mathrm{conv}_r(E_1)$ contains points from $\Delta(abc) \subset B(y, \rho)$. Since $\mathrm{conv}_r(E_1) \subset \mathrm{conv}_r(E)$ for $E_1 \subset E$, E cannot be r-convex either, as claimed. □

Remark 2.3. Given a triangle $\Delta(abc)$ with circumradius R, let $E \subset \partial\Delta$ be a compact set, which contains the vertices a, b, and c. It follows from the above proof and monotonicity of the r-convex hull, that for $r > R$ the intersection of $\mathrm{conv}_r(E)$ and Δ contains a nonempty open set.

To extend the above result, we say that a compact set E has finite global curvature if

$$r_g(E) := \inf\{R(abc)\} > 0, \tag{2.1}$$

where the infimum is taken over all possible triangles with vertices in E. Clearly, (2.1) holds for finite sets. When E is a Jordan rectifiable curve, the value $r_g^{-1}(E)$ is known as the global curvature of E; see [34].

Proposition 2.4. *Each compact set E with finite global curvature is r-convex for some $r > 0$, and*

$$r_0(E) = r_g(E).$$

Proof. Assume that for some $r > r_g(E)$ the set E is r-convex. Take a triangle $\Delta(a_0b_0c_0)$ with $a_0, b_0, c_0 \in E$ so that $r > R(a_0b_0c_0)$. As was shown in the proof of Proposition 2.2, the r-convex hull of this triangle (and, moreover, the r-convex hull of E itself) contains an open subset of $\Delta(a_0b_0c_0)$. But it is easily seen from (2.1) that E has empty interior. This contradiction shows that $r_0(E) \le r_g(E)$.

There remains only to show the converse inequality. We show that each point $z\in\mathbb{C}\backslash E$ can be covered with a disk $B\subset\mathbb{C}\backslash E$ of radius at least $r_g(E)$. Define

$$\rho_z := \sup\{r : z\in B(x,r)\subset\mathbb{C}\backslash E\}. \tag{2.2}$$

A compactness argument shows that there is a disk $B(x_z,\rho_z)$ with

$$z\in B(x_z,\rho_z)\subset\mathbb{C}\backslash E.$$

If $\partial B(x_z,\rho_z)\cap E$ contains at least 3 different points, then $\rho_z\ge r_g(E)$, as needed. Assume that $\partial B(x_z,\rho_z)\cap E=\{\zeta_1\}$, or $\partial B(x_z,\rho_z)\cap E=\{\zeta_1,\zeta_2\}$, and the points ζ_1 and ζ_2 do not belong to a diameter of the circle. Then we can shift the disk in an appropriate direction (perpendicular to the interval $[\zeta_1,\zeta_2]$ towards the center of the circle), and inflate it a bit to obtain a bigger disk with the same property, which contradicts the maximality (2.2) of ρ_z.

Hence we can restrict attention to the case where $x_z=\rho_z$, $\partial B(\rho_z,\rho_z)\cap E=\{a,b\}$, $a=0$ and $b=2\rho_z$ (after an affine transformation of the plane). Let $G=\{z : 0\le \mathrm{Re} z\le 2\rho_z\}$.

Assume first that there is a sequence $c_n=x_n+iy_n\in G\cap E$ with $c_n\to a$ or $c_n\to b$ (with no loss of generality let the first relation hold). We want to show that $R(abc_n)\to\rho_z$. To this end, we apply the explicit formula

$$R^{-2}(z_1z_2z_3)=\sum_{\pi}\frac{1}{(z_{\pi(1)}-z_{\pi(2)})\overline{(z_{\pi(1)}-z_{\pi(3)})}}=\frac{4\mathrm{Im}^2\,(z_1-z_2)\overline{(z_2-z_3)}}{|(z_1-z_2)(z_1-z_3)(z_2-z_3)|^2} \tag{2.3}$$

for the circumradius $R(z_1z_2z_3)$, given in [27], where the sum is taken over all permutations of $\{1,2,3\}$. Since $c_n\in G\backslash B(\rho_z,\rho_z)$, $c_n\to 0$, it is easy to see that $x_n/y_n\to 0$ as $n\to\infty$. It follows from (2.3) that

$$\begin{aligned} R^{-2}(abc_n) &= \frac{16\rho_z^2\,y_n^2}{4\rho_z^2|2\rho_z-x_n-iy_n|^2(x_n^2+y_n^2)}=\frac{4y_n^2}{((2\rho_z-x_n)^2+y_n^2)(x_n^2+y_n^2)}\\ &=\frac{4}{((2\rho_z-x_n)^2+y_n^2)(x_n^2/y_n^2+1)}\to\frac{1}{\rho_z^2},\end{aligned}$$

as claimed. Hence $\rho_z\ge r_g(E)$.

If there exists no such sequence c_n, the disk can be shifted and inflated, as above, which contradicts maximality of ρ_z. The proof is complete. □

Proposition 2.5. *Each C^2-smooth Jordan curve (arc) has finite global curvature.*

Proof. Assume on the contrary, that $r_g(\Gamma)=0$ where Γ is a C^2-smooth Jordan curve or arc. Then there is a sequence of triangles $\Delta(a_nb_nc_n)$ with $a_n,b_n,c_n\in\Gamma$ such that $R(a_nb_nc_n)\to 0$ as $n\to\infty$. By taking subsequences, if needed, we have $a_n\to a\in\Gamma$, and so $b_n\to a$ and $c_n\to a$ as $n\to\infty$.

On the other hand, we will show that

$$\lim_{n\to\infty}R^{-1}(a_nb_nc_n)=\tau(a)<\infty, \tag{2.4}$$

where $\tau(a)$ is the curvature of Γ at a, which will lead to contradiction. Let

$$\Gamma = \{z(t) = x(t) + iy(t)\}, \quad (a_n b_n c_n) = (z(t_1) z(t_2) z(t_3)), \quad a = z(0).$$

We apply (2.3) again to obtain

$$\begin{aligned} \mathrm{Im}\,(z(t_1) - z(t_2))\overline{(z(t_2) - z(t_3))} &= (y(t_1) - y(t_2))(x(t_2) - x(t_3)) \\ &\quad - (x(t_1) - x(t_2))(y(t_2) - y(t_3)), \end{aligned}$$

so

$$\begin{aligned} \frac{\mathrm{Im}\,(z(t_1) - z(t_2))\overline{(z(t_2) - z(t_3))}}{(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)} &= \frac{[t_1 t_2]_y\, [t_2 t_3]_x - [t_1 t_2]_x\, [t_2 t_3]_y}{t_1 - t_3} \\ &= [t_1 t_2 t_3]_y\, [t_2 t_3]_x - [t_1 t_2 t_3]_x\, [t_2 t_3]_y, \end{aligned}$$

where

$$[t_i t_k]_f := \frac{f(t_i) - f(t_k)}{t_i - t_k}, \quad [t_1 t_2 t_3]_f := \frac{[t_1 t_2]_f - [t_2 t_3]_f}{t_1 - t_3}$$

are divided differences of the first and second order, respectively. The limit relation

$$\lim_{t_i \to 0} [t_1 t_2 t_3]_f = \frac{1}{2}\, f''(0),$$

provided f is a C^2-smooth function at the origin, is one of the basic properties of divided differences (see, e.g., [28], page 12). Hence

$$\lim_{t_i \to 0} \frac{\mathrm{Im}\,(z(t_1) - z(t_2))\overline{(z(t_2) - z(t_3))}}{(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)} = \frac{y''(0)x'(0) - x''(0)y'(0)}{2}\,,$$

and finally, by (2.3) and the definition of the curvature,

$$\lim_{t_i \to 0} R^{-1}(z(t_1) z(t_2) z(t_3)) = \frac{|y''(0)x'(0) - x''(0)y'(0)|}{|z'(0)|^3}\,.$$

This is (2.4), as claimed. □

Remark 2.6. Proposition 2.5 is a particular case of the much more sophisticated Theorem 1 (iii) in [34], which claims that Γ has finite global curvature if and only if its arc length parametrization $\tau(s)$ is C^1, and τ' satisfies the Lipschitz condition with Lipschitz constant $r_g^{-1}(E)$.

Yet another example arises in the theory of elliptic equations in domains with nonsmooth boundaries [1].

Definition 2.7. A planar domain Ω with boundary $\partial\Omega$ is said to satisfy the uniform ball condition if there is $r > 0$ so that for each $x \in \partial\Omega$ there is a ball B of radius r such that

$$B \subset \Omega, \quad x \in \partial B.$$

Let Γ be a Jordan curve, and let $\mathbb{C}\backslash\Gamma = \Omega_i \cup \Omega_o$ be the interior and exterior domains of Γ. We say that Γ is a BC-curve if both Ω_i and Ω_o satisfy the uniform ball condition. A Jordan arc γ is a BC-arc if there is a BC-curve $\Gamma \supset \gamma$.

It is easy to see that if E is an r-convex compact set then $\Omega = \mathbb{C}\backslash E$ satisfies the uniform ball condition. Indeed, let $x \in \partial\Omega$. There is a sequence of points $z_n \in \Omega$ so that $z_n \to x$ as $n \to \infty$. Take the corresponding sequence of disks B_n of radius r, $z_n \in B_n \subset \Omega$. Then a certain subsequence of B_n converges to B in Definition 2.7.

Proposition 2.8. *A Jordan curve (arc) is r-convex for some $r > 0$ if and only if it is a BC-curve (arc).*

Proof. Due to the above remark we need to show that each BC-curve (arc) is r-convex for some $r > 0$. So let Γ be a BC-curve, let $z \in \Omega_i$, and suppose $d(z) = d(z, \Gamma) < r$. Take $\zeta \in \Gamma$ with $|z - \zeta| = d(z)$ and the "supporting" disks B_i and B_0 of radius r at the point ζ as in Definition 2.7. Since $B_i \subset \Omega_i$ and $B_o \subset \Omega_o$, the disks touch each other at ζ. The disk $B(z, d(z)) \subset \Omega_i$ passes through ζ, hence it is necessarily contained in B_i and touches B_i at ζ. So $z \in B_i$, as needed. The argument for $z \in \Omega_o$ is the same.

As for the BC arc γ, take the BC curve $\Gamma \supset \gamma$. Let $z \in \Gamma\backslash\gamma$. Then the inner supporting disk B_i at z is disjoint from γ, so it can be shifted appropriately so that z is contained in its shift, which is still disjoint from γ. □

The simple example $E = \{i/n\} \cup \{1/n\} \cup \{0\}$, $n \geq 1$, exhibits a set E, which is not r-convex, but such that $\mathbb{C}\backslash E$ satisfies the uniform ball condition.

Given a compact set E, consider the unbounded open set $\Omega_t := \{z \in \mathbb{C} : d(z, E) > t\}$ for $t \geq 0$. It is clear that $\{\Omega_t\}$ forms a monotone decreasing family of sets.

Let $\Theta_t \subseteq \Omega_t$ be the unbounded component of Ω_t. It is not hard to construct a compact set E so that $\Theta_0 = \Omega_0$, i.e., $\mathbb{C}\backslash E$ is connected, but $\Theta_t \neq \Omega_t$ for all $t > 0$. We show that this is not the case for r-convex sets, and the situation is stable for small enough t.

Theorem 2.9. *Let E be an r-convex compact set for $r > 0$, and let $\Theta_0 = \Omega_0$. Then there is $t_0 = t_0(E)$ such that $0 < t_0 \leq r/4$, and $\Theta_t = \Omega_t$ for $0 \leq t \leq t_0$.*

Proof. Define $S = S(E) := \max_{\zeta\in E} |\zeta|$, and assume that $r \leq S(E)$. The proof is split into several steps.

Step 1. The set $\hat{\Omega} := \Omega_r \cap B(0, 2S)$ is relatively compact, so it contains a finite $r/2$-net $Z = \{z_j\}_{j=1}^N$,

$$Z \subset \hat{\Omega}, \quad \mathrm{dist}(z', Z) \leq \frac{r}{2} < r, \quad \forall z' \in \hat{\Omega}.$$

Since $\Omega_0 = \Theta_0 = \mathbb{C}\backslash E$ is connected, we can find paths $\Gamma_j : [0, \infty) \to \Theta_0$ with

$$\Gamma_j(0) = z_j, \quad \Gamma_j(\tau) \to \infty, \quad \tau \to \infty; \quad j = 1, \dots, N.$$

We put $\delta := \frac{1}{2} \min_{j,\tau} \mathrm{dist}(\Gamma_j(\tau), E) > 0$, so that $\Gamma_j \subset \Theta_\delta$, $1 \leq j \leq N$.

Step 2. Let $z' \in \hat{\Omega}$, then there is $z_k \in Z$, $1 \le k \le N$ such that $|z' - z_k| < r$. In other words, $z' \in B(z_k, r)$, $z_k \in B(z', r)$. Put

$$B_1 := B(z_k, r) \cup B(z', r), \quad \{\xi_\pm\} := \partial B(z_k, r) \cap \partial B(z', r).$$

Since both z' and z_k are in Ω_r, the closure $\overline{B}_1$ is contained in Ω_0, so

$$\begin{aligned} \operatorname{dist}([z', z_k], E) &> \operatorname{dist}([z', z_k], \partial B_1) = \operatorname{dist}([z', z_k], \{\xi_\pm\}) \\ &= \sqrt{r^2 - |z' - z_k|^2/4} > \frac{\sqrt{3}}{2}\, r. \end{aligned}$$

Now, take $t_0 := \min(\delta, r/4)$, so by Step 1, $[z', z_k] \subset \Omega_t$ and $\Gamma_k \subset \Omega_t$ for $t \le t_0$. Hence $[z', z_k] \cup \Gamma_k \subset \Omega_t$, and as the set on the left-hand side is a path from z' to infinity, we conclude

$$[z', z_k] \cup \Gamma_k \subset \Theta_t \Rightarrow z' \in \Theta_t, \quad \forall t \le t_0.$$

Clearly, $B^c(0, 2S) \subset \Theta_t$ for such t, so, finally, $\Omega_r \subset \Theta_t$ for $t \le t_0$.

Step 3. Assume that for some η, $0 < \eta \le t_0$, the statement of the theorem is wrong, so Ω_η has a bounded component D such that $D \cap \Theta_\eta = \emptyset$. We want to show that

$$d(z) \le \sqrt{2}\, \eta, \quad \forall z \in D. \tag{2.5}$$

Let $z \in D$. Note that $d(z) \le r$, for otherwise $z \in \Omega_r \subset \Theta_\eta$ by Step 2, and hence $z \in D \cap \Theta_\eta$, which is impossible. By the definition of r-convexity there exists $z' \in \Omega$ such that $z \in B(z', r) \subset \Omega_0$, so $z' \in \Omega_r \subset \Theta_\eta$, and, in particular, $z' \ne z$. Hence the segment $[z', z]$ meets the boundary ∂D, so there is a point $\zeta \in [z', z]$ with $d(\zeta) = \eta$, and we conclude that

$$\operatorname{dist}([z', z], E) \le \eta. \tag{2.6}$$

We examine the configuration of two disks, $B(z', r)$ and $B(z, d(z))$, each of which belongs to Ω_0. As the circle $\partial B(z, d(z))$ contains points from E, it is clear that the closed disk $\overline{B(z, d(z))}$ cannot lie inside $B(z', r)$. Hence either the smaller disk $B(z, d(z))$ touches the bigger one from within, and in this case the point ζ of contact is in E (which implies $d(z) = \operatorname{dist}([z', z], E)$, and (2.5) follows from (2.6)), or the disks have a proper intersection. Write

$$B_2 := B(z', r) \cup B(z, d(z)) \subset \Omega_0, \quad \{\xi_\pm\} := \partial B(z', r) \cap \partial B(z, d(z)).$$

Then

$$\operatorname{dist}([z', z], E) \ge \operatorname{dist}([z', z], \partial B_2) = \operatorname{dist}([z', z], \xi_+) = \operatorname{dist}([z', z], \xi_-) \ge h, \tag{2.7}$$

where h is the length of the altitude from the vertex ξ_+ in the triangle $\Delta(z', z, \xi_+)$. If this altitude crosses the side $[z', z]$ then

$$\begin{aligned} &\sqrt{r^2 - h^2} + \sqrt{d^2(z) - h^2} = |z - z'| < r, \\ &\qquad\qquad 2h^2 > d^2(z) + 2\sqrt{(r^2 - h^2)(d^2(z) - h^2)} > d^2(z), \end{aligned}$$

so $d(z) < \sqrt{2}h$, and (2.5) follows from (2.6) and (2.7). If the altitude crosses the extension of the side $[z', z]$, one has $d(z) = \operatorname{dist}([z', z], E)$, and again (2.5) holds.

The inclusion $z \in D \subset \Omega_\eta$ means $d(z) > \eta$, so we obtain the two-sided bound

$$\eta < r' := \sup_{z \in D} d(z) \le \sqrt{2}\,\eta. \tag{2.8}$$

Let $\{z_n\} \subset D$ so that $d(z_n) \to r'$. We can assume $z_n \to z_0$, and hence there is a point $z_0 \in D$ with $\eta < d(z_0) = r' \le \sqrt{2}\,\eta$.

Step 4. We show that there is a triangle $\Delta(abc)$ with circumradius $R(\Delta) < 4\eta$ such that

$$\Delta(abc) \cap E = \emptyset, \quad E_1 := \overline{\Delta(abc)} \cap E \supset \{a, b, c\}. \tag{2.9}$$

Take the point z_0 from Step 3 and consider the disk $B(z_0, r')$. Its boundary has nonempty intersection with E. If the circle $\partial B(z_0, r')$ contains 3 different points from E, then in view of (2.8) we are done. Assume that $\partial B(z_0, r') \cap E = \{\zeta_1\}$, or $\partial B(z_0, r') \cap E = \{\zeta_1, \zeta_2\}$, and the points ζ_1 and ζ_2 do not belong to a diameter of the circle. The same argument as in the proof of Proposition 2.4 shows that such configurations cannot occur.

There remains only the case where $z_0 = 0$, $\partial B(z_0, r') \cap E = \{a, b\}$, $a = ir'$ and $b = -ir'$ (after an appropriate affine transformation of the plane). Set

$$G := \{z = x + iy : \ 0 < x \le r', \ |y| \le r'\}.$$

Then $\overline{G \cap E} \neq \emptyset$, since otherwise the circle could be shifted to the right to have $\overline{B(z_0', r')} \cap E = \emptyset$, which, as we have already seen, contradicts the maximality of r'. Let h be the nonzero number of least magnitude such that the triangle $\Delta(abc_h)$, $c_h = r' + ih$, contains points from E. The number h exists since by assumption $c_0 = r' \notin E$, and $0 < |h| \le r'$ (if there are two options, h and $-h$, we take the positive one). Clearly, such points from E belong to the side ac_h for $h > 0$ (bc_h for $h < 0$). If we choose the point $c \in E$ on the corresponding side, then (2.9) holds. The triangle $\Delta(abc)$ is either acute or rectangular. For its sides we have by (2.8)

$$M := \max(|ab|, |ac|, |bc|) \le \sqrt{5}\,r' \le \sqrt{10}\,\eta, \tag{2.10}$$

and by the known upper bound for the circumradius of such a triangle $R(\Delta) \le M < 4\eta$.

Step 5. The choice of $t_0 = \min(\delta, r/4)$ implies $R(\Delta) < 4\eta \le 4t_0 \le r$. By Proposition 2.2 (see Remark 2.3 after its proof), the set E_1 in (2.9) is not r-convex, and $\mathrm{conv}_r(E_1) \cap \Delta(abc) \neq \emptyset$. Hence, $\mathrm{conv}_r(E) \cap \Delta(abc) \neq \emptyset$, which contradicts the r-convexity of E, and so the assumption made in Step 3 must be wrong for $r \le S(E)$.

To remove the assumption $r \le S(E)$, note that if $r > S(E)$, then E is r_1-convex with $r_1 = S(E)$. So for the value t_0 one has $0 < t_0 \le r_1/4 < r/4$, as needed. The proof is complete. □

Note that for $E = \{\zeta_1, \dots, \zeta_N\}$ the result is obvious with

$$0 \le t \le t_1(E) := \frac{1}{2}\,\delta(E), \quad \delta(E) := \min_{i \neq k} |\zeta_i - \zeta_k|. \tag{2.11}$$

3. Lower bounds for Green's functions

We will be dealing with domains $\Omega = \overline{\mathbb{C}}\backslash E$ where E a compact set in $\mathbb{C}$.

Definition 3.1. The Green's function for a domain Ω is a map $G_\Omega : \ \Omega \times \Omega \to (-\infty, \infty]$, such that, for each $w \in \Omega$,

(i) $G_\Omega(\cdot, w)$ is harmonic on $\Omega\backslash\{w\}$, and bounded from above and below outside each neighborhood of w;

(ii) $G_\Omega(w, w) = \infty$, and as $z \to w$,

$$G_\Omega(z, w) = -\log|z - w| + O(1), \quad w \neq \infty,$$
$$G_\Omega(z, w) = \log|z| + O(1), \quad w = \infty;$$

(iii) $G_\Omega(z, w) \to 0$, as $z \to \zeta$, for nearly every $\zeta \in \partial\Omega$.

Let us list some basic properties of the Green's functions in the form we need them later on (see, e.g., Section 4.4 in [33]):

(1) If $\partial\Omega$ is nonpolar, then there exists a unique Green's function G_Ω for Ω;

(2) $G_\Omega(z, w) = G_\Omega(w, z) > 0$, moreover, if Ω' is a relatively compact (in $\overline{\mathbb{C}}$) open subset of Ω, then $\min_{z,w\in\Omega'} G_\Omega(z, w) = C(\Omega, \Omega') > 0$;

(3) If $\Omega' \subset \Omega''$ are domains in $\overline{\mathbb{C}}$ with nonpolar boundaries, then

$$G_{\Omega'}(z, w) \le G_{\Omega''}(z, w), \quad z, w \in \Omega'.$$

The notion of Harnack distance proves useful for our reasoning (see [33], pp. 14–15).

Definition 3.2. Let D be a domain in $\overline{\mathbb{C}}$. Given $z_1, z_2 \in D$, the Harnack distance between z_1 and z_2 is the smallest number $\tau_D(z_1, z_2)$ so that for every positive harmonic function h on D,

$$\tau_D^{-1}(z_1, z_2)\, h(z_2) \le h(z_1) \le \tau_D(z_1, z_2)\, h(z_2).$$

It is known that

1. $\tau_D(z, w) = \tau_D(w, z) \ge 1$ and $\tau_D(z, z) = 1$;

2. $\tau_D(z_1, z_3) \le \tau_D(z_1, z_2)\, \tau_D(z_2, z_3)$ for $z_1, z_2, z_3 \in D$;

3. τ_D is a continuous function of both variables, in particular, if D_1 is a relatively compact (in $\overline{\mathbb{C}}$) open subset of D, then $\max_{z,w\in D_1} \tau_D(z, w) = C(D, D_1) < \infty$.

Given a compact set E, we recall the notation $\Omega_t = \{z \in \overline{\mathbb{C}} : d(z) > t\}$ (we view Ω_t as an open subset of $\overline{\mathbb{C}}$). If E is an r-convex compact set with connected complement, then, by Theorem 2.9, we know that Ω_t is a subdomain of $\overline{\mathbb{C}}$ for sufficiently small t. Its boundary $\partial\Omega_t = \{z : d(z) = t\}$ is nonpolar (since it splits the plane), so the Green's function G_t for Ω_t exists and is unique.

The main technical tool is the following lower bound for $G_t(z, \infty)$. In what follows $C = C(E)$ stands for different positive constants which depend only on E, and the particular values of which are immaterial.

Lemma 3.3. *Let E be an r-convex compact set with connected complement $\Omega = \overline{\mathbb{C}}\backslash E$. Then for $0 < t \le t_0$, where $t_0 \le r/4$ is defined in Theorem 2.9,*

$$G_{t/3}(z,\infty) \ge C\min\{1,\, d(z)\}, \quad z \in \Omega_t. \tag{3.1}$$

Proof. By Theorem 2.9, we have $\Omega_t = \Theta_t$ for $0 < t \le t_0$.

Assume first that $d(z) > t_0$, so $z \in \Omega_{t_0}$. By properties (2) and (3),

$$G_{t/3}(z,\infty) \ge G_{t_0/3}(z,\infty) \ge C \ge C\min\{1,\, d(z)\},$$

as needed.

For the rest of the proof we assume that $z \in \Omega_t$ and $d(z) \le t_0$, so $t < d(z) \le t_0$. By r-convexity, $z \in B(z',r) \subset \Omega$, and since $2t_0 < r$ (see Theorem 2.9), the following chain of inequalities can be checked easily:

$$r > |z - z'| \ge d(z') - d(z) \ge r - t_0 > t_0 \ge d(z) > t. \tag{3.2}$$

Define

$$r_1 := |z - z'| - \frac{t}{3}, \quad r_2 := d(z) - \frac{t}{3},$$

so $2t/3 < r_2 < r_1 < |z - z'|$. It follows from (3.2) that the disks $B(z',r_1)$ and $B(z,r_2)$ satisfy

(a) $B(z',r_1) \cup B(z,r_2) \subset \Omega_{t/3}$;

(b) $z' \notin B(z,r_2), \quad z \notin B(z',r_1)$;

(c) $B(z',r_1) \cap B(z,r_2/2) \ne \emptyset$. Indeed,

$$r_1 + \frac{r_2}{2} = |z - z'| - \frac{t}{3} + \frac{1}{2}\Big(d(z) - \frac{t}{3}\Big) > |z - z'|.$$

Let $L := \partial B(z',r_1) \cap B(z,3r_2/4)$ be the arc of $\partial B(z',r_1)$ inside $B(z,3r_2/4)$. A simple argument from planar geometry shows that property (c) implies the lower bound $|L| > r_2/2$ for the length of L.

We proceed with the bounds for the Green's functions. By properties (a) and (b), the function $G_{t/3}(\cdot,z)$ is harmonic and positive in the disk $B(z',r_1)$. As $r_1 < |z - z'| < r$, the mean value theorem provides

$$G_{t/3}(z',z) = \frac{1}{2\pi r_1}\int_{\partial B(z',r_1)} G_{t/3}(\zeta,z)\, m(d\zeta) \ge \frac{1}{2\pi r}\int_L G_{t/3}(\zeta,z)\, m(d\zeta).$$

Since $B(z,r_2) \subset \Omega_{t/3}$, and the Green's function increases with the domain, we have

$$G_{t/3}(u,v) \ge G_{B(z,r_2)}(u,v), \quad u,v \in B\,(z,r_2)\,.$$

The latter can be computed explicitly as

$$G_{B(z,r_2)}(z,v) = \log\left|\frac{r_2}{v - z}\right| \ge \log\frac{4}{3}, \quad v \in B\Big(z, \frac{3r_2}{4}\Big).$$

Hence $G_{t/3}(z,\zeta) \ge \log\frac{4}{3}$ for $\zeta \in L$, so, taking into account $r_2 = d(z) - t/3 > 2d(z)/3$, we obtain the lower bound

$$G_{t/3}(z',z) \ge \frac{\log\frac{4}{3}}{2\pi r}\,|L| > \frac{\log\frac{4}{3}}{4\pi r}\,r_2 > \frac{\log\frac{4}{3}}{6\pi r}\,d(z). \tag{3.3}$$

To pass from z' to ∞, we invoke the Harnack distance. Let $D = \Omega_{2t_0}$ be a domain in $\overline{\mathbb{C}}$ which depends only on E and such that $D \subset \Omega_{t/3}$. It is clear that $z \notin D$ (by the assumption $d(z) \le t_0$), so the function $h_{t,z}(\zeta) := G_{t/3}(z,\zeta)$ is positive and harmonic in D. Next, $2t_0 < r$ implies Ω_r is a relatively compact subset of D, so $z' \in \Omega_r \subset D$, and, by the definition of the Harnack distance with $z_1 = \infty$ and $z_2 = z'$,

$$\tau_D^{-1}(\infty, z')\, G_{t/3}(z,z') \le G_{t/3}(z,\infty).$$

By property (3) of the Harnack distance, $\min_{z'\in\Omega_r} \tau_D^{-1}(\infty,z') = C > 0$, and hence, by (3.3),

$$G_{t/3}(z,\infty) \ge C d(z) \ge C \min\{1,\, d(z)\}, \quad z \in \Omega_t,$$

as claimed. The proof is complete. □

Remark 3.4. Assume that E is a nonpolar r-convex compact set with connected complement. Then the Green's function $G = G_0$ exists and is unique, and it easily follows from Lemma 3.3 that

$$G(z,\infty) \ge C \min\{1,\, d(z)\}, \quad z \in \Omega.$$

For similar bounds for the Green's functions of a bounded domain with C^2 boundary, see formula (2.8) in [38]. Note that our result is proved under no assumptions on the smoothness of the boundary.

4. Proof of the main result and its consequences

We consider subharmonic functions and their Riesz measures. Let $\mathcal{D}$ be a domain in $\overline{\mathbb{C}}$ such that its boundary $\partial\mathcal{D}$ is nonpolar, and let v be a subharmonic function on $\mathcal{D}$, $v \not\equiv -\infty$, which has a harmonic majorant on $\mathcal{D}$. Let $\mu = 1/(2\pi)\Delta v$ be its Riesz measure. By the fundamental Riesz decomposition theorem (RDT) (see, e.g., Theorem 4.5.4 in [33])

$$v(z) = u(z) - \int_{\mathcal{D}} G(z,\zeta)\,\mu(d\zeta), \quad z \in \mathcal{D},$$

u is the least harmonic majorant of v on $\mathcal{D}$ and G is the Green's function of $\mathcal{D}$.

We apply this result to subharmonic functions on $\Omega = \overline{\mathbb{C}}\backslash E$ where E is an r-convex compact set, with $\mathcal{D} = \Omega_t$ for $t \le t_0$ from Lemma 3.3, so its boundary is nonpolar, and $G = G_t$. We assume that the subharmonic function v on Ω is subject to the growth and normalization conditions (1.3), and we assume that ψ

is a positive and monotone decreasing function on $\mathbb{R}_+$. Hence v has a harmonic majorant u_t on $\mathcal{D}$, and so

$$v(z) = u_t(z) - \int_{\Omega_t} G_t(z,\zeta)\,\mu(d\zeta), \quad z \in \Omega_t. \tag{4.1}$$

Proof of Theorem 1.2. By (1.3), v is bounded above on Ω_t, and

$$v(z) \le K_v\,\psi(t), \quad z \in \Omega_t, \quad t > 0, \tag{4.2}$$

so the least harmonic majorant u_t exists and satisfies the same bound (4.2). Moreover, (4.1) with $z = \infty$ gives

$$\int_{\Omega_t} G_t(\infty,\zeta)\,\mu(d\zeta) \le K_v\,\psi(t), \quad 0 < t \le t_0. \tag{4.3}$$

Next, it follows from the hypothesis of the theorem that $\varphi_1' \ge 0$ a.e. on some interval $[0,\delta]$ (we assume $\delta \le t_0$). We decompose the integral on the left-hand side of (1.5) as

$$\int_\Omega \varphi(d(\zeta))\,\mu(d\zeta) \le \Big\{ \int_{\Omega\setminus\Omega_\delta} + \int_{\Omega_\delta\cap B(0,6S+1)} + \int_{B^c(0,6S+1)} \Big\} \varphi(d(\zeta))\,\mu(d\zeta)$$
$$= I_1 + I_2 + I_3, \quad S = \max_{\zeta\in E} |\zeta|.$$

We begin with the bound for

$$I_1 = \int_{\Omega\setminus\Omega_\delta} \varphi_1(d(\zeta))\,\sigma(d\zeta), \quad \sigma(d\zeta) := d(\zeta)\,\mu(d\zeta).$$

Put

$$H_\delta(t) := \int_{\Omega_t\setminus\Omega_\delta} d(\zeta)\,\mu(d\zeta) = \int_{\Omega_t\setminus\Omega_\delta} \sigma(d\zeta) = \sigma(\{\zeta : t < d(\zeta) \le \delta\}).$$

The following result known as the layer cake representation (LCR) theorem (see Theorem 1.13 in [26]), plays a key role in the next step of the proof.

Theorem (LCR). *Let ν be a measure on the Borel sets of the positive real line $\mathbb{R}_+$ such that*

$$\xi(t) := \nu([0,t)), \quad \xi(0) = 0$$

is finite for every $t > 0$. Let (X,Σ,σ) be a measure space and let f be any non-negative measurable function on X. Then

$$\int_X \xi(f(x))\,\sigma(dx) = \int_0^\infty \sigma(\{x : f(x) > t\})\,\nu(dt).$$

We apply this with $X = \Omega\setminus\Omega_\delta$, $\nu(dt) = \varphi_1'(t)dt$ and $f(x) = d(x)$:

$$I_1 = \int_{\Omega\setminus\Omega_\delta} \varphi_1(d(\zeta))\,\sigma(d\zeta) = \int_0^\delta \varphi_1'(t)\,H_\delta(t)\,dt.$$

For $t < d(\zeta) \le \delta$ one has $d(\zeta) \le (1+\delta)\min\{1, d(\zeta)\} \le C G_{t/3}(\zeta,\infty)$ in view of Lemma 3.3. Here and in the rest of the proof C stands for a positive constant which depends on E, φ, and ψ, as in (1.5). So by (4.3),

$$H_\delta(t) \le C \int_{\Omega_t} G_{t/3}(\zeta,\infty)\,\mu(d\zeta) \le C \int_{\Omega_{t/3}} G_{t/3}(\zeta,\infty)\,\mu(d\zeta) \le C\,K_v\,\psi\Big(\frac{t}{3}\Big),$$

and, finally,

$$I_1 \le C\,K_v \int_0^\delta \varphi_1'(t)\,\psi\Big(\frac{t}{3}\Big)\,dt. \tag{4.4}$$

Since $d(\zeta) \le |\zeta| + S \le 7S+1$ for $\zeta \in B(0, 6S+1)$, the bound for I_2 is

$$\begin{aligned} I_2 &= \int_{\Omega_\delta \cap B(0,6S+1)} \varphi_1(d(\zeta))\,d(\zeta)\,\mu(d\zeta) \le \max_{\delta \le y \le 7S+1} \varphi_1(y) \int_{\Omega_\delta} d(\zeta)\,\mu(d\zeta) \\ &\le C \int_{\Omega_\delta} \min\{1,\, d(\zeta)\}\,\mu(d\zeta) \le C \int_{\Omega_{\delta/3}} G_{\delta/3}(\zeta,\infty)\,\mu(d\zeta) \le C\,K_v\,\psi\Big(\frac{\delta}{3}\Big), \end{aligned}$$

and so

$$I_2 \le C\,K_v. \tag{4.5}$$

The bound for I_3 is standard, and has nothing to do with the subtle Lemma 3.3. The LCR theorem and the inclusion $B^c(0,6S+1) \subset \Omega_{5S+1}$ give

$$I_3 \le \int_{\Omega_{5S+1}} \varphi(d(\zeta))\,\mu(d\zeta) = \int_{5S+1}^\infty \varphi'(t) H(t)\,dt, \quad H(t) := \int_{\Omega_t} \mu(d\zeta). \tag{4.6}$$

For $t \ge 5S+1$ we put

$$R_t := \frac{2}{3}\,(t-S) \ge \frac{2}{3}(4S+1), \quad R_t - S = \frac{2t-5S}{3} \ge \frac{t}{3},$$

and apply again the RDT in the form

$$v(z) = \tilde{u}(z) - \int_{|\zeta|>R_t} \tilde{G}(z,\zeta)\,\mu(d\zeta), \quad |z| > R_t,$$

where $\tilde{G}$ is the Green's function of the domain $\{\zeta : |\zeta| > R_t\}$ and $\tilde{u}$ is the least harmonic majorant of v on this domain. Since $d(z) \ge R_t - S$ for $|z| > R_t$, the assumptions on v and ψ imply

$$\tilde{u}(z) \le K_v\,\psi(R_t - S), \quad |z| > R_t,$$

and, as above,

$$\int_{|\zeta|>R_t} \tilde{G}(\infty,\zeta)\,\mu(d\zeta) \le K_v\,\psi(R_t - S) \le K_v\,\psi\Big(\frac{t}{3}\Big). \tag{4.7}$$

The function $\tilde{G}(\infty,\zeta)$ is known to have the explicit form $\tilde{G}(\infty,\zeta) = \log|\zeta| - \log|R_t|$, so, by (4.7), we conclude

$$\log\frac{3}{2}\int_{|\zeta|>\frac{3}{2}R_t}\mu(d\zeta) \le \int_{|\zeta|>\frac{3}{2}R_t}\log\left|\frac{\zeta}{R_t}\right|\mu(d\zeta)$$
$$\le \int_{|\zeta|>R_t}\log\left|\frac{\zeta}{R_t}\right|\mu(d\zeta) \le K_v\,\psi\Bigl(\frac{t}{3}\Bigr).$$

Next, note that

$$\Bigl\{\zeta : |\zeta| > \frac{3}{2}R_t\Bigr\} \supset \Bigl\{\zeta : d(\zeta) > \frac{3}{2}R_t + S\Bigr\} = \Omega_t,$$

so

$$H(t) = \int_{\Omega_t}\mu(d\zeta) \le C\,K_v\,\psi\Bigl(\frac{t}{3}\Bigr),$$

and, finally, in view of (4.6), we have

$$I_3 \le C\,K_v\int_{5S+1}^{\infty}\varphi'(t)\,\psi\Bigl(\frac{t}{3}\Bigr)\,dt. \tag{4.8}$$

The main statement of Theorem 1.2 now follows from (4.4), (4.5), and (4.8).

The case of bounded subharmonic functions is not formally covered by the main result since $\varphi_1(0) = 1 \ne 0$ for $\varphi(x) = \min\{1,x\}$. Fortunately, (1.6) is a direct consequence of (4.3) and Lemma 3.3. Indeed, for $0 < t < t_0$

$$CK_v \ge \int_{\Omega_t} G_t(\infty,\zeta)\,\mu(d\zeta) \ge C\int_{\Omega_t}\min\{1,\,d(\zeta)\}\,\mu(d\zeta),$$

and there remains only to let $t \to 0$. The proof is complete. □

Corollary 4.1. *Suppose that for a subharmonic function v estimate (1.3) holds with $\psi(t) = t^{-q}$, $q > 0$. Then, for each $\varepsilon > 0$,*

$$\int_{\Omega}\varphi(d(\zeta))\,\mu(d\zeta) \le C(E,q,\varepsilon)\,K_v, \tag{4.9}$$

with

$$\varphi(x) = x^{q+1/2}\,(\min\{x,1/x\})^{\varepsilon+1/2} = \begin{cases} x^{q+1+\varepsilon}, & x \le 1, \\ x^{q-\varepsilon}, & x > 1, \end{cases} \qquad 0 < \varepsilon < q.$$

In some instances, in addition to the hypothesis of Theorem 1.2, the support of the Riesz measure μ is bounded. Such a situation occurs when $v = \log|f|$ for an analytic function f on Ω satisfying $f(\infty) = 1$ (see Section 5). Now only the first term in (1.4) matters, so we obtain:

Corollary 4.2. *In addition to the assumptions of Theorem 1.2, let* $\operatorname{supp}\mu \subset B(0, R_\mu)$*, and, instead of (1.4), suppose*

$$\int_0 \varphi_1'(t)\,\psi\Big(\frac{t}{3}\Big)\,dt < \infty.$$

Then

$$\int_\Omega \varphi(d(\zeta))\,\mu(d\zeta) \le C(E, \psi, \varphi, R_\mu)\,K_v.$$

Consider the case of finite sets E, where the bound for the Green's function in (3.1) and the main result can be refined. We formulate this for the special bound as in Corollary 4.1, although the general case of Theorem 1.2 can be handled in the same fashion.

Theorem 4.3. *Let* $E = \{\zeta_1, \ldots, \zeta_N\}$ *be a finite set and let* v *be a subharmonic function on* $\Omega = \overline{\mathbb{C}}\backslash E$ *so that*

$$v(z) \le \frac{K_v}{d^q(z)}, \quad v(\infty) = 0; \quad q > 0, \quad z \in \Omega. \tag{4.10}$$

Then there are positive constants $k = k(E) > 1$ *and* $t_2 = t_2(E)$*, defined below in (4.16), such that*

$$G_t(z, \infty) > \frac{\log 2}{N} > 0, \quad z \in \Omega_{kt}, \quad t \le t_2, \tag{4.11}$$

and for each $0 < \varepsilon < q$

$$\int_\Omega \varphi(d(\zeta))\,\mu(d\zeta) \le C(E, q, \varepsilon)\,K_v, \quad \varphi(x) = \begin{cases} x^{q+\varepsilon}, & x \le 1\,; \\ x^{q-\varepsilon}, & x > 1\,. \end{cases}$$

If, in addition, $\operatorname{supp}\mu$ *is bounded then*

$$\int_\Omega d^{q+\varepsilon}(\zeta)\,\mu(d\zeta) < \infty. \tag{4.12}$$

Proof. Put

$$m_j := \prod_{i\ne j} |\zeta_i - \zeta_j|, \quad C := 2^{N-1} \max_j m_j. \tag{4.13}$$

The function

$$v_t(z) := \frac{1}{N}\Big(\sum_{j=1}^N \log|z-\zeta_j| - \log t - \log C\Big) \tag{4.14}$$

is subharmonic on $\mathbb{C}$ (and harmonic on Ω), and $v_t(z) = \log|z| + O(1)$, as $z \to \infty$. For

$$0 \le t \le t_1(E) = \frac{1}{2}\,\delta(E), \quad \delta(E) := \min_{i\ne k} |\zeta_i - \zeta_k|$$

(see (2.11)), on each circle $|z-\zeta_n|=t$, $1\le n\le N$, one has

$$v_t(z)=\frac{1}{N}\Big(\sum_{j\ne n}\log|z-\zeta_j|-\log C\Big).$$

Since $|z-\zeta_j|\le|z-\zeta_n|+|\zeta_n-\zeta_j|=t+|\zeta_n-\zeta_j|$,

$$\begin{aligned} v_t(z) &\le \frac{1}{N}\Big(\sum_{j\ne n}\log(|\zeta_j-\zeta_n|+t)-\log C\Big)\\ &\le \frac{1}{N}\Big((N-1)\log 2+\sum_{j\ne n}\log|\zeta_n-\zeta_j|-\log C\Big)\le 0 \end{aligned}$$

in view of the choice of C. Hence $u_t(z)=v_t(z)-G_t(z,\infty)$ is subharmonic on Ω_t and

$$\limsup_{z\to\zeta}u_t(z)\le 0,\quad \zeta\in\partial\Omega_t,\quad \limsup_{z\to\infty}\frac{u_t(z)}{\log|z|}=0,$$

so by the Phragmen–Lindelöf principle (see Corollary 2.3.3 in [33]) $u_t\le 0$, or

$$v_t(z)\le G_t(z,\infty),\quad z\in\Omega_t. \tag{4.15}$$

On the other hand, put

$$k:=1+2C\Big(\frac{2}{\delta(E)}\Big)^{N-1}>1,\quad t_2:=\frac{t_1}{k}. \tag{4.16}$$

For $z\in\Omega_{kt}$ and $t\le t_2$ we have for some l,m, $1\le l,m\le n$, that

$$\min_{1\le i\le n}|z-\zeta_i|=|z-\zeta_l|>kt,\ \ \min_{i\ne l}|z-\zeta_i|=|z-\zeta_m|\ge|\zeta_l-\zeta_m|-|z-\zeta_l|\ge\delta(E)-kt,$$

so

$$\begin{aligned} v_t(z) &= \frac{1}{N}\Big(\sum_{j\ne l}\log|z-\zeta_j|+\log|z-\zeta_l|-\log t-\log C\Big)\\ &> \frac{1}{N}\big((N-1)\log(\delta(E)-kt)+\log kt-\log t-\log C\big)\\ &\ge \frac{1}{N}\Big((N-1)\log\frac{\delta(E)}{2}+\log k-\log C\Big)>\frac{\log 2}{N}, \end{aligned}$$

by the choice of k and C. Finally,

$$G_t(z,\infty)\ge v_t(z)>\frac{\log 2}{N}>0,\quad z\in\Omega_{kt},\quad t\le t_2,$$

as needed.

The rest of the proof follows the same line of reasoning as the the proof of Theorem 1.2, using the LCR theorem, with Lemma 3.3 replaced with (4.11). □

To show that Corollary 4.1 and Theorem 4.3 are optimal in some sense, we proceed with the following simple result.

Lemma 4.4. *Let $E \subset \mathbb{C}$ be an arbitrary compact set, which does not split the plane, let D be a relatively compact (in the sense of $\overline{\mathbb{C}}$) subdomain of $\Omega = \overline{\mathbb{C}} \backslash E$, and suppose $\infty \in D$. Let v be a subharmonic and continuous (in the sense of $\overline{\mathbb{C}}$), nonnegative function on Ω. Then the least harmonic majorant u for D exists, and*

$$v_{\min} := \min_{\zeta \in \partial D} v(\zeta) \le u(z) \le \max_{\zeta \in \partial D} v(\zeta) =: v_{\max}, \quad z \in D. \tag{4.17}$$

Proof. By assumption, v is nonnegative and bounded on D, so the least harmonic majorant u exists, and it is nonnegative and bounded.

To prove the right inequality, note that v is continuous on $\overline{D}$, and so

$$\limsup_{z \to \zeta \in \partial D} v(z) = v(\zeta) \le v_{\max}.$$

By the maximum principle $v \le v_{\max}$, so $u \le v_{\max}$.

To prove the left inequality, note that

$$\liminf_{z \to \zeta} u(z) \ge \liminf_{z \to \zeta} v(z) = v(\zeta) \ge v_{\min}.$$

Put $V = -u + v_{\min}$. This is a harmonic and bounded function on D, and $\limsup_{z \to \zeta} V(z) \le 0$, $\zeta \in \partial D$. Again, by the maximum principle, $V \le 0$ in D, as needed. The proof is complete. □

For the class of subharmonic functions v satisfying (1.3) with $\psi(t) = t^{-q}$, $q > 0$, there is an obvious extremal element $\hat{v}(z) = d^{-q}(z)$. This function is subharmonic and continuous on Ω, and it is quite natural to expect that it provides divergence of integrals in (4.9).

We apply Lemma 4.4 to $\hat{v}$. By the RDT,

$$0 = \hat{v}(\infty) = \hat{u}(\infty) - \int_D G_D(z, \infty)\, \hat{\mu}(dz), \quad \hat{\mu} = \frac{1}{2\pi} \Delta \hat{v},$$

and so, by Lemma 4.4,

$$\big[\max_{\zeta \in \partial D} d(\zeta)\big]^{-q} \le \int_D G_D(z, \infty)\, \hat{\mu}(dz) \le \big[\min_{\zeta \in \partial D} d(\zeta)\big]^{-q}. \tag{4.18}$$

Two types of domains D are of particular interest.

1. As above in Section 2, let Θ_t be the unbounded component of the set $\Omega_t = \{z : \ d(z) > \ t\}$. Then $d(\zeta) = t$ on $\partial \Theta_t$, so, by (4.18),

$$\int_{\Theta_t} G_{\Theta_t}(z, \infty)\, \hat{\mu}(dz) = t^{-q}. \tag{4.19}$$

2. Let $D = D_t = \{|z| > t\}$ where $t > S = \max_{\zeta \in E} |\zeta|$. Then, for $|z| \ge t$,

$$\frac{t - S}{t} |z| \le d(z) \le |z| + S, \tag{4.20}$$

$G_{D_t}(z,\infty)=\log\frac{|z|}{t}$, and (4.18) takes the form

$$(t+S)^{-q}\le\int_{D_t}\log\frac{|z|}{t}\,\hat\mu(dz)\le(t-S)^{-q}\,. \tag{4.21}$$

We mention two important consequences of (4.21). First, let $t>\tau>S$. Then

$$\begin{aligned}\int_{D_t}\hat\mu(dz)&\le\Bigl(\log\frac{t}{\tau}\Bigr)^{-1}\int_{D_t}\log\frac{|z|}{\tau}\,\hat\mu(dz)\le\Bigl(\log\frac{t}{\tau}\Bigr)^{-1}\int_{D_\tau}\log\frac{|z|}{\tau}\,\hat\mu(dz)\\ &\le\Bigl(\log\frac{t}{\tau}\Bigr)^{-1}(\tau-S)^{-q}<\infty.\end{aligned} \tag{4.22}$$

Next,

$$\int_{D_t}\log|z|\,\hat\mu(dz)\le(t-S)^{-q}+\log t\int_{D_t}\hat\mu(dz)<\infty. \tag{4.23}$$

We show now that Corollary 4.1 is false for the function $\hat v$ and $\varepsilon<0$.

Theorem 4.5. *Let $E\subset\mathbb{C}$ be an arbitrary compact set, which does not split the plane and let $\hat v(z)=d^{-q}(z)$, $q>0$. Then, for each $0<\varepsilon<q$,*

$$I_\pm:=\int_\Omega d^{q\pm\varepsilon}(z)\,\hat\mu(dz)=+\infty. \tag{4.24}$$

Proof. Define $M:=B(0,S+1)\backslash E=B(0,S+1)\bigcap\Omega$. We actually prove that

$$\int_{D_{S+1}}d^{q+\varepsilon}(z)\,\hat\mu(dz)=\int_M d^{q-\varepsilon}(z)\,\hat\mu(dz)=+\infty,\qquad D_{S+1}=\{|z|>S+1\}.$$

We begin with I_+. By (4.20) with $t=S+1$ we have, for $|z|\ge S+1$,

$$d^{q+\varepsilon}(z)\ge\frac{|z|^{q+\varepsilon}}{(S+1)^{q+\varepsilon}}\ge C_1(E,q,\varepsilon)\,|z|^q\log|z|,$$

so that

$$\int_{D_{S+1}}d^{q+\varepsilon}(z)\,\hat\mu(dz)\ge C_1(E,q,\varepsilon)\int_{D_{S+1}}|z|^q\log|z|\,\hat\mu(dz). \tag{4.25}$$

Let $\sigma(dz)$ be the restriction of $\log|z|\,\hat\mu(dz)$ to D_{S+1}. The LCR theorem gives

$$\begin{aligned}\int_{D_{S+1}}|z|^q\,\sigma(dz)&=q\int_0^\infty t^{q-1}\,dt\int_{D_t\cap D_{S+1}}\log|z|\,\hat\mu(dz)\\ &=(S+1)^q\int_{D_{S+1}}\log|z|\,\hat\mu(dz)+q\int_{S+1}^\infty t^{q-1}\,dt\int_{D_t}\log|z|\,\hat\mu(dz),\end{aligned}$$

so, by (4.25),

$$\int_{D_{S+1}}d^{q+\varepsilon}(z)\,\hat\mu(dz)\ge C_2(E,q,\varepsilon)\int_{S+1}^\infty t^{q-1}\,dt\int_{D_t}\log|z|\,\hat\mu(dz).$$

But $G_{D_t}(z,\infty) = \log|z| - \log t < \log|z|$, and it follows from (4.21) that

$$\int_{D_t} \log|z|\,\hat\mu(dz) \ge \int_{D_t} G_{D_t}(z,\infty)\,\hat\mu(dz) \ge (t+S)^{-q},$$

which implies

$$I_+ \ge \int_{D_{S+1}} d^{q+\varepsilon}(z)\,\hat\mu(dz) = +\infty,$$

as claimed.

The domain Θ_x plays a key role in estimating I_-. Let $z\in\Theta_x$. Then for every $z_0\in E$ one has

$$|z-z_0| \ge d(z) > x, \qquad \frac{|z-z_0|}{x} > 1,$$

so the function $h(z) = \log(|z-z_0|/x)$ is harmonic on Θ_x and

$$h(z)\ge 0, \quad z\in\overline{\Theta}_x; \quad h(z) = \log|z| + O(1), \quad z\to\infty.$$

Hence by the maximum principle

$$\log\frac{|z-z_0|}{x} - G_{\Theta_x}(z,\infty) \ge 0, \quad z\in\Theta_x. \tag{4.26}$$

Define $M_x := B(0,S+1)\cap\Omega_x$ and $N_x := B(0,S+1)\cap\Theta_x \subset M_x$. If $z\in N_x$ and $x<1$, then (4.26) implies

$$G_{\Theta_x}(z,\infty) < \log\frac{2S+1}{x} < C_3(E,\varepsilon)\,x^{-\varepsilon}. \tag{4.27}$$

We apply again the LCR theorem to obtain

$$\begin{aligned}\int_M d^{q-\varepsilon}(z)\,\hat\mu(dz) &= (q-\varepsilon)\int_0^{S+1} x^{q-\varepsilon-1}dx\int_{M_x}\hat\mu(dz)\\ &\ge (q-\varepsilon)\int_0^1 x^{q-\varepsilon-1}dx\int_{N_x}\hat\mu(dz).\end{aligned}$$

By (4.27),

$$\int_M d^{q-\varepsilon}(z)\,\hat\mu(dz) \ge C_4(E,q,\varepsilon)\int_0^1 x^{q-1}dx\int_{N_x} G_{\Theta_x}(z,\infty)\,\hat\mu(dz).$$

Next,

$$\int_{N_x} G_{\Theta_x}(z,\infty)\,\hat\mu(dz) = \int_{\Theta_x} G_{\Theta_x}(z,\infty)\,\hat\mu(dz) - \int_{\Theta_x\cap\overline{D}_{S+1}} G_{\Theta_x}(z,\infty)\,\hat\mu(dz).$$

The first integral on the right-hand side equals x^{-q} due to (4.19). For the second one, we have, by (4.26), (4.22), and (4.23),

$$\begin{aligned}\int_{\Theta_x\cap\overline{D}_{S+1}} G_{\Theta_x}(z,\infty)\,\hat\mu(dz) &\le \int_{\Theta_x\cap\overline{D}_{S+1}} \log\frac{|z-z_0|}{x}\,\hat\mu(dz) \le \int_{\overline{D}_{S+1}} \log\frac{2|z|}{x}\,\hat\mu(dz)\\ &= \int_{\overline{D}_{S+1}} \log|z|\,\hat\mu(dz) + \log\frac{2}{x}\int_{\overline{D}_{S+1}}\hat\mu(dz) \le C_5(E)\Big(1+\log\frac{2}{x}\Big).\end{aligned}$$

Finally,

$$\int_{N_x} G_{\Theta_x}(z,\infty)\,\hat\mu(dz) \ge x^{-q} - C_5(E)\Big(1+\log\frac{2}{x}\Big) \ge C_6(E)\,x^{-q}$$

for sufficiently small x, and so

$$I_- \ge \int_M d^{q-\,\varepsilon}(z)\,\hat\mu(dz) = +\infty.$$

The proof is complete. □

It follows from (4.24) (compare with Corollary 4.1) that

$$\int_\Omega \hat\varphi(d(z))\,\hat\mu(dz) = +\infty, \quad \hat\varphi(x) = \begin{cases} x^{q-\,\varepsilon}, & x\le 1\,; \\ x^{q+\,\varepsilon}, & x>1\,. \end{cases}$$

It turns out that Corollary 4.1 is false for the function $\hat v$ even for $\varepsilon = 0$ as long as we consider particular sets E.

Example. Let $E_0 = [0,1]$, $v_0(z) = d^{-2}(z,E_0)$, and $\mu_0 = \frac{1}{2\pi}\Delta v_0$. By Corollary 4.1,

$$\int_{\Omega_0} \varphi_0(d(\zeta))\,\mu_0(d\zeta) < \infty, \quad \Omega_0 = \mathbb{C}\backslash E_0, \quad \varphi_0(x) = \begin{cases} x^{3+\varepsilon}, & x\le 1\,, \\ x^{2-\varepsilon}, & x>1\,, \end{cases} \qquad \forall \varepsilon>0.$$

To show that the integral diverges for $\varepsilon = 0$ we compute the Riesz measure μ_0 of v_0 explicitly. Indeed, now $\Omega_0 = \mathbb{C}_1\cup\mathbb{C}_2\cup\mathbb{C}_3$, where

$$\mathbb{C}_1 = \{z: 0\le x\le 1,\ y\ne 0\}, \quad \mathbb{C}_2 = \{z: x<0\}, \quad \mathbb{C}_3 = \{z: x>1\}, \quad z = x+iy.$$

We apply the well-known equality $\Delta|F|^2 = 4|F'|^2$, for an analytic function F to obtain

$$v_0(z) = \begin{cases} y^{-2}, & z\in\mathbb{C}_1\,, \\ |z|^{-2}, & z\in\mathbb{C}_2\,, \\ |z-1|^{-2}, & z\in\mathbb{C}_3\,, \end{cases} \qquad \Delta v_0 = \begin{cases} 6y^{-4}, & z\in\mathbb{C}_1\,, \\ 4|z|^{-4}, & z\in\mathbb{C}_2\,, \\ 4|z-1|^{-4}, & z\in\mathbb{C}_3\,. \end{cases}$$

For $p>0$ we have

$$\int_{\Omega_0} d^p(z)\,\mu_0(dz) = \sum_{j=1}^{3}\int_{\mathbb{C}_j} d^p(z)\,\mu_0(dz).$$

The first integral

$$I_1 := \int_{\mathbb{C}_1} d^p(z)\,\mu_0(dz) = \frac{6}{\pi}\int_0^1 dx \int_0^\infty \frac{dy}{y^{4-p}} = +\infty$$

for $p=3$. The second integral

$$I_2 := \int_{\mathbb{C}_2} d^p(z)\,\mu_0(dz) = \frac{4}{\pi}\int_{-\infty}^0 dx\int_0^\infty \frac{dy}{(x^2+y^2)^{2-p/2}} = +\infty$$

for $p=2$. The computation for I_3 is similar.

We complete the section with the converse result for analytic functions (see [13]).

Proposition 4.6. *Let E be a compact subset of $\mathbb{C}$ and let $Z = \{z_n\}$ be a sequence of points in $\Omega = \mathbb{C}\backslash E$ so that*

$$K := \sum_{n\geq 1} d^q(z_n) < \infty, \quad q \geq 1.$$

Then there is an analytic function f on Ω with zero set $Z(f) = Z$ and $f(\infty) = 1$, such that

$$\log|f(z)| \leq \frac{C_q\, K}{d^q(z)}. \tag{4.28}$$

Proof. We begin with the well known Weierstrass prime factor of order $p \geq 0$,

$$W(z,p) = (1-z)\exp\Big(\sum_{k=1}^{p}\frac{z^k}{k}\Big), \quad p \geq 1, \quad W(z,0) = 1-z, \quad z \in \mathbb{C},$$

and its bounds (see, e.g., Lemma 4.3.1 in [25])

$$|W(z,p) - 1| \leq |z|^{p+1}, \quad |z| \leq 1, \tag{4.29}$$

$$\log|W(z,p)| \leq A_p|z|^p, \quad |z| \geq \frac{1}{3}, \quad A_p = 3e(2 + \log(p+1)). \tag{4.30}$$

Denote by $e_n \in E$ one of the points closest to z_n, i.e., such that $d(z_n) = |z_n - e_n|$. Put

$$f(z) := \prod_{n\geq 1} W(u_n(z), p), \quad u_n(z) = \frac{z_n - e_n}{z - e_n}\,, z \in \Omega,$$

where $p \geq 0$ is chosen so that $q - 1 \leq p < q$, and write

$$f(z) = \Pi_1(z)\cdot\Pi_2(z), \quad \Pi_j(z) = \prod_{n\in\Lambda_j} W(u_n(z), p), \quad j = 1, 2,$$

where

$$\Lambda_1 = \Lambda_1(z) = \{n : |u_n(z)| \leq 1\}, \quad \Lambda_2 = \Lambda_2(z) = \{n : |u_n(z)| > 1\}.$$

Since $u_n(z) \to 0$ for each $z \in \Omega$, the product Π_2 is finite. By (4.29)

$$\sum_{n\in\Lambda_1}|W(u_n(z), p) - 1| \leq \sum_{n\in\Lambda_1}|u_n(z)|^{p+1} \leq \sum_{n\in\Lambda_1}|u_n(z)|^q \leq \frac{K}{d^q(z)}\,,$$

so the product Π_1 converges absolutely and uniformly in Ω. Moreover,

$$\log|\Pi_1(z)| \leq \sum_{n\in\Lambda_1}|W(u_n(z), p) - 1| \leq \frac{K}{d^q(z)}\,.$$

For the second product, by (4.30),

$$\log|\Pi_2(z)| \leq A_p\sum_{n\in\Lambda_2}|u_n(z)|^p \leq A_p\sum_{n\in\Lambda_2}|u_n(z)|^q \leq \frac{A_p\, K}{d^q(z)}\,,$$

which proves (4.28). The equality $Z(f) = Z$ is obvious by construction. □

5. Applications to perturbation theory of linear operators

We recall some rudiments of the spectral theory of linear operators on Hilbert space, related to the structure of the spectrum (see, e.g., Section IV.5.6 in [24]). A bounded linear operator T on the infinite-dimensional Hilbert space $\mathcal{H}$ is said to be a Fredholm operator if its kernel and cokernel are both finite-dimensional subspaces. A complex number λ lies in the essential spectrum $\sigma_{\rm ess}(T)$ of the operator T if $T-\lambda$ is not a Fredholm operator. The essential spectrum is known to be a nonempty closed subset of the spectrum $\sigma(T)$, and its complement $\mathcal{F}(T)=\mathbb{C}\backslash\sigma_{\rm ess}(T)$ is called the Fredholm domain of T (it is not necessarily connected). Clearly, the resolvent set $\rho(T)=\mathbb{C}\backslash\sigma(T)\subset\mathcal{F}(T)$.

The set of all isolated eigenvalues of finite algebraic multiplicity is referred to as the discrete spectrum $\sigma_d(T)=\{\lambda_j\}$. Each eigenvalue is counted according to its algebraic multiplicity. Although $\sigma_{\rm ess}(T)\cap\sigma_d(T)=\emptyset$, the entire spectrum is not in general exhausted by their union. Indeed, write

$$\mathcal{F}(T)=\bigcup\nolimits_{j\ge 0}\mathcal{F}_j(T),$$

where $\mathcal{F}_j(T)$ are the connected components of $\mathcal{F}(T)$ and $\mathcal{F}_0$ is the unbounded component (the outer domain). Then either $\mathcal{F}_j\subset\sigma(T)$, or $\mathcal{F}_j\cap\sigma(T)\subset\sigma_d(T)$ (the latter always occurs for $j=0$). That $\mathcal{F}(T)$ is connected ($\mathcal{F}(T)=\mathcal{F}_0(T)$) implies (the union is disjoint)

$$\sigma(T)=\sigma_{\rm ess}(T)\,\dot{\bigcup}\,\sigma_d(T). \tag{5.1}$$

The fundamental theorem of Weyl (see Theorem IV.5.35 in [24]) is an outstanding result in perturbation theory. Its version for bounded operators states that the essential spectrum is stable under compact perturbations, that is, for any bounded operator A_0 and compact operator B

$$\sigma_{\rm ess}(A)=\sigma_{\rm ess}(A_0),\qquad A=A_0+B. \tag{5.2}$$

Under certain conditions (see below) relation (5.1) holds for the spectrum $\sigma(A)$ of the perturbed operator as well, and all accumulation points of $\sigma_d(A)$ belong to $\sigma_{\rm ess}(A_0)$. We aim here to find the quantitative rate of convergence of the eigenvalues of A in the form

$$\sum_{\lambda\in\sigma_d(A)}\Phi(d(\lambda))\le C\,\|B\|^q_{\mathcal{S}_q},\qquad d(\lambda)=\mathrm{dist}(\lambda,\sigma(A_0)),\qquad q\ge 1,$$

provided B is contained in $\mathcal{S}_q$, the Schatten–von Neumann operator ideal.

Our main assumptions on the unperturbed operator A_0 are:

(i) $\sigma_{\rm ess}(A_0)$ does not split the plane;

(ii) $\sigma(A_0)$ is an r-convex compact set;

(iii) The resolvent $R(\lambda,A_0)=(A_0-\lambda)^{-1}$ is subject to the bound

$$\|R(\lambda,A_0)\|\le\Psi(d(\lambda)),\qquad \lambda\notin\sigma(A_0), \tag{5.3}$$

where Ψ is a monotone decreasing function on $\mathbb{R}_+$ with $\Psi(0)=\infty$ and $\Psi(\infty)=0$.

Note that conditions (i) and (ii) are certainly fulfilled whenever $\sigma(A_0) \subset \mathbb{R}$ or $\sigma(A_0) \subset \mathbb{T}$ and $\sigma(A_0) \neq \mathbb{T}$. Condition (iii) is not really a restriction, as one can put

$$\Psi(x) = \sup\{\|R(\lambda, A_0)\| : \quad d(\lambda) \geq x\}.$$

However such a choice of Ψ is very much implicit. There are many operators, for which (5.3) holds with explicit function Ψ. These include hyponormal operators (see Theorem 3.10.2 in [32]) and spectral operators (in the sense of Dunford) of finite degree [11] (with $\Psi(x) = x^{-s}$, $s > 0$). For normal operators A_0 equality holds in (5.3) with $\Psi(x) = x^{-1}$. Another typical example is (see [30], [15])

$$\Psi(x) = \frac{C_1}{x} \exp\left(\frac{C_2}{x^2}\right). \tag{5.4}$$

A key analytic tool in perturbation theory is the (regularized) perturbation determinant

$$g_q(\lambda) := \det{}_{\lceil q \rceil}(I + BR(\lambda, A_0)), \quad B = A - A_0 \in \mathcal{S}_q,$$

where $\lceil q \rceil = \min\{n \in \mathbb{N} \colon n \geq q\}$, thanks to its properties (see, e.g., [36], Section XI.9 of [10] or Section IV.3 of [16]):

1. g_q is analytic on $\overline{\mathbb{C}} \backslash \sigma(A_0)$, $g_q(\infty) = 1$;
2. λ is a zero of g_q of multiplicity k if and only if $\lambda \in \sigma_d(A) \backslash \sigma(A_0)$ with algebraic multiplicity k;
3. $\log|g_q(\lambda)| \leq C_q \|B\|_{\mathcal{S}_q}^q \|R(\lambda, A_0)\|^q$, $\lambda \in \overline{\mathbb{C}} \backslash \sigma(A_0)$.

We are in a position to present the main spectral consequences of Theorem 1.2 and Corollary 4.2.

Theorem 5.1. *Given a bounded linear operator A_0 subject to conditions* (i)–(iii), *and $B \in \mathcal{S}_q$, $q \geq 1$, let Φ be a positive and absolutely continuous function on $[0, \infty)$ such that $\Phi_1(t) = t^{-1}\Phi(t)$ is monotone increasing in some neighborhood of the origin, $\Phi_1(0) = 0$, and*

$$\int_0 \Phi_1'(t)\, \Psi^q\left(\frac{t}{3}\right) dt + \int^{\infty} \Phi'(t)\, \Psi^q\left(\frac{t}{3}\right) dt < \infty. \tag{5.5}$$

Then

$$\sum_{\lambda \in \sigma_d(A)} \Phi\left(d(\lambda)\right) \leq C(\sigma(A_0), \Psi, \Phi, q)\, \|B\|_{\mathcal{S}_q}^q. \tag{5.6}$$

Proof. Since $\sigma_{\text{ess}}(A_0) = \sigma_{\text{ess}}(A)$ does not split the plane, we see that

$$\sigma(A_0) = \sigma_{\text{ess}}(A_0) \dot{\bigcup} \sigma_d(A_0), \quad \sigma(A) = \sigma_{\text{ess}}(A) \dot{\bigcup} \sigma_d(A), \tag{5.7}$$

and so both $\sigma(A_0)$ and $\sigma(A)$ do not split the plane.

We apply Theorem 1.2 with $E = \sigma(A_0)$ to the subharmonic function

$$v(z) = \log|g_q(z)|, \quad z \in \rho(A_0) \cup \{\infty\}.$$

In view of property (3) of perturbation determinants, and condition (iii), inequality (1.3) holds with $K_v = C_q \|B\|^q_{\mathcal{S}_q}$ and $\psi = \Psi^q$. The Riesz measure μ of v is now a discrete and integer-valued measure supported on $Z(g_q)$, and $\mu\{\lambda\}$ equals the multiplicity of the zero of g_q at λ (the algebraic multiplicity of the eigenvalue $\lambda(A)$). The only problem is that in (5.7) $\sigma_d(A_0)$ is in general nonempty, and, what is more to the point, the set $\sigma_d(A_0) \cap \sigma_d(A)$ can be nonempty as well, and this part of $\sigma_d(A)$ is not controlled by the zero set of the perturbation determinant.[2] Anyway, since $\Phi(0) = 0$, Theorem 1.2 yields

$$\sum_{\lambda\in\sigma_d(A)} \Phi\left(d(\lambda)\right) = \sum_{\lambda\in\sigma_d(A)\setminus\sigma_d(A_0)} \Phi\left(d(\lambda)\right) \le C(\sigma(A_0), \Psi, \Phi, q)\, \|B\|^q_{\mathcal{S}_q},$$

as claimed. □

If $\sigma_{\text{ess}}(A_0)$ splits the plane, then (see Remark 1.3 after Theorem 1.2) we can argue as above, with the resolvent set $\rho(A_0)$ replaced by the outer domain $\mathcal{F}_0(A_0)$, to obtain the inequality

$$\sum_{\lambda\in\sigma_d(A)\cap\mathcal{F}_0(A_0)} \Phi\left(d(\lambda)\right) \le C(\sigma(A_0), \Psi, \Phi, q)\, \|B\|^q_{\mathcal{S}_q}.$$

The question arises naturally, whether condition (i) can be relaxed to

(i') $\sigma(A_0)$ does not split the plane.

The answer is in general negative. Indeed, assume that for A_0 we have $\sigma(A_0) = \overline{\mathbb{D}}$, $\sigma_{\text{ess}}(A_0) = \partial\mathbb{D}$ (e.g., A_0 is adjoint to the shift operator in H^2). Then A can be constructed in such a way that

$$\sigma(A) = \partial\mathbb{D} \,\dot\cup\, \sigma_d(A),$$

and the portion of $\sigma_d(A)$ inside $\mathbb{D}$ cannot be controlled.

In the perturbation theory setting the support of the Riesz measure (the zero set of the perturbation determinant) is bounded, $\operatorname{supp}\mu \subset B(0, R_\mu)$, so Corollary 4.2 applies. It implies that the second term in (5.5) can be dropped. On the other hand, the constant C on the right-hand side depends now on R_μ. It is easy to see that the value R_μ is controlled by the operator norm $\|B\|$. Indeed, it is proved in Lemma 8.4.2 of [15] that, under condition (5.3),

$$\max_{\zeta\in\sigma(A)} d(\zeta) \le x(\Psi, \|B\|^{-1}),$$

[2]As a matter of fact, the Weinstein–Aronszajn formula says that the order of a zero (pole) of g_p at the point $\lambda \in \sigma_d(A)$ equals $\nu(\lambda(A)) - \nu(\lambda(A_0)) \in \mathbb{Z}$, the difference of the algebraic multiplicities of the eigenvalue λ.

where $x(\Psi, a)$, $a > 0$, is the largest solution of the equation $\Psi(x) = a$. So one can take

$$R_\mu = \sup_{\lambda\in\sigma(A_0)} |\lambda| + x(\Psi, \|B\|^{-1}),$$

as needed. Hence

$$\int_0 \Phi_1'(t)\,\Psi^q\Big(\frac{t}{3}\Big)\,dt < \infty \Longrightarrow \sum_{\lambda\in\sigma_d(A)} \Phi\,(d(\lambda)) \le C(\sigma(A_0), \Psi, \Phi, q, \|B\|)\,\|B\|^q_{\mathcal{S}_q} < \infty.$$

Example 1. Let A_0 be a bounded linear operator with a real spectrum, $\sigma(A_0) \subset \mathbb{R}$, and such that condition (5.3) holds with $\Psi(x) = x^{-p}$, $p > 0$. Now both $\sigma_{\text{ess}}(A_0)$ and $\sigma(A_0)$ are compact subsets of the real line, so they are r-convex and do not split the plane. So for $A = A_0 + B$, $B \in \mathcal{S}_q$, and each $\varepsilon > 0$, the bounds

$$\sum_{\lambda\in\sigma_d(A)} \Phi\,(d(\lambda)) \le C(\sigma(A_0), p, q, \varepsilon)\,\|B\|^q_{\mathcal{S}_q}, \quad \Phi(x) = \begin{cases} x^{pq+1+\varepsilon}, & x \le 1\,; \\ x^{pq-\varepsilon}, & x > 1\,, \end{cases} \tag{5.8}$$

and

$$\sum_{\lambda\in\sigma_d(A)} d^{pq+1+\varepsilon}(\lambda) < \infty \tag{5.9}$$

hold. In particular, if W is a bounded linear operator with imaginary component from $\mathcal{S}_q$, relations (5.8) and (5.9) are true with $p = 1$ and

$$A_0 = W_R = \frac{W + W^*}{2}, \quad B = W_I = \frac{W - W^*}{2i}\,.$$

A stronger result for self-adjoint A_0 is in [20]. Its direct application to operators A_0 similar to a self-adjoint operator, that is, $A_0 = T^{-1}A_1T$, $A_1 = A_1^*$, would lead to a constant C on the right-hand side and depending on the transform T. In (5.8) this constant depends only on the *spectrum* of A_0.

Example 2. The same argument works for unitary (or similar to unitary) operators A_0 such that there is $\zeta \in \mathbb{T}\cap\rho(A_0)$. In particular, let V be an $\mathcal{S}_q$-quasiunitary operator, that is, $I - V^*V \in \mathcal{S}_q$, and $\zeta \in \mathbb{T} \cap \rho(V)$. Then its Cayley transform $W = i(\zeta + V)(\zeta - V)^{-1}$ satisfies

$$W_I = (\bar\zeta - V^*)^{-1}\{I - V^*V\}(\zeta - V)^{-1} \in \mathcal{S}_q.$$

Note that $W + i = 2i\zeta(\zeta - A)^{-1}$ is invertible, and $V = \zeta(W + i)^{-1}(W - i)$. It is easy to see that

$$V = U + B, \quad B \in \mathcal{S}_q, \quad U = \zeta(W_R + i)^{-1}(W_R - i)$$

is a unitary operator with $\sigma(U) \ne \mathbb{T}$, so the bound similar to (5.9) holds with $A_0 = U$ and $A = V$.

Example 3. In the Hilbert space $L^2[0,1]$ consider an operator

$$[Af](x) = a_0(x)f(x) + \int_0^1 K(x,y)f(y)\,dy$$

with the Hilbert–Schmidt kernel K, i.e., $K \in L^2([0,1]\times[0,1])$. We assume that the function a_0 is complex valued, continuous on $[0,1]$, and the arc $\gamma = \{a_0(x) : 0 \le x \le 1\}$ is Jordan and either a BC-arc (see Definition 2.7) or has finite global curvature (in particular, C^2-smooth). The multiplication operator $A_0 f = a_0 f$ is normal, and its spectrum $\sigma(A_0) = \gamma$ is the r-convex compact set with connected complement (see Propositions 2.4 and 2.8). As in (5.8) we have

$$\sum_{\lambda\in\sigma_d(A)} \Phi\left(d(\lambda,\gamma)\right) \le C(\gamma,\varepsilon)\,\|K\|^2_{\mathcal{S}_2}, \quad \Phi(x) = \begin{cases} x^{3+\varepsilon}, & x \le 1; \\ x^{2-\varepsilon}, & x > 1. \end{cases} \tag{5.10}$$

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