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On the existence of almost-periodic solutions for the 2D dissipative Euler equations

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Abstract. In this paper we study the two-dimensional dissipative Euler equations in a smooth and bounded domain. In the presence of a sufficiently large dissipative term (or equivalently a sufficiently small external force) precise uniform estimates on the modulus of continuity of the vorticity are proved. These allow us to show existence of Stepanov almost-periodic solutions.

1. Introduction

In this paper we prove some results related with the long-time behavior of the Euler equations (with dissipation) for incompressible fluids in two space dimensions, aimed at proving existence of almost-periodic solutions. For the Euler equations it is well known that in the 2D case it is possible to prove, for smooth enough data, existence and uniqueness of smooth solution, for all positive times; see also the discussion in the next section for certain less standard results. It is also clear that without any smoothing or dissipation, one cannot expect to have uniform boundedness of the energy and of other interesting quantities such as the enstrophy or higher norms of the velocity. In order to study general properties as attractors or existence of almost-periodic solutions (where uniform bounds are sought) we consider the *dissipative Euler equations*

$$
\begin{aligned}
\partial_t u + \chi\, u + (u\cdot\nabla)\, u + \nabla\pi &= f && \text{in } (0,\infty)\times\Omega,\\
\nabla\cdot u &= 0 && \text{in } (0,\infty)\times\Omega,\\
u\cdot n &= 0 && \text{on } (0,\infty)\times\Gamma,\\
u_0(0,x) &= u_0(x) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^2$ is a bounded open set with smooth boundary Γ, n is the unit outward normal vector on Γ, the vector $u(t,x) = (u_1(t,x), u_2(t,x))$ is the velocity of the fluid, $\pi(t,x)$ is the kinematic pressure, and $f = f(t,x)$ is the external force field.

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The *damping* term χu (with a constant $\chi > 0$) models the friction along the bottom in some 2D oceanic models (when this system is considered in a bounded domain; in that case, the system is called the viscous Charney–Stommel barotropic ocean circulation model of the gulf stream) or the Rayleigh friction in the planetary boundary layer (with space-periodic boundary conditions). The positive constant χ is the Rayleigh friction coefficient (or the Ekman pumping/dissipation constant) or also the sticky viscosity, when the model is used to study motion in the presence of rough boundaries; see for instance Gallavotti [23]. Early existence results can be found in Barcilon, Constantin, and Titi [3], while links between the driven and damped 2D Navier–Stokes equations, attractors, and statistical solutions are proved in Ilyin, Miranville, and Titi [25]; Constantin and Ramos [18]; and Constantin, Tarfulea, and Vicol [19]. In recent years the present model has been considered by a number of authors, see for instance [11], [13], [16], [17], [24]. The system (1.1) represents (probably) the "weakest" dissipative modification of the Euler equations and results on the long-time behavior of the damped/driven Navier–Stokes equations do not pass directly to the limit as the "viscosity goes to zero", hence a completely different treatment is required to study the problem without viscosity. This is why here we use some special topologies, which are not derived from the classical Hölder or Sobolev norms.

The main result we will prove is the existence of almost-periodic solutions in the sense of Stepanov, (see [10]) with values in $L^2(\Omega)$, under certain restrictions on the relative sizes of the external force and of the dissipation term; see Theorem 5.1 for the precise statement. To this end we need to show precise estimates, uniform in time, for the vorticity. The boundedness of the vorticity, although sufficient to show uniqueness of weak solutions, is not enough to prove results of *asymptotic stability*, which is one of the main points generally needed to prove existence of almost-periodic solutions; see Amerio and Prouse [1]. For dissipative equations this is now well established (see also the recent results in [9] for an inverse problem) but the Euler system does not directly satisfy the assumptions needed to use abstract results, and this motivates using a stronger topology. In particular, the regularity needed to quantitatively estimate the difference between two solutions over large time-intervals seems to be the represented by the supremum norm (with respect to the x variable) of the gradient of velocity. The topology of Hölder spaces seems to be poorly suited to this problem, hence we resort to something quite sharp, the Dini norm of the vorticity field. We point out that the use of this topology on continuous functions goes back to Beirão da Veiga [5] in the context of global well-posedness of the 2D Euler equations, while in questions of stability the role of Dini-continuous vorticity was first recognized by Koch [29], even if the application to almost-periodic solutions and some of the techniques we apply here are, so far as we know, original.

2. Notation and preliminary facts

Here and in the sequel, we suppose, without loss of generality, the diameter of the bounded set Ω to be one. To avoid technical complications, we assume also that Ω is simply connected, referring to the cited bibliography for how to modify the proofs to deal also with this case. Adhering to the notation standard in mathematical fluid mechanics, let $\mathcal{V}$ denote the space of infinitely differentiable vector fields v on Ω with compact support strictly contained in Ω and satisfying the constraint $\nabla \cdot v = 0$. We introduce the space H of measurable vector fields $v : \Omega \to \mathbb{R}^2$ which are square integrable, divergence free, and tangential to the boundary Γ:

$$H := \big\{ v \in [L^2(\Omega)]^2 : \ \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \Gamma \big\}.$$

In H the normal trace is well defined in $H^{-1/2}(\Gamma)$ and, moreover, H is a separable Hilbert space with the inner product of $[L^2(\Omega)]^2$, denoted in the sequel by $\langle \cdot, \cdot \rangle$, and corresponding norm $\| \, . \, \|_2$; see for instance [33]. (This space is also the closure of $\mathcal{V}$ with respect to the norm $\| \cdot \|_2$.) As usual we will also denote by $\| \, . \, \|_p$ the L^p-norm with respect to the space variables belonging to Ω. Let $V \subset H$ be the following subspace:

$$V := \big\{ v \in [H^1(\Omega)]^2 : \ \nabla \cdot v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \Gamma \big\}.$$

The space V is a separable Hilbert space with the inner product induced by $[H^1(\Omega)]^2$ and its natural norm denoted by $\| \cdot \|_{1,2}$. We also introduce the trilinear form on V, defined as

$$b(u, v, w) := \int_\Omega (u \cdot \nabla) \, v \cdot w \, dx.$$

Since we study time evolution problems, given a Banach space X, for $p \in [1, \infty)$ we denote the usual Bochner spaces $L^p(0, T; X)$ with associated norm $\|f\|^p_{L^p(0,T;X)} := \int_0^T \|f(s)\|^p_X \, ds$, (the lower upper bound of $\|f(s)\|_X$ if $p = \infty$) while $L^p_{\text{loc}}(X)$ is the space of measurable functions $\mathbb{R} \mapsto X$ belonging to $L^p(T_1, T_2; X)$, for any $T_1 \le T_2 \in \mathbb{R}$.

The definition of weak solution (see [13]) for the system (1.1), is the following.

Definition 2.1. We say that a function u is a weak solution to (1.1) on $[0, \infty)$, provided that the following four properties hold true:

$$u \in C([0, \infty); H) \cap L^\infty_{\text{loc}}(0, \infty; V) \quad \text{with} \quad \partial_t u \in L^2_{\text{loc}}(0, \infty; V'), \tag{2.1a}$$

for almost every $t \ge t_0 \ge 0$ and for all $v \in \mathcal{V}$,

$$\|u(t)\|_2^2 + 2\chi \int_{t_0}^t \|u(s)\|_2^2 \, ds \le \|u(t_0)\|_2^2 + \int_{t_0}^t \langle f(s), u(s) \rangle \, ds, \tag{2.1b}$$

$$\|u(t)\|_{1,2}^2 \le \|u(t_0)\|_{1,2}^2 \, \mathrm{e}^{-\chi(t - t_0)} + \frac{1}{\chi} \int_{t_0}^t \|f(s)\|_{1,2}^2 \, \mathrm{e}^{-\chi(t-s)} \, ds, \tag{2.1c}$$

$$\langle u(t) - u(t_0), v \rangle + \int_{t_0}^t \big(\chi \, \langle u(s), v \rangle + b(u(s), u(s), v) \big) \, ds = \int_{t_0}^t \langle f(s), v \rangle \, ds. \tag{2.1d}$$

The following existence theorem is proved in [13], by adapting the well known technique developed by Yudovich [35], which is based on approximation by a special Navier–Stokes system and by using a priori estimates in $L^2(\Omega)$ on both velocity and vorticity, obtained from the momentum equation and from (2.2) below.

Theorem 2.1. *Given $u_0 \in V$ and $f \in L^2_{\mathrm{loc}}(0,\infty;V)$, there exists a weak solution for* (1.1). *Such a weak solution is unique if* $\operatorname{curl} u_0 \in L^\infty(\Omega)$ *and* $\operatorname{curl} f \in L^1_{\mathrm{loc}}(0,\infty;L^\infty(\Omega))$.

In the context of existence and uniqueness of solutions in classes broader than of bounded vorticity we want also to recall the recent results by Bernicot and Hmidi [6], Azzam and Bedrossian [2], and references therein.

We now recall the definition of some other functional spaces that will be frequently used in the sequel. We denote by $L^p_{\mathrm{uloc}}(X)$, the Banach space of *uniformly locally p-integrable functions* on $\mathbb{R}$, defined for $1 \le p < \infty$, by

$$L^p_{\mathrm{uloc}}(X) := \Big\{u:\ \mathbb{R}\to X,\ u \in L^p_{\mathrm{loc}}(\mathbb{R};X):\ \sup_{t\in\mathbb{R}} \int_t^{t+1} \|u(s)\|^p_X\, ds < \infty\Big\},$$

endowed with the norm

$$\|u\|_{L^p_{\mathrm{uloc}}(X)} := \Big[\sup_{t\in\mathbb{R}} \int_t^{t+1} \|u(s)\|^p_X\, ds\Big]^{1/p}.$$

We give now the precise definition of the almost-periodic functions we will use.

Definition 2.2. We say that a function $f \in L^2_{\mathrm{uloc}}(X)$ is Stepanov 2-almost-periodic (or simply Stepanov almost-periodic) if the set of its translates is relatively compact in the $L^2_{\mathrm{uloc}}(X)$-topology. The space of Stepanov almost-periodic functions will be denoted by $\mathcal{S}^2(X)$

The condition that $f \in \mathcal{S}^2(X)$ is given explicitly as follows: $f \in L^2_{\mathrm{uloc}}(X)$ and for any sequence $\{r_m\}$ there is a subsequence $\{r_{m_k}\}$ and a function $\widetilde{f} \in L^2_{\mathrm{uloc}}(X)$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s+r_{m_k}) - \widetilde{f}(s)\|_2^2\, ds \to 0.$$

Results about and further properties of Stepanov spaces can be found for instance in the classical book by Besicovitch [10].

In the study of 2D Euler equations fundamental estimates are obtained by the analysis of the transport equation for the vorticity. In the case of the dissipative system (1.1) the equation satisfied by the vorticity $\xi = \operatorname{curl} u := \partial_1 u_2 - \partial_2 u_1$ is

$$\partial_t \xi + (u\cdot\nabla)\,\xi + \chi\,\xi = \phi, \tag{2.2}$$

where $\phi := \operatorname{curl} f$. By a change of variables we can also write

$$\partial_t \eta + (u\cdot\nabla)\,\eta = \phi\ \mathrm{e}^{\chi t},$$

with $\eta := \xi\,\mathrm{e}^{\chi t}$, recovering a transport equation, without zero order terms.

Since we work with space-time functions, we also define $\overline{\Omega}_T := [0,T]\times\overline{\Omega}$ and we use the following notation: for a given $T>0$,

$$|||f|||_{L^\infty} := \sup_{(x,t)\in\overline{\Omega}_T} |f(t,x)|.$$

What makes the two-dimensional Euler equations very special is that the connection between velocity and vorticity can be made very explicit by the use of the stream-function; it is particularly clean in the case of a simply-connected domain. Let $\psi := -\Delta^{-1}\theta$ be the solution of the Poisson equation with homogeneous Dirichlet data

$$\begin{cases} -\Delta\psi = \theta & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma. \end{cases}$$

Then the vector $v := \nabla^{\perp}\psi := (-\partial_2\psi, \partial_1\psi)$ satisfies

$$\begin{cases} \operatorname{curl} v = \theta & \text{in } \Omega, \\ \nabla \cdot v = 0 & \text{in } \Omega, \\ v \cdot n = 0 & \text{on } \Gamma, \end{cases}$$

and for this reason we use the notation $v = \nabla^{\perp}(-\Delta^{-1}\theta) := \operatorname{curl}^{-1}\theta$.

The use of the vorticity equation, being a nonlocal transport equation, is also at the basis of the classical existence results of classical solutions, dating back to Lichtenstein, Hölder, Wolibner, Leray, Schaeffer, and Kato. See also the historical account in Brezis and Browder [15], §11.

As will be clear later on, in order to prove sharp estimates on the growth of the vorticity, we will use a particular topology, namely that of Dini-continuous functions $C_D(\overline{\Omega}) \subset C(\overline{\Omega})$. This space is the subset of continuous functions $f : \overline{\Omega} \to \mathbb{R}^n$ such that

$$\|f\|_{C_D} := \|f\|_{L^\infty} + [f]_{C_D} := \|f\|_{L^\infty} + \int_0^1 \omega(f,\sigma)\,\frac{d\sigma}{\sigma} < \infty,$$

where, for $\sigma > 0$, the quantity $\omega(f,\sigma)$ denotes the *modulus of continuity* of f, defined by

$$\omega(f,\sigma) := \sup\big\{|f(x) - f(y)| \text{ with } x, y \in \overline{\Omega},\ |x-y| < \sigma\big\}.$$

As will be clear in the next section, the main reason for using this space is that there holds the potential-theoretic estimate:

$$\exists\, C_0 = C_0(\Omega) > 0 : \quad \|\nabla u\|_\infty \leq C_0 \|\operatorname{curl} u\|_{C_D}, \tag{2.3}$$

where $\operatorname{curl} u$ is the vorticity. Some classical (well-known) results dating back to Dini [21] imply in fact that the second derivatives of $-\Delta^{-1}f$ are bounded (more precisely they are also continuous) if $f \in C_D(\overline{\Omega})$, while the simple boundedness of f is not enough (recall that $-\Delta^{-1}f$ is the solution of the Poisson problem with vanishing Dirichlet data and right-hand side equal to f.) We do not exclude further extensions to other functional settings such as Besov or multiplier spaces as in Vishik [34] or Koch and Sickel [30], however here we are not interested in these kind of technicalities, but rather focus on a functional setting that is properly defined also in the case of a domain with boundary.

3. A basic estimate on the Dini norm of the vorticity

We start proving existence and uniqueness of *strong* solutions to the dissipative Euler equations.

Definition 3.1. We say that a vector field u is a strong solution to (1.1) in $[0,T]$ if $u \in C([0,T];C(\overline{\Omega}))$ is divergence-free and tangential to the boundary, $\operatorname{curl} u \in C([0,T];C(\overline{\Omega}))$, $\partial_t u \in L^1(0,T;L^2(\Omega))$, $\pi \in L^1(0,T;W^{1,2}(\Omega))$, u is a weak solution and, in addition,

$$\|\operatorname{curl} u(t)\|_\infty \leq \|\operatorname{curl} u_0\|_\infty + \int_0^t \|\operatorname{curl} f(s)\|_\infty \, \mathrm{e}^{\chi s} \, ds \quad \forall\, t \in [0,T].$$

These solutions are called "strong solutions" since they are unique and depend continuously on the data, but not classical, since a priori $\nabla u \in C([0,T];L^p(\Omega))$ for all $p < \infty$, but ∇u may be not pointwise bounded. The proof is an easy adaption of the sharp results of Hadamard well-posedness proved in [5]. Nevertheless, since we will use these results (which are a sort of endpoint for the well-posedness of the Euler equations), and they are not easily found in literature, we sketch the proof and we make some remarks in order to make the presentation self-contained.

The main theorem on existence and uniqueness for *strong* solutions is the following, which is proved below after some preliminary lemmas.

Theorem 3.1. *Let* $u_0 \in H$ *with* $\operatorname{curl} u_0 \in C(\overline{\Omega})$. *Assume also that* $f \in L^1(0,T;H)$ *with* $\operatorname{curl} f \in L^1(0,T;C(\overline{\Omega}))$, *and* $\chi > 0$. *Then, there exists a unique strong solution of the dissipative Euler equations in* $[0,T]$.

By using a classical approach (see the discussion in [15] and other remarks in [8]) the proof is based on a representation formula for the vorticity, by means of characteristics $U(t,s,x)$, which are solutions of the Cauchy problem for the ordinary differential equation

$$\begin{cases} \dfrac{dU(t,s,x)}{dt} = u(t,U(t,s,x)), \\ U(s,s,x) = x, \end{cases}$$

where $(t,s,x) \in [0,T]^2 \times \overline{\Omega}$, while u is the velocity field sought. From the solution of above family of Cauchy problems one can easily infer (see Eq. (7) in [29]) that

$$|\nabla U(t,s,x)| \leq \mathrm{e}^{\left| \int_s^t \sup_{x\in\Omega} |\nabla u(\tau,x)| \, d\tau \right|} \quad \forall\, (t,s,x) \in [0,T]^2 \times \overline{\Omega}. \tag{3.1}$$

Moreover, (see Kato [27]) the following potential-theoretic estimates for the characteristics hold true. If $\xi \in L^\infty(\overline{\Omega}_T)$, and if $u = \operatorname{curl}^{-1} \xi$, then there is $c_1 > 0$ (depending only on Ω, and hence independent of T) such that

$$\begin{aligned} &|||u|||_{L^\infty} \leq c_1 \, |||\xi|||_{L^\infty}, \\ &|u(t,x) - u(t,y)| \leq c_1 \, |||\xi|||_{L^\infty} \, |x-y| \log\left(\frac{\mathrm{e}}{|x-y|}\right) \quad \forall\, t \in [0,T], \ \forall\, x \neq y. \end{aligned} \tag{3.2}$$

Further, it is well known that in presence of bounded vorticity, characteristics are uniquely defined and are Hölder continuous, see e.g. [5], and they satisfy

$$
\begin{aligned}
(3.3)\quad & |U(t,s,x)-U(t_1,s_1,x_1)| \\
& \qquad \le c_1\,|||\xi|||_{L^\infty}|t-t_1| + \mathrm{e}\,(1+c_1\,|||\xi|||_{L^\infty})\,(|x-x_1|^\alpha+|s-s_1|^\alpha),
\end{aligned}
$$

where the exponent is defined by $\alpha := \mathrm{e}^{-c_1|||\xi|||_{L^\infty}T}$.

To construct the strong solution u to (1.1) we consider the Banach space

$$X := \big\{\theta : \overline{\Omega}_T \to \mathbb{R} :\ \theta \in C(\overline{\Omega}_T)\big\},$$

and we define a map $J : X \to X$ by

$$[J\theta](t,x) := \xi_0(U[\theta](0,t,x))\,\mathrm{e}^{-\chi t} + \int_0^t \phi(s,U[\theta](s,t,x))\,\mathrm{e}^{-\chi(t-s)}\,ds,$$

where $\xi_0 = \operatorname{curl} u_0$ is the initial vorticity, while $\phi = \operatorname{curl} f$, $u[\theta] = \operatorname{curl}^{-1}\theta$, and the characteristics $U[\theta](t,s,x)$ are constructed tracing the trajectories by using the field $u[\theta]$.

We first show that this mapping has a fixed point, then that this fixed point is the vorticity of a strong solution of the dissipative Euler equations. This solution is also a weak solution and uniqueness follows by standard results on weak solutions to the Euler equations with bounded vorticity. We split the proof in two lemmas, following step-by-step the approach in [5].

Lemma 3.1. *Define the convex set*

$$\mathcal{K} := \big\{\theta \in X :\ |||\theta|||_{L^\infty} \le \mathcal{R}\big\},$$

where $\mathcal{R} := \|\operatorname{curl} u_0\|_\infty + \int_0^T \|\operatorname{curl} f(s)\|_\infty e^{\chi s}\,ds$. *Then* $J(\mathcal{K}) \subset \mathcal{K}$ *and, moreover,* $J(\mathcal{K})$ *is an equicontinuous family of functions in* $\overline{\Omega}_T$.

Proof. The bound $|||J\theta|||_{L^\infty} \le \mathcal{R}$ is obvious as also is the equicontinuity of the family $\xi_0(U[\theta](0,t,x))\,\mathrm{e}^{-\chi t}$ (in fact ξ_0 is continuous on $\overline{\Omega}$ and there is a composition with the uniformly continuous $U[\theta]$ as follows by using (3.3)).

For the integral appearing in the definition of $J\theta$ we write

$$
\begin{aligned}
\Big| \int_0^t & \phi(s,U[\theta](s,t,x))\,\mathrm{e}^{-\chi(t-s)}\,ds - \int_0^{t_1} \phi(s,U[\theta](s,t_1,x_1))\,\mathrm{e}^{-\chi(t_1-s)}\,ds \Big| \\
&\le \Big| \int_{t_1}^t \|\phi(s)\|_\infty\,\mathrm{e}^{\chi s}\,ds \Big| + |\mathrm{e}^{-\chi t}-\mathrm{e}^{-\chi t_1}| \int_0^T \|\phi(s)\|_\infty\,\mathrm{e}^{\chi s}\,ds \\
&\qquad + \int_0^t |\phi(s,U[\theta](s,t,x)) - \phi(s,U[\theta](s,t_1,x_1))|\,\mathrm{e}^{\chi s}\,ds.
\end{aligned}
$$

The first and second term from the right-hand side clearly go to zero uniformly as $t_1 \to t$, by the absolute continuity of the integral and the continuity of the exponential function. For the last term observe that the function

$$\varpi(s,\epsilon) := \sup_{|z-z_1|<\epsilon} \mathrm{e}^{\chi s}\,|\phi(s,z)-\phi(s,z_1)|,$$

satisfies the bound $\varpi(s,\epsilon) \leq 2\,\mathrm{e}^{\chi s}\|\phi(s)\|_\infty$ and $\varpi(s,\epsilon) \to 0$ as $\epsilon \to 0$ for almost every $s \in [0,T]$. Hence, we can apply the Lebesgue dominated convergence theorem to show that its integral can be made as small as we wish. Combining this with the continuity of characteristics gives the claim. For further details, see the proof of Theorem 2.1 in [5]. □

We then use some compactness results to employ a fixed-point argument.

Lemma 3.2. *The mapping J has a fixed point.*

Proof. In order to show existence of a fixed point we just need to show that the mapping J is continuous with respect to the $L^\infty(\overline{\Omega}_T)$ topology. (This result, or some of its variations, seems to be folklore and its proof is sketched in Theorem 2.2 of [5] and Lemma 2.8 of [27]. For the reader's convenience we include an elementary and complete alternative proof.)

By the compactness of the mapping, which is ensured by equicontinuity and the Arzelà–Ascoli theorem, it follows that all the other hypotheses of the Schauder fixed point theorem are satisfied.

To this end, let $\{\theta_m\}_m \subset \mathcal{K}$ be such that $\theta_m \to \theta$ uniformly in $\overline{\Omega}_T$. The unique function u_m such that $u_m = \operatorname{curl}^{-1}\theta_m$ satisfies $u_m \to u$ uniformly in $\overline{\Omega}_T$. We show now that $U_m(t,s,x) \to U(t,s,x)$ uniformly in $[0,T]^2 \times \overline{\Omega}$, where U_m is a solution of

$$\begin{cases} \dfrac{dU_m(t,s,x)}{dt} = u_m(t, U_m(t,s,x)), \quad t \in [0,T], \\ U_m(s,s,x) = x, \quad s \in [0,T]. \end{cases}$$

Fix some $\epsilon \in (0,1)$. Then, there exists $N = N(\epsilon) \in \mathbb{N}$ such that

$$\sup_{(x,t)\in\overline{\Omega}_T} |u_n(t,x) - u(t,x)| < \epsilon, \quad \forall n > N.$$

Define $\zeta_{n,s}(t) := |U_n(t,s,x) - U(t,s,x)|^2$ and observe that, for $n > N$, and by using (3.2) and the bound $\mathcal{R}$ on the elements of the set $\mathcal{K}$,

$$\begin{aligned} &\Big|\frac{dU_n(t,s,x)}{dt} - \frac{dU(t,s,x)}{dt}\Big| \\ &\quad \leq |u_n(U_n(t,s,x)) - u(U_n(t,s,x))| + |u(U_n(t,s,x)) - u(U(t,s,x))| \\ &\quad \leq \epsilon + c_1\mathcal{R}|U_n(t,s,x) - U(t,s,x)| \log\Big(\frac{\mathrm{e}}{|U_n(t,s,x) - U(t,s,x)|}\Big). \end{aligned}$$

For some $\lambda \in (0,1)$ (sufficiently small in a manner to be made precise later),

$$\tau_n := \inf\{t > s : \ \zeta_{n,s}(t) \geq \lambda^2\}.$$

Note that τ_n is strictly larger than s, since $\zeta_{n,s}(s) = 0$ and $\zeta_{n,s}$ is a continuous function of its arguments. We then obtain, in $[s,\tau_n]$,

$$\Big|\frac{d\zeta_{n,s}}{dt}\Big| \leq 2\lambda\,\epsilon + c_2\,\zeta_{n,s}\log\Big(\frac{\mathrm{e}}{\zeta_{n,s}}\Big).$$

We define $Z_{n,s}(t) := 2\lambda\,\epsilon/c_2 + \zeta_{n,s}(t)$ and, with simple calculations after optimization in $\varepsilon \in (0,1]$, we can see that for

$$0 < \lambda < \lambda_0 := \sqrt{\frac{1}{\mathrm{e}} + \frac{1}{c_2^2}} - \frac{1}{c_2},$$

there holds for $s < t < \tau_n$,

$$2\lambda\,\epsilon + c_2\,\zeta_{n,s}(t)\log\Big(\frac{\mathrm{e}}{\zeta_{n,s}(t)}\Big) \le c_2\,Z_{n,s}(t)\log\Big(\frac{\mathrm{e}}{Z_{n,s}(t)}\Big).$$

We recall the fact that, from the differential inequality

$$y'(t) \le C\,y(t)\log\Big(\frac{\mathrm{e}}{y(t)}\Big), \quad \text{with} \quad y(0) = y_0,$$

we have by direct integration

$$y(t) \le \mathrm{e}\Big(\frac{y_0}{\mathrm{e}}\Big)^{\mathrm{e}^{-C\,t}} \qquad t \ge s,$$

consequently applying this to the function $Z_{n,s}$ we have

$$Z_{n,s}(t) \le \mathrm{e}\Big(\frac{2\lambda\,\epsilon}{c_2\,\mathrm{e}}\Big)^{\mathrm{e}^{-c_2\,t}} \le \mathrm{e}\Big(\frac{2\lambda\,\epsilon}{c_2\,\mathrm{e}}\Big)^{\mathrm{e}^{-c_2\,T}} \qquad \forall\, t \in [s, \min\{\tau_n, T\}],$$

provided that $0 < \epsilon < \epsilon_0$, where $\epsilon_0 := \min\{1, \frac{c_2\,\mathrm{e}}{2\lambda_0}\}$. Hence, we obtain

$$\zeta_{n,s}(t) \le \frac{2\lambda\,\epsilon}{c_2} + \mathrm{e}\Big(\frac{2\lambda\,\epsilon}{c_2\,\mathrm{e}}\Big)^{\mathrm{e}^{-c_2\,T}} \qquad \forall\, t \in [s, \min\{\tau_n, T\}]. \tag{3.4}$$

Since $2\lambda\,\epsilon/c_2 + \mathrm{e}\big(2\lambda\,\epsilon/(c_2\,\mathrm{e})\big)^{\mathrm{e}^{-c_2 T}}$ is monotonically increasing in λ, the quantity $\zeta_{n,s}(t)$ is bounded by the value assumed at $\lambda = \lambda_0$. Hence we can choose $0 < \epsilon_1 < \epsilon_0$ so small such that

$$\frac{2\lambda_0\,\epsilon}{c_2} + \mathrm{e}\Big(\frac{2\lambda_0\,\epsilon}{c_2\,\mathrm{e}}\Big)^{\mathrm{e}^{-c_2\,T}} \le \lambda^2 \quad \forall\, \epsilon \in (0, \epsilon_1).$$

This shows that, for small enough $\epsilon > 0$, the same bound (3.4) holds for *all* $s \in [0,T]$, for *all* $t \in [s,T]$, and for *all* $x \in \overline{\Omega}$. Consequently, $\zeta_{n,s}$ goes to zero uniformly when ϵ goes to zero. The same reasoning can be used also for $t \in [0,s]$. Hence we obtain that U_m converges uniformly to U in $[0,T]^2 \times \overline{\Omega}$.

Finally, from the definition of J (being a composition of uniformly continuous functions) it follows that if $\theta_m \to \theta$, then $J\theta_m \to J\theta$ uniformly. □

We can now prove the existence result for strong solutions.

Proof of Theorem 3.1. The fixed point $\xi \in \mathcal{K}$ of the map J satisfies $\xi = J\xi$. That is,

$$\xi(t,x) = \xi_0(U(0,t,x))\,\mathrm{e}^{-\chi t} + \int_0^t \phi(s, U(s,t,x))\,\mathrm{e}^{-\chi(t-s)}\,ds, \quad t \in [0,T]. \tag{3.5}$$

By a standard argument (adapting for instance that in Lemmas 2.3-2.4 of [5], and Lemma 2.4 of [27]) we obtain that $x \mapsto U(t,s,x)$ is measure preserving (since $\nabla \cdot u = 0$, where $u := \operatorname{curl}^{-1} \xi$). Multiplying (3.5) by a smooth test function Ψ, integrating over $(0,T)\times\Omega$, and making a change of variables in the multiple integral yields that the scalar ξ satisfies

$$\int_0^T \int_\Omega \Big[\xi \frac{\partial \Psi}{\partial t} + (\xi\, u)\cdot\nabla\Psi - \chi\,\xi\,\Psi + \phi\,\Psi\Big]\, dx\, dt = 0 \quad \forall\,\Psi \in C_0^\infty((0,T)\times\Omega),$$

and u is (also) a weak solution of the dissipative Euler equations. Finally the basic uniqueness results as in Yudovich [35] and Bardos [4] (see also Bessaih and Flandoli, [12], [13], for the dissipative case) show that the solution is unique. □

We prove now the fundamental estimate needed to prove the existence of almost-periodic solutions to the dissipative Euler equations. The main point is a uniform (in time) bound for the Dini norm of the vorticity. To this end we recall that the existence of *classical* (since now all terms are defined pointwise) solutions to the Euler equations such that $\xi \in C([0,T]; C_D(\overline{\Omega}))$ is not new. This appeared first in Beirão da Veiga (see Theorem 1.4 in [5]) and again and in an independent way (with a slightly-different proof) in Koch (see Theorem 2 in [29]). We do not reproduce here the proof, which is also in this case based on the representation formula (3.5) and the Schauder fixed point theorem, but we give just the main point, which is a uniform estimate for the Dini norm of the vorticity.

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, assume that* $\operatorname{curl} u_0 \in C_D(\overline{\Omega})$ *and that* $\operatorname{curl} f \in L^1_{\text{loc}}(0,\infty; C_D(\overline{\Omega}))$. *Then, for any* $T>0$, *the unique strong solution of the dissipative Euler equations is such that* $\operatorname{curl} u \in C([0,T]; C_D(\overline{\Omega}))$.

Moreover, if $\operatorname{curl} f \in L^\infty(0,\infty; C_D(\overline{\Omega}))$, *then there exists* $\chi_0 > 0$ (*depending on the initial datum* ξ_0, *the force* f, *and the domain* Ω; *see* (3.8)) *such that if* $\chi > \chi_0$, *then*

$$\sup_{t\geq 0} \|\operatorname{curl} u(t)\|_{C_D} \leq C < \infty. \tag{3.6}$$

where $C = C(\xi_0, f, \chi, \Omega)$.

Proof. We already know from Theorem 3.1 the existence and uniqueness of a *strong* solution corresponding to the data (u_0, f), for any $\chi > 0$. In particular, adapting Theorem 1.4 in [5] and Theorem 2 in [29], it is straightforward to show that the solution is such that $\operatorname{curl} u \in C([0,T]; C_D(\overline{\Omega}))$. For the reader's convenience we recall that the main point is to check that the fixed point of the mapping J satisfies $\xi = J\xi \in C([0,T]; C_D(\overline{\Omega}))$. This allows to use the Schauder fixed point argument in the topology of $L^\infty(\overline{\Omega}_T)$.

We show now that, in presence of a large enough dissipative constant χ, the representation formula allows us to obtain uniform bounds on the Dini norm of the vorticity over all positive times. For any given $T > 0$, ξ is the fixed point of the mapping J, hence it satisfies (3.5). Next, we give a uniform bound for the Dini norm of ξ. First the $L^\infty(\Omega)$ bound

$$\|\xi(t)\|_\infty \le \|\xi_0\|_\infty \,\mathrm{e}^{-\chi t} + \sup_{t\ge 0}\|\phi(t)\|_\infty \,\frac{1-\mathrm{e}^{-\chi t}}{\chi} \qquad \forall\, t \ge 0,$$

is straightforward and it is shown also in [13]. In the following calculations we assume that there is a unique solution such that $\xi \in C([0,T]; C_D(\overline{\Omega}))$. This implies that U is Lipschitz continuous (especially in the space variable) and that its Lipschitz norm is bounded by the Dini norm of ξ. We will work on a given interval $[0,T]$ and then we will show that the estimates are independent of T, for large enough $\chi > 0$.

We estimate the Dini-continuity of $\eta = \xi\,\mathrm{e}^{\chi t}$, where ξ is the vorticity of the solution, hence such that $\xi = J\xi$ on $[0,T]$. Observe that, from the equation satisfied by η we have the representation formula

$$\eta(t,x) = \xi_0(U(0,t,x)) + \int_0^t \phi(s, U(s,t,x))\,\mathrm{e}^{\chi s}\,ds,$$

and clearly

$$\|\eta(t)\|_\infty \le \|\xi_0\|_\infty + \sup_{t\ge 0}\|\phi(t)\|_\infty \,\frac{\mathrm{e}^{\chi t}-1}{\chi}.$$

Moreover, we observe that $[\eta(t)]_{C_D} = [\xi(t)]_{C_D}\mathrm{e}^{\chi t}$, as follows easily from the definition. We estimate the Dini seminorm of η as follows:

$$\begin{aligned}
[\eta(t)]_{C_D} &:= \int_0^1 \sup_{|x-y|\le\rho} |\eta(t,x)-\eta(t,y)|\,\frac{d\rho}{\rho} \\
&\le \int_0^1 \sup_{|x-y|\le\rho} |\xi_0(U(0,t,x))-\xi_0(U(0,t,y))|\,\frac{d\rho}{\rho} \\
&\quad + \int_0^t\int_0^1 \sup_{|x-y|\le\rho} |\phi(s,U(s,t,x))-\phi(s,U(s,t,y))|\,\mathrm{e}^{\chi s}\,\frac{d\rho}{\rho}\,ds \\
&=: B_1 + B_2.
\end{aligned} \tag{3.7}$$

Next, we estimate separately B_1 and B_2. For the first term, making a change of variable by means of the unitary diffeomorphism $U(0,t,x)$, we have that

$$\begin{aligned}
B_1 &\le \int_0^1 \sup_{|x-y|\le\rho\|\nabla U(0,t,\cdot)\|_\infty} |\xi_0(x)-\xi_0(y)|\,\frac{d\rho}{\rho} \\
&\le \int_0^1 \sup_{|x-y|\le\rho} |\xi_0(x)-\xi_0(y)|\,\frac{d\rho}{\rho} + 2\,\|\xi_0\|_\infty \int_1^{\|\nabla U(0,t,\cdot)\|_\infty} \frac{d\rho}{\rho} \\
&\le [\xi_0]_{C_D} + 2\,\|\xi_0\|_\infty \,\log\|\nabla U(0,t,\cdot)\|_\infty,
\end{aligned}$$

(where the term $2\|\xi_0\|_\infty \log\|\nabla U(0,t,\cdot)\|_\infty$ is set to zero if $\|\nabla U(0,t,\cdot)\|_\infty < 1$), and, by appealing to (3.1), we obtain

$$
\begin{aligned}
B_1 &\leq [\xi_0]_{C_D} + 2\,\|\xi_0\|_\infty \int_0^t \|\nabla u(s)\|_\infty\, ds \leq [\xi_0]_{C_D} + 2\,C_0\,\|\xi_0\|_\infty \int_0^t \|\xi(s)\|_{C_D}\, ds \\
&\leq [\xi_0]_{C_D} + 2\,C_0\,\|\xi_0\|_\infty \int_0^t \|\eta(s)\|_{C_D}\, \mathrm{e}^{-\chi s}\, ds.
\end{aligned}
$$

For the term B_2, by making the change of variables by means of $U(s,t,x)$, we have that

$$
\begin{aligned}
B_2 &\leq \int_0^t \int_0^1 \sup_{|x-y|\leq \rho\|\nabla U(s,t,\cdot)\|_\infty} |\phi(s,x)-\phi(s,y)|\, \frac{d\rho}{\rho}\, \mathrm{e}^{\chi s}\, ds \\
&\leq \int_0^t [\phi(s)]_{C_D(\overline{\Omega})} \mathrm{e}^{\chi s}\, ds + 2\,\|\phi(s)\|_\infty \int_0^t \int_1^{\|\nabla U(s,t,\cdot)\|_\infty} \frac{d\rho}{\rho}\, \mathrm{e}^{\chi s}\, ds \\
&\leq \sup_{t\geq 0} [\phi(t)]_{C_D} \int_0^t \mathrm{e}^{\chi s} ds + 2 \sup_{t\geq 0} \|\phi(t)\|_\infty \int_0^t \log \|\nabla U(s,t,\cdot)\|_\infty\, \mathrm{e}^{\chi s}\, ds \\
&\leq \sup_{t\geq 0} [\phi(t)]_{C_D} \int_0^t \mathrm{e}^{\chi s} ds + 2 \sup_{t\geq 0} \|\phi(t)\|_\infty \int_0^t \log \|\nabla U(s,t,\cdot)\|_\infty\, \mathrm{e}^{\chi s}\, ds \\
&\leq \sup_{t\geq 0} [\phi(t)]_{C_D} \int_0^t \mathrm{e}^{\chi s}\, ds + 2 \sup_{t\geq 0} \|\phi(t)\|_\infty \int_0^t \int_s^t \|\nabla u(\tau)\|_\infty\, \mathrm{e}^{\chi s}\, d\tau\, ds.
\end{aligned}
$$

Changing the order of integration in the last integral we have

$$
\begin{aligned}
B_2 &\leq \sup_{t\geq 0} [\phi(t)]_{C_D} \int_0^t \mathrm{e}^{\chi s} ds + 2 \sup_{t\geq 0} \|\phi(t)\|_\infty \int_0^t \int_0^\tau \|\nabla u(\tau)\|_\infty\, \mathrm{e}^{\chi s}\, ds\, d\tau \\
&\leq \sup_{t\geq 0} [\phi(t)]_{C_D} \frac{\mathrm{e}^{\chi t}-1}{\chi} + 2\,C_0 \sup_{t\geq 0} \|\phi(t)\|_\infty \int_0^t \int_0^\tau \|\xi(\tau)\|_{C_D}\, \mathrm{e}^{\chi s}\, ds\, d\tau \\
&\leq \sup_{t\geq 0} [\phi(t)]_{C_D} \frac{\mathrm{e}^{\chi t}-1}{\chi} + 2\,C_0 \sup_{t\geq 0} \|\phi(t)\|_{L^\infty} \int_0^t \|\xi(\tau)\|_{C_D} \frac{\mathrm{e}^{\chi \tau}-1}{\chi}\, d\tau \\
&\leq \sup_{t\geq 0} [\phi(t)]_{C_D} \frac{\mathrm{e}^{\chi t}}{\chi} + \frac{2\,C_0}{\chi} \sup_{t\geq 0} \|\phi(t)\|_\infty \int_0^t \|\eta(\tau)\|_{C_D}\, d\tau.
\end{aligned}
$$

Collecting all the estimates we get the inequality

$$
\|\eta(t)\|_{C_D} \leq \|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} \mathrm{e}^{\chi t} + 2\,C_0 \left[\|\xi_0\|_{C_D} + \frac{\Phi}{\chi}\right] \int_0^t \|\eta(s)\|_{C_D}\, ds,
$$

where $\Phi := \sup_t \|\phi(t)\|_{C_D}$. By using the Gronwall lemma we get

$$
\begin{aligned}
\|\eta(t)\|_{C_D} \leq &\left[\|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} - \frac{2\Phi\chi}{\chi^2 - 2\,C_0\,(\Phi + \|\xi_0\|_{C_D}\chi)}\right] \mathrm{e}^{\frac{2C_0 t(\Phi+\|\xi_0\|_{C_D}\chi)}{\chi}} \\
&+ \frac{2\Phi\chi}{\chi^2 - 2\,C_0\,(\Phi + \|\xi_0\|_{C_D}\chi)}\, \mathrm{e}^{t\chi}\,,
\end{aligned}
$$

and consequently

$$\|\xi(t)\|_{C_D} \le \Big[\|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} - \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)}\Big] e^{\,t\,\frac{2C_0(\Phi+\|\xi_0\|_{C_D}\chi)}{\chi} - \chi}$$
$$+ \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)},$$

which is uniformly bounded on $[0\infty)$ if

$$2\,C_0\,\Phi + 2\,C_0\,\|\xi_0\|_{C_D}\chi - \chi^2 < 0,$$

that is if

$$\chi > \chi_0 := C_0\,\|\xi_0\|_{C_D} + \sqrt{C_0^2\,\|\xi_0\|^2_{C_D} + 2\,C_0\,\Phi}. \tag{3.8}$$

□

Remark 3.1. To obtain directly continuity of the mapping J and also uniform estimates, the Hölder topology seems not to work. The reader can also compare with Remark. 2.2 in [5], and also the related observation in [29], page 494, on the noncontinuity of $C^{1,\alpha}$-under simple rigid rotations. The fixed point and other arguments also require handling these topologies, especially when looking for properties valid for arbitrary positive times. The connection between continuity of the mapping $t \mapsto u(t)$, the growth in a critical way of different norms (Dini, Hölder, and Sobolev), and the long-time behavior is addressed in [29]. Moreover, in the recent work of Kiselev and Šverák [28] it is shown that for the Euler equations (that is in the case $\chi = 0$) it is possible to find smooth initial data producing solutions with sharp growth in derivatives of the vorticity, such that exponential growth for $\|\nabla u(t)\|_\infty$ follows.

Remark 3.2. Especially in connection with the existence of attractors, hence with uniform bounds together with a semigroup condition, a similar approach is used in [8], by employing other arguments, closely related with the Hadamard well-posedness. Results concerning the existence of certain strong global-attractors were announced in [7].

4. Existence of solutions defined on the entire real line

This section is devoted to proving the existence of weak solutions to (1.1) defined on the entire real axis. To do so, we follow the analysis carried out in Section 3 of [31] and in Section 1 of [13] to obtain the following result.

Theorem 4.1. *Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V)$ and that $\operatorname{curl} f \in L^\infty(\mathbb{R}; C_D(\overline{\Omega}))$. Then, if $\chi > \chi_1(f,\Omega) := \sqrt{2C_0\Phi} > 0$ (see (3.8), with $\Phi := \|\operatorname{curl} f\|_{L^\infty(\mathbb{R};C_D)}$), there exists a weak solution $\widetilde{u}$ to (1.1), defined on $\mathbb{R}$, which satisfies*

$$\sup_{t\in\mathbb{R}} \|\nabla \widetilde{u}(t)\|_\infty \le C_2, \tag{4.1}$$

with $C_2 := C_0C_1$, where the constants C_0 and C_1 are given in (2.3) and (4.3), respectively.

Proof. We consider the system (1.1) in $[-k, \infty)$, $k \in \mathbb{N}$, with initial datum $u_k(-k) = 0$ (and so $\xi_k(-k) \equiv 0$). Arguing as in the proof of Theorem 3.2, we get the existence of a unique strong solution u_k to (1.1), on the interval $[-k, \infty)$, such that $\operatorname{curl} u_k \in C([-k, \infty); C_D(\overline{\Omega}))$.

As a further consequence of the results in Theorem 3.2, it follows that if $\chi > \chi_1$, then

$$\sup_{t \geq -k} \| \operatorname{curl} u_k(t) \|_{C_D} \leq C_1 < \infty, \tag{4.2}$$

where

$$C_1(f, \chi, \Omega) := \frac{2\,\Phi\,\chi}{\chi^2 - 2\,C_0\,\Phi} \tag{4.3}$$

is the constant $C = C(\xi_0, f, \chi, \Omega)$, introduced in (3.6), in the case when $\xi_0 = 0$.

Now we set

$$\widetilde{u}_k(t) := \begin{cases} u_k(t) & \text{for } t \in [-k, \infty), \\ 0 & \text{for } t \in (-\infty, -k]. \end{cases}$$

Clearly, relation (4.2) remains true for $\widetilde{u}_k$, $k \in \mathbb{N}$, and, by appealing to the inequality (2.3), we get

$$\frac{1}{C_0} \sup_{t \in \mathbb{R}} \|\nabla \widetilde{u}_k(t)\|_\infty \leq \sup_{t \in \mathbb{R}} \| \operatorname{curl} \widetilde{u}_k(t) \|_{C_D} \leq C_1.$$

In particular, we have that $\nabla \widetilde{u}_k$ is uniformly bounded in $\mathbb{R} \times \Omega$ by C_2. Therefore, there is a subsequence of $\widetilde{u}_k$ (labeled again $\widetilde{u}_k$) and a function with $\nabla \widetilde{u} \in L^\infty(\mathbb{R}; L^\infty(\Omega))$ such that

$$\nabla \widetilde{u}_k \rightharpoonup \nabla \widetilde{u} \quad \text{in } L^\infty(\mathbb{R}; L^\infty(\Omega))\text{-weak}^\star, \tag{4.4}$$

and, due to the weak* lower semicontinuity of the norm, we also get

$$\sup_{t \in \mathbb{R}} \|\nabla \widetilde{u}(t)\|_\infty \leq C_2.$$

Next, we show that $\widetilde{u}$ is a solution to (1.1) in the distributional sense and that it satisfies the relations (2.1a)-(2.1d). In such a way, we will retrieve the existence of a weak solution to (1.1), defined on $\mathbb{R}$, with the property that $\|\nabla \widetilde{u}(t)\|_\infty$ is uniformly bounded on the whole real line. This latter fact will be crucial in order to prove the existence of $\mathcal{S}^2(H)$-almost-periodic solutions to (1.1)

Let $L > 0$ be an arbitrary number. By using (2.1d) for the sequence $\widetilde{u}_k$ we get

$$\begin{aligned} |\langle \widetilde{u}_k(t) - \widetilde{u}_k(s), \varphi \rangle| \leq \chi \int_s^t |\langle \widetilde{u}_k(\tau), \varphi \rangle|\, d\tau + \int_s^t |b(\widetilde{u}_k(\tau), \widetilde{u}_k(\tau), \varphi)|\, d\tau \\ + \int_s^t |\langle f(\tau), \varphi \rangle|\, d\tau, \end{aligned}$$

for all $\varphi \in \mathcal{V}$, and for all $-k \leq s \leq t \leq L$. By the boundedness of $\nabla \widetilde{u}_k$ in $L^\infty(\mathbb{R}; L^\infty(\Omega))$, and the hypotheses on f, it follows that $\widetilde{u}_k(t) - \widetilde{u}_k(s)$ is bounded in $L^2_{\text{loc}}(-\infty, L; V')$. In particular, the sequence $\widetilde{u}_k$ is bounded in $L^2_{\text{loc}}(-\infty, L; V) \cap W^{1,2}_{\text{loc}}(-\infty, L; V')$.

By using classical compactness arguments, we can extract a subsequence (still written as $\widetilde{u}_k$) such that

$$
\begin{aligned}
&\widetilde{u}_k \to \widetilde{u} \text{ in } L^2(-L,L;H)\text{-strong},\\
&\widetilde{u}_k \rightharpoonup \widetilde{u} \text{ in } L^\infty(-L,L;V)\text{-weak}^\star,\\
&\widetilde{u}_k \rightharpoonup \widetilde{u} \text{ in } L^2(-L,L;V)\text{-weak},\\
&\partial_t\widetilde{u}_k \rightharpoonup \partial_t\widetilde{u} \text{ in } L^2(-L,L;V')\text{-weak},\\
&\exists\, E\subset[-L,L] \text{ of zero Lebesgue measure s.t.} \,\forall\, t\in[-L,L]\setminus E,\ \widetilde{u}_k(t)\to\widetilde{u}(t) \text{ in } H,
\end{aligned}
$$

and the limit $\widetilde{u}$ coincides with that in (4.4), due to the uniqueness of the limit for the convergence in distribution. Moreover, by using standard interpolation theory (see, e.g. [13], [33]), it follows that $\widetilde{u}\in C(\mathbb{R};H)$, and so condition (2.1a) is satisfied.

As a consequence of the strong convergence of $\widetilde{u}_k$ to $\widetilde{u}$ in $L^2_{\rm loc}(\mathbb{R};H)$, for any compact interval $[-L,L]\subseteq\mathbb{R}$, we can pass to the limit in equation (2.1d), proving that $\widetilde{u}$ is solution to (1.1) in the space $\mathcal{D}'(\mathbb{R};V')$ of distributions.

Now, take inequality (2.1b) for $\widetilde{u}_k$, i.e.,

$$
\|\widetilde{u}_k(t)\|_2^2+\chi\int_{-k}^t\|u_k(s)\|_2^2\,ds\le\int_{-k}^t|\langle f(s),\widetilde{u}_k\rangle|\,ds\qquad\text{for a.e. } t\in[-k,L]. \tag{4.5}
$$

Using again the strong convergence of $\widetilde{u}_k$ to $\widetilde{u}$ in $L^2_{\rm loc}(\mathbb{R};H)$, passing to the limit on both sides of (4.5), it follows that the left-hand side of (4.5) converges to $\|\widetilde{u}(t)|_2^2+\chi\int_{-k}^t\|\widetilde{u}(s)\|_2^2\,ds$ and the right-hand-side converges to $\int_{-k}^t\langle f(s),\widetilde{u}(s)\rangle\,ds$. Then, for all $k\in\mathbb{N}$,

$$
\|\widetilde{u}(t)\|_2^2+\chi\int_{-k}^t\|\widetilde{u}(s)\|_2^2\,ds\le\int_{-k}^t|\langle f(s),\widetilde{u}(s)\rangle|\,ds\qquad\text{for a.e. } t\in[-k,\infty].
$$

Thus, $\widetilde{u}$ satisfies (2.1b).

Finally, relation (2.1c) easily follows by exploiting the same argument used in Section 3 of [13] and the solution to elliptic problem for $\widetilde{u}=\operatorname{curl}^{-1}\widetilde{\xi}$, as explained in the previous section. □

The previous result leads to the definition below.

Definition 4.1. Provided that $f\in L^2_{\rm loc}(\mathbb{R};V)$, we say that a weak solution u of the dissipative Euler equation (1.1) is "global" if it satisfies (2.1a) on $\mathbb{R}$, and the properties (2.1b)–(2.1d) hold for almost every $t,t_0\in\mathbb{R}$, with $t\ge t_0$.

Remark 4.1. Since u is tangential to the boundary, and Ω is bounded then the Poincaré inequality holds. Consequently, from $\nabla u\in L^\infty(\mathbb{R}\times\Omega)$ it follows also that u is uniformly bounded.

4.1. Some remarks on uniform bounds in Hilbert spaces

Simpler and more standard techniques can be used to show the following uniform bounds, which are nevertheless too weak for the existence of almost-periodic solutions. We report them for the reader's convenience and also to show in a different way some related estimates, which (unlike those in the previous section) hold true for any positive χ. We point out that they are useless to show certain asymptotic equivalence properties, that is to quantitatively estimate the difference of two solutions starting from different initial data, explaining the critical role of the functional setting we use and of the restrictions on the dissipation constant χ.

Lemma 4.1. *In addition to the hypotheses of Theorem 2.1, assume that* $f \in L^2_{\text{uloc}}(0,\infty;H)$. *Then, weak solutions* u *to* (1.1) *are defined for all* $t \geq 0$, *they belong to* $L^\infty(0,\infty;H)$, *and the estimate*

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2\, e^{-\chi t} + \frac{3}{\chi^2}\,\|f\|^2_{L^2_{\text{uloc}}(0,\infty;H)}, \quad t \in [0,\infty), \tag{4.6}$$

holds.

Proof. Consider the dissipative Euler equations (1.1). Using u as a test function, we get the inequality

$$\frac{d}{dt}\|u\|_2^2 \leq -\chi\,\|u\|_2^2 + \frac{1}{\chi}\,\|f\|_2^2.$$

Notice that the calculations can be made rigorous by considering the same equations on Galerkin approximate functions, or using the fact that the solution is a weak solution over $[0,T]$, for all positive T. Set $z(t) := \|u(t)\|_2^2$ and $\beta(t) := \|f(t)\|_2^2$ (in particular $\beta \in L^1_{\text{uloc}}(0,\infty)$). Now, to estimate z in $L^\infty(0,\infty)$, we follow a more or less classical argument, as in Proposition 2.1 of [31]. Suppose there exists $\overline{t} \in [0,\infty[$ such that $z(\overline{t}) \leq z(\overline{t}+1)$. Then, it follows that

$$0 \leq z(\overline{t}+1) - z(\overline{t}) = \int_{\overline{t}}^{\overline{t}+1} \partial_t z(s)\,ds \leq -\chi \int_{\overline{t}}^{\overline{t}+1} z(s)\,ds + \frac{1}{\chi}\int_{\overline{t}}^{\overline{t}+1} \beta(s)\,ds,$$

that is

$$\chi \int_{\overline{t}}^{\overline{t}+1} z(s)\,ds \leq \frac{1}{\chi}\int_{\overline{t}}^{\overline{t}+1} \beta(s)\,ds \leq \frac{1}{\chi}\,\|\beta\|_{L^1_{\text{uloc}}(\mathbb{R}^+)}.$$

Observe now that for every $\tau, \sigma \in [\overline{t}, \overline{t}+1]$, there holds

$$|z(\tau) - z(\sigma)| \leq \int_{\overline{t}}^{\overline{t}+1} \left| -\chi z(s) + \frac{1}{\chi}\beta(s) \right| ds \leq \frac{2}{\chi}\|\beta\|_{L^1_{\text{uloc}}(\mathbb{R}^+)}.$$

By the integral mean value theorem, it follows that there exists $\zeta \in (\overline{t}, \overline{t}+1)$ such that $z(\zeta) = \int_{\overline{t}}^{\overline{t}+1} z(s)\,ds$, so we obtain

$$\begin{aligned} z(\overline{t}) \leq z(\overline{t}+1) &\leq |z(\overline{t}+1) - z(\zeta)| + \int_{\overline{t}}^{\overline{t}+1} z(s)\,ds \\ &\leq \frac{2}{\chi}\,\|\beta\|_{L^1_{\text{uloc}}(\mathbb{R}^+)} + \frac{1}{\chi^2}\,\|\beta\|_{L^1_{\text{uloc}}(\mathbb{R}^+)} \leq \frac{3}{\chi^2}\,\|\beta\|_{L^1_{\text{uloc}}(\mathbb{R}^+)}, \end{aligned}$$

and the above estimate holds for every $\overline{t} \in [0,\infty)$ such that $z(\overline{t}) \leq z(\overline{t}+1)$. Instead, in the case when $z(\overline{t}) > z(\overline{t}+1)$, one repeats the same procedure for $z(\overline{t}-1)$ and $z(\overline{t})$. Continuing in this manner, we need to estimate $z(t)$ on $[0,1]$. The estimate in $[0,1]$ follows by applying the Gronwall inequality. In this way we find (4.6). $\square$

By using the same approach, one can easily show also the following result.

Lemma 4.2. *In addition to the hypotheses of Theorem 2.1, assume that* $f \in L^2_{\text{uloc}}(0,\infty;V)$. *Then, weak solutions* u *to* (1.1) *belong to* $L^\infty(0,\infty;V)$, *and the following estimate holds:*

$$\|u(t)\|_{1,2}^2 \leq \|u_0\|_{1,2}^2 \, e^{-\chi t} + \frac{3}{\chi^2} \, \|f\|^2_{L^2_{\text{uloc}}(\mathbb{R}^+;V)}, \quad t \in [0,\infty).$$

If $f \in L^2_{\text{uloc}}(V)$, Lemmas 4.1 and 4.2 then suffice to show, by the same argument with the initial value problem in $[-k,\infty)$ and then letting $k \to \infty$, that

$$\exists\, C > 0: \quad \|u(t)\|_{1,2} \leq C \quad \forall\, t \in \mathbb{R}.$$

The same argument as before implies then the following result.

Theorem 4.2. *In addition to the hypotheses of Theorem 2.1 assume that* $f \in L^2_{\text{uloc}}(\mathbb{R};V)$. *Then, there exists a weak solution* $\widetilde{u}$ *to* (1.1), *defined on* $\mathbb{R}$, *such that*

$$\sup_{t\in\mathbb{R}} \|\widetilde{u}(t)\|_{1,2} \leq C < \infty.$$

The reason why this result is useless is that the estimate for $u \in V$ does not imply any kind of uniqueness. Bounded vorticity is enough to obtain uniqueness, but to estimate in a uniform way the difference of two solutions a bound on the gradient in $L^\infty(\Omega)$ seems necessary and the larger space in which we are able to prove this result is that of Dini-continuous vorticities.

5. Existence of almost-periodic solutions

We finally prove existence of almost-periodic solutions, under the natural assumption that the external force field f is in $\mathcal{S}^2(H)$ and is also such that $\operatorname{curl} f \in L^\infty(\mathbb{R}; C_D(\overline{\Omega}))$. With these hypotheses we will show that the global weak solution built in Theorem 4.1 is $\mathcal{S}^2(H)$-almost-periodic as well, but a restriction on the size of χ is needed.

To reach this goal, some preliminary facts are needed. Let f and $\widehat{f}$ be two external force fields satisfying the hypotheses of Theorem 4.1, and let u and $\widehat{u}$ be the associated global weak solutions constructed as in Theorem 4.1. Denote the differences by $w := u - \widehat{u}$ and $g := f - \widehat{f}$. Taking the difference of the equations satisfied by u and $\widehat{u}$ we get

$$\partial_t w + \chi\, w + \nabla(\pi - \widehat{\pi}) = -(u \cdot \nabla)\, w - (w \cdot \nabla)\, \widehat{u} + g.$$

Now, taking the L^2-product with w we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \chi\|w(t)\|_2^2 &\leq |b(w(t), \widehat{u}(t), w(t))| + \|w(t)\|\,\|g(t)\| \\ &\leq \|w(t)\|_2^2\,\|\nabla \widehat{u}(t)\|_\infty + \frac{\chi}{2}\,\|w(t)\|_2^2 + \frac{1}{2\chi}\,\|g(t)\|_2^2 \\ &\leq C_2\,\|w(t)\|_2^2 + \frac{\chi}{2}\,\|w(t)\|_2^2 + \frac{1}{2\chi}\,\|g(t)\|_2^2,\end{aligned}$$

where we used the inequality (4.1). Hence, via standard manipulations we get

$$\|w(t)\|_2^2 \leq \|w_0\|_2^2 \,\mathrm{e}^{(2C_2-\chi)(t-t_0)} + \frac{1}{\chi}\int_{t_0}^{t} \|g(\tau)\|_2^2 \,\mathrm{e}^{(2C_2-\chi)(t-\tau)}\, d\tau, \tag{5.1}$$

where $w_0 = u_0 - \widehat{u}_0$, and $t, t_0 \in \mathbb{R}$ with $t \geq t_0$.

Remark 5.1. Let u be a global weak solution constructed as in Theorem 4.1. Given the external force field f as in the hypotheses, it is always possible to choose the parameter χ large enough such that $\chi > \sqrt{2\,C_0\Phi}$, so that from the existence result there follows

$$\sup_{t\in\mathbb{R}} \|\nabla u(t)\|_\infty \leq C_2.$$

Thus, to have $2C_2 - \chi < 0$, it suffices to take

$$\chi > \sqrt{6\,C_0\,\Phi} > \sqrt{2\,C_0\,\Phi}.$$

For the reminder of this section we always assume that $2C_2 - \chi$ is strictly negative. We are now ready to prove our main result.

Theorem 5.1. *Suppose that the hypotheses of Theorem 4.1 are satisfied and also that $f \in \mathcal{S}^2(H)$. Moreover, suppose that $\chi > \chi_2 := \sqrt{6\,C_0\Phi}$. Then, there exists a weak solution u of (1.1) such that $u \in \mathcal{S}^2(H)$.*

Proof. We prove that the global solution u of (1.1), constructed as in the previous section, belongs to $\mathcal{S}^2(H)$. As usual we argue by contradiction; see, for instance, Foias [22], for early results on the Navier–Stokes equations with "large viscosity", instead of the large dissipation used here (notice that in that case the condition on the viscosity is used to ensure global regularity for the three-dimensional problem). There is a sequence $\{h_m\}$ and a function $\widetilde{f}$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s+h_m) - \widetilde{f}(s)\|_2^2 \, ds \to 0, \tag{5.2}$$

and there exist a sequence $\{t_k\}$, two subsequences $\{h_{m_k}\}$ and $\{h_{n_k}\}$ (of $\{h_m\}$), and a constant $\delta_0 > 0$ such that

$$0 < \delta_0 \leq \int_{t_k}^{t_k+1} \|u(s+h_{m_k}) - u(s+h_{n_k})\|_2^2 \, ds \quad \forall k \in \mathbb{N}. \tag{5.3}$$

Since f is $\mathcal{S}^2(H)$-almost-periodic, by relation (5.2), one has that there exist f_1^* and f_2^* such that

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s+t_k+h_{m_k}) - f_1^*(s)\|_2^2 \, ds \to 0, \\ &\sup_{t\in\mathbb{R}} \int_t^{t+1} \|f(s+t_k+h_{n_k}) - f_2^*(s)\|_2^2 \, ds \to 0. \end{aligned} \tag{5.4}$$

This holds up to a subsequence $\{k'\}$ of $\{k\}$ (still denoted by $\{k\}$) hence by taking $\{t_{k'}\}$, $\{h_{m_{k'}}\}$ and $\{h_{n_{k'}}\}$. Applying the triangle inequality twice, it can be easily proved that $f_1^* = f_2^* =: f^*$ (for details see Theorem 4.1 in [31]).

For any $k \in \mathbb{N}$ we can construct two global solutions $u_1^k(r) := u(r + t_k + h_{m_k})$ and $u_2^k(r) := u(r+t_k+h_{n_k})$, with $r \in \mathbb{R}$, corresponding to the external force fields $f_1^k(r) := f(r + t_k + h_{m_k})$ and $f_2^k(r) := f(r + t_k + h_{n_k})$. Hence, relation (5.3) can be rewritten as

$$\delta_0 \le \int_{t_k}^{t_k+1} \|u_1^k(s-t_k) - u_2^k(s-t_k)\|_2^2 \, ds = \int_0^1 \|u_1^k(s) - u_2^k(s)\|_2^2 \, ds. \tag{5.5}$$

Observe that, under our hypotheses,

$$\sup_{t\in\mathbb{R}} \|\nabla u_i^k(t)\|_\infty \le C_2 < \infty, \qquad \text{for } i = 1, 2,$$

where C_2 is given in (4.1).

Following the lines of reasoning in the proof of Theorem 4.1, from u_1^k and u_2^k we can extract subsequences (still labeled u_1^k and u_2^k) strongly converging in $L^2_{\text{loc}}(\mathbb{R}; H)$ to the global weak solutions u_1 and u_2, respectively. Thus, passing to the limit in (5.5), we get

$$\delta_0 \le \int_0^1 \|u_1(s) - u_2(s)\|_2^2 \, ds. \tag{5.6}$$

On the other hand, exploiting inequality (5.1), we get (recall that $\chi - 2C_2 > 0$)

$$\begin{aligned}
\int_0^1 \|u_1^k(s) - u_2^k(s)\|_2^2 \, ds &\le \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 \int_0^1 \mathrm{e}^{(2C_2-\chi)(s-t_0)} \, ds \\
&\quad + \frac{1}{\chi} \int_0^1 ds \int_{t_0}^s \|f_1^k(\tau) - f_2^k(\tau)\|_2^2 \, \mathrm{e}^{(2C_2-\chi)(s-\tau)} \, d\tau \\
&\le \frac{1}{\chi - 2C_2} \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)t_0} \left(1 - \mathrm{e}^{-(\chi-2C_2)}\right) \\
&\quad + \frac{1}{\chi} \int_0^1 \mathrm{e}^{-(\chi-2C_2)s} \, ds \int_{t_0}^1 \|f_1^k(\tau) - f_2^k(\tau)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)\tau} \, d\tau,
\end{aligned}$$

and consequently

$$\begin{aligned}
\int_0^1 \|u_1^k(s) - u_2^k(s)\|_2^2 \, ds \le \frac{1}{\chi - 2C_2} \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)t_0} \\
+ \frac{1}{\chi} \int_{t_0}^1 \|f_1^k(s) - f_2^k(s)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)s} \, ds.
\end{aligned} \tag{5.7}$$

Here, without loss of generality, we can assume that $t_0 \le 0$. Recall that $\|u_i^k\|_2$ is bounded uniformly. Then, fix $t_0 < 0$ small enough, such that there holds

$$\frac{1}{\chi - 2C_2} \|u_1^k(t_0) - u_2^k(t_0)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)t_0} < \frac{\delta_0}{4}.$$

In order to estimate the second term on the right-hand side of (5.7), we employ a well-known argument used for instance in Lemma 4.1 of [26]. Given $t_0 \leq 0$ determined from the previous inequality, let $M \in \mathbb{N}$ be such that $t_0 + (M-1) \leq 1 \leq t_0 + M$. Therefore, we have that

$$\begin{aligned}
&\int_{t_0}^{1} \|f_1^k(s) - f_2^k(s)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)s} \, ds \\
&\qquad \leq \sum_{m=1}^{M} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)s} \, ds \\
&\qquad \leq \sum_{m=1}^{M} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)(m+2-M)} \, ds \\
&\qquad = \mathrm{e}^{(\chi-2C_2)(2-M)} \sum_{m=1}^{M} \mathrm{e}^{(\chi-2C_2)m} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 \, ds,
\end{aligned}$$

where we used that $\chi - 2C_2 > 0$ and also that by the definition of M, $t_0 \leq 2 - M$. Hence, adding to both sides $m \in \mathbb{N}$, the upper bound for the exponential in the interval $[t_0 + (m-1), t_0 + m]$ follows.

Next, by using the explicit expression for the summation of a geometric sum, we obtain

$$\begin{aligned}
&\mathrm{e}^{(\chi-2C_2)(2-M)} \sum_{m=1}^{M} \mathrm{e}^{(\chi-2C_2)m} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 \, ds, \\
&\quad \leq \mathrm{e}^{(\chi-2C_2)(2-M)} \max_{m=1,\dots,M} \int_{t_0+m-1}^{t_0+m} \|f_1^k(s) - f_2^k(s)\|_2^2 \, ds \cdot \sum_{m=1}^{M} \mathrm{e}^{(\chi-2C_2)m} \\
&\quad \leq \frac{\mathrm{e}^{(\chi-2C_2)(M+1)} - 1}{\mathrm{e}^{\chi-2C_2} - 1} \, \mathrm{e}^{(\chi-2C_2)(2-M)} \sup_{t \geq t_0} \int_{t}^{t+1} \|f_1^k(s) - f_2^k(s)\|_2^2 \, ds \\
&\quad \leq \frac{\mathrm{e}^{3(\chi-2C_2)(M+1)}}{\mathrm{e}^{\chi-2C_2}} \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|f_1^k(s) - f_2^k(s)\|_2^2 \, ds.
\end{aligned}$$

Next, recall that, due to (5.4), f_i^k, for $i = 1, 2$, converges to f^* in $L^2_{\mathrm{uloc}}(H)$, as k goes to ∞. Hence, by collecting the estimates and for fixed $t_0 \leq 0$ and for k large enough, we obtain

$$\begin{aligned}
&\frac{1}{\chi} \int_{t_0}^{1} \|f_1^k(s) - f_2^k(s)\|_2^2 \, \mathrm{e}^{(\chi-2C_2)s} \, ds \\
&\qquad \leq \frac{1}{\chi} \frac{\mathrm{e}^{3(\chi-2C_2)(M+1)}}{\mathrm{e}^{\chi-2C_2}} \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|f_1^k(s) - f_2^k(s)\|_2^2 \, ds < \frac{\delta_0}{4}.
\end{aligned}$$

Hence, by collecting all the estimates, we get

$$\int_{0}^{1} \|u_1^k(s) - u_2^k(s)\|_2^2 \, ds < \frac{\delta_0}{2},$$

and since u_i^k converges strongly in L^2 to u_i (and also almost everywhere up to a redefinition on a subset $E \subset \mathbb{R}$ of Lebesgue measure zero), we obtain that

$$\int_0^1 \|u_1(s) - u_2(s)\|_2^2 \, ds \le \frac{\delta_0}{2},$$

contradicting (5.6), and the assertion is proved. □

5.1. Further regularity of almost-periodic solutions

By using a classical characterization of Stepanov almost-periodic functions and a theorem of Dafermos [20] we can prove also the following easy corollary.

Corollary 5.1. *Suppose that the hypotheses of Theorem 5.1 are satisfied. Then, there exists a weak solution u of (1.1) such that $u \in \mathcal{S}^2(V \cap W^{1,q}(\Omega))$, for all $q < \infty$, and* $\operatorname{curl} u \in \mathcal{S}(C(\overline{\Omega}))$.

The proof of Corollary 5.1 is based first on a characterization of Stepanov almost-periodicity in terms of Bohr–Bochner almost-periodicity; see Bochner [14]. To this end, recall that if we set $t \mapsto u^t(s) = u(t+s)$, with $s \in [0,1]$, then for $u \in L^2_{\text{uloc}}(\mathbb{R}; X)$ we can define the map $t \mapsto u_* := u^t$, which belongs to $C(\mathbb{R}; L^2(0,1;X))$. Then, $u \in \mathcal{S}^2(X)$ (that is u is Stepanov almost-periodic with values in X), if and only if $u_* \in AP(\mathbb{R}; L^2(0,1;X))$, that is u_* is Bohr–Bochner almost-periodic with values in $L^2(0,1;X)$ (Recall that a function is Bohr–Bochner almost-periodic if it continuous and its translates are relatively compact in the C^0-topology.)

Further, we will apply the following lemma due to Dafermos [20].

Lemma 5.1. *Let Y and Z be complete metric spaces, continuously embedded in a Hausdorff space W. Suppose that*

$$u: \ \mathbb{R} \to Y \cap Z$$

is almost-periodic in Y and its range is relatively compact in Z. Then u is almost-periodic in Z. (Here almost-periodicity is in the sense of Bohr–Bochner.)

Next, we will need the following compactness result in the style of Aubin–Lions (in particular we use a version valid for non reflexive spaces proved by Dubinskiĭ; see Simon [32])

Lemma 5.2. *Let be given three Banach spaces $Y_1 \hookrightarrow\hookrightarrow X \hookrightarrow Y_2$ (that is, the first inclusion is compact and the second continuous), the set F of functions $f\colon [0,T] \to Y_1$ such that there exists $C > 0$*

$$F := \{f \in L^2(0,T;Y_1), \ f_t \in L^2(0,T;Y_2): \ \|f\|_{L^2(0,T;Y_1)} + \|f_t\|_{L^2(0,T;Y_2)} \le C\}$$

is relatively compact in $L^2(0,T;X)$.

Proof of Corollary 5.1. We observe that, by easy computations, we have

$$u_t \in L^2_{\text{uloc}}(\mathbb{R}; L^2(\Omega)).$$

In fact, by testing the equation (1.1) with u_t and by the Young inequality we get

$$\frac{1}{2}\,\|u_t\|_2^2+\frac{\chi}{2}\,\frac{d}{dt}\|u\|_2^2\leq\|u\|_2^2\,\|\nabla u\|_\infty^2+\|f\|_2^2.$$

Hence, by using the previously obtained bounds for $\|u\|_2$ and $\|\nabla u\|_\infty$, and integrating over a generic interval $[t,t+1]$, we obtain that

$$u_t\in L^2_{\mathrm{uloc}}(\mathbb{R};H),$$

since $\nabla\cdot u_t=0$ and $(u_t\cdot n)_{|\Gamma}=0$.

Defining the Banach space

$$\mathcal{E}(\Omega):=\left\{v:\,\Omega\to\mathbb{R}^2:\ v\in V\cap C(\overline{\Omega}),\ \nabla v\in C(\overline{\Omega}),\ \operatorname{curl}v\in C_D(\overline{\Omega})\right\},$$

we can observe that we are in the following situation with respect to the time translates:

$$v_*(s)\in C(\mathbb{R};L^2(0,1;\mathcal{E}(\Omega)))\quad (v_*)_t(s)\in C(\mathbb{R};L^2(0,1;H)).$$

We use the compactness result from Lemma 5.2 with $Y_1=\mathcal{E}(\Omega)$, $X=\mathcal{F}(\Omega)$, and $Y_2=H$, where

$$\mathcal{F}(\Omega):=\left\{v:\,\Omega\to\mathbb{R}^2:\ v\in H\cap C(\overline{\Omega}),\ \nabla v\in L^q(\Omega)\ \forall q<\infty,\ \operatorname{curl}v\in C(\overline{\Omega})\right\}.$$

We briefly show the compactness of the inclusion $\mathcal{E}(\Omega)\hookrightarrow\hookrightarrow\mathcal{F}(\Omega)$. Let $\{f_n\}$ be a bounded sequence in $\mathcal{E}(\Omega)$. Then

$$\exists\,C:\quad \|f_n\|_{V\cap L^\infty}+\|\nabla f_n\|_{L^\infty}+\|\operatorname{curl}f_n\|_{C_D}\leq C\quad\forall n\in\mathbb{N}.$$

We recall now that the embedding of $C_D(\overline{\Omega})$ into $C(\overline{\Omega})$ is compact, since (see [29], page 498) for x close enough to y

$$|\phi(x)-\phi(y)|\leq\frac{\|\phi\|_{C_D}}{|\log|x-y||}\quad\forall\,\phi\in C_D(\Omega).$$

Hence we have equicontinuity and the Arzelà–Ascoli theorem applies. With this observation and by using classical Rellich–Kondrachov results on Sobolev spaces, we can extract a subsequence (relabelled as $\{f_n\}$) and find $f\in\mathcal{F}(\Omega)$ such that

$$\begin{aligned}
&f_n\rightharpoonup f && V\cap W^{1,q}(\Omega),\ \forall\,q<\infty,\\
&f_n\overset{*}{\rightharpoonup} f && W^{1,\infty}(\Omega),\\
&f_n\to f && H\cap C^{0,\alpha}(\overline{\Omega}),\ \forall\,\alpha<1,\\
&\operatorname{curl}f_n\to\operatorname{curl}f && L^\infty(\Omega),\\
&\nabla f_n\to\nabla f && L^q(\Omega),\ \forall q<\infty,
\end{aligned}$$

where in particular, we used that the L^q-norm of the gradient, for all $q<\infty$, can be controlled with those of the curl and the divergence (which vanishes), for functions tangential to the boundary. This is a by-product of the representation formulas coming from the potential theory. Hence, recalling that $u\in\mathcal{S}^2(H)$ by Theorem 5.1, all the hypotheses of Lemma 5.1 are satisfied with $Y=X=L^2(0,1;H)$ and $Z=L^2(0,1;\mathcal{F}(\Omega))$, concluding the proof. □

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