



Extensions of finite cyclic group actions on bordered surfaces

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Abstract. We study the question of the extendability of the action of a finite cyclic group on a compact bordered Klein surface (either orientable or non-orientable). This extends previous work by the authors for group actions on unbordered surfaces. It is shown that if such a cyclic action is realised by means of a non-maximal NEC signature, then the action always extends. For a given integer $g \geq 2$, we determine the order of the largest cyclic group that acts as the full automorphism group of a bordered surface of algebraic genus g , and the topological type of the surfaces on which the largest action takes place. In addition, we calculate the smallest algebraic genus of a bordered surface on which a given cyclic group acts as the full automorphism group of the surface. For this, we deal separately with orientable and non-orientable surfaces, and we also determine the topological type of the surfaces attaining the bounds.

Introduction

A natural extension of the definition of a compact Riemann surface, which has no boundary and is orientable, is to allow surfaces with non-empty boundary and to endow them with a dianalytic structure –that is, a structure whose transition functions are either analytic or the composite of complex conjugation with an analytic function. This structure yields the concept of *Klein surface*, using the terminology introduced by Alling and Greenleaf in [1]. Under the well-known functorial correspondence between compact Klein surfaces and real algebraic curves explained in [1], the bordered surfaces (the ones we will consider here) correspond to real algebraic curves with real points.

Let S be a compact bordered Klein surface of algebraic genus greater than 1. An automorphism of S is a homeomorphism $v: S \rightarrow S$ which is dianalytic in local coordinates. The group $\text{Aut}(S)$ of all such homeomorphisms is called the full automorphism group of S , and it is finite because S has algebraic genus greater

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than 1. A group G is said to act on S if it is a subgroup of $\text{Aut}(S)$. Here we consider the question of whether a cyclic group acting on S is the full group of all automorphisms of S . The same question for Riemann surfaces and for unbordered non-orientable Klein surfaces was considered in [6] and [4] respectively.

Given a finite group G acting effectively as a group of automorphisms on a compact surface S , it is in general a difficult task to decide whether G is the full group $\text{Aut}(S)$ of all automorphisms of S or just a proper subgroup of $\text{Aut}(S)$. A fruitful technique to deal with this problem is the combinatorial theory of non-euclidean crystallographic (NEC) groups. Let us briefly explain how to use it.

A compact bordered surface S of algebraic genus greater than 1 can be represented as the quotient H/Λ of the hyperbolic plane H under the action of a surface NEC group Λ . A finite group G then acts as a group of automorphisms of S if and only if G is isomorphic to Γ/Λ for some NEC group Γ containing Λ as a normal subgroup. If $G \neq \text{Aut}(S)$, then Γ is properly contained with finite index in some other NEC group Γ' , which also normalises Λ . The converse holds as well, and accordingly, the above question is closely related to the finite-index extendability of NEC groups.

The extendability of Γ depends mainly on the geometry of a fundamental region for Γ . In particular, although Γ could be contained in an NEC group Γ' normalising Λ , the group Γ might be abstractly isomorphic to a *maximal* NEC group –that is, to a group which is not contained as a subgroup of finite index in any other NEC group. If this happens and $f : \Gamma \rightarrow \Gamma'$ is such an isomorphism, then $f(\Gamma)/f(\Lambda)$ is the group of all automorphisms of the surface $H/f(\Lambda)$. For some signatures, however, it can happen that every NEC group Γ with signature σ is properly contained in another NEC group Γ' with finite index, and the dimensions of the Teichmüller spaces of Γ and Γ' coincide. Such a signature σ is called *non-maximal*, and the pair (σ, σ') of signatures of Γ and Γ' is called a *normal pair* if Γ is normal in Γ' , and *non-normal* otherwise.

This question was originally analysed for Fuchsian signatures by Greenberg in [12] and answered completely by Singerman in [21]. Using the list of non-maximal Fuchsian signatures produced by Singerman, the first author produced a complete list of normal pairs of NEC signatures in [3], and subsequently Estévez and Izquierdo gave the list of the non-normal ones in [8]. These lists play a key role in this paper, since it follows from the above remarks that if a finite group G can be written, as above, as Γ/Λ and the signature of Γ does not appear in either list, then G is the group of all automorphisms of some surface homeomorphic to H/Λ . If, on the other hand, the signature of Γ is non-maximal, then the action of G in any surface homeomorphic to H/Λ could possibly be extended. The main result of this paper shows that such an extension always occurs when the group G is cyclic.

The paper is organised as follows. In Section 1 we recall basic facts and introduce notation for NEC groups and bordered surfaces to be used in the paper. In Section 2 we analyse the pairs (σ, σ') of NEC signatures (from Table 1 in Section 1) in terms of the possible extension of the action of a cyclic group. The analysis produces our main result, namely Theorem 2.1. Some applications are described in the subsequent sections. In Section 3 we calculate, for a given $g \geq 2$, the order

of the largest cyclic group that acts as the full automorphism group of a bordered surface of algebraic genus g . The topological type of the surfaces attaining this bound is also determined. This bound turns out to be the same as for unbordered non-orientable surfaces, and we show that, in some cases, surfaces of both types attaining the bound come together as the Klein surfaces associated to two symmetries on the same Riemann surface. In Section 4 we determine for each n the smallest algebraic genus of a bordered surface on which the cyclic group of order n acts as the full automorphism group of the surface. We call this the *full real genus* of the cyclic group, following the definition by May [20] of the real genus as the smallest algebraic genus of a bordered surface on which the cyclic group acts effectively. We deal separately with orientable and non-orientable surfaces, which leads to the concepts of *full real orientable* and *full real non-orientable* genus of a group.

Let us mention that the concept analogous to the real genus for Riemann surfaces is called the symmetric genus of a group G (if we allow only orientation preserving automorphisms then we are led to the concept of strong symmetric genus). These are classic topics in Riemann and Klein surface theories which have catalysed a large amount of research. Let us mention, for Riemann surfaces, the seminal paper [13] by Harvey on cyclic groups, and [11], [14], [15], and [18], among many others. For the real genus, see [17], [9], and [10], for instance.

Using the above mentioned functorial correspondence, all the results in this paper can be translated into the language of real algebraic curves and birational transformations among them. Each boundary component of a surface S corresponds to an oval of its associated real curve \mathcal{C} , and S is orientable if and only if \mathcal{C} disconnects its complexification.

Throughout the paper, a bordered surface will mean a compact bordered Klein surface of algebraic genus greater than 1.

1. Preliminaries

We briefly recall the main facts about NEC groups to be used in the paper. For a general account of this topic, we refer the reader to Section 0.2 in [7]. An NEC group is a cocompact discrete subgroup of the group of orientation preserving or reversing isometries of the hyperbolic plane H . The *signature* of an NEC group Γ , as introduced by Macbeath in [16], is a collection of symbols and non-negative integers, of the form

$$(1.1) \quad \sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}).$$

The integers m_1, \dots, m_r are called *proper periods*; each bracket $(n_{i1}, \dots, n_{is_i})$ is a *period cycle*, and the integers n_{ij} are called *link periods*. An empty set of proper periods (where $r = 0$) will be denoted by $[-]$, an empty period-cycle (where $s_i = 0$) by $(-)$.

The signature of Γ gives some topological features of the projection $H \rightarrow H/\Gamma$, and also provides a presentation of Γ . The generators are as follows:

- Elliptic elements x_i , for $1 \leq i \leq r$;
- Reflections c_{i0}, \dots, c_{is_i} , for $1 \leq i \leq k$;
- Orientation-preserving elements e_i , for $1 \leq i \leq k$;
- Hyperbolic elements a_i and b_i , for $1 \leq i \leq \gamma$, if the sign is +;
- Glide reflections d_i , for $1 \leq i \leq \gamma$, if the sign is -;

and the defining relations are these:

- $x_i^{m_i} = 1$ for $1 \leq i \leq r$;
- $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$ for $1 \leq j \leq s_i$, for $1 \leq i \leq k$;
- $c_{is_i} = e_i c_{i0} e_i^{-1}$ for $1 \leq i \leq k$;
- $x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_\gamma b_\gamma a_\gamma^{-1} b_\gamma^{-1} = 1$ if the sign is +;
- $x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_\gamma^2 = 1$ if the sign is -.

A set of generators satisfying these conditions will be called a *set of canonical generators*.

The orientation-preserving elements of Γ constitute the *canonical Fuchsian subgroup* Γ^+ . These are the elements expressible as words in the canonical generators of Γ containing an even number of occurrences of the reflections c_{ij} and glide reflections d_i . Such words may be called *orientable words*, while those containing an odd number of the c_{ij} and d_i may be called *non-orientable words*. In particular, Γ^+ has index 1 or 2 in Γ . If the index is 1 (or equivalently, $\Gamma = \Gamma^+$), then Γ is a Fuchsian group, while if the index is 2 then Γ is called a *proper* NEC group.

The area of a fundamental region for an NEC group Γ with signature (1.1) is $2\pi\mu(\Gamma)$, where

$$\mu(\Gamma) = \alpha\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right),$$

with $\alpha = 2$ if the sign is + and $\alpha = 1$ otherwise. The expression $\mu(\Gamma)$ is usually called the *reduced area* of Γ . If Γ' is a subgroup of finite index in Γ , then Γ' is also an NEC group, and its area is given by the *Riemann-Hurwitz formula*:

$$\mu(\Gamma') = [\Gamma : \Gamma'] \cdot \mu(\Gamma).$$

The *algebraic genus* g of a compact surface of topological genus γ with k boundary components, is defined as

$$g = \begin{cases} 2\gamma + k - 1 & \text{if the surface is orientable,} \\ \gamma + k - 1 & \text{otherwise.} \end{cases}$$

It follows from the uniformisation theorem that any bordered surface of algebraic genus $g \geq 2$ and $k > 0$ boundary components is of the form H/Λ , for some proper NEC group Λ with signature

$$(\gamma; \pm; [-]; \{(-), \cdot^k, (-)\}),$$

where the sign is + if the surface is orientable, and – otherwise. We will call each NEC group with such a signature a *bordered surface NEC group*. A finite group G acts (faithfully) as a group of automorphisms of a bordered surface if and only if there exist an NEC group Γ and an epimorphism $\theta: \Gamma \rightarrow G$ whose kernel is a bordered surface NEC group Λ . In this case, the surface is H/Λ , and we say (for short) that θ is a *smooth epimorphism*, and that G acts with signature $\sigma(\Gamma)$.

The next lemma imposes necessary and sufficient restrictions on the signature of an NEC group Γ for it to admit a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ onto a cyclic group; see Theorems 2.4.2 and 2.4.4 in [7].

Lemma 1.1. *An NEC group Γ contains a bordered surface NEC group Λ as a normal subgroup with cyclic factor group Γ/Λ if and only if the signature of Γ has some period cycle and each period cycle is either empty or formed just by an even number of link periods equal to 2. In particular, if the index $|\Gamma : \Lambda|$ is odd then all the period cycles of Γ are empty.*

Lemma 1.1 rules out all of the non-normal pairs of NEC signatures, and 23 of the 36 normal pairs, namely the pairs 1, 2, 5, 9, 10, 12, 13, 16, 17, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35 and 36 listed in Table 3 in [8]. Also the normal pairs 7 and 15 can occur only for $t = 2$, while the normal pair 18 can occur only for $u = 2$. (Actually there are typographic errors in the entries for the pairs 3, 17, 18 and 26 in Table 3 in [8], and the 28th pair should not be listed at all, since the real Teichmüller dimension of Fuchsian groups with signature $(2; +; [-]; \{-\})$ is 6 while that of NEC groups with signature $(0; +; [-]; \{(2, 2, 2, 2, 2, 2)\})$ is 3.) This leaves the 13 normal pairs in Table 1 below to be analysed.

	Signature $\sigma = \sigma(\Gamma)$	Signature $\sigma' = \sigma(\Gamma')$	$ \Gamma' : \Gamma $
Case 1	$(2; -; [-]; \{(-)\})$	$(0; +; [2, 2]; \{(2, 2)\})$	2
Case 2	$(1; +; [-]; \{(-)\})$	$(0; +; [2, 2, 2]; \{(-)\})$	2
Case 3	$(1; -; [t]; \{(-)\})$	$(0; +; [2]; \{(2, 2, t)\})$	2
Case 4	$(1; -; [-]; \{(2, 2)\})$	$(0; +; [2]; \{(2, 2, 2)\})$	2
Case 5	$(1; -; [-]; \{(-), (-)\})$	$(0; +; [2]; \{(2, 2, 2, 2)\})$	2
Case 6	$(0; +; [-]; \{(-), (-), (-)\})$	$(0; +; [-]; \{(2, 2, 2, 2, 2, 2)\})$	2
Case 7	$(0; +; [t]; \{(-), (-)\})$	$(0; +; [-]; \{(2, 2, 2, 2, t)\})$	2
Case 8	$(0; +; [-]; \{(2, 2), (-)\})$	$(0; +; [-]; \{(2, 2, 2, 2, 2)\})$	2
Case 9	$(0; +; [t]; \{(2, 2)\}) \quad t \geq 3$	$(0; +; [-]; \{(2, 2, 2, t)\})$	2
Case 10	$(0; +; [t, u]; \{(-)\}), \quad \max(t, u) \geq 3$	$(0; +; [-]; \{(2, 2, t, u)\})$	2
Case 11	$(0; +; [t, t]; \{(-)\}) \quad t \geq 3$	$(0; +; [t]; \{(2, 2)\})$	2
Case 12	$(0; +; [t, t]; \{(-)\}) \quad t \geq 3$	$(0; +; [t, 2]; \{(-)\})$	2
Case 13	$(0; +; [t, t]; \{(-)\}) \quad t \geq 3$	$(0; +; [-]; \{(2, 2, 2, t)\})$	4

TABLE 1. Normal pairs of NEC signatures to be analysed.

2. Analysis of cases

For each normal pair (σ, σ') in Table 1, we consider the possibility of some extension of a smooth epimorphism $\theta: \Gamma \rightarrow C_n$ to a smooth epimorphism $\theta': \Gamma' \rightarrow G'$, where Γ is a NEC group with signature σ , and Γ' is a NEC group with signature σ' , such that Γ can be embedded as a subgroup of Γ' with finite index m , and G' is a finite group containing C_n as a subgroup of index m . Observe that under these circumstances, $\ker \theta = \ker \theta'$ and so G' also acts on the surface $H/\ker \theta$. Two embeddings $i_1, i_2: \Gamma \rightarrow \Gamma'$ are said to be *equivalent* if there exists an automorphism $\beta \in \text{Aut}(\Gamma')$ such that $\beta i_1 = i_2$. Observe that the extendability of θ does not depend on the representative chosen in the equivalence class of the embedding. In the cases that follow, inequivalent embeddings were found and analysed with the help of the MAGMA system [2].

We will use the following standard notation: $|G : K|$ denotes the index of a subgroup K in a group G , and $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$ of any pair (a, b) of elements of a group, while a^x denotes the conjugate $x^{-1}ax$ of a by the element x (or the image of a under the automorphism x , depending on the context). Note that the index $|\Gamma' : \Gamma|$ is 2 in all cases other than case 13, where it is 4.

Cases 1, 2, 3, 5, 6, 7, 10, 11, 12 and 13 are the same as cases 3, 4, 5, 6, 9, 10, 11, 12, 13 and 15 in [4] respectively, where it was shown that in the first seven of these, the cyclic action with signature σ always extends to an action with signature σ' , while in the last three, it does not always extend. In each of these last three cases, however, the NEC group Γ has signature $(0 ; +; [t, t]; \{(-)\})$, and the action always extends in the way prescribed in case 10, with $u = t$. Hence we need only consider extendability in the three remaining cases from Table 1, namely cases 4, 8 and 9.

Case 4: $\sigma = (1; -; [-]; \{(2, 2)\})$, $\sigma' = (0; +; [2]; \{(2, 2, 2)\})$.

The group Γ is generated by two involutions a_0 and a_1 and a third element d such that $(a_0a_1)^2 = (a_1d^2a_0d^{-2})^2 = 1$, while Γ' is generated by four involutions x, c_0, c_1, c_2 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2xc_0x)^2 = 1$. An embedding of Γ in Γ' is given by taking $d = c_2x$, $a_0 = c_0$ and $a_1 = c_1$. There are two other embeddings, but those are equivalent to this one under outer automorphisms of Γ' . Conjugation by x gives an involutory automorphism, with $d^x = d^{-1}$, $a_0^x = da_0d^{-1}$ and $a_1^x = d^{-1}a_1d$. Hence in any extension of $\theta: \Gamma \rightarrow C_n$ to a smooth epimorphism $\theta': \Gamma' \rightarrow G'$ with $|G' : C_n| = 2$, conjugation by the image of x must invert the images of all the generators of Γ , and so G' is the dihedral group D_n . Such an extension is always possible, and therefore the given action of C_n is never maximal.

Case 8: $\sigma = (0; +; [-]; \{(2, 2), (-)\})$, $\sigma' = (0; +; [-]; \{(2, 2, 2, 2, 2)\})$.

Here Γ is generated by three involutions a_0, a_1, a_2 and a fourth element e such that $(a_0a_1)^2 = (a_1ea_0e^{-1})^2 = [a_2, e] = 1$, while Γ' is generated by five involutions c_0, c_1, c_2, c_3, c_4 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = (c_3c_4)^2 = (c_4c_0)^2 = 1$. An embedding of Γ in Γ' is given by taking $e = c_3c_1$, $a_0 = c_0$, $a_1 = c_4$ and $a_2 = c_2$. Note that there are other possible embeddings, but all of them are equivalent

under a dihedral group of 10 outer automorphisms of Γ' (preserving the generating set $\{c_0, c_1, c_2, c_3, c_4\}$). Conjugation by c_1 gives an involutory automorphism with $e^{c_1} = e^{-1}$, $a_0^{c_1} = a_0$, $a_1^{c_1} = e^{-1}a_1e$ and $a_2^{c_1} = a_2$. As above, an extension is always possible to an epimorphism from Γ' onto D_n , with c_1 being taken to any element of $D_n \setminus C_n$.

Case 9: $\sigma = (0; +; [t]; \{(2, 2)\})$, $\sigma' = (0; +; [-]; \{(2, 2, 2, t)\})$, $t \geq 3$.

Here Γ is generated by two involutions a_0 and a_1 and a third element x of order t such that $(a_0a_1)^2 = (a_1xa_0x^{-1})^2 = 1$, while Γ' is generated by four involutions c_0, c_1, c_2, c_3 such that $(c_0c_1)^2 = (c_1c_2)^2 = (c_2c_3)^2 = 1$ and the product c_3c_0 has order $t \geq 3$. An embedding of Γ in Γ' is given by taking $x = c_0c_3$, $a_0 = c_2$ and $a_1 = c_1$. There is another embedding with $x = c_3c_0$, $a_0 = c_1$ and $a_1 = c_2$, but this is equivalent under an outer automorphism of Γ' . Conjugation by c_0 gives an involutory automorphism with $x^{c_0} = x^{-1}$, $a_0^{c_0} = xa_0x^{-1}$ and $a_1^{c_0} = a_1$. Again, an extension is always possible to an epimorphism from Γ' onto D_n , with c_0 being taken to any element of $D_n \setminus C_n$.

Thus we have proven the following.

Theorem 2.1. *Every effective action of a finite cyclic group with non-maximal signature on a bordered surface S extends to the action of a larger group on S .*

Remark 2.2. Under the hypothesis above, the action of the cyclic group C_n extends in all cases to an action of the dihedral group D_n (at least) on the same surface. This is shown above for cases 4, 8 and 9, and in [4] for the other ten cases.

Remark 2.3. Theorem 2.1 contrasts with the situation for cyclic actions on unbordered orientable surfaces – that is, on Riemann surfaces. For these surfaces, a cyclic group acting with a non-maximal Fuchsian signature does not always extend; see Theorem 4.1 in [6].

3. Largest cyclic full groups of automorphisms

It is well known that the maximum order of a cyclic group of automorphisms acting on a bordered surface of algebraic genus g is $2g + 2$ if g is even and $2g$ if g is odd. This was proved by May in [19], and then refined in Theorem 3.2.18 in [7], where the authors distinguish between orientable and non-orientable surfaces, and orientation preserving and orientation reversing automorphisms of orientable surfaces. If $\theta: \Gamma \rightarrow C_{2g+2}$ or C_{2g} is a smooth epimorphism realising such a maximal cyclic action, then the signature of Γ is

- $(0; +; [g + 1]; \{(2, 2)\})$ or $(0; +; [2, g + 1]; \{(-)\})$ if g is even (and the image is C_{2g+2}),
- $(0; +; [2g]; \{(2, 2)\})$ or $(0; +; [2, 2g]; \{(-)\})$ if g is odd (and the image is C_{2g}).

It is also known that such a maximal cyclic action on a bordered surface always extends to a larger group of automorphisms; see [5]. We can now give a direct proof of this result, as a corollary of Theorem 2.1.

Corollary 3.1. *Let S be a compact bordered surface of algebraic genus $g \geq 2$ admitting an automorphism v of maximum possible order (namely, $2g+2$ if g is even, or $2g$ if g is odd). Then the full automorphism group of S properly contains $\langle v \rangle$.*

Proof. This is a consequence of the fact that such a maximal cyclic group $\langle v \rangle$ acts with signature of the form $(0; +; [t]; \{(2, 2)\})$ or $(0; +; [t, u]; \{(-)\})$, and these are non-maximal signatures, occurring in cases 9 and 10 of Table 1, and dealt with in Section 2. □

It is worth mentioning that the full automorphism group of such a surface is a dihedral 2-extension of $\langle v \rangle$, except for three surfaces of odd genus and one of even genus; see [5].

Next, it is a natural question to ask what is the largest order of a cyclic group that acts as the full group of automorphisms of some bordered surface S of given genus g . We answer this question in Theorem 3.3 below, where the topological types of the surfaces attaining the bounds are also given. For a fixed value of g , the *topological type* of a surface is encoded in the symbol εk , where ε is $+$ if the surface is orientable, or $-$ if it is non-orientable, and k is the number of its boundary components.

If $\theta : \Gamma \rightarrow C_n$ is the smooth epimorphism corresponding to the cyclic action of C_n on $S = H/\ker \theta$, then the topological type of S can be obtained from the signature of $\ker \theta$. Namely, the orientability of S coincides with the sign of $\sigma(\ker \theta)$ and the number of its boundary components is the number of (empty) period cycles of $\sigma(\ker \theta)$. Results in Sections 2.1 and 2.3 in [7] can be used to determine both parameters, but for the reader’s convenience we display in Lemma 3.2 the important things we will use here.

Lemma 3.2. *Let $\theta : \Gamma \rightarrow C_n$ be a smooth epimorphism with n even.*

- 1) *If Γ is orientable then $\ker \theta$ is orientable if and only if no non-orientable word of Γ with respect to $\ker \theta$ belongs to $\ker \theta$. (A word of Γ with respect to $\ker \theta$ is a composition of canonical generators of Γ , none of which belongs to $\ker \theta$.)*

If Γ is non-orientable then $\ker \theta$ is non-orientable if and only if $\ker \theta$ contains either a glide reflection (from the canonical generators of Γ) or a non-orientable word.

- 2) *Each period cycle of $\sigma(\Gamma)$ of the form $(2, 2^r, 2)$ produces $rn/2$ empty period cycles in $\sigma(\ker \theta)$.*

Let C be an empty period cycle in $\sigma(\Gamma)$, and let c_0 and e be its associated canonical generators. If $\theta(c_0) \neq 1$ then C produces no period cycle in $\sigma(\ker \theta)$, while if $\theta(c_0) = 1$ then C produces n/m empty period cycles in $\sigma(\ker \theta)$, where m is the order of $\theta(e)$.

Theorem 3.3. *For every integer $g \geq 2$, the order of the largest cyclic group that acts as the full group of automorphisms of a bordered surface of algebraic genus g is given in Table 2, together with all possibilities for the corresponding signature of the cyclic action and the topological type of the surfaces on which the action takes place.*

Genus	Largest n	Possible signatures	Topological type
$g \equiv 1 \pmod{4}$	$g + 1$	$(0; +; [\frac{g+1}{2}]; \{(2, 2, 2, 2)\})$	$+(g + 1)$
		$(0; +; [2, 2, \frac{g+1}{2}]; \{(-)\})$	$+2$
		$(0; +; [2, \frac{g+1}{2}]; \{(2, 2)\})$	$-\frac{g+1}{2}$
g even	g	$(0; +; [g]; \{(2, 2, 2, 2)\})$	$-g$
		$(0; +; [2, 2, g]; \{(-)\})$	$+1$
		$(0; +; [2, g]; \{(2, 2)\})$	$-\frac{g}{2}$
		or $(0; +; [-]; \{(2, \cdot^6, 2)\})$ when $g = 2$	$+3$
		or $(0; +; [3, 3]; \{(2, 2)\})$ when $g = 6$	$+3$
		or $(0; +; [2, 3, 3]; \{(-)\})$ when $g = 6$	$+3$ or $+1$
		or $(0; +; [2, 3, 4]; \{(-)\})$ when $g = 12$	$+1$
		or $(0; +; [3, 4]; \{(2, 2)\})$ when $g = 12$	-6
		or $(0; +; [3, 5]; \{(2, 2)\})$ when $g = 30$	$+15$
		or $(0; +; [2, 3, 5]; \{(-)\})$ when $g = 30$	$+1$
		$g \equiv 3 \pmod{4}$	$g - 1$
$(0; +; [-]; \{(-), (2, 2, 2, 2)\})$	$+(g + 1), +(g - 1),$ $-(g - 1)$ or $-g$		
$(1; -; [-]; \{(2, 2, 2, 2)\})$	$+(g - 1)$		
$(0; +; [2, 2]; \{(-), (-)\})$	$+2, +4, -1$ or -2		
$(1; -; [2, 2]; \{(-)\})$	-2		
$(0; +; [2]; \{(-), (2, 2)\})$	$-\frac{g-1}{2}, -\frac{g+1}{2}$ or $-\frac{g+3}{2}$		
$(1; -; [2]; \{(2, 2)\})$	$-\frac{g-1}{2}$		
or $(0; +; [-]; \{(2, \cdot^8, 2)\})$ when $g = 3$	$+4$		
or $(0; +; [2]; \{(2, \cdot^6, 2)\})$ when $g = 3$	-3		
or $(0; +; [2, 2]; \{(2, \cdot^4, 2)\})$ when $g = 3$	-2		
or $(0; +; [2, 2, 2]; \{(2, 2)\})$ when $g = 3$	-1		
or $(0; +; [2, 2, 2, 2]; \{(-)\})$ when $g = 3$	$+2$		
or $(0; +; [2, 3, 6]; \{(-)\})$ when $g = 7$	$+6$ or $+2$		
or $(0; +; [3, 6]; \{(2, 2)\})$ when $g = 7$	-3		

TABLE 2. Largest C_n as full automorphism group of a bordered surface of genus g .

Proof. If $\theta: \Gamma \rightarrow C_n$ is a smooth epimorphism then, by Lemma 1.1,

$$\sigma = \sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(-), \cdot^k, (-), (2, \cdot^{r_1}, 2), \dots, (2, \cdot^{r_d}, 2)\})$$

where r_i is even for $i = 1, \dots, d$ and $k + d \geq 1$, with $d = 0$ if n is odd. By the Riemann–Hurwitz formula, $\mu(\Gamma) = (g - 1)/n$. We first find all possible signatures σ for which $n \geq g - 1$, that is, $\mu(\sigma) \leq 1$. Since $1 - 1/m_i \geq 1/2$ for all $i = 1, \dots, r$,

we have

$$(3.1) \quad \alpha\gamma + k + d + \frac{r}{2} + \frac{r_1 + \dots + r_d}{4} \leq 3,$$

where $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise. A straightforward calculation shows there are 26 different types of NEC signatures that satisfy these conditions, namely the following:

- | | | |
|---|---|-------------------------------------|
| $(2; -; [-]; \{(-)\})$, | $(1; +; [-]; \{(-)\})$, | $(1; -; [m]; \{(-)\})$, |
| $(1; -; [-]; \{(2, 2)\})$, | $(1; -; [-]; \{(-), (-)\})$, | $(0; +; [-]; \{(-), (-), (-)\})$, |
| $(0; +; [m]; \{(-), (-)\})$, | $(0; +; [-]; \{(-), (2, 2)\})$, | $(0; +; [m]; \{(2, 2)\})$, |
| $(0; +; [m_1, m_2]; \{(-)\})$, | $(0; +; [m]; \{(2, 2, 2, 2)\})$, | $(0; +; [-]; \{(2, 2), (2, 2)\})$, |
| $(0; +; [-]; \{(-), (2, 2, 2, 2)\})$, | $(1; -; [-]; \{(2, 2, 2, 2)\})$, | $(0; +; [-]; \{(2, \dots, 2)\})$, |
| $(0; +; [2]; \{(2, 2, 2, 2, 2, 2)\})$, | $(0; +; [2, 2]; \{(2, 2, 2, 2, 2)\})$, | $(0; +; [2, 2, 2]; \{(2, 2)\})$, |
| $(0; +; [2, 2, 2, 2]; \{(-)\})$, | $(0; +; [-]; \{(2, 2, 2, 2, 2, 2)\})$, | $(0; +; [2]; \{(-), (2, 2)\})$, |
| $(1; -; [2]; \{(2, 2)\})$, | $(0; +; [2, 2]; \{(-), (-)\})$, | $(1; -; [2, 2]; \{(-)\})$, |
| $(0; +; [m_1, m_2]; \{(2, 2)\})$, | $(0; +; [m_1, m_2, m_3]; \{(-)\})$. | |

Among these, 10 are non-maximal NEC signatures, in which case Theorem 2.1 shows that the cyclic action is never the full group of automorphisms of the surface. We deal with the remaining 16 signature types below. Since each of them is maximal, we may choose a maximal NEC group Γ with such signature, so that if $\theta: \Gamma \rightarrow C_n$ is a smooth epimorphism then the maximality of Γ prevents θ from being extended to a larger group Γ' . Accordingly, this guarantees that C_n acts as the full group of automorphisms of the bordered surface $H/\ker \theta$.

If C_n acts with signature of the form $(0; +; [m]; \{(2, 2, 2, 2)\})$ then C_n is generated by elements of orders m and 2 and so either $m = n$ or $m = n/2$, the latter case occurring only if $n/2$ is odd. If $\sigma(\Gamma) = (0; +; [n]; \{(2, 2, 2, 2)\})$ then $n = g$, since $\mu(\Gamma) = 1 - 1/n$. There exists a unique smooth epimorphism $\theta: \Gamma \rightarrow C_n = \langle v \rangle$ (up to automorphism of C_n) given by $\theta(x_1) = v = \theta(e)^{-1}$, $\theta(c_i) = v^{n/2}$ for $i = 0, 2, 4$ and $\theta(c_i) = 1$ for $i = 1, 3$, where $\{x_1, c_0, c_1, c_2, c_3, c_4, e\}$ is a set of canonical generators of Γ . We observe that $x_1^{n/2} c_0$ is a non-orientable word in $\ker \theta$, and so C_n acts as full group on non-orientable surfaces. As to the number of boundary components, Lemma 3.2 shows that the number of (empty) period cycles of $\sigma(\ker \theta)$ equals n . Hence for g even, the cyclic group C_g acts with signature $(0; +; [g]; \{(2, 2, 2, 2)\})$ as the full group of automorphisms on surfaces of topological type $-g$. It will follow from the analysis of the other cases we consider below that no larger cyclic action occurs, and this gives the first row for the case where g is even, in Table 2.

If $\sigma(\Gamma) = (0; +; [n/2]; \{(2, 2, 2, 2)\})$ (with $n/2$ odd) then $n = g + 1$, since $\mu(\Gamma) = 1 - 2/n$. In this case C_n acts on orientable surfaces since any word with an odd number of proper canonical reflections is mapped by θ onto an odd power of v and so there is no non-orientable word in $\ker \theta$. (A canonical reflection $c \in \Gamma$ is proper (with respect to $\ker \theta$) if $c \notin \ker \theta$.) Again Lemma 3.2 implies that the number of empty period cycles in $\sigma(\ker \theta)$ is n . Consequently, for $g \equiv 1 \pmod{4}$ the cyclic group C_{g+1} acts with signature $(0; +; [(g + 1)/2]; \{(2, 2, 2, 2)\})$ as the full group of automorphisms on surfaces of topological type $+(g + 1)$. It will follow from the analysis of the other cases we consider below that for these values of g no larger cyclic action occurs, and this gives the first row of Table 2.

If $\sigma(\Gamma) = (0; +; [-]; \{(2, 2), (2, 2)\})$ or $(0; +; [-]; \{(-), (2, 2, 2, 2)\})$, then n is even because C_n contains elements of order two, and $n = g - 1$ because $\mu(\Gamma) = 1$. We do not have to consider the case $g \equiv 1 \pmod{4}$, since for these values of g we have just shown the existence of a larger cyclic action. Hence we may assume that $n/2$ is odd. The image $\theta(e_1)$ of the canonical generator e_1 (which coincides with $\theta(e_2)^{-1}$) must have order n or $n/2$, in order for θ to be an epimorphism. In the first case, $e_1^{n/2}c_{20}$ is a non-orientable word in $\ker \theta$, while in the second, any word with an odd number of proper canonical reflections is mapped by θ onto an odd power of v and so there is no non-orientable word in $\ker \theta$. If $\sigma(\Gamma) = (0; +; [-]; \{(2, 2), (2, 2)\})$, then the number of empty period cycles in $\sigma(\ker \theta)$ is n , irrespective of the order of $\theta(e_1)$, and so C_{g-1} acts with this signature on topological types $+(g-1)$ and $-(g-1)$. On the other hand, if $\sigma(\Gamma) = (0; +; [-]; \{(-), (2, 2, 2, 2)\})$, then the number of empty period cycles in $\sigma(\ker \theta)$ depends on the orders of $\theta(c_{10})$ and $\theta(e_1)$; see the second part of Lemma 3.2. If $\theta(c_{10}) \neq 1$ then the number of empty period cycles of $\sigma(\ker \theta)$ is $g - 1$, while if $\theta(c_{10}) = 1$ then this number equals g when $\theta(e_1)$ has order n , and $g + 1$ when $\theta(e_1)$ has order $n/2$. Together with the analysis of orientability, this implies that C_{g-1} acts with signature $(0; +; [-]; \{(-), (2, 2, 2, 2)\})$ on topological types $+(g-1)$, $-(g-1)$, $-g$ or $+(g+1)$. It will follow from the analysis of the other cases we consider below that no larger cyclic action occurs, and so this gives the first two rows for the case $g \equiv 3 \pmod{4}$ in Table 2.

If $\sigma(\Gamma) = (1; -; [-]; \{(2, 2, 2, 2)\})$, then n is even and $n = g - 1$. As above, we do not have to consider the case $g \equiv 1 \pmod{4}$, so $g \equiv 3 \pmod{4}$. The image of the canonical glide reflection d_1 must have order n or $n/2$, but in either case there is no non-orientable word in $\ker \theta$ because $n/2$ is odd. Thus C_{g-1} acts with this signature on orientable surfaces, with $n = g - 1$ boundary components according to Lemma 3.2.

If $\sigma(\Gamma) = (0; +; [-]; \{(2, \cdot^8, \cdot, 2)\})$, $(0; +; [2]; \{(2, \cdot^6, \cdot, 2)\})$, $(0; +; [2, 2]; \{(2, 2, 2, 2)\})$, $(0; +; [2, 2, 2]; \{(2, 2)\})$ or $(0; +; [2, 2, 2, 2]; \{(-)\})$, then $n = 2$ because C_n is generated by involutions, and $n = g - 1$ (and so $g = 3$) because $\mu(\Gamma) = 1$ in all cases. By Lemma 3.2, the actions occur on surfaces with 4, 3, 2, 1 and 2 boundary components, respectively. In the first and last cases there is no non-orientable word in $\ker \theta$, and so C_2 acts as full group with signature $(0; +; [-]; \{(2, \cdot^8, \cdot, 2)\})$ or $(0; +; [2, 2, 2, 2]; \{(-)\})$ on orientable surfaces. In the remaining cases, x_1c_0 is a non-orientable word in $\ker \theta$, and so C_2 acts with these signatures on non-orientable surfaces.

Similarly, if $\sigma(\Gamma) = (0; +; [-]; \{(2, \cdot^6, \cdot, 2)\})$, then $n = 2$ and $g = 2$ (since $\mu(\Gamma) = \frac{1}{2}$), and C_2 acts on orientable surfaces with 3 boundary components.

If $\sigma(\Gamma) = (0; +; [2]; \{(-), (2, 2)\})$ or $(1; -; [2]; \{(2, 2)\})$, then n is even and $n = g - 1$, and again we do not have to consider the case $g \equiv 1 \pmod{4}$. If c is a proper canonical reflection in Γ then x_1c is a non-orientable word in $\ker \theta$ for any θ . Hence any action of C_n with either signature occurs only on non-orientable surfaces. The number of boundary components is $n/2 = (g - 1)/2$ for the second signature, and $(g - 1)/2$, $(g + 1)/2$ or $(g + 3)/2$ for the first one, depending on the orders of the images of c_{10} and e_1 .

If $\sigma(\Gamma) = (0; +; [2, 2]; \{(-), (-)\})$, then n is even and $n = g - 1$, and once again we do not have to consider the case $g \equiv 1 \pmod{4}$. If the two canonical reflections of Γ belong to $\ker \theta$ then there is no non-orientable word in $\ker \theta$, but if one of them, say c_{10} , does not belong to $\ker \theta$, then $x_1 c_{10}$ is a non-orientable word in $\ker \theta$. The order of $\theta(e_1)$ is n or $n/2$, and the number of empty period cycles of $\sigma(\ker \theta)$ is 2 or 4 in the former case, and 1 or 2 in the latter.

Similarly, if $\sigma(\Gamma) = (1; -; [2, 2]; \{(-)\})$ then n is even, and $n = g - 1$, so we may assume that $g \equiv 3 \pmod{4}$, and hence $n/2$ is odd. The image $\theta(d_1)$ of the canonical glide reflection d_1 must have order n or $n/2$. In either case, $\theta(e_1) = \theta(d_1)^{-2}$ has order $n/2$, and so $\sigma(\ker \theta)$ has two empty period cycles. With regard to orientability, $\ker \theta$ always contains a non-orientable word, such as $d_1^{n/2} x_1$ when $\theta(d_1)$ has order n , and $d_1^{n/2}$ when $\theta(d_1)$ has order $n/2$.

Next, for NEC groups Γ with signature type $(0; +; [m_1, m_2]; \{(2, 2)\})$ we have $\mu(\Gamma) = 3/2 - 1/m_1 - 1/m_2$, and the only pairs (m_1, m_2) with $2 \leq m_1 \leq m_2$ which give $\mu(\Gamma) \leq 1$ are the following: $(2, m)$ for any $m \geq 2$, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(3, 6)$ and $(4, 4)$. In all cases, $n = \text{lcm}(m_1, m_2, 2)$, since C_n is generated by elements of orders m_1 , m_2 and 2; for example, in the first case $m = n$ or $m = n/2$, with the latter value occurring only when $n/2$ is odd. For signature $(0; +; [4, 4]; \{(2, 2)\})$ we have $n = 4$ and $g = 5$, and we can rule this out because C_4 is not the largest full cyclic group acting in genus five. With regard to orientability of the surface on which C_n acts, we observe that if one of the proper periods m_i is even, then $x_i^{m_i/2} c_0$ is a non-orientable word in $\ker \theta$ for any θ , where x_i is an elliptic canonical generator of order m_i . Otherwise, if both proper periods m_1 and m_2 are odd, then any word with an odd number of proper canonical reflections is mapped by θ onto an odd power of v , and so there is no non-orientable word in $\ker \theta$. Summarising, the analysis of orientability for actions with signature type $(0; +; [m_1, m_2]; \{(2, 2)\})$ splits into the following cases:

Actions on non-orientable surfaces	Actions on orientable surfaces
$(0; +; [2, n]; \{(2, 2)\})$, $n = g$ even;	$(0; +; [3, 3]; \{(2, 2)\})$, $n = g = 6$;
$(0; +; [2, n/2], \{(2, 2)\})$, $n = g + 1 \equiv 2 \pmod{4}$;	$(0; +; [3, 5]; \{(2, 2)\})$, $n = g = 30$;
$(0; +; [3, 4]; \{(2, 2)\})$, $n = g = 12$;	
$(0; +; [3, 6]; \{(2, 2)\})$, $n = 6$, $g = 7$.	

In all cases, the actions occur on surfaces with $n/2$ boundary components, by Lemma 3.2.

Finally, for NEC groups Γ with signature type $(0; +; [m_1, m_2, m_3]; \{(-)\})$, we have $\mu(\Gamma) = 2 - 1/m_1 - 1/m_2 - 1/m_3$, and the only triples (m_1, m_2, m_3) with $2 \leq m_1 \leq m_2 \leq m_3$ such that $\mu(\Gamma) \leq 1$ are the following: $(2, 2, m)$ where $m \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$. Note also that the unique canonical generating reflection c_0 of Γ has to be killed by θ . This implies that $n = \text{lcm}(m_1, m_2, m_3)$, since C_n is generated by elements of orders m_1 , m_2 and m_3 . For signatures $(0; +; [2, 4, 4]; \{(-)\})$ and $(0; +; [3, 3, 3]; \{(-)\})$, we have $n = g - 1 = 4$ and $n = g - 1 = 3$ respectively, and we can rule both of these out since C_4 is not the largest full cyclic group acting in genus five, and C_3 is not the largest in genus four.

For the remaining signatures, the values of n and g are as follows:

$$\begin{aligned} (0; +; [2, 2, n]; \{(-)\}) \quad n=g \text{ even}; & \quad (0; +; [2, 2, n/2], \{(-)\}) \quad n=g + 1 \equiv 2 \pmod{4}; \\ (0; +; [2, 3, 3]; \{(-)\}) \quad n=g=6; & \quad (0; +; [2, 3, 4]; \{(-)\}) \quad n = g = 12; \\ (0; +; [2, 3, 5]; \{(-)\}) \quad n=g=30; & \quad (0; +; [2, 3, 6]; \{(-)\}) \quad n = 6, g = 7. \end{aligned}$$

With regard to orientability, c_0 is the unique orientation reversing canonical generator of Γ and so $\ker \theta$ contains no non-orientable word. The number of the boundary components equals n/m where m is the order of $\theta(e) = \theta(x_1x_2x_3)^{-1}$. For signatures $(0; +; [2, 2, n]; \{(-)\})$ and $(0; +; [2, 2, n/2]; \{(-)\})$, where necessarily $\theta(x_1) = \theta(x_2)$, this order is n or $n/2$ respectively. Thus C_n acts with signature $(0; +; [2, 2, n]; \{(-)\})$ on surfaces of topological type $+1$, and with signature $(0; +; [2, 2, n/2]; \{(-)\})$ on surfaces of topological type $+2$. For signature $(0; +; [2, 3, 3]; \{(-)\})$ we have $\theta(x_1)=v^3$, $\theta(x_2) = v^{\pm 2}$ and $\theta(x_3) = v^{\pm 2}$, and so $\theta(e) = v, v^3$ or v^5 . This gives one or three boundary components. For signature $(0; +; [2, 3, 4]; \{(-)\})$ we have $\theta(x_1) = v^6$, $\theta(x_2) = v^{\pm 4}$ and $\theta(x_3) = v^{\pm 3}$, so $\theta(e) = v^{\pm 1}$ or $v^{\pm 5}$, and there is just one boundary component. For signature $(0; +; [2, 3, 5]; \{(-)\})$ we have $\theta(x_1) = v^{15}$, $\theta(x_2) = v^{\pm 10}$ and $\theta(x_3) = v^{\pm 6}$ or $v^{\pm 12}$, so $\theta(e) = v^{\pm 1}, v^{\pm 7}, v^{\pm 11}$ or $v^{\pm 13}$, and again this gives one boundary component. Finally, for signature $(0; +; [2, 3, 6]; \{(-)\})$ we have $\theta(x_1) = v^3$, $\theta(x_2) = v^{\pm 2}$ and $\theta(x_3) = v^{\pm 1}$, so $\theta(e) = 1$ or $v^{\pm 2}$, giving six or two boundary components. \square

As a consequence of Remark 2.2 and the proof of Theorem 3.3 we also have the following.

Corollary 3.4. *If the cyclic group C_n acts faithfully on a bordered surface S of algebraic genus $g \geq 2$, and either $n > g + 1$ with $g \equiv 1 \pmod{4}$, or $n > g$ with g is even, or $n > g - 1$ with $g \equiv 3 \pmod{4}$, then also the dihedral group D_n acts faithfully on S .*

In [4] it was shown that the order of the largest cyclic group that acts as the full group of automorphisms on an *unbordered* non-orientable surface of algebraic genus g is $g + 1$ if $g \equiv 1 \pmod{4}$, g if g is even, and $g - 1$ if $g \equiv 3 \pmod{4}$. These are the same bounds as for bordered surfaces, according to Theorem 3.3. Proposition 3.5 below shows examples of the phenomenon that a bordered and an unbordered surface attaining the same bound may come together as the Klein surfaces associated with two different symmetries on the same compact Riemann surface. Here we recall the definition of a symmetry of a Riemann surface X as an antianalytic involution $\tau : X \rightarrow X$. Given such a τ , the orbit space X/τ admits a dianalytic structure which makes it a compact Klein surface. Moreover, all such surfaces arise in this way; see [1]. In addition, the full group $\text{Aut}(X/\tau)$ of automorphisms of the Klein surface X/τ is isomorphic to the centraliser in $\text{Aut}^+(X)$ of τ , where $\text{Aut}^+(X)$ stands for the group of analytic automorphisms of X .

Proposition 3.5. *For every integer $g \geq 2$ there exists a compact Riemann surface X of genus g with two symmetries τ_1 and τ_2 such that the full group of automorphisms of the associated Klein surfaces X/τ_i is cyclic of the largest possible order, with one of the surfaces being bordered, and the other one unbordered.*

Proof. For each $g \geq 2$, take $n = g + 1$, g or $g - 1$, according to whether $g \equiv 1 \pmod{4}$, or g is even, or $g \equiv 3 \pmod{4}$ respectively. We will provide an epimorphism ϕ from a maximal NEC group Γ of the appropriate signature to $C_n \times C_2 = \langle v, t \mid v^n = t^2 = [v, t] = 1 \rangle$, such that $\ker \phi$ is a surface Fuchsian group (a surface NEC group with no orientation reversing element). The orbit space $X = H/\ker \phi$ of the hyperbolic plane under the action of $\ker \phi$ is then a compact Riemann surface. The maximality of Γ assures that $C_n \times C_2$ is the full group $\text{Aut}^\pm(X)$ of analytic and antianalytic automorphisms of X , and the epimorphism ϕ will be defined so that the subgroup of analytic automorphisms will be cyclic of order n . This way, the Riemann surface X will have exactly two non-conjugate symmetries, say τ_1 and τ_2 , and the full group $\text{Aut}(X/\tau_i)$ of each associated Klein surface will be cyclic of the largest possible order. For each symmetry τ_i , we can write $\langle \tau_i \rangle = \Lambda_i/\ker \phi$ where $\Lambda_i = \phi^{-1}(\langle \tau_i \rangle)$ is a surface NEC group uniformising X/τ_i , since $X/\tau_i = (H/\ker \phi)/(\Lambda_i/\ker \phi) = H/\Lambda_i$. Hence the topological type of X/τ_i can be read from the sign and the number of empty period cycles in the signature of Λ_i . Results from Sections 2.1 and 2.3 of [7] can be used to determine both parameters of $\sigma(\Lambda_i)$.

If $g \equiv 1 \pmod{4}$, then we take a maximal NEC group Γ with signature $\sigma(\Gamma) = (0; +; [2, 2, \frac{g+1}{2}]; \{(-)\})$ and define $\phi : \Gamma \rightarrow C_{g+1} \times C_2 = \langle v \rangle \times \langle t \rangle$ by setting

$$\phi(x_1) = \phi(x_2) = v^{(g+1)/2}, \quad \phi(x_3) = v^2, \quad \phi(e_1) = v^{-2}, \quad \phi(c_0) = t.$$

It is clear that $\ker \phi$ is a surface Fuchsian group, and so $X = H/\ker \phi$ is a compact Riemann surface, of genus g by the Riemann–Hurwitz formula. Let $\tau_1 := t$ and $\tau_2 := tv^{(g+1)/2}$ be the two symmetries of X . The preimage $\phi^{-1}(\langle \tau_1 \rangle)$ is a surface NEC group with no non-orientable word (with respect to Γ), and whose signature has two empty period cycles, and so the Klein surface X/τ_1 has topological type $+2$. The preimage $\phi^{-1}(\langle \tau_2 \rangle)$ is a surface NEC group containing the non-orientable word x_1c_0 , and whose signature has no period cycle, and it follows that the Klein surface X/τ_2 is unbordered and non-orientable.

If g is even, then we take a maximal NEC group Γ with signature $\sigma(\Gamma) = (0; +; [2, 2, g]; \{(-)\})$, and define $\phi : \Gamma \rightarrow C_g \times C_2 = \langle v \rangle \times \langle t \rangle$ by setting

$$\phi(x_1) = \phi(x_2) = v^{g/2}, \quad \phi(x_3) = v, \quad \phi(e_1) = v^{-1}, \quad \phi(c_0) = t.$$

As above, it is easy to check that $X = H/\ker \phi$ is a compact Riemann surface of genus g with two symmetries $\tau_1 := t$ and $\tau_2 := tv^{g/2}$, such that X/τ_1 has topological type $+1$ while X/τ_2 is unbordered and non-orientable.

If $g \equiv 3 \pmod{4}$, then we take a maximal NEC group Γ with signature $\sigma(\Gamma) = (0; +; [2, 2]; \{(-), (-)\})$, and define $\phi : \Gamma \rightarrow C_{g-1} \times C_2 = \langle v \rangle \times \langle t \rangle$ by setting

$$\phi(x_1) = \phi(x_2) = v^{(g-1)/2}, \quad \phi(e_1) = v, \quad \phi(e_2) = v^{-1}, \quad \phi(c_{10}) = \phi(c_{20}) = t.$$

The two symmetries of the genus g compact Riemann surface $X = H/\ker \phi$ are $\tau_1 := t$ and $\tau_2 := tv^{(g-1)/2}$. In this case, the topological type of X/τ_1 is $+2$, while X/τ_2 is unbordered and non-orientable.

Also if $g \equiv 3 \pmod{4}$, we can take a maximal NEC group Γ with signature $\sigma(\Gamma) = (1; -; [2, 2]; \{(-)\})$, and define $\phi : \Gamma \rightarrow C_{g-1} \times C_2 = \langle v \rangle \times \langle t \rangle$ by setting

$$\phi(d) = v, \quad \phi(x_1) = \phi(x_2) = t, \quad \phi(e_1) = v^{-2}, \quad \phi(c_0) = v^{(g-1)/2}.$$

The orbit space $X := H/\ker \phi$ is a compact Riemann surface of genus g , with two symmetries $\tau_1 := v^{(g-1)/2}$ and $\tau_2 := tv^{(g-1)/2}$ such that X/τ_1 has topological type -2 while X/τ_2 is unbordered and non-orientable. □

Remark 3.6. In analogy with the notion of q -hyperellipticity on Riemann surfaces, a compact Klein surface S of algebraic genus $g \geq 2$ is said to be q -hyperelliptic if it admits an involutory automorphism φ such that the quotient surface S/φ has algebraic genus q . If $q = 0$ then S is hyperelliptic, while 1-hyperelliptic surfaces are usually called *elliptic-hyperelliptic*. In [4] it is shown that the unbordered and non-orientable surface $S = X/\tau$ occurring in Proposition 3.5 is hyperelliptic if $g \not\equiv 3 \pmod{4}$, and elliptic-hyperelliptic but not hyperelliptic if $g \equiv 3 \pmod{4}$, for all $g \neq 3, 6, 7, 12, 30$. Viewing φ as an analytic automorphism of X , we can consider the orbit space X/φ , which is a compact Riemann surface. It is easy to see that the genus of X/φ is the same as the algebraic genus of the Klein surface S/φ . Hence the compact Riemann surface X occurring in Proposition 3.5 is hyperelliptic if $g \not\equiv 3 \pmod{4}$, and elliptic-hyperelliptic but not hyperelliptic if $g \equiv 3 \pmod{4}$, for all $g \neq 3, 6, 7, 12, 30$.

Remark 3.7. The Riemann surface X in the statement of Proposition 3.5 is not unique within its genus, but is a member of an infinite family of Riemann surfaces with the same property. The reason this happens is that the Teichmüller dimension of the maximal Fuchsian groups Γ in the proof of the proposition is positive.

4. The full real genus of a cyclic group

The *real genus* of a finite group G is the minimum algebraic genus of any compact bordered surface S on which G acts effectively as a group of automorphisms; this was defined by May in [20]. For surfaces of genus $g \geq 2$, this parameter has been completely determined for all cyclic G ; see Chapter 3 in [7]. For instance, if $n \equiv 2 \pmod{4}$ with $n > 2$ then the real genus of C_n is $n/2 - 1$, attainable via a smooth epimorphism $\theta : \Gamma \rightarrow C_n$ where Γ has signature $(0; +; [n/2]; \{(2, 2)\})$ or $(0; +; [2, n/2]; \{(-)\})$. These signatures correspond to cases 9 and 10 of Section 2, and so Theorem 2.1 implies that the smooth epimorphism θ always extends to a larger group action. Hence for all such n , whenever C_n acts on a bordered surface of the minimum genus, the full automorphism group of the surface is strictly larger than C_n . A natural question arises from these observations: *Given a finite group G , what is the minimum algebraic genus of a bordered surface S on which G acts effectively as the full automorphism group of S ?* We call this number the *full real genus* of the group G .

It is interesting to point out that when $n \equiv 2 \pmod{4}$ and $n > 2$, the above real genus $n/2 - 1$ is attained for orientable surfaces, whereas the minimum algebraic

genus of a non-orientable surface on which C_n acts is $n/2$. This makes it sensible to split the concept of real genus into two, by defining the *real orientable genus* and (respectively) the *real non-orientable genus* of a group G as the minimum algebraic genus of the compact bordered orientable surfaces and the non-orientable surfaces on which G acts. Then in turn, we may define the *full real orientable genus* and the *full real non-orientable genus* of G as the minimum algebraic genus of the compact bordered orientable surfaces and (respectively) the non-orientable surfaces on which G acts effectively as the full automorphism group. In this section, we compute the full real orientable genus and the full real non-orientable genus of the cyclic group C_n , for all n . In addition, we also determine the number of boundary components of the surfaces for which these minima are attained.

Observe that the cyclic group C_2 acts on every surface of algebraic genus 2, since these are hyperelliptic. Moreover, most surfaces of genus 2 have no non-trivial automorphisms other than the hyperelliptic involution, and it follows that the full real genus (orientable or non-orientable) of C_2 is 2. Note that by *most* surfaces, we mean all (classes of) surfaces in each of the four connected components of the moduli space of bordered surfaces of genus 2, except for those in a subvariety of codimension one. Accordingly, from now on we will assume that $n > 2$.

Suppose the finite group G acts on a bordered surface H/Λ of algebraic genus g , and let $\theta: \Gamma \rightarrow G$ be the corresponding smooth epimorphism, with $\ker \theta = \Lambda$. Then since the area $2\pi\mu(\Lambda)$ of the surface NEC group Λ is $2\pi(g - 1)$, the Riemann–Hurwitz formula $\mu(\Lambda) = |G|\mu(\Gamma)$ gives

$$g = 1 + |G|\mu(\Gamma).$$

Hence in order to find the full real genus of G , we have to minimise the area of Γ among all NEC groups admitting a non-extendable smooth epimorphism $\theta: \Gamma \rightarrow G$.

For $G = C_n$, Theorem 2.1 shows that Γ cannot have any of the signatures σ occurring in the second column of Table 1. We find the only signatures for which the full real orientable genus can be attained, in Lemma 4.1 below. The analogous result for the full real non-orientable genus is given in Lemma 4.6.

4.1. The full real orientable genus of a cyclic group

Lemma 4.1. *If the full real orientable genus of C_n ($n > 2$) is attained by means of a smooth epimorphism $\theta: \Gamma \rightarrow C_n$, then the signature of Γ is of the form $(0; +; [m_1, m_2, m_3]; \{(-)\})$ or $(0; +; [n/2]; \{(2, 2, 2, 2)\})$, with the latter occurring only when $n/2$ is odd.*

Proof. We first show that the reduced area $\mu(\Gamma)$ of Γ is bounded above by

$$(4.1) \quad \mu(\Gamma) \leq 2 - \frac{2}{p} - \frac{1}{n},$$

where p is the smallest prime divisor of n . To show this, we choose a maximal NEC group Δ with signature $\sigma(\Delta) = (0; +; [p, p, n]; \{(-)\})$ and define a smooth epimorphism $\theta: \Delta \rightarrow C_n = \langle v \rangle$ by setting

$$\theta(x_1) = \theta(x_2) = v^{n/p}, \quad \theta(x_3) = v, \quad \theta(e) = v^{-2n/p-1}, \quad \theta(c) = 1.$$

By the maximality of Δ , it follows that C_n acts as the full automorphism group of the bordered surface $H/\ker\theta$. We observe that $H/\ker\theta$ is orientable, because the only orientation reversing canonical generator of Δ belongs to $\ker\theta$; see Theorems 2.1.2 and 2.1.3 in [7]. Hence $\mu(\Gamma) \leq \mu(\Delta) = 2 - 2/p - 1/n$, which gives inequality (4.1).

Next, we look for NEC groups Γ uniformising an action of C_n as the full group of a bordered orientable surface, such that the reduced area $\mu(\Gamma)$ satisfies the above inequality. We know by Lemma 1.1 that Γ has signature

$$\sigma = \sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(-), \dots, (-), (2, r_1, 2), \dots, (2, r_d, 2)\})$$

where r_i is even for $i = 1, \dots, d$, and $k + d \geq 1$, with $d = 0$ if n is odd.

Let us first consider odd values of n . Since each m_i is a divisor of n we get $m_i \geq p$ for all i , and so

$$\alpha\gamma + k - 2 + r\left(1 - \frac{1}{p}\right) \leq \alpha\gamma + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) = \mu(\Gamma) \leq 2 - \frac{2}{p} - \frac{1}{n} < 2 - \frac{2}{p}.$$

This gives $\alpha\gamma + k + (r(p-1)+2)/p < 4$, and then since $k \geq 1$ we get $\alpha\gamma + k = 1, 2$ or 3 . Actually $\alpha = 2$, since for n odd, $\ker\theta$ and Γ have the same sign; see Theorem 2.1.2 in [7].

Now a straightforward calculation shows that there are five different types of NEC signatures that satisfy these conditions, namely the following:

$$(0; +; [m_1, m_2]; \{(-)\}), \quad (0; +; [m_1, m_2, m_3]; \{(-)\}), \quad (0; +; [m]; \{(-), (-)\}), \\ (0; +; [-]; \{(-), (-), (-)\}), \quad (1; +; [-]; \{(-)\}).$$

Of these, all but $(0; +; [m_1, m_2, m_3]; \{(-)\})$ are non-maximal NEC signatures, and in those cases, Theorem 2.1 shows that the cyclic action is never the full group of automorphisms of the surface. Hence in particular, the lemma holds for all odd values of n .

For even values of n , we have

$$\alpha\gamma + k + d - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i}\right) + \frac{r_1 + \dots + r_d}{4} = \mu(\Gamma) \leq 2 - \frac{2}{p} - \frac{1}{n} = 1 - \frac{1}{n} < 1.$$

If $d \geq 2$ then $(r_1 + \dots + r_d)/4 \geq 1$ because each r_i is even, and it follows that $\mu(\Gamma) \geq \alpha\gamma + k + \sum(1 - 1/m_i) + 1 \geq 1$, which contradicts the above inequality. If $d = 0$ then $k \geq 1$ and $\alpha\gamma + k = 1$ or 2 . Then if $\alpha = 2$, we can repeat the same calculations as in the case n odd to conclude that $(0; +; [m_1, m_2, m_3]; \{(-)\})$ is the only maximal NEC signature which satisfies these conditions. On the other hand, if $\alpha = 1$, then we have $\gamma = k = r = 1$, and this gives $(1; -; [m]; \{(-)\})$, which is a non-maximal NEC signature.

Finally, if $d = 1$ then, since $1 - 1/m_i \geq 1/2$ for all i we get

$$\alpha\gamma + k - 1 + \frac{r}{2} + \frac{r_1}{4} \leq \mu(\Gamma) < 1$$

and so $\alpha\gamma + k = 0$ or 1 . A straightforward calculation then reveals there are eight different types of NEC signatures satisfying these conditions, namely the following:

$$\begin{aligned} (0; +; [m]; \{(2, 2)\}), & \quad (0; +; [-]; \{(-), (2, 2)\}), & \quad (1; -; [-]; \{(2, 2)\}), \\ (0; +; [3, 3]; \{(2, 2)\}), & \quad (0; +; [3, 5]; \{(2, 2)\}), & \quad (0; +; [m]; \{(2, 2, 2, 2)\}), \\ (0; +; [3, 4]; \{(2, 2)\}), & \quad (0; +; [2, m]; \{(2, 2)\}). \end{aligned}$$

The three NEC signatures in the first row are non-maximal, and so Theorem 2.1 rules them out. Also if the signature of Γ is one of the two occurring in the last row, then there exists no smooth epimorphism $\theta: \Gamma \rightarrow C_n$ with orientable kernel, by part of the proof of Theorem 3.3. The same happens if Γ has signature $(0; +; [m]; \{(2, 2, 2, 2)\})$ with m even; on the other hand, if m is odd, then $n = 2m$ since C_n is generated by elements of order m and 2 , and then there does exist a smooth epimorphism with orientable kernel.

If $\sigma(\Gamma) = (0; +; [3, 3]; \{(2, 2)\})$ or $(0; +; [3, 5]; \{(2, 2)\})$ then $n = 6$ or 30 respectively, and there do exist smooth epimorphisms with orientable kernel in both cases. The full real orientable genus of C_6 , respectively C_{30} , however, cannot be attained by an NEC group with signature $(0; +; [3, 3]; \{(2, 2)\})$, respectively $(0; +; [3, 5]; \{(2, 2)\})$, since its area is larger than the area of $(0; +; [3]; \{(2, 2, 2, 2)\})$, respectively $(0; +; [15]; \{(2, 2, 2, 2)\})$. This proves the lemma for even values of n . \square

Conditions on the signature $\sigma(\Gamma)$ of an NEC group Γ that are necessary and sufficient for it to admit a smooth epimorphism $\theta: \Gamma \rightarrow C_n$, with $\ker \theta$ orientable, can be obtained from Section 3.1 in [7]. For signature type $(0; +; [m_1, m_2, m_3]; \{(-)\})$, one requires only that $\text{lcm}(m_1, m_2, m_3) = n$. This condition is necessary since C_n has to be generated by elements $\theta(x_1)$, $\theta(x_2)$ and $\theta(x_3)$ of orders m_1 , m_2 and m_3 , respectively; and conversely, if $\text{lcm}(m_1, m_2, m_3) = n$ then by setting $\theta(x_i) = v^{n/m_i}$ for $i = 1, 2, 3$ and $\theta(c) = 1$, we get a smooth epimorphism $\theta: \Gamma \rightarrow C_n = \langle v \rangle$ with $\ker \theta$ orientable. For signature $(0; +; [n/2]; \{(2, 2, 2, 2)\})$, no extra condition is required as long as $n/2$ is odd, as observed in the proof of Lemma 4.1.

Hence in order to find the full real orientable genus of C_n , we have to minimise the reduced area $\mu(\sigma) = 2 - (1/m_1 + 1/m_2 + 1/m_3)$ among all NEC signatures of the form $(0; +; [m_1, m_2, m_3]; \{(-)\})$ under the condition $\text{lcm}(m_1, m_2, m_3) = n$. This was carried out in Lemma 4.2 in [4], and for the reader's convenience, we restate the result here, as follows.

Lemma 4.2. *Let $n = p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$ be the prime-power decomposition of n , such that $p_1 < p_2 < \dots < p_s$. Then the maximum value of $1/m_1 + 1/m_2 + 1/m_3$ over all triples (m_1, m_2, m_3) such that $2 \leq m_1 \leq m_2 \leq m_3$ and $\text{lcm}(m_1, m_2, m_3) = n$ is attained by:*

- (a) (p_1, p_2, p_3) , when n is of the form $p_1 p_2 p_3$ with $3 < p_1 < p_2 < p_3 < \frac{p_1(p_2-1)}{p_2-p_1}$,
- (b) $(p_1, p_1, n/p_1)$, when $s > 1$ and $e_1 = 1$ and n is not of the form in (a), and
- (c) (p_1, p_1, n) otherwise.

If $n/2$ is odd then the minimum reduced area is attained by either signature $(0; +; [2, 2, n/2]; \{(-)\})$ (see case (b) above) or $(0; +; [n/2]; \{(2, 2, 2, 2)\})$ (see

Lemma 4.1) since both have the same reduced area. Using Lemmas 4.1, 4.2 and the Riemann–Hurwitz formula, we find that the full real orientable genus of C_n is as given in Theorem 4.3, for all $n \geq 2$.

Theorem 4.3. *Let $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ be the prime-power decomposition of n , such that $p_1 < p_2 < \cdots < p_s$. Then the full real orientable genus of the cyclic group C_n is*

- (a) $\frac{2p_1 p_2 p_3 - p_1 p_2 - p_1 p_3 - p_2 p_3 + 1}{p_2 - p_1}$, when $n = p_1 p_2 p_3$ with $3 < p_1 < p_2 < p_3 <$
- (b) $2n - 2n/p_1 - p_1 + 1$ when $s > 1$ and $e_1 = 1$ and n is not of the form in (a), and
- (c) $2n - 2n/p_1$ otherwise.

Remark 4.4. Note that when n is even and $n > 2$, the full real orientable genus of C_n is $n - 1$ if $n \equiv 2 \pmod{4}$, and n if $n \equiv 0 \pmod{4}$. For these values of n , in Theorem 4.7 we will see that the full real orientable genus is equal to the full real non-orientable genus.

The number of boundary components (and hence the topological type) of the surfaces attaining the full real orientable genus is almost completely determined by the signature with which C_n acts, as the proof of the next proposition shows.

Proposition 4.5. *The number of boundary components of a surface S attaining the full real orientable genus of a cyclic group of order n is either 1, p or $g + 1$, where p is the smallest prime divisor of n .*

Proof. If n is as in part (a) of Theorem 4.3, then $\sigma(\Gamma) = (0; +; [p_1, p_2, p_3]; \{(-)\})$, assuming the same notation as in Lemma 4.1. In this case, the number of boundary components of S is $k = n/m$, where m is the order of $\theta(e) = \theta(x_1 x_2 x_3)^{-1}$; see Theorem 2.3.1 in [7]. It is clear that $\theta(x_1 x_2 x_3)$ has order $p_1 p_2 p_3 = n$, and so $k = 1$ for any such n .

If n is as in part (b) of Theorem 4.3, then $\sigma(\Gamma) = (0; +; [n/2]; \{(2, 2, 2, 2)\})$ with $n/2$ odd, or $\sigma(\Gamma) = (0; +; [p_1, p_1, n/p_1]; \{(-)\})$. In the first case, Lemma 3.2 implies that $k = n$, and then since for $n/2$ odd, the full real orientable genus g is $n - 1$, we have $k = g + 1$. In the second case, $k = n/m$ where m is the order of $\theta(x_1 x_2 x_3)$; in this case, if $\theta(x_1 x_2)$ is trivial then $m = n/p_1$ and so $k = p_1$, while otherwise $\theta(x_1 x_2)$ has order p_1 and then since $\gcd(p_1, n/p_1) = 1$ the order of $\theta(x_1 x_2 x_3)$ is n , and so $k = 1$.

Finally, if n is as in part (c) of Theorem 4.3, then $\sigma(\Gamma) = (0; +; [p_1, p_1, n]; \{(-)\})$. Writing $v = \theta(x_3)$ we have $\theta(x_1 x_2 x_3) = v^{\frac{n}{p}(\alpha_1 + \alpha_2) + 1}$ for some α_1 and α_2 , both coprime with p_1 . This element has order $m = n/d$ where $d = \gcd(\frac{n}{p}(\alpha_1 + \alpha_2) + 1, n)$. Now if p^2 divides n , then p divides n/p and so $d = 1$, which gives $m = n$ and $k = n/m = 1$; otherwise $n = p$ and then $d = 1$ or p , so $k = n/m = d = 1$ or p as well. □

4.2. The full real non-orientable genus of a cyclic group

Lemma 4.6. *For $n > 2$, if the full real non-orientable genus of C_n is attained by means of a smooth epimorphism $\theta: \Gamma \rightarrow C_n$, then the signature of Γ is of the form*

$$(1; -; [m_1, m_2]; \{(-)\})$$

whenever n is odd, and

$$(0; +; [2, m]; \{(2, 2)\}), \quad (0; +; [n]; \{(2, 2, 2, 2)\}) \quad \text{or} \quad (0; +; [3, 4]; \{(2, 2)\})$$

whenever n is even, with the last of these signatures occurring only when $n = 12$.

Proof. For n odd, we choose a maximal NEC group Δ with signature $(1; -; [p, p]; \{(-)\})$, where p is the smallest prime divisor of n , and define $\theta: \Delta \rightarrow C_n = \langle v \rangle$ by setting

$$\theta(d) = v, \quad \theta(x_1) = \theta(x_2) = v^{n/p}, \quad \theta(e) = v^{-2-2n/p}, \quad \theta(c) = 1.$$

For n even, we let Δ be a maximal NEC group with signature $(0; +; [2, n]; \{(2, 2)\})$, and take

$$\theta(x_1) = v^{n/2}, \quad \theta(x_2) = v, \quad \theta(e) = v^{-n/2-1}, \quad \theta(c_0) = \theta(c_2) = v^{n/2}, \quad \theta(c_1) = 1.$$

Observe that each surface $H/\ker \theta$ is bordered (since $\ker \theta$ contains one canonical reflection of Δ), and also is non-orientable (since $\ker \theta$ contains a non-orientable word $w \in \Delta$, namely $w = d^{n/p}x_1^{-1}$ for n odd, and $w = x_1c_0$ for n even; see Theorem 2.1.3 in [7]). Accordingly,

$$(4.2) \quad \mu(\Gamma) \leq \mu(\Delta) = \begin{cases} 2 - 2/p & \text{if } n \text{ is odd;} \\ 1 - 1/n & \text{if } n \text{ is even.} \end{cases}$$

Next we look for maximal NEC groups Γ uniformising an action of C_n as the full group of a non-orientable bordered surface, such that the reduced area $\mu(\Gamma)$ satisfies the corresponding inequality in (4.2). Lemma 1.1 implies that Γ has signature

$$\sigma = \sigma(\Gamma) = (\gamma; \pm; [m_1, \dots, m_r]; \{(-), \overset{k}{\cdot}, (-), (2, r_1, 2), \dots, (2, r_d, 2)\})$$

where r_i is even for all i , and $d = 0$ if n is odd, and $k + d \geq 1$.

For n odd, we know that $\ker \theta$ and Γ have the same sign; see Theorem 2.1.2 in [7]. It is now easy to see (by the arithmetic arguments made in Lemma 4.1) that $\gamma + k = 2$ or 3 . A straightforward calculation shows there are four different types of NEC signatures that satisfy these conditions, namely the following:

$$(1; -; [m_1, m_2]; \{(-)\}), \quad (1; -; [m]; \{(-)\}), \quad (1; -; [-]; \{(-), (-)\}), \quad (2; -; [-]; \{(-)\}).$$

All but the first one of these are non-maximal NEC signatures, and so this proves the lemma for the case where n is odd.

For n even, we have $d = 0$ or 1 , because if $d \geq 2$ then we have $\mu(\Gamma) \geq 1$; see the proof of Lemma 4.1. If $d = 0$ then $\alpha\gamma + k = 1$ or 2 . If $\alpha\gamma + k = 1$, we find that $\sigma(\Gamma) = (0; +; [m_1, \dots, m_r]; \{(-)\})$, but there is no smooth epimorphism from an NEC group Γ with this signature that has non-orientable kernel. Hence $\alpha\gamma + k = 2$. This implies that $r = 1$, and so $\sigma(\Gamma) = (0; +; [m]; \{(-), (-)\})$ or $(1; -; [m]; \{(-)\})$; but these are both non-maximal NEC signatures, and so Theorem 2.1 rules them out. For $d = 1$, the same argument as used in the proof of Lemma 4.1 shows that the only maximal signatures that can occur in this case are

$$(0; +; [3, 3]; \{(2, 2)\}), \quad (0; +; [3, 5]; \{(2, 2)\}), \quad (0; +; [m]; \{(2, 2, 2, 2)\}), \\ (0; +; [2, m]; \{(2, 2)\}), \quad (0; +; [3, 4]; \{(2, 2)\}).$$

If $\sigma(\Gamma) = (0; +; [3, 3]; \{(2, 2)\})$ or $(0; +; [3, 5]; \{(2, 2)\})$, then by the proof of Theorem 3.3, any smooth epimorphism $\theta : \Gamma \rightarrow C_n$ has orientable kernel, so we may discard these two cases. The same happens to $(0; +; [m]; \{(2, 2, 2, 2)\})$ if m is odd; on the other hand, if m is even (and then $m = n$ because C_n is generated by elements of orders m and 2), then there does exist a smooth epimorphism with non-orientable kernel. If $\sigma(\Gamma) = (0; +; [2, m]; \{(2, 2)\})$, then $n = m$ if m is even while $n = 2m$ if m is odd, and if $\sigma(\Gamma) = (0; +; [3, 4]; \{(2, 2)\})$ then $n = 12$; it is also easy to see that there do exist smooth epimorphisms with non-orientable kernel, for each of these two signatures. Note that the full real non-orientable genus of C_{12} can be attained by an NEC group with signature $(0; +; [3, 4]; \{(2, 2)\})$, $(0; +; [12]; \{(2, 2, 2, 2)\})$ or $(0; +; [2, 12]; \{(2, 2)\})$ since all three of these signatures have the same area. □

Conditions for an NEC group Γ with given signature to admit a smooth epimorphism $\theta : \Gamma \rightarrow C_n$ with $\ker \theta$ non-orientable are obtainable from Section 3.1 in [7]. For signature type $(1; -; [m_1, m_2]; \{(-)\})$ with n odd, one requires only that n is divisible by each of m_1 and m_2 . This condition is clearly necessary, and is also sufficient, because if we set $\theta(d) = v$, $\theta(x_1) = v^{n/m_1}$, $\theta(x_2) = v^{n/m_2}$ and $\theta(c) = 1$, then we get a smooth epimorphism $\theta : \Gamma \rightarrow C_n = \langle v \rangle$, with $\ker \theta$ non-orientable (since it contains the non-orientable word $x_1 d^{-n/m_1}$). Clearly, the minimum reduced area of all such signatures is $2 - 2/p$, attained by $(1; -; [p, p]; \{(-)\})$ where p is the smallest prime divisor of n . For signatures $(0; +; [2, n/2]; \{(2, 2)\})$, $(0; +; [2, n]; \{(2, 2)\})$ and $(0; +; [n]; \{(2, 2, 2, 2)\})$, no extra condition is required, provided that for the first one, n is even with $n/2$ odd. The reduced area for the first one is $1 - 2/n$, while the areas for the last two are both $1 - 1/n$. This latter is also the reduced area of $(0; +; [3, 4]; \{(2, 2)\})$, which is valid for $n = 12$.

All of this, when together with the Riemann–Hurwitz formula, gives the following.

Theorem 4.7. *The full real non-orientable genus of the cyclic group C_n is*

- (a) $2n - 2n/p + 1$ where p is the smallest prime divisor of n , if n is odd;
- (b) n if $n \equiv 0 \pmod{4}$;
- (c) $n - 1$ if $n \equiv 2 \pmod{4}$.

Next, we determine the number of boundary components (and hence the topological type) of the surfaces attaining the full real non-orientable genus. This number k is completely determined by n if $n \equiv 2 \pmod{4}$ or n is odd, as the proof of the next proposition shows.

Proposition 4.8. *The number of boundary components of a surface S attaining the full real non-orientable genus g of the cyclic group C_n is $g/2$, $(g + 1)/2$ or g when n is even, and is 1 or p , where p is the smallest prime divisor of n , when n is odd.*

Proof. If $n \equiv 2 \pmod{4}$ then $\sigma(\Gamma) = (0; +; [2, n/2]; \{(2, 2)\})$, where we are using the same notation as in Lemma 4.1. Hence the number of boundary components of S is $k = n/2$; see Lemma 3.2. Since for these values of n the full real non-orientable genus g is $n - 1$, we find that $k = n/2 = (g + 1)/2$.

Next, if $n \equiv 0 \pmod{4}$, then $\sigma(\Gamma) = (0; +; [2, n]; \{(2, 2)\})$, $(0; +; [n]; \{(2, 2, 2, 2)\})$ or $(0; +; [3, 4]; \{(2, 2)\})$, with the last of these occurring only when $n = 12$. The number of boundary components is $k = n/2$ for the first and the last signatures, and $k = n$ for the second. Thus $k = g/2$ or $k = g$ (because $g = n$ for these values of n).

We assume from now on that n is odd. Then $\sigma(\Gamma) = (1; -; [p, p]; \{(-)\})$ and in this case $k = n/m$ where m is the order of $\theta(e) = \theta(d_1^2 x_1 x_2)^{-1}$. Observe that C_n is generated by $\theta(d_1)$, $\theta(x_1)$ and $\theta(x_2)$, of orders r (say), p and p . It follows that $r = n$ or $r = n/p$, with the latter occurring only if p^2 does not divide n .

If p^2 divides n , then p^2 must divide r and so $r = n$. Then taking $v = \theta(d_1)$, we have $\theta(e) = \theta(d_1^2 x_1 x_2) = v^{2 + \frac{n}{p}(\alpha_1 + \alpha_2)}$ for some α_1 and α_2 , both coprime with p . This element has order n/d , where $d = \gcd(2 + \frac{n}{p}(\alpha_1 + \alpha_2), n)$, which is 1 since n is odd and p divides n/p . Hence $m = n$ and $k = 1$ in this case.

Similarly, if p^2 does not divide n , but $r = n$, then $\theta(e) = \theta(d_1^2 x_1 x_2)$ has order n/d where $d = \gcd(2 + \frac{n}{p}(\alpha_1 + \alpha_2), n)$, and this must be 1 or p . Both possibilities can occur, for instance by taking $\alpha_1 = 1 = -\alpha_2$ to get $d = 1$, and using the fact that p and n/p are coprime to choose $\alpha_1 + \alpha_2$ (and then α_1 and α_2) such that $2 + \frac{n}{p}(\alpha_1 + \alpha_2)$ is divisible by p , to get $d = p$. Thus $m = n$ or n/p , and $k = 1$ or p .

Finally, if p^2 does not divide n , and $r = n/p$, then the orders of $\theta(d_1^2)$ and $\theta(x_1 x_2)$ are n/p and either 1 or p , and hence are coprime, so the order of $\theta(e) = \theta(d_1^2 x_1 x_2)$ is either n/p or n . Thus $k = n/m = p$ or 1 in this case too. \square

As noted in Remark 4.4, the full real orientable genus and the full real non-orientable genus coincide for even values of n . For odd values of n , it is easy to see that the full real non-orientable genus $2n - 2n/p_1 + 1$ is larger than the corresponding quantities occurring in parts (a), (b) and (c) of Theorem 4.3. As a consequence, we have the following.

Corollary 4.9. *The full real genus of the cyclic group C_n is equal to its full real orientable genus, as given in Theorem 4.3.*

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