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# Lattice points in rotated convex domains

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**Abstract.** If $\mathcal{B} \subset \mathbb{R}^d$ ($d \geqslant 2$) is a compact convex domain with a smooth boundary of finite type, we prove that for almost every rotation $\theta \in SO(d)$ the remainder of the lattice point problem, $P_{\theta\mathcal{B}}(t)$, is of order $O_\theta(t^{d-2+2/(d+1)-\zeta_d})$ with a positive number $\zeta_d$. Furthermore we extend the estimate of the above type, in the planar case, to general compact convex domains.

## 1. Introduction

Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geqslant 2$) be a compact convex domain, which contains the origin in its interior and has a smooth boundary $\partial\mathcal{B}$. The number of lattice points $\mathbb{Z}^d$ in the dilated domain $t\mathcal{B}$ is approximately $|t\mathcal{B}|$ (i.e., the volume (area if $d = 2$) of $t\mathcal{B}$) and the lattice point problem is to study the remainder, $P_\mathcal{B}(t)$, in the equation

$$P_\mathcal{B}(t) = \#(t\mathcal{B} \cap \mathbb{Z}^d) - |\mathcal{B}|t^d \quad \text{for } t \geqslant 1.$$

A trivial estimate gives $P_\mathcal{B}(t) = O(t^{d-1})$.

If $\partial\mathcal{B}$ has everywhere positive (Gaussian) curvature, a standard estimate is

$$P_\mathcal{B}(t) = O(t^{d-2+2/(d+1)}),$$

which can be readily obtained by a combination of the Poisson summation formula and (nowadays standard) oscillatory integral estimates (see Hlawka [6]). Over the years this result has been improved by many authors and the best bounds up-to-date are due to Huxley [8] in the planar case and the author [4] in the higher dimensional case. For a survey on historical results the reader is referred to Ivić, Krätzel, Kühleitner, and Nowak [11].

While the above case is relatively well understood, the general case when the (Gaussian) curvature is allowed to vanish is not.

Let us first consider the $d \geqslant 3$ case of vanishing curvature. Partial results indicate that the remainder may become much larger. For example, Randol [24]

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considered the super spheres

$$\mathcal{B} = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : |x_1|^\omega + |x_2|^\omega + \cdots + |x_d|^\omega \leqslant 1\right\}$$

for even integer $\omega \geqslant 3$, and proved that

$$P_{\mathcal{B}}(t) = \begin{cases} O(t^{d-2+2/(d+1)}) & \text{for } \omega \leqslant d+1, \\ O(t^{(d-1)(1-1/\omega)}) & \text{for } \omega > d+1, \end{cases}$$

and this estimate is the best possible when $\omega > d+1$. Krätzel [12] extended this result to odd $\omega \geqslant 3$ and gave an asymptotic formula

$$P_{\mathcal{B}}(t) = H(t)t^{(d-1)(1-1/\omega)} + O(t^{\Theta}) \tag{1.1}$$

with an explicit $\Theta < (d-1)(1-1/\omega)$ and $H(t)$ continuous and periodic (see Krätzel [13] for more details). We observe that the remainder $P_{\mathcal{B}}(t)$ becomes extremely large as $\omega \to \infty$.

This observation is supported by the study of more examples, and special attention is paid to specific convex domains in $\mathbb{R}^3$. See Krätzel [16] and Krätzel and Nowak [17], [18], in which they proved, among other results, asymptotic formulas of $P_{\mathcal{B}}(t)$ with explicit representations of the main terms given.

For general domains with boundary points of Gaussian curvature zero, our knowledge is still very poor. Partial results in $\mathbb{R}^3$ are available in Krätzel [14], [15], Peter [22], Popov [23], and Nowak [21] (with the latter two papers focusing on bodies of rotation). Under a variety of assumptions, they provide $O$-estimates (or asymptotic formulas) of $P_{\mathcal{B}}(t)$, and evaluate the contributions (to $P_{\mathcal{B}}(t)$) of different types of boundary points of Gaussian curvature zero. Their results show that the size of $P_{\mathcal{B}}(t)$ depends on certain properties of the boundary points of Gaussian curvature zero and whether the slope of the normal at such a point is rational or irrational. In particular $P_{\mathcal{B}}(t)$ may become extremely large and a substantial contribution to it is due to the neighborhoods of those boundary points of Gaussian curvature zero at which the normal has a rational direction.

However after a rotation of the domain there may be no such points, hence we can expect a better estimate. For example one may consider rotations of a compact convex domain $\mathcal{B}$ with a smooth boundary of finite type (Here we say that the boundary $\partial\mathcal{B}$ is *of finite type* if at every point $x \in \partial\mathcal{B}$, every one dimensional tangent line to $\partial\mathcal{B}$ at $x$ makes finite order of contact with $\partial\mathcal{B}$. If $\partial\mathcal{B}$ is of finite type, the maximum order of contact over all $x \in \partial\mathcal{B}$ and all tangent lines to $x \in \partial\mathcal{B}$ is called the *type* of $\partial\mathcal{B}$. We will always assume below that the type is $\geqslant 3$ since if the type is two then we recover the case of nonvanishing (Gaussian) curvature). For such domains Iosevich, Sawyer, and Seeger [10] proved that there is $r > 2$ so that

$$P_{\mathcal{B}_\theta}(t) = O_\theta\left(t^{d-2+2/(d+1)} \log^{1/r}(2+t)\right) \quad \text{for a.e. } \theta \in SO(d), \tag{1.2}$$

where $\mathcal{B}_\theta = \theta\mathcal{B}$ denotes the rotated domain $\{\theta x : x \in \mathcal{B}\}$. Results of type (1.2) with the same exponent $d-2+2/(d+1)$ can be found in Randol [25] for convex domains with an analytic boundary, and in Colin de Verdière [3] for general (not necessarily convex) domains if $d \leqslant 7$.

It is then natural to ask whether one can prove a result of type (1.2) with an exponent $d-2+2/(d+1)-c$ for some positive $c$. We make a progress in this direction and prove the following theorem with a $c>0$ depending only on the dimension $d$.

**Theorem 1.1.** *Let $\mathcal{B}\subset\mathbb{R}^d$ $(d\geqslant 3)$ be a compact convex domain containing the origin in its interior. If the boundary is a smooth hypersurface of finite type then*

$$P_{\mathcal{B}_\theta}(t)=O_\theta(t^{d-2+2/(d+1)-\zeta_d})\quad \text{for a.e. } \theta\in SO(d),$$

*where $\zeta_d>0$ is defined as*

$$\zeta_d=\begin{cases}\dfrac{2(d-2)(d-1)d}{(d+1)(6d^5+118d^4+109d^3-210d^2-119d+82)} & \text{for } 3\leqslant d\leqslant 4,\\[2ex] \dfrac{(d-3)(d-1)d}{2d^6+49d^5+123d^4-9d^3-167d^2-52d+30} & \text{for } d\geqslant 5.\end{cases}\tag{1.3}$$

This result is an easy consequence of the following theorem.

**Theorem 1.2.** *Let $\mathcal{B}\subset\mathbb{R}^d$ $(d\geqslant 3)$ be a compact convex domain containing the origin in its interior. If the boundary is a smooth hypersurface of finite type $\omega$ then*

$$\sup_{t\geqslant 2}|P_{\mathcal{B}_\theta}(t)|/\big(t^{d-2+2/(d+1)-\zeta_d-\sigma(d,\omega)}\log^b(t)\big)\in L^1(SO(d)),$$

*where $b>1$, $\zeta_d$ is given by (1.3), and $\sigma(d,\omega)>0$ is defined as*

$$\sigma(d,\omega)=\begin{cases}\dfrac{4d(6d^5+100d^4-230d^3-193d^2+496d-172)}{(6d^5+118d^4+109d^3-210d^2-119d+82)\cdot\square} & \text{for } 3\leqslant d\leqslant 4,\\[2ex] \dfrac{2d(2d^5+39d^4-105d^3-205d^2+377d-96)}{(2d^5+47d^4+76d^3-85d^2-82d+30)\cdot\triangle} & \text{for } d\geqslant 5,\end{cases}\tag{1.4}$$

*with*

$$\begin{aligned}\square={}&6(\omega-2)d^5+118(\omega-2)d^4+109(\omega-2)d^3\\&-6(35\omega-71)d^2+(246-119\omega)d+(82\omega-156)\end{aligned}\tag{1.5}$$

*and*

$$\begin{aligned}\triangle={}&2(\omega-2)d^5+47(\omega-2)d^4+76(\omega-2)d^3\\&+(172-85\omega)d^2+(166-82\omega)d+(30\omega-56).\end{aligned}\tag{1.6}$$

The proof of Theorem 1.2 relies on the following analysis result (implied by Svensson's Theorem 4.1 in [28]): if $\mathcal{B}\subset\mathbb{R}^d$ $(d\geqslant 3)$ is a compact convex domain and its boundary is a smooth hypersurface of finite type[1] $\omega$, then

$$\Phi\in L^p(S^{d-1})\quad \text{for any } p<2+2/(d-1)(\omega-2),\tag{1.7}$$

[1]The restriction on the size of $\omega$ given by Svensson's Theorem 4.1 in [28] can be removed under current assumptions.

where

$$\Phi(\xi) = \sup_{r>0} r^{(d+1)/2} |\widehat{\chi}_{\mathcal{B}}(r\xi)|, \qquad \xi \in S^{d-1}. \tag{1.8}$$

For a general convex domain $\mathcal{B}$ with a smooth boundary, (1.7) is not necessarily true, however, we have (due to Varchenko's Theorem 8 in [29]) that

$$r^{(d+1)/2} |\widehat{\chi}_{\mathcal{B}}(r\xi)| \in L^2(S^{d-1}).$$

By using this result we can readily modify the proof of Theorem 1.2 and prove the following theorem, which improves similar results contained on page 285 of Randol [25] and Varchenko's Theorem 7 in [29] in terms of the estimate.

**Theorem 1.3.** *Let $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 3)$ be a compact convex domain containing the origin in its interior. If the boundary is a smooth hypersurface, then*

$$|P_{\mathcal{B}_\theta}(t)|/t^{d-2+2/(d+1)-\zeta_d} \in L^1(SO(d)),$$

*where $\zeta_d$ is given by (1.3).*

Let us now consider the $d = 2$ case of vanishing curvature, in which we have a better understanding than in the higher dimensional case. We refer the interested readers to Ivić, Krätzel, Kühleitner, and Nowak [11] and the author [5] for an introduction to related results.

For general convex planar domains we know $\Phi \in L^{2,\infty}(S^1)$ (see Brandolini, Colzani, Iosevich, Podkorytov, and Travaglini's Theorem 0.3 in [2]). By using this result and the same method used in the proofs of Theorem 1.2 and 1.3, we are able to extend our previous result for convex planar domains of finite type in Theorem 1.1 in [5] to the following result for convex planar domains with no curvature assumption on the boundary (with even a better estimate, due to an improved estimate of certain nonvanishing determinants given in Lemma 3.5 below).

**Theorem 1.4.** *If $\mathcal{B}$ is a compact convex planar domain with a smooth boundary containing the origin in its interior, then*

$$\sup_{t\geqslant 2} |P_{\mathcal{B}_\theta}(t)|/\big(t^{2/3-\zeta_2}\log^b(t)\big) \in L^1(SO(2)),$$

*where $b > 1$ and $\zeta_2 = 1/2859$. In particular,*

$$P_{\mathcal{B}_\theta}(t) = O_\theta\big(t^{2/3-\zeta_2}\log^b(t)\big) \quad \textit{for a.e. } \theta \in SO(2).$$

This theorem improves Iosevich's Theorem 0.2 in [9] and Brandolini, Colzani, Iosevich, Podkorytov, and Travaglini's Theorem 0.1 in [2] (in terms of the estimate). If we assume that the boundary is of finite type then we have the following better estimate (again due to the improved result given in Lemma 3.5 below), which improves Theorem 1.2 in [5].

**Theorem 1.5.** *If $\mathcal{B}$ is a compact convex planar domain with a smooth boundary of finite type $\omega$ containing the origin in its interior, then*

$$\sup_{t\geqslant 2} |P_{\mathcal{B}_\theta}(t)|/\left(t^{2/3-\zeta_2-\sigma(2,\omega)}\log^b(t)\right) \in L^1(SO(2)),$$

*where $b>1$, $\zeta_2 = 1/2859$, and*

$$\sigma(2,\omega) = \frac{616}{953(953\omega-1848)}.$$

*In particular,*

$$P_{\mathcal{B}_\theta}(t) = O_\theta(t^{2/3-\zeta_2}) \quad \textit{for a.e. } \theta\in SO(2).$$

**Remark 1.6.** Our main idea originates from Iosevich, Sawyer, and Seeger [10] (see pp. 168-169) and Müller [20]. Our main tools used in this paper are from the oscillatory integral theory and the classical Van der Corput's method of exponential sums (namely, the $A$- and $B$-processes). To prove our estimate of exponential sums (see Proposition 5.1 below) we use an $A^qB$-process. If we use more $A$- and $B$-processes we may achieve further improvement at the cost of more technical difficulties.

*Notations.* We use the usual Euclidean norm $|x|$ for a point $x\in\mathbb{R}^d$. $B(x,r)$ represents the Euclidean ball centered at $x$ with radius $r$, and its dimension will be clear from the context. The norm of a matrix $A\in\mathbb{R}^{d\times d}$ is given by $\|A\| = \sup_{|x|=1}|Ax|$. We set $e(f(x)) = \exp(2\pi i f(x))$, $\mathbb{Z}^d_* = \mathbb{Z}^d\setminus\{0\}$, and $\mathbb{R}^d_* = \mathbb{R}^d\setminus\{0\}$. The Fourier transform of $f\in L^1(\mathbb{R}^d)$ is given by $\widehat{f}(\xi) = \int f(x)e(-\langle x,\xi\rangle)\,dx$.

We fix $\chi_0$ to be a smooth cut-off function whose value is 1 on $B(0,1/2)$ and 0 on the complement of $B(0,1)$. For a set $E\subset\mathbb{R}^d$ and a positive number $a$, we define $E_{(a)}$ to be the larger set

$$E_{(a)} = \left\{x\in\mathbb{R}^d : \mathrm{dist}(E,x) < a\right\}.$$

We use the differential operators

$$D^\nu_x = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}\cdots\partial x_d^{\nu_d}} \quad \left(\nu = (\nu_1,\ldots,\nu_d)\in\mathbb{N}_0^d,\ |\nu| = \sum_{i=1}^d \nu_i\right)$$

and the gradient operator $\nabla_x$. We often omit the subscript if no ambiguity occurs.

*Structure of the paper.* We first establish some preliminaries in §2-4 mainly for compact convex domains with no curvature assumption on the boundary. We then prove an estimate of exponential sums in §5, which will be needed in the next section. In §6 we give a proof of Theorem 1.2, in which the problem is reduced to the estimate of two sums (Sum I and II). The estimate of Sum I that we give essentially works for general compact convex domains, while the curvature condition on the boundary is used in the estimate of Sum II. Since it is easy to modify the proof of Theorem 1.2 to prove the other theorems, we only provide brief proofs of Theorem 1.3 in §6 and Theorem 1.4 and 1.5 in §7. At last we collect some standard analysis results in Appendix A.

## 2. Some geometric facts

Assume $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a compact convex domain and its boundary is a smooth hypersurface. For a point $x \in \partial\mathcal{B}$, let $K(x)$ be the (Gaussian) curvature of $\partial\mathcal{B}$ at $x$. Define

$$(\partial\mathcal{B})_+ = \big\{x \in \partial\mathcal{B} : K(x) > 0\big\} \quad \text{and} \quad (\partial\mathcal{B})_0 = \big\{x \in \partial\mathcal{B} : K(x) = 0\big\},$$

thus

$$\partial\mathcal{B} = (\partial\mathcal{B})_+ \biguplus (\partial\mathcal{B})_0.$$

The Gauss map of $\partial\mathcal{B}$, denoted by $\vec{n}$, maps each boundary point $x \in \partial\mathcal{B}$ to a unit exterior normal $\vec{n}(x) \in S^{d-1}$. Define

$$S_+^{d-1} = \vec{n}((\partial\mathcal{B})_+) \quad \text{and} \quad S_0^{d-1} = \vec{n}((\partial\mathcal{B})_0),$$

thus

$$S^{d-1} = S_+^{d-1} \biguplus S_0^{d-1}.$$

Note that the restriction of $\vec{n}$ to $(\partial\mathcal{B})_+$, namely

$$\vec{n}|(\partial\mathcal{B})_+ : (\partial\mathcal{B})_+ \longrightarrow S_+^{d-1} \subset S^{d-1},$$

is bijective. For $\xi \neq 0$ with $\xi/|\xi| \in S_+^{d-1}$ let $x(\xi) := \vec{n}^{-1}(\xi/|\xi|)$ be the unique point on $\partial\mathcal{B}$ where the unit exterior normal is $\xi/|\xi|$. Hence $K_\xi = K(x(\xi))$ is well defined for such points $\xi$.

For nonzero $\xi$ with $\xi/|\xi| \in \theta S_+^{d-1}$ let $x^\theta(\xi) = \theta x(\theta^t \xi)$ and $K_\xi^\theta = K_{\theta^t\xi}$. Then $x^\theta(\xi)$ is the unique point on $\partial\mathcal{B}_\theta$ where the exterior normal is $\xi$ and $K_\xi^\theta$ is the curvature of $\partial\mathcal{B}_\theta$ at $x^\theta(\xi)$.

**Lemma 2.1.** *Assume $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a compact convex domain and its boundary is a smooth hypersurface. Then there exists a constant $c_1 > 0$ (depending only on $\mathcal{B}$) such that, for any $\xi \in S_+^{d-1}$, if $\eta \in B(\xi, c_1(K_\xi)^2) \subset \mathbb{R}^d$ then $\eta/|\eta| \in S_+^{d-1}$ and*

$$K_\xi/2 \leqslant K_\eta \leqslant 3K_\xi/2.$$

*Proof.* For any $\xi \in S_+^{d-1}$ it follows from the mean value theorem that there exists a constant $c$ (depending only on $\mathcal{B}$) such that

$$K_\xi/2 \leqslant K(y) \leqslant 3K_\xi/2 \qquad \text{if} \quad y \in B(x(\xi), cK_\xi) \cap \partial\mathcal{B}.$$

It is a consequence of Lemma A.1 that the Gauss map is bijective from a subset of $B(x(\xi), cK_\xi) \cap \partial\mathcal{B}$ onto a subset of $S^{d-1}$ containing $B(\xi, c'(K_\xi)^2) \cap S^{d-1}$ where the constant $c'$ depends only on $\mathcal{B}$. Then the lemma follows easily. □

**Lemma 2.2.** *Assume $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a compact convex domain and its boundary is a smooth hypersurface. Then*

$$\big|\,\vec{n}\big(\big\{x \in \partial\mathcal{B} : K(x) < \delta\big\}\big)\big| \leqslant C_\mathcal{B}\,\delta\,\big|\,\big\{x \in \partial\mathcal{B} : 0 < K(x) < \delta\big\}\big|,$$

*where the absolute value denotes the induced Lebesgue measure on $S^{d-1}$ and $\partial\mathcal{B}$.*

*Proof.* Note that

$$\{x \in \partial\mathcal{B} : K(x) < \delta\} = \{x \in \partial\mathcal{B} : 0 < K(x) < \delta\} \biguplus (\partial\mathcal{B})_0.$$

We first have

$$|S_0^{d-1}| = 0$$

due to Sard's theorem (see p. 286 of Lang [19]). Hence it suffices to prove

$$\big|\,\vec{n}\big(\{x \in \partial\mathcal{B} : 0 < K(x) < \delta\}\big)\big| \leqslant C_{\mathcal{B}}\,\delta\,\big|\,\{x \in \partial\mathcal{B} : 0 < K(x) < \delta\}\big|.$$

By using a standard technique found in the proof of certain covering lemma of Vitali type (see Stein [27]), we reduce the above estimate to

$$|\vec{n}(B)| \leqslant C_{\mathcal{B}}\delta|B|,$$

where $B \subset \{x \in \partial\mathcal{B} : 0 < K(x) < \delta\}$ is a ball in $\partial\mathcal{B}$. However this last estimate follows from the equality $d\sigma = K(x)dA$ where $dA$ is the volume element of $\partial\mathcal{B}$ at the point $x \in \partial\mathcal{B}$ and $d\sigma$ the volume element of $S^{d-1}$ at the point $\vec{n}(x) \in S^{d-1}$ (see p. 47 of [1]; this equality can also be verified by using local coordinate charts). This finishes the proof. □

**Lemma 2.3.** *Assume $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a compact convex domain and its boundary is a smooth hypersurface of finite type $\omega$. Then*

$$\big|\,\{x \in \partial\mathcal{B} : K(x) < \delta\}\big| \leqslant C_{\mathcal{B}}\,\delta^{1/(d-1)(\omega-2)}.$$

*Proof.* By using a compactness argument and local coordinates we may only regard $K$ as a function of $x'$ in a neighborhood $B(0, C_0)$ of 0 in $\mathbb{R}^{d-1}$ for some constant $C_0$. We may assume that $K$, $\partial K/\partial x_1, \ldots, \partial^h K/\partial x_1^h$ (with $h = (d-1)(\omega-2)$) do not vanish simultaneously (see p. 19 of Svensson [28]). We then apply Svensson's Lemma 3.3 in [28] to $K$ in $x_1$-direction, which yields

$$\big|\,\{x_1 : |x_1| \leqslant C_0, K(x') < \delta\}\big| \leqslant C_{\mathcal{B}}\,\delta^{1/h},$$

and the trivial estimate in $x_2, \ldots, x_{d-1}$-directions. Thus the desired estimate follows. □

## 3. Nonvanishing $d \times d$ determinants

In this section we always assume that $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a compact convex domain and its boundary is a smooth hypersurface.

The support function of $\mathcal{B}$ is given by $H(\xi) = \sup_{y\in\mathcal{B}}\langle \xi, y\rangle$ for any nonzero $\xi \in \mathbb{R}^d$. In particular $H(\xi) = \langle \xi, x(\xi)\rangle$ for any nonzero $\xi$ with $\xi/|\xi| \in S_+^{d-1}$. It is positively homogeneous of degree one, i.e., $H(\lambda\xi) = \lambda H(\xi)$ if $\lambda > 0$. The results in this section are mainly stated for unit vectors, but we can easily remove this restriction by using the homogeneity of $H$.

The next two lemmas can be easily proved by using local coordinates, implicit differentiation, and induction, hence we omit the proof.

**Lemma 3.1.** *$H$ is smooth at every $\xi \in S_+^{d-1}$ and satisfies*

$$D^\nu H(\xi) \lesssim 1 \qquad \text{for } \ 0 \leqslant |\nu| \leqslant 1$$

*and*

$$D^\nu H(\xi) \lesssim (K_\xi)^{3-2|\nu|} \qquad \text{for } \ |\nu| \geqslant 2,$$

*where the implicit constants may depend only on $|\nu|$ and $\mathcal{B}$.*

**Remark 3.2.** For $\theta \in SO(d)$, we will denote the support function of $\mathcal{B}_\theta$ by $H_\theta(\xi) = \sup_{y \in \mathcal{B}_\theta} \langle \xi, y \rangle$. Since $H_\theta(\xi) = H(\theta^t \xi)$, we can easily get bounds for $H_\theta$ in the same form as in the above lemma (with $S_+^{d-1}$ and $K_\xi$ replaced by $\theta S_+^{d-1}$ and $K_\xi^\theta$ respectively).

**Lemma 3.3.** *For $\xi \in S_+^{d-1}$ the eigenvalues of the matrix $\nabla^2_{\xi\xi} H(\xi)$ are $0, \beta_1, \dots, \beta_{d-1}$, where $\{\beta_j^{-1}\}_{j=1}^{d-1}$ are principle curvatures of $\partial\mathcal{B}$ at $x(\xi)$.*

Given $d$ vectors $v_1, \dots, v_d \in \mathbb{R}^d$, by writing $V = (v_1, \dots, v_d)$ we mean $V$ is the matrix in $\mathbb{R}^{d\times d}$ with column vectors $v_1, \dots, v_d$. If $y \neq 0$ we define $F_\theta(u_1, \dots, u_d) = H_\theta(y + \sum_{l=1}^d u_l v_l)$, $u_l \in \mathbb{R}$ $(l = 1, \dots, d)$. For $q \in \mathbb{N}$ let

$$h_q^\theta(y, v_1, \dots, v_d) = \det\left(g_{i,j}^\theta(y, v_1, \dots, v_d)\right)_{1 \leqslant i,j \leqslant d},$$

where

$$g_{i,j}^\theta(y, v_1, \dots, v_d) = \frac{\partial^{q+2} F_\theta}{\partial u_1 \partial u_i \partial u_j \partial u_d^{q-1}}(0).$$

The following lemma is a higher dimensional analogue of Lemma 3.4 in [5], which enables us to apply the method of stationary phase later in the estimate of certain exponential sums. We will follow Müller's method used to prove his Lemma 3 in [20].

**Lemma 3.4.** *If $d \geqslant 3$, for every $\xi \in \theta S_+^{d-1}$ there exist $d$ linearly independent vectors $v_l = v_l(\xi, \theta) \in \mathbb{Z}^d$ $(l = 1, \dots, d)$ such that*

$$\begin{aligned} &|v_1| \asymp (K_\xi^\theta)^{-d-2q-8+1/(d-1)}, \\ &|v_l| \asymp (K_\xi^\theta)^{-d-2q-5+1/(d-1)} \qquad (l = 2, \dots, d), \\ &|\det(V)| \asymp (K_\xi^\theta)^{d(-d-2q-5+1/(d-1))}, \\ &\|V^{-1}\| \lesssim (K_\xi^\theta)^{d+2q+2-1/(d-1)}, \end{aligned} \tag{3.1}$$

*where $V = (v_1, \dots, v_d)$. Furthermore there exists a constant $c_2 > 0$ (depending only on $q$ and $\mathcal{B}$) such that, for $\eta \in B(\xi, c_2 (K_\xi^\theta)^{d+2q+7-1/(d-1)})$,*

$$|h_q^\theta(\eta, v_1, \dots, v_d)| \gtrsim (K_\xi^\theta)^{(-d-2q-5+1/(d-1))d(q+2)-3d+5-1/(d-1)}. \tag{3.2}$$

*The constants implicit in (3.1) and (3.2) depend only on $q$ and $\mathcal{B}$.*

*Proof.* Let $\xi \in \theta S_+^{d-1}$ be arbitrarily fixed.

*Step* 1. Let $p_1 = \xi$. We first choose $d-1$ vectors $p_2, \ldots, p_d \in S^{d-1}$ such that $P = (p_1, \ldots, p_d) \in \mathbb{R}^{d\times d}$ is an orthogonal matrix. Let $\widetilde{H}_\theta(y) = H_\theta(Py)$. Then $\widetilde{H}_\theta$ is positively homogeneous of degree one and smooth at $e_1$. Since the matrix $\nabla^2 \widetilde{H}_\theta(e_1)$ is similar to $\nabla^2 H_\theta(\xi)$ it follows from Lemma 3.3 that the eigenvalues of $\nabla^2 \widetilde{H}_\theta(e_1)$ are 0, $\beta_1, \ldots, \beta_{d-1}$, where $\{\beta_j^{-1}\}_{j=1}^{d-1}$ are principle curvatures of $\partial\mathcal{B}_\theta$ at $x^\theta(\xi)$. Without loss of generality we assume $\beta_1 = \max_{1\leqslant j\leqslant d-1} \beta_j$, therefore $\beta_1 \geqslant (K_\xi^\theta)^{-1/(d-1)}$.

Set $A = \nabla^2 \widetilde{H}_\theta(e_1)$. $A$ is a symmetric matrix of rank $d-1$ with vanishing first row and column (due to the homogeneity of $\widetilde{H}_\theta$; see the proof of Müller's Lemma 3 in [20]). Choose a system of orthonormal eigenvectors $w'_1, \ldots, w'_{d-1}$ of $A$, whose first components vanish, such that the eigenvalue of $w'_j$ is $\beta_j$. For $\alpha > 1$ denote

$$w_l = \begin{cases} w'_1 + \alpha e_1 & \text{if} \quad l = 1, \\ w'_l & \text{if} \quad 2 \leqslant l \leqslant d-1, \\ e_1 & \text{if} \quad l = d. \end{cases}$$

Then $Aw_l = \beta_l w'_l$ $(l = 1, \ldots, d-1)$ and $w_1$ is orthogonal to $w'_l$ $(l = 2, \ldots, d-1)$. We also have $|w_1| \asymp \alpha$, $|w_l| = 1$ $(l = 2, \ldots, d)$, and $|\det(W)| = 1$ where $W = (w_1, \ldots, w_d)$. Let $v_l^* = Pw_l$. Then $|v_1^*| \asymp \alpha$, $|v_l^*| = 1$ $(l = 2, \ldots, d)$, and $|\det(V^*)| = 1$ where $V^* = (v_1^*, \ldots, v_d^*)$. We claim that if $\alpha = C_{q,\mathcal{B}}(K_\xi^\theta)^{-3}$ with a sufficiently large $C_{q,\mathcal{B}}$ then

$$\left| h_q^\theta(\xi, v_1^*, \ldots, v_d^*) \right| \gtrsim (K_\xi^\theta)^{-3d+5-1/(d-1)} \tag{3.3}$$

with $F_\theta(u_1, \ldots, u_d) = H_\theta(\xi + \sum_{l=1}^d u_l v_l^*)$.

This claim can be proved by a straightforward computation (given below). Note that $F_\theta(u_1, \ldots, u_d) = \widetilde{H}_\theta(e_1 + \sum_{l=1}^d u_l w_l)$ and we will use this formula to compute $g_{i,j}^\theta(\xi, v_1^*, \ldots, v_d^*) =: b_{i,j}^\theta(\alpha)$. If $1 \leqslant i, j \leqslant d-1$,

$$b_{i,j}^\theta(0) = (\nabla\cdot w'_1)(\nabla\cdot w'_i)(\nabla\cdot w'_j)\partial_{y_1}^{q-1}\widetilde{H}_\theta(e_1) \lesssim (K_\xi^\theta)^{-3}. \tag{3.4}$$

The last inequality is due to the homogeneity of $\widetilde{H}_\theta$ (see the proof of Müller's Lemma 3 in [20]) and Remark 3.2.

If $i = 1$, $1 \leqslant j \leqslant d-1$, then

$$b_{1,j}^\theta(\alpha) = b_{1,j}^\theta(0) + 3\alpha(-1)^q q!\beta_1\delta_{1j}, \tag{3.5}$$

where $\delta_{ij}$ is the Kronecker notation.

If $2 \leqslant i, j \leqslant d-1$, then

$$b_{i,j}^\theta(\alpha) = b_{i,j}^\theta(0) + \alpha(-1)^q q!\beta_j\delta_{ij}. \tag{3.6}$$

If $1 \leqslant i \leqslant d$, $j = d$, then

$$b_{i,d}^\theta(\alpha) = (-1)^q q!\beta_1\delta_{1i}. \tag{3.7}$$

Using formulas (3.6) and (3.7), we get

$$\begin{aligned}\big|\, h_q^\theta(\xi, v_1^*, \ldots, v_d^*)\big| &= (q!\beta_1)^2 \big|\det(b_{i,j}^\theta(\alpha))_{2\leqslant i,j\leqslant d-1}\big| \\ &= (q!\beta_1)^2 \big|\det\big(b_{i,j}^\theta(0) + \alpha(-1)^q q!\beta_j\delta_{ij}\big)_{2\leqslant i,j\leqslant d-1}\big| \\ &= \beta_1 (K_\xi^\theta)^{-3d+5}\big|\, q!^d C_{q,\mathcal{B}}^{d-2} + O(C_{q,\mathcal{B}}^{d-3})\big|,\end{aligned}$$

where we have used (3.4), $\beta_j \gtrsim 1$, and $\prod \beta_j = (K_\xi^\theta)^{-1}$ to get the last equality. Since $\beta_1 \geqslant (K_\xi^\theta)^{-1/(d-1)}$, we get (3.3) if $C_{q,\mathcal{B}}$ is sufficiently large.

*Step* 2. For any $N \in \mathbb{N}$, there exist $v_l \in \mathbb{Z}^d$ $(l = 1, \ldots, d)$ such that $|v_l^{**} - v_l^*| \leqslant \sqrt{d}/N$ where $v_l^{**} = v_l/N$. If $N \geqslant C(K_\xi^\theta)^{-3}$ then $|v_1^{**}| \asymp (K_\xi^\theta)^{-3}$, $|v_l^{**}| \asymp 1$ $(l = 2, \ldots, d)$, and $|\det(V^{**})| \asymp 1$ where $V^{**} = (v_1^{**}, \ldots, v_d^{**})$.

Assume $N$ is the smallest integer not less than $C'(K_\xi^\theta)^{-d-2q-5+1/(d-1)}$ with $C'$ chosen below and $\eta \in B(\xi, c_2 r^\theta(\xi))$ with $r^\theta(\xi) = (K_\xi^\theta)^{d+2q+7-1/(d-1)}$ and $c_2 \leqslant c_1$, where $c_1$ is the constant appearing in Lemma 2.1. By the mean value theorem, Lemma 2.1, and Remark 3.2, we get

$$\begin{aligned}&\big|\, g_{i,j}^\theta(\xi, v_1^*, \ldots, v_d^*) - g_{i,j}^\theta(\eta, v_1^{**}, \ldots, v_d^{**})\big| \\ &\qquad\lesssim \begin{cases} (K_\xi^\theta)^{-2q-10}(N^{-1} + c_2(K_\xi^\theta)^{-2} r^\theta(\xi)) & \text{if} \quad i = j = 1, \\ (K_\xi^\theta)^{-2q-7}(N^{-1} + c_2(K_\xi^\theta)^{-2} r^\theta(\xi)) & \text{if} \quad i = 1, j \geqslant 2, \\ (K_\xi^\theta)^{-2q-4}(N^{-1} + c_2(K_\xi^\theta)^{-2} r^\theta(\xi)) & \text{if} \quad i \geqslant 2, j \geqslant 2. \end{cases}\end{aligned}$$

These estimates, together with the bounds of $g_{i,j}^\theta(\xi, v_1^*, \ldots, v_d^*)$'s (given by (3.4), (3.5), (3.6), and (3.7)), lead to

$$\big|\, h_q^\theta(\xi, v_1^*, \ldots, v_d^*) - h_q^\theta(\eta, v_1^{**}, \ldots, v_d^{**})\big| \lesssim (K_\xi^\theta)^{-4d-2q}\big(N^{-1} + c_2(K_\xi^\theta)^{-2} r^\theta(\xi)\big).$$

If $C'$ is sufficiently large and $c_2$ is sufficiently small, it then follows from (3.3) that

$$\big|\, h_q^\theta(\eta, v_1^{**}, \ldots, v_d^{**})\big| \gtrsim (K_\xi^\theta)^{-3d+5-1/(d-1)}.$$

The desired bound (3.2) now follows from the equality

$$\big|\, h_q^\theta(\eta, v_1, \ldots, v_d)\big| = N^{d(q+2)}\big|\, h_q^\theta(\eta, v_1^{**}, \ldots, v_d^{**})\big|.$$

All bounds in (3.1) are easy to get. □

For $d = 2$ case Lemma 3.4 in [5] gives a similar result but in a nicer form. That lemma can be proved by using the same method. In particular, the bound $g_{11}(\xi, v_1, v_2) \lesssim (K_\xi^\theta)^{-2q-1}$ is used in its proof, but later we find a better bound of $g_{11}$, namely $g_{11}(\xi, v_1, v_2) \lesssim (K_\xi^\theta)^{-3}$ (just like the bound (3.4) in the above proof). By using the latter bound without modifying too much of the proof of Lemma 3.4 in [5], we are able to prove the following improved result, which eventually leads to our estimates in Theorem 1.4 and 1.5.

**Lemma 3.5.** *If $d = 2$, for every $\xi \in \theta S_+^{d-1}$ there exist two orthogonal vectors $v_i = v_i(\xi, \theta) \in \mathbb{Z}^2$ $(i = 1, 2)$ such that*

$$|v_1| = |v_2| \asymp (K_\xi^\theta)^{-2q-2} \quad \text{and} \quad \|V^{-1}\| \lesssim (K_\xi^\theta)^{2q+2}, \tag{3.8}$$

*where $V = (v_1, v_2)$. Furthermore there exists a constant $c_2 > 0$ (depending only on $q$ and $\mathcal{B}$) such that, for $\eta \in B(\xi, c_2(K_\xi^\theta)^{2q+4})$,*

$$\left| h_q^\theta(\eta, v_1, v_2) \right| \gtrsim (K_\xi^\theta)^{-4q^2-12q-10}. \tag{3.9}$$

*The constants implicit in (3.8) and (3.9) depend only on $q$ and $\mathcal{B}$.*

## 4. The Fourier transform of certain indicator functions

In this section we will establish an asymptotic formula of the Fourier transform of the indicator function $\chi_{\mathcal{B}}$ for convex domains $\mathcal{B}$ in $\mathbb{R}^d$, which generalizes the results in Section 4 of [5].

**Lemma 4.1.** *Assume $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a compact convex domain and its boundary is a smooth hypersurface. Let $x_\pm$ be given points on $\partial\mathcal{B}$ with $\vec{n}(x_+) = -\vec{n}(x_-)$ and $K_\pm$ the (Gaussian) curvature at them. Then there exist two positive constants $c$ and $c_3$ (both depending only on $\mathcal{B}$) such that*

$$\left| \langle \vec{n}(x), \vec{n}(x_+) \rangle \right| \leqslant 1 - cr^2(\min(K_+, K_-))^4 \tag{4.1}$$

*for every $r \leqslant c_3$ and $x \in \partial\mathcal{B} \setminus \big(B(x_+, rK_+) \cup B(x_-, rK_-)\big)$.*

*Proof.* It suffices to assume that $K_\pm \neq 0$ otherwise it is trivial. It follows from Lemma A.1 that there exists a constant $c_3 > 0$ (depending only on $\mathcal{B}$) such that, for every $r \leqslant c_3$, the Gauss map is bijective from $B(x_+, rK_+) \cap \partial\mathcal{B}$ and $B(x_-, rK_-) \cap \partial\mathcal{B}$ to two subsets of $S^{d-1}$ which contain $B(\vec{n}(x_+), c'rK_+^2) \cap S^{d-1}$ and $B(\vec{n}(x_-), c'rK_-^2) \cap S^{d-1}$ respectively where the constant $c' > 0$ depends only on $\mathcal{B}$. Then the lemma follows easily with $c = 2c'^2/\pi^2$. □

**Theorem 4.2.** *Assume $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 2)$ is a compact convex domain and its boundary is a smooth hypersurface. Let $n_l$ $(l = 1, \dots, d)$ be the $l^{th}$ component of the Gauss map of $\partial\mathcal{B}$ and $dS$ the induced Lebesgue measure on $\partial\mathcal{B}$. For any $\xi \in S_+^{d-1} \cap (-S_+^{d-1})$ we have*

$$\begin{aligned}\widehat{n_l dS}(\lambda\xi) = \big(&e((d-1)/8)\,\xi_l (K_\xi)^{-1/2} e(-\lambda H(\xi)) \\ &+ e(-(d-1)/8)(-\xi_l)(K_{-\xi})^{-1/2} e(\lambda H(-\xi))\big)\lambda^{-(d-1)/2} \\ &+ O\big(\lambda^{-(d+1)/2}\delta^{-(d+5)/2} + \lambda^{-N}\delta^{-4N}\big)\end{aligned}$$

*for $\lambda > 0$, where $H(\xi) = \sup_{y\in\mathcal{B}}\langle y, \xi\rangle$, $N \in \mathbb{N}$, and $\delta = \min(K_\xi, K_{-\xi})$. The implicit constant depends only on $N$ and $\mathcal{B}$.*

*Proof.* We will only prove the case $d \geqslant 3$ below, while the case $d = 2$ is easier and can be handled in the same way. Note that there exists a $C_0 > 0$ such that, for any $x \in \partial\mathcal{B}$, the boundary $\partial\mathcal{B}$ in a neighborhood of $x$ can be parametrized by

$$\begin{aligned} \vec{r}(u,x) &= x + \sum_{j=1}^{d-1} u_j\, \vec{t}_j(x) + h(u,x)(-\vec{n}(x)), \\ &\text{for} \quad u = (u_1, \ldots, u_{d-1}) \in B_0 = B(0, C_0) \subset \mathbb{R}^{d-1}, \end{aligned} \tag{4.2}$$

where $\{\vec{t}_j(x)\}_1^{d-1}$ is an orthonormal basis of the tangent plane of $\partial\mathcal{B}$ at $x$ (we require that the basis $\{\vec{t}_1(x), \ldots, \vec{t}_{d-1}(x), -\vec{n}(x)\}$ has the same orientation as $\{e_1, \ldots, e_d\}$) and $h(\cdot\,, x) \in C^\infty(B_0)$ such that $h(0,x) = 0$, $\nabla_u h(0,x) = 0$, and $\det \nabla^2_{uu} h(0,x) = K(x)$.

For any fixed $\xi \in S^{d-1}_+ \cap (-S^{d-1}_+)$ decompose $n_l$ as a sum

$$n_l = \psi_1 + \psi_2 + \psi_3$$

where

$$\psi_1(x,\xi) = n_l(x)\chi_0\Big(\frac{x - x(\xi)}{c_4 K_\xi}\Big) \quad \text{and} \quad \psi_2(x,\xi) = n_l(x)\chi_0\Big(\frac{x - x(-\xi)}{c_4 K_{-\xi}}\Big),$$

where $c_4 > 0$ is determined below and $\chi_0$ is the fixed cut-off function (see §1).

We first estimate $\widehat{\psi_1 dS}$ (while $\widehat{\psi_2 dS}$ is handled in the same way). Applying the parametrization (4.2) at $x(\xi)$ yields

$$\widehat{\psi_1 dS}(\lambda\xi) = e\big(-\lambda\langle \xi, x(\xi)\rangle\big) \int \tau(u,\xi)\, e\big(\lambda h(u, x(\xi))\big)\, du, \tag{4.3}$$

where $\tau(u,\xi) = \psi_1(\vec{r}(u, x(\xi)), \xi)(1 + |\nabla_u h(u, x(\xi))|^2)^{1/2}$ such that

$$\tau(\cdot\,, \xi) \in C_c^\infty(B(0, c_4 K_\xi))$$

and

$$|D_u^\nu \tau(u,\xi)| \leqslant C(c_4 K_\xi)^{-|\nu|}.$$

By a change of variable the integral in (4.3), denoted by $\Delta(\xi)$, is

$$\Delta(\xi) = K_\xi^{d-1} \int \tau(K_\xi u, \xi)\, e\big(\lambda h(K_\xi u, x(\xi))\big)\, du.$$

Applying a quantitative version of the Morse lemma (see the proof of Sogge and Stein's Lemma 2 in [26]) we can find an $\alpha_1 > 0$ and a smooth invertible mapping $u \mapsto v$ from $B(0, \alpha_1)$ to a neighborhood of the origin in $v$-space, so that $|D_u^\nu v| \leqslant C$, $|D_v^\nu u| \leqslant C$, $\det(\nabla_v u(0)) = 1$, and

$$h(K_\xi u, x(\xi)) = K_\xi^2\big(\mu_1 v_1^2 + \cdots + \mu_{d-1} v_{d-1}^2\big)/2, \quad u \in B(0, \alpha_1),$$

where $\mu_1, \ldots, \mu_{d-1}$ are the eigenvalues of the matrix $\nabla^2_{uu} h(0, x(\xi))$. Let $c_4 \leqslant \alpha_1$. Then

$$\Delta(\xi) = K_\xi^{d-1} \int \tilde{\tau}(v,\xi)\, e\big(\lambda K_\xi^2(\mu_1 v_1^2 + \cdots + \mu_{d-1} v_{d-1}^2)/2\big)\, dv,$$

where $\tilde{\tau}(v,\xi) = \tau(K_\xi u(v),\xi)\,|\det(\nabla_v u)|$. Applying Lemma A.3 to the integral above yields an asymptotic expansion, which in turn gives

$$\widehat{\psi_1 dS}(\lambda\xi) = e((d-1)/8)\,\xi_l(K_\xi)^{-1/2}\,e(-\lambda H(\xi))\lambda^{-(d-1)/2} + O\big(\lambda^{-(d+1)/2}K_\xi^{-(d+5)/2}\big).$$

The estimate $\widehat{\psi_3 dS} = O(\lambda^{-N}\delta^{-4N})$ follows from Lemma 4.1 and integration by parts (see p. 350 of Stein [27] for a similar argument). This finishes the proof. □

As a consequence of the Gauss–Green formula we get:

**Corollary 4.3.** *Assume $\mathcal{B} \subset \mathbb{R}^d$ ($d \geqslant 2$) is a compact convex domain and its boundary is a smooth hypersurface. For any $\xi \in S_+^{d-1} \cap (-S_+^{d-1})$ we have*

$$\begin{aligned}\widehat{\chi}_{\mathcal{B}}(\lambda\xi) = \big(&(2\pi)^{-1}e((d+1)/8)(K_\xi)^{-1/2}e(-\lambda H(\xi))\\ &+ (2\pi)^{-1}e(-(d+1)/8)(K_{-\xi})^{-1/2}e(\lambda H(-\xi))\big)\lambda^{-(d+1)/2}\\ &+ O\big(\lambda^{-(d+3)/2}\delta^{-(d+5)/2} + \lambda^{-N-1}\delta^{-4N}\big) \qquad \text{for } \lambda > 0,\end{aligned}$$

*where $H(\xi) = \sup_{y\in\mathcal{B}}\langle y,\xi\rangle$, $N \in \mathbb{N}$, and $\delta = \min(K_\xi, K_{-\xi})$. The implicit constant depends only on $N$ and $\mathcal{B}$.*

## 5. Estimate of exponential sums

In this section we will prove a higher dimensional analogue of Proposition 5.2 in [5] by using the same method.

Let $M_* > 1$ and $T > 0$ be parameters. We consider $d$-dimensional exponential sums of the form

$$S(T, M_*; G, F) = \sum_{m\in\mathbb{Z}^d} G(m/M_*)\,e(-TF(m/M_*)),$$

where $G\colon \mathbb{R}^d \to \mathbb{R}$ is smooth, compactly supported, and bounded above by a constant, and $F\colon \Omega \subset \mathbb{R}^d \to \mathbb{R}$ is smooth on an open convex domain $\Omega$ such that

$$\mathrm{supp}(G) \subset \Omega \subset c_0 B(0,1),$$

where $c_0 > 0$ is a fixed constant.

**Proposition 5.1.** *Let $d \geqslant 3$, $q \in \mathbb{N}$, $Q = 2^q$, and $0 < K < 1$ be a parameter. Assume that*

$$\mathrm{dist}(\mathrm{supp}(G), \Omega^c) \gtrsim K^{d+2q+13-1/(d-1)}, \tag{5.1}$$

*for all $\nu \in \mathbb{N}_0^d$ and $y \in \Omega$,*

$$D^\nu G(y) \lesssim K^{-(d+2q+13-1/(d-1))|\nu|}, \tag{5.2}$$

$$D^\nu F(y) \lesssim \begin{cases} K^{-6|\nu|} & \text{if } \ 0 \leqslant |\nu| \leqslant 1,\\ K^{3-8|\nu|} & \text{if } \ |\nu| \geqslant 2,\end{cases} \tag{5.3}$$

*and for* $\mu = (1, 0, \ldots, 0, q-1) \in \mathbb{N}_0^d$

$$\big|\det(\nabla^2 D^\mu F(y))\big| \gtrsim K^{-3(q+3)d+5-1/(d-1)}. \tag{5.4}$$

*If*

$$M_* \geqslant K^{-4(5q+4)d-37+4/(d-1)} \tag{5.5}$$

*and*

$$T \geqslant K^{-I/(Q(d-1)d^2)} M_*^{q+2/Q-2/d} \tag{5.6}$$

*with*

$$I = 2(5q+4)(2Q-3)d^4 + (-35+25q+40Q+2qQ)d^3 \\ + (60+5q-Q-17qQ)d^2 + (6-60Q-5qQ)d - 6Q,$$

*then*

$$S(T, M_*; G, F) \lesssim [K^{-(2(5q+4)d^3+(3q+19)d^2-(13q+24)d-6)/(d(d-1))} \\ \cdot T M_*^{2(Q-1)d+2Q-q-2}]^{d/(2Q+2(Q-1)d)}. \tag{5.7}$$

*The constant implicit in* (5.7) *depends only on* $d$, $q$, $c_0$, *and the constants implicit in* (5.1), (5.2), (5.3), *and* (5.4).

*Proof.* Let $H$ be a parameter satisfying

$$1 < H \leqslant c_5\, K^{(6(5q+4)d^3-5(5q-7)d^2-5(q+12)d-6)/(2(d-1)d)} M_* \tag{5.8}$$

with $c_5 < 1$ chosen (later) to be sufficiently small. Then $H \leqslant M_*$. We apply to $S(T, M_*; G, F)$ the iterated one-dimensional Weyl–Van der Corput inequality with $r_1 = e_1$ and $r_j = e_d$ $(j = 2, \ldots, q)$ (see Lemma 2.2 in [4] for this inequality and notations like $G_q$, $F_q$, $\mathscr{H}$, and $\Omega_q$ that we will use below). Then we need to estimate $S_4 := S(\mathscr{H} T M_*^{-q}, M_*; G_q, F_q)$. Applying the Poisson summation formula followed by a change of variables yields

$$S_4 = \sum_{p\in\mathbb{Z}^d} K^{8d} M_*^d \int_{\mathbb{R}^d} \Psi_q(z)\, e\big(-\mathscr{H} T M_*^{-q} F_q(K^8 z) + K^8 M_* \langle p, z\rangle\big)\, dz,$$

where $\Psi_q(z) = G_q(K^8 z)$. It is obvious that

$$\operatorname{supp}(\Psi_q) \subset K^{-8}\Omega_q \subset c_0\, K^{-8}\, B(0,1). \tag{5.9}$$

By (5.1) we also have

$$\operatorname{dist}(\operatorname{supp}(\Psi_q), (K^{-8}\Omega_q)^c) \gtrsim K^{d+2q+5-1/(d-1)}. \tag{5.10}$$

By the assumption (5.3) there exists a constant $A_1$ such that

$$|\nabla_z(F_q(K^8 z))| \leqslant (A_1/2)\, K^{3-8q}.$$

We split $S_4$ into two parts:

$$S_4 = \sum_{|p|<A_1K^{-8q-5}\mathscr{H}TM_*^{-q-1}} + \sum_{|p|\geqslant A_1K^{-8q-5}\mathscr{H}TM_*^{-q-1}} =: S_5 + R_5.$$

It is not hard to prove[2], by integration by parts (Lemma A.2), that

$$R_5 \lesssim K^{-(d+2q+13-1/(d-1))(d+1)} M_*^{-1}. \tag{5.11}$$

Define $\lambda_1 = K^{3-8q}\mathscr{H}TM_*^{-q}$ and

$$\Phi_q(z,p) = \big(\mathscr{H}TM_*^{-q}F_q(K^8z) - K^8M_*\langle p,z\rangle\big)/\lambda_1,$$

then

$$S_5 = K^{8d}M_*^d \sum_{|p|<A_1K^{-8}\lambda_1M_*^{-1}} \int_{\mathbb{R}^d} \Psi_q(z)\, e(-\lambda_1\Phi_q(z,p))\, dz. \tag{5.12}$$

To estimate $S_5$ we discuss in two cases.

CASE 1. $\lambda_1 \geqslant K^{-4(5q+4)d-29+4/(d-1)}$.

For all $z \in K^{-8}\Omega_q$, by (5.2), (5.3), and (5.4), we get

$$D_z^\nu \Psi_q(z) \lesssim K^{-(d+2q+5-1/(d-1))|\nu|}, \tag{5.13}$$

$$D_z^\nu \Phi_q(z,p) \lesssim \begin{cases} K^{-8} & \text{for} \quad \nu = 0, \\ 1 & \text{for} \quad |\nu| \geqslant 1, \end{cases} \tag{5.14}$$

and

$$|\det\big(\nabla_{zz}^2\Phi_q(z,p)\big)| \gtrsim K^{(5q+4)d+5-1/(d-1)}. \tag{5.15}$$

To prove this lower bound (5.15) we first note, by using the definition of $F_q$ and the mean value theorem, that for $\mu = (1,0,\dots,0,q-1) \in \mathbb{N}_0^d$

$$\frac{\partial^2}{\partial z_{l_1}\partial z_{l_2}}\big(\Phi_q(z,p)\big) = K^{8q+13}\frac{\partial^2 D^\mu F}{\partial x_{l_1}\partial x_{l_2}}(K^8z) + O\Big(K^{-8}\frac{H}{M_*}\Big).$$

The two terms on the right are $\lesssim 1$ and $c_5K^{(5q+4)d+5-1/(d-1)}$ respectively (implied by (5.3) and (5.8)). Thus

$$\det\big(\nabla_{zz}^2(\Phi_q(z,p))\big) = K^{(8q+13)d}\det(\nabla^2D^\mu F) + O\big(c_5K^{(5q+4)d+5-1/(d-1)}\big).$$

By (5.4), we get (5.15) if we pick a sufficiently small $c_5$.

With (5.9), (5.10), (5.13), (5.14), and (5.15), we can estimate the integrals in $S_5$. Let us fix an arbitrary $p \in \mathbb{Z}^d$ with $|p| < A_1K^{-8}\lambda_1M_*^{-1}$.

[2] Check the proof of Lemma 5.4 in [5] for a similar argument.

We first need to estimate the number of critical points of the phase function $\Phi_q(z,p)$. Denote $\widetilde{p} = K^8 M_* p/\lambda_1$ and $F(z) = K^{8q-3}\nabla_z(F_q(K^8 z))$, then $\nabla_z \Phi_q(z,p) = F(z) - \widetilde{p}$ and the critical points are determined by the equation

$$F(z) = \widetilde{p} \quad \text{for } z \in K^{-8}\Omega_q.$$

The bounds (5.14) and (5.15) imply that the mapping $F$ and its components $F_j$ satisfy

$$D^\nu F_j(z) \lesssim 1 \quad \text{for} \quad |\nu| \leqslant 2, \quad j = 1, \ldots, d,$$

and

$$|\det(\nabla_z F(z))| \gtrsim K^{(5q+4)d+5-1/(d-1)}.$$

By (5.10), we know that $\mathrm{supp}(\Psi_q)$ is strictly smaller than the domain $K^{-8}\Omega_q$ and the distance between their boundary is larger than $a_1 K^{d+2q+5-1/(d-1)}$ for some positive constant $a_1$. Let $r_0 = a_1 K^{d+2q+5-1/(d-1)}/2$. By Taylor's formula, there exists a positive constant $a_2$ ($< a_1/2$) such that if $\tilde{z}$ is a critical point in $(\mathrm{supp}(\Psi_q))_{(r_0)}$,[3] then, for any $z \in B(\tilde{z}, a_2 K^{d+2q+5-1/(d-1)})$,

$$|\nabla_z \Phi_q(z,p)| \gtrsim K^{(5q+4)d+5-1/(d-1)}|z-\tilde{z}|. \tag{5.16}$$

Applying Lemma A.1 to $F$ with $r_0$ as above yields two positive constants $a_3$ ($< a_2/2$) and $a_4$ such that if

$$r_1 = a_3\, K^{(5q+4)d+5-1/(d-1)} \quad \text{and} \quad r_2 = a_4\, K^{2((5q+4)d+5-1/(d-1))},$$

then $F$ is bijective from $B(z, 2r_1)$ to an open set containing $B(F(z), 2r_2)$ for any $z \in (\mathrm{supp}(\Psi_q))_{(r_0)}$. It follows, simply by a size estimate, that the number of critical points in $(\mathrm{supp}(\Psi_q))_{(r_1)}$ is $\lesssim (K^{-8}/r_1)^d \lesssim K^{-((5q+4)d+13-1/(d-1))d}$.

For the $p$ that we have fixed, let $Z_j$ ($j = 1, \ldots, J(p)$) be all critical points in $(\mathrm{supp}(\Psi_q))_{(r_1)}$ of the phase function $\Phi_q(z,p)$ and $\chi_j(z) = \chi_0((z - Z_j)/(c_6 r_1))$ with $c_6$ chosen below. Then the integral in $S_5$ can be decomposed as

$$\int \Psi_q(z)\, e(-\lambda_1 \Phi_q(z,p))\, dz = S_6 + R_6, \tag{5.17}$$

where

$$S_6 = \sum_{j=1}^{J(p)} \int \chi_j(z)\Psi_q(z)\, e\big(-\lambda_1 \Phi_q(z,p)\big)\, dz$$

and

$$R_6 = \int \Big(1 - \sum_{j=1}^{J(p)} \chi_j(z)\Big)\Psi_q(z)\, e\big(-\lambda_1 \Phi_q(z,p)\big)\, dz.$$

It follows from integration by parts (Lemma A.2) and (5.16) that[4]

$$R_6 \lesssim K^{-8d-4((5q+4)d+7-1/(d-1))N}\lambda_1^{-N}. \tag{5.18}$$

[3] Check §1 for the definition of this notation.

[4] Check the proof of Lemma 5.5 in [5] for a similar argument.

As for $S_6$, for each $1 \leqslant j \leqslant J(p)$, let $\phi_j(z,p) = \Phi_q(z,p) - \Phi_q(Z_j,p)$. By Lemma A.4, if $c_6$ is sufficiently small then

$$\Big| \int \chi_j(z)\Psi_q(z)\, e\big( - \lambda_1 \Phi_q(z,p)\big)\, dz \Big|$$

$$= \Big| \int \chi_j(z)\Psi_q(z)\, e\big( - \lambda_1 \phi_j(z,p)\big)\, dz \Big| \lesssim K^{-((5q+4)d+5-1/(d-1))/2} \lambda_1^{-d/2}. \tag{5.19}$$

Hence

$$S_6 \lesssim K^{-8d-((5q+4)d+5-1/(d-1))(d+1/2)} \lambda_1^{-d/2}. \tag{5.20}$$

Noticing that we have assumed $\lambda_1 \geqslant K^{-4(5q+4)d-29+4/(d-1)}$ in the case 1, it is then easy to check that the bound (5.18) of $R_6$ is less than the bound (5.20) of $S_6$ if $N$ is sufficiently large. Hence, by (5.12), (5.17), (5.18), and (5.20), we get the following bound of $S_5$:

$$\begin{aligned} S_5 &\lesssim K^{8d} M_*^d \big( (K^{-8}\lambda_1 M_*^{-1})^d + 1 \big) K^{-8d-((5q+4)d+5-1/(d-1))(d+1/2)} \lambda_1^{-d/2}, \\ &\lesssim K^{-((5q+4)d+5-1/(d-1))(d+1/2)} (K^{-8d} \lambda_1^{d/2} + M_*^d \lambda_1^{-d/2}). \end{aligned} \tag{5.21}$$

CASE 2. $\lambda_1 < K^{-4(5q+4)d-29+4/(d-1)}$.

Within this range of $\lambda_1$, the assumption (5.5) implies $K^{-8}\lambda_1 M_*^{-1} < 1$, hence the trivial estimate of $S_5$ (together with (5.9) and (5.13)) yields

$$S_5 \lesssim M_*^d \leqslant K^{-(4(5q+4)d+29-4/(d-1))d/2} M_*^d \lambda_1^{-d/2}. \tag{5.22}$$

Combining the bounds of $S_5$ from cases 1 and 2 (namely, (5.21) and (5.22)) yields

$$\begin{aligned} S_5 \lesssim\ & K^{-8d-((5q+4)d+5-1/(d-1))(d+1/2)} \lambda_1^{d/2} \\ & + K^{-(4(5q+4)d+29-4/(d-1))d/2} M_*^d \lambda_1^{-d/2}. \end{aligned}$$

Note that this bound of $S_5$ is larger than the bound (5.11) of $R_5$ no matter whether $\lambda_1 \leqslant 1$ or $\lambda_1 > 1$. It follows that

$$\begin{aligned} S_4 = S_5 + R_5 \lesssim\ & K^{-(4q+13/2)d-((5q+4)d+5-1/(d-1))(d+1/2)} (\mathscr{H} T M_*^{-q})^{d/2} \\ & + K^{-2d((5q+4)d+8-2q-1/(d-1))} (\mathscr{H} T M_*^{-q-2})^{-d/2}, \end{aligned}$$

where we have already used the definition of $\lambda_1$.

Plugging this bound of $S_4$ into the Weyl–Van der Corput inequality that we used at the beginning gives

$$\begin{aligned} |S(T, M_*; G, F)|^Q \lesssim\ & M_*^{dQ} H^{-1} + H^{(1-1/Q)d} T^{d/2} M_*^{d(Q-1-q/2)} \\ & \cdot K^{-(4q+13/2)d-((5q+4)d+5-1/(d-1))(d+1/2)} + \mathrm{E}, \end{aligned} \tag{5.23}$$

where

$$\mathrm{E} = K^{-2d((5q+4)d+8-2q-1/(d-1))} H^{-2+2/Q} T^{-d/2} M_*^{d(Q+q/2)}.$$

In order to balance the first two terms on the right side of (5.23) we let

$$H = c_5\big(K^{\frac{2(5q+4)d^3+(3q+19)d^2-(13q+24)d-6}{2(d-1)}}T^{-d/2}M_*^{(q/2+1)d}\big)^{Q/(Q+(Q-1)d)}.$$

We then need to check that (5.8) is satisfied with this choice of $H$. First, $H > 1$ since we can assume

$$T < c_7\, K^{(2(5q+4)d^3+(3q+19)d^2-(13q+24)d-6)/(d(d-1))} M_*^{q+2} \tag{5.24}$$

with a sufficiently small $c_7$ (otherwise the trivial bound of $S(T, M_*; G, F)$, i.e., $M_*^d$, is better than (5.7)). Second, the assumption (5.6) implies the second inequality in (5.8).

With the choice of $H$ as above and (5.24), we get

$$\begin{aligned} &H^{-2+2/Q}T^{-d/2}M_*^{d(Q+q/2)} \\ &\qquad\leqslant K^{-\frac{2(5q+4)d^3+(3q+19)d^2-(13q+24)d-6}{2(d-1)}} M_*^{(Q-1)d}H^{d-1}. \end{aligned} \tag{5.25}$$

It then follows from (5.8) and (5.25) that

$$\mathrm{E} \leqslant M_*^{Qd}H^{-1}.$$

Applying this bound to (5.23) finally yields the desired bound (5.7). □

## 6. The $\mathbb{R}^d$ ($d \geqslant 3$) case

By a very standard argument, Theorem 1.2 follows easily from the following lemma (see Lemma 6.1 and 6.2 in [5], p. 26–27 of Iosevich [9], or p. 168–169 of Iosevich, Sawyer, and Seeger [10] for this argument).

**Lemma 6.1.** *Let $\mathcal{B} \subset \mathbb{R}^d$ ($d \geqslant 3$) be a compact convex domain and $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \rho(y)\,dy = 1$. If the boundary is a smooth hypersurface of finite type $\omega$, then, for $j \in \mathbb{N}$, we have*

$$\int_{SO(d)} \sup_{2^{j-1}\leqslant t\leqslant 2^{j+2}} t^{2d/(d+1)+\zeta_d+\sigma(d,\omega)}\Big|\sum_{k\in\mathbb{Z}_*^d} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\widehat{\rho}(\varepsilon k)\Big|\,d\theta \lesssim 1, \tag{6.1}$$

*where $d\theta$ is the normalized Haar measure on $SO(d)$, $\zeta_d$ and $\sigma(d,\omega)$ are given by (1.3) and (1.4) respectively, and*

$$\varepsilon = \varepsilon(j, d, \omega) = 2^{-j\alpha(d,\omega)},$$

$$\alpha(d,\omega) = \begin{cases} 1 - 2[6(\omega-2)d^4 + 112(\omega-2)d^3 - 4(\omega-2)d^2 \\ \quad + (410-203\omega)d + 82\omega - 156]/\square & \text{for} \quad 3 \leqslant d \leqslant 4, \\ 1 - [4(\omega-2)d^4 + 90(\omega-2)d^3 + 61(\omega-2)d^2 \\ \quad - (227\omega - 456)d + 60\omega - 112]/\triangle & \text{for} \quad d \geqslant 5, \end{cases}$$

*with □ and △ given by (1.5) and (1.6) respectively. The implicit constant depends only on $\mathcal{B}$.*

*Proof.* Let $t \in [2^{j-1}, 2^{j+2}]$ and $\delta = \delta(j, d, \omega) = 2^{-j\beta(d,\omega)}$ with

$$\beta(d,\omega) = \begin{cases} 2(\omega-2)d(d-1)(d-2)/\square & \text{for} \quad 3 \leqslant d \leqslant 4, \\ (\omega-2)d(d-1)(d-3)/\triangle & \text{for} \quad d \geqslant 5. \end{cases}$$

For any $\theta \in SO(d)$ we have the following splitting:

$$\sum_{k \in \mathbb{Z}^d_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\widehat{\rho}(\varepsilon k) = \text{Sum I}\,(t,\varepsilon,\delta,\theta) + \text{Sum II}\,(t,\varepsilon,\delta,\theta),$$

where

$$\text{Sum I}\,(t,\varepsilon,\delta,\theta) = \sum_{k \in D_1(\delta,\theta)} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\,\widehat{\rho}(\varepsilon k),$$
$$\text{Sum II}\,(t,\varepsilon,\delta,\theta) = \sum_{k \in D_2(\delta,\theta)} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\,\widehat{\rho}(\varepsilon k),$$

and $D_1(\delta,\theta)$ and $D_2(\delta,\theta)$ are two regions defined as follows:

$$D_2(\delta,0) = \big\{\xi \in \mathbb{R}^d_* : \xi/|\xi| \text{ or } -\xi/|\xi| \in \vec{n}\big(\{x \in \partial\mathcal{B} : K(x) < \delta\}\big)\big\},$$
$$D_1(\delta,0) = \mathbb{R}^d_* \setminus D_2(\delta,0), \quad D_1(\delta,\theta) = \theta D_1(\delta,0), \quad \text{and} \quad D_2(\delta,\theta) = \theta D_2(\delta,0).$$

The estimate (6.1) follows from the next two claims. Notice that the finite type condition is only used in the estimate of Sum II and the size estimate of $|D_2(\delta,0)|$.

**Claim 6.2.**

$$\int_{SO(d)} \sup_{2^{j-1} \leqslant t \leqslant 2^{j+2}} t^{2d/(d+1)+\zeta_d+\sigma(d,\omega)} \big|\,\text{Sum II}\,(t,\varepsilon,\delta,\theta)\big|\, d\theta \lesssim 1$$

*with an implicit constant depending only on $\mathcal{B}$.*

**Claim 6.3.**

$$\int_{SO(d)} \sup_{2^{j-1} \leqslant t \leqslant 2^{j+2}} t^{2d/(d+1)+\zeta_d+\sigma(d,\omega)} \big|\,\text{Sum I}\,(t,\varepsilon,\delta,\theta)\big|\, d\theta \lesssim 1$$

*with an implicit constant depending only on $\mathcal{B}$.*

*Proof of Claim 6.2.*

$$\text{L.H.S.} \lesssim (2^j)^{2d/(d+1)-(d+1)/2+\zeta_d+\sigma(d,\omega)} \sum_{k \in \mathbb{Z}^d_*} |k|^{-(d+1)/2} |\widehat{\rho}(\varepsilon k)|(*),$$

where

$$(*) := \int_{SO(d)} 1_{D_2(\delta,\theta)}(k)\Big(\sup_{2^{j-1} \leqslant t \leqslant 2^{j+2}} |tk|^{(d+1)/2} |\widehat{\chi}_{\mathcal{B}_\theta}(tk)|\Big)\, d\theta.$$

Recalling the definition (1.8) of the function $\Phi$, we get

$$\begin{aligned}(*) &\lesssim \int_{S^{d-1}\cap D_2(\delta,0)} \Phi(\xi)\,d\xi \lesssim \int_{S^{d-1}\cap D_2(\delta,0)} (K_\xi)^{-1/2} + (K_{-\xi})^{-1/2} + 1\,d\xi \\ &\lesssim \int_{\{\xi\in S^{d-1}:K_\xi<\delta\}} (K_\xi)^{-1/2}\,d\xi \lesssim \int_{\{x\in\partial\mathcal{B}:K(x)<\delta\}} K(x)^{1/2}\,dA(x) \\ &\lesssim \delta^{1/2+1/(d-1)(\omega-2)}.\end{aligned}$$

In the above estimate of $(*)$ we have used Svensson's estimate of $\Phi(\xi)$ for finite type domains (see p. 19 of [28]), the symmetry of $D_2(\delta,0)$, a change of variables, and Lemma 2.3. Hence,

$$\text{L.H.S.} \lesssim (2^j)^{2d/(d+1)-(d+1)/2+\zeta_d+\sigma(d,\omega)}\delta^{1/2+1/(d-1)(\omega-2)}\varepsilon^{-(d-1)/2} \lesssim 1. \qquad \square$$

*Proof of Claim* 6.3. Note if $\xi\in D_1(\delta,\theta)$ then $K^\theta_{\pm\xi}\geqslant\delta$. Applying Corollary 4.3 to Sum I yields

$$\text{Sum I}\,(t,\varepsilon,\delta,\theta) = (2\pi)^{-1}e\big((d+1)/8\big)S_1 + (2\pi)^{-1}e\big(-(d+1)/8\big)\widetilde{S}_1 + R_1, \tag{6.2}$$

where

$$S_1(t,\varepsilon,\delta,\theta) = t^{-(d+1)/2}\sum_{k\in D_1(\delta,\theta)} |k|^{-(d+1)/2}(K^\theta_k)^{-1/2}\widehat{\rho}(\varepsilon k)\,e(-tH_\theta(k)),$$

$$\widetilde{S}_1(t,\varepsilon,\delta,\theta) = t^{-(d+1)/2}\sum_{k\in D_1(\delta,\theta)} |k|^{-(d+1)/2}(K^\theta_{-k})^{-1/2}\widehat{\rho}(\varepsilon k)\,e(tH_\theta(-k)),$$

and

$$R_1 \lesssim \delta^{-2(d+1)}t^{-(d+3)/2}\big(\varepsilon^{-(d-3)/2}+\log(\varepsilon^{-1})\big) \lesssim t^{-2d/(d+1)-\zeta_d-\sigma(d,\omega)}. \tag{6.3}$$

We will only estimate $S_1$ since $\widetilde{S}_1$ is similar. Denote $\mathscr{C}_1=\{\xi\in\mathbb{R}^d : 1/2\leqslant|\xi|\leqslant 2\}$. Let us introduce a dyadic decomposition and a partition of unity.

Assume $\varphi\in C_0^\infty(\mathbb{R}^d)$ is a real radial function such that $\operatorname{supp}(\varphi)\subset\mathscr{C}_1$, $0\leqslant\varphi\leqslant 1$, and

$$\sum_{l_0=-\infty}^{\infty}\varphi\Big(\frac{\xi}{2^{l_0}}\Big)=1 \quad\text{for}\quad \xi\in\mathbb{R}^d\setminus\{0\}.$$

Denote

$$S_{1,M} = \sum_{k\in D_1(\delta,\theta)} \varphi(M^{-1}k)|k|^{-(d+1)/2}(K^\theta_k)^{-1/2}\widehat{\rho}(\varepsilon k)\,e(-tH_\theta(k)),$$

then

$$S_1 = t^{-(d+1)/2}\sum_{l_0=0}^{\infty} S_{1,2^{l_0}}. \tag{6.4}$$

We will estimate $S_{1,M}$ for a fixed $M = 2^{l_0}$, $l_0 \in \mathbb{N}_0$. Let $q \in \mathbb{N}$. For each $\xi \in S_+^{d-1}$ there exists a cone

$$\mathfrak{C}(\xi, 2r(\xi)) := \bigcup_{l>0} l\, B(\xi, 2r(\xi)) \subset \mathbb{R}^d,$$

where $r(\xi) = c_2(K_\xi)^{d+2q+7-1/(d-1)}/2$ and $c_2$ is the constant appearing in the statement of Lemma 3.4. Note that Lemma 2.1 implies that $K_\eta \asymp K_\xi$ if $\eta \in \mathfrak{C}(\xi, 2r(\xi))$. From the family of cones $\{\mathfrak{C}(\xi, r(\xi)/2) : \xi \in S_+^{d-1}\}$, we can choose, by a Vitali procedure, a sequence $\{\mathfrak{C}(\xi_i, r(\xi_i)/2)\}_{i=1}^\infty$ such that these cones still cover $S_+^{d-1}$ and that $\{\mathfrak{C}(\xi_i, r(\xi_i))\}_{i=1}^\infty$ satisfies the bounded overlap property. Denote

$$\mathfrak{C}_i^\theta = \theta\, \mathfrak{C}(\xi_i, r(\xi_i)).$$

Then the collection $\{\mathfrak{C}_i^\theta\}_{i=1}^\infty$ forms an open cover of $\theta S_+^{d-1}$. We can construct a partition of unity $\{\psi_i\}_{i=1}^\infty$ such that

(i) $\sum_i \psi_i \equiv 1$ on $\theta S_+^{d-1}$, and $\psi_i \in C_0^\infty(\mathfrak{C}_i^\theta)$;

(ii) each $\psi_i$ is positively homogeneous of degree zero;

(iii) $|D^\nu \psi_i| \lesssim_{|\nu|} (K_{\xi_i})^{-(d+2q+7-1/(d-1))|\nu|}$ on $\mathscr{C}_1$.

From the family $\{\mathfrak{C}_i^\theta\}_{i=1}^\infty$ we can find a subfamily $\{\mathfrak{C}_i^\theta\}_{i\in\mathscr{A}}$ which covers $D_1(\delta, \theta)$, where $\mathscr{A} = \mathscr{A}(\delta)$ is an index set such that $i \in \mathscr{A}$ if and only if $\mathfrak{C}_i^\theta$ intersects $D_1(\delta, \theta)$. Since $r(\xi_i) \gtrsim \delta^{d+2q+7-1/(d-1)}$ for any $i \in \mathscr{A}$, a size estimate gives that $\#\mathscr{A} \lesssim \delta^{-(d+2q+7-1/(d-1))(d-1)}$. Define

$$S_{1,M}^* = \sum_{i\in\mathscr{A}} S_{2,i}, \tag{6.5}$$

where

$$S_{2,i} = \sum_{k\in\mathbb{Z}^d} U_i^\theta(k)\, e(-tH_\theta(k))$$

and

$$U_i^\theta(k) = \psi_i(M^{-1}k)\, \varphi(M^{-1}k)|k|^{-(d+1)/2}(K_k^\theta)^{-1/2}\widehat{\rho}(\varepsilon k).$$

Instead of $S_{1,M}$ we will estimate $S_{1,M}^*$. It turns out that the error

$$R_{2,M} = S_{1,M} - S_{1,M}^* \tag{6.6}$$

is relatively small and this will be clear at the end of this proof.

To estimate $S_{1,M}^*$ we will estimate $S_{2,i}$ for any fixed $i \in \mathscr{A}$. By Lemma 3.4 and the homogeneity of $H_\theta$, there exist $d$ linearly independent vectors $v_j \in \mathbb{Z}^d$ ($j = 1, \dots, d$) such that if $\eta \in \cup_{1/4\leqslant l\leqslant 4}\, lB(\theta\xi_i, 2r(\xi_i))$ then

$$|h_q^\theta(\eta, v_1, \dots, v_d)| \gtrsim (K_{\xi_i})^{(-d-2q-5+1/(d-1))d(q+2)-3d+5-1/(d-1)}. \tag{6.7}$$

Let $L=[\mathbb{Z}^d:\mathbb{Z}v_1\oplus\mathbb{Z}v_2\oplus\cdots\oplus\mathbb{Z}v_d]$ be the index of the lattice spanned by $v_1,\dots,v_d$ in the lattice $\mathbb{Z}^d$. Then there exist vectors $b_l\in\mathbb{Z}^d$ $(l=1,\dots,L)$ such that

$$\mathbb{Z}^d=\biguplus_{l=1}^{L}\big(\mathbb{Z}v_1+\cdots+\mathbb{Z}v_d+b_l\big).$$

It follows from Lemma 3.4 that

$$L=|\det(v_1,\dots,v_d)|\asymp(K_{\xi_i})^{d(-d-2q-5+1/(d-1))}$$

and

$$|b_l|\lesssim(K_{\xi_i})^{-d-2q-8+1/(d-1)}.$$

Let $N>d/2$ be an arbitrarily fixed natural number. We have

$$\begin{aligned}S_{2,i}&=\sum_{l=1}^{L}\sum_{m\in\mathbb{Z}^d}U_i^\theta\Big(\sum_{j=1}^{d}m_jv_j+b_l\Big)\,e\Big(-tH_\theta\Big(\sum_{j=1}^{d}m_jv_j+b_l\Big)\Big)\\ &=(K_{\xi_i})^{-1/2}M^{-(d+1)/2}(1+M\varepsilon)^{-N}\sum_{l=1}^{L}S(T,M_*;G_l,F_l),\end{aligned}\tag{6.8}$$

where $T=tM$, $M_*=(K_{\xi_i})^{d+2q+2-1/(d-1)}M$,

$$G_l(y)=(K_{\xi_i})^{1/2}M^{(d+1)/2}(1+M\varepsilon)^NU_i^\theta(M_*Vy+b_l),$$

and

$$F_l(y)=H_\theta(M^{-1}(M_*Vy+b_l)),$$

where $V=(v_1,\dots,v_d)$.

We consider the function $F_l$ restricted to the convex domain

$$\Omega_l=\Big\{y\in\mathbb{R}^d:M^{-1}(M_*Vy+b_l)\in\bigcup_{1/4\leqslant l\leqslant 4}l\,B(\theta\xi_i,2r(\xi_i))\Big\}.$$

The support of $G_l$ satisfies

$$\operatorname{supp}(G_l)\subset\{y\in\mathbb{R}^d:M^{-1}(M_*Vy+b_l)\in\overline{\mathscr{C}_1\cap\mathfrak{C}_i^\theta}\}\subset\Omega_l.$$

We apply to $S(T,M_*;G_l,F_l)$ Proposition 5.1 with $G=G_l$, $F=F_l$, $K=K_{\xi_i}$, and $\Omega=\Omega_l$. And we only compute below the case $d\geqslant 5$ with $q=1$ (while the case $3\leqslant d\leqslant 4$ with $q=2$ can be handled in the same way).

Since $1\gtrsim K_{\xi_i}\gtrsim\delta$ if $i\in\mathscr{A}$, there exist positive constants $C_2$ and $C_3$ such that the assumptions of Proposition 5.1 are satisfied if $M\in\mathscr{I}_1$ where $\mathscr{I}_1$ is an interval defined by

$$\mathscr{I}_1=\big[C_3\delta^{-37d-41+5/(d-1)},C_2\delta^{(14d^4+66d^3+61d^2-144d-12)/(2(d-2)(d-1)d)}t^{d/(d-2)}\big].$$

This follows from Lemma 3.4, (6.7) and the following facts: if

$$(K_{\xi_i})^{-d-2q-8+1/(d-1)}\lesssim M$$

then $\Omega_l \subset c_0 B(0,1)$ for a constant $c_0$ (depending only on $q$, $\mathcal{B}$);

$$\operatorname{dist}\Big(\Big(\bigcup_{1/4\leqslant l\leqslant 4} lB(\theta\xi_i, 2r(\xi_i))\Big)^c, \overline{\mathscr{C}_1 \cap \mathfrak{C}_i^\theta}\Big) \geqslant c_2(K_{\xi_i})^{d+2q+7-1/(d-1)}/8;$$

and

$$D^\nu U_i^\theta \lesssim (K_{\xi_i})^{-(d+2q+7-1/(d-1))|\nu|-1/2} M^{-|\nu|-(d+1)/2}(1+M\varepsilon)^{-N}.$$

Thus by Proposition 5.1 we get

$$S(T, M_*; G_l, F_l) \lesssim (K_{\xi_i})^{d^2-13d/2-6+9/(d+2)} t^{d/(2d+4)} M^{d-d/(d+2)}. \tag{6.9}$$

Then by using (6.5), (6.8), (6.9), $K_{\xi_i} \gtrsim \delta$, and bounds of $\#\mathscr{A}$ and $L$, we get

$$\begin{aligned} S^*_{1,M} \lesssim\ & \delta^{-d^2-43d/2+9/2+1/(d-1)+9/(d+2)} \\ & \cdot t^{d/(2d+4)} M^{(d-1)/2-d/(d+2)}(1+M\varepsilon)^{-N}. \end{aligned} \tag{6.10}$$

Now we can estimate $S_1$. By (6.4) and (6.6) we get

$$S_1 = t^{-(d+1)/2}\Big(\sum_{l_0\in\{n\in\mathbb{N}_0: 2^n\in\mathscr{I}_1\}} S^*_{1,2^{l_0}} + R_2 + R_3\Big), \tag{6.11}$$

where

$$R_2 = \sum_{l_0\in\{n\in\mathbb{N}_0: 2^n\in\mathscr{I}_1\}} R_{2,2^{l_0}} \quad \text{and} \quad R_3 = \sum_{l_0\in\{n\in\mathbb{N}_0: 2^n\in\mathscr{I}_1^c\}} S_{1,2^{l_0}}.$$

Using the bound (6.10) of $S^*_{1,M}$ we get

$$\begin{aligned} &\sum_{l_0\in\{n\in\mathbb{N}_0: 2^n\in\mathscr{I}_1\}} S^*_{1,2^{l_0}} \\ &\qquad \lesssim \delta^{-d^2-43d/2+9/2+1/(d-1)+9/(d+2)} t^{d/(2d+4)} \varepsilon^{-(d-1)/2+d/(d+2)}. \end{aligned} \tag{6.12}$$

Hence Claim 6.3 follows from (6.2), (6.3), (6.11), (6.12), sizes of $\delta$ and $\varepsilon$, and the following estimates of[5] $R_2$ and $R_3$:

$$\int_{SO(d)} \sup_{2^{j-1}\leqslant t\leqslant 2^{j+2}} t^{2d/(d+1)+\zeta_d+\sigma(d,\omega)} t^{-(d+1)/2}|R_2|\, d\theta \lesssim 1; \tag{6.13}$$

$$\sup_{2^{j-1}\leqslant t\leqslant 2^{j+2}} t^{2d/(d+1)+\zeta_d+\sigma(d,\omega)} t^{-(d+1)/2}|R_3| \lesssim 1. \tag{6.14}$$

Inequality (6.13) follows from Lemma 2.2 and 2.3 if we notice that

$$|R_{2,M}| \lesssim \delta^{-1/2} \sum_{k\in D_2(\delta,\theta)} |\varphi(M^{-1}k)||k|^{-(d+1)/2}|\widehat{\rho}(\varepsilon k)|;$$

[5]The method used here to estimate $R_2$ is different from what we used in [5]. More precisely, we estimate the integral of $R_2$ rather than $R_2$ itself. Here we need the size estimate of $|D_2(\delta, 0)|$, and this is the only place in the estimate of Sum I where the finite type condition is used.

and (6.14) is true since we have, by trivial estimates,

$$|R_3| \lesssim \delta^{-5d^2-7d/2+9} + \delta^{-1/2} t^{d(d-1)/(2(d-2))-N_1 d/(d-2)} \varepsilon^{-N_1},$$

for any integer $N_1 > (d-1)/2$. □

Just like that Lemma 6.1 implies Theorem 1.2, the following lemma implies Theorem 1.3. Its proof is essentially the same as above, however, we now use, in the estimate of Sum II, Hölder's inequality and Varchenko's Theorem 8 in [29] instead of Svensson's estimate of $\Phi(\xi)$ (as we mentioned in §1) and $\delta = t^{-\beta(d,\infty)}$ with $\beta(d,\infty) = \lim_{\omega\to\infty} \beta(d,\omega)$.

**Lemma 6.4.** *Let $\mathcal{B} \subset \mathbb{R}^d$ $(d \geqslant 3)$ be a compact convex domain and $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \rho(y)\, dy = 1$. If the boundary is a smooth hypersurface then*

$$\int_{SO(d)} t^{2d/(d+1)+\zeta_d} \Big| \sum_{k\in\mathbb{Z}^d_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\, \widehat{\rho}(\varepsilon k) \Big|\, d\theta \lesssim 1,$$

*where $d\theta$ is the normalized Haar measure on $SO(d)$, $\zeta_d$ is given by (1.3), and*

$$\varepsilon = t^{-\alpha(d,\infty)}$$

*with*

$$\alpha(d,\infty) = \begin{cases} 1 - \dfrac{12d^4+224d^3-8d^2-406d+164}{6d^5+118d^4+109d^3-210d^2-119d+82} & \text{for} \quad 3 \leqslant d \leqslant 4, \\[2mm] 1 - \dfrac{4d^4+90d^3+61d^2-227d+60}{2d^5+47d^4+76d^3-85d^2-82d+30} & \text{for} \quad d \geqslant 5. \end{cases}$$

**Remark 6.5.** Note that $\alpha(d,\infty) = \lim_{\omega\to\infty} \alpha(d,\omega)$.

## 7. The $\mathbb{R}^2$ case

To prove Theorem 1.4 and 1.5 the key step is to prove the following $\mathbb{R}^2$ analogues of Lemma 6.1.

**Lemma 7.1.** *Let $\zeta_2 = 1/2859$, let $\mathcal{B} \subset \mathbb{R}^2$ be a compact convex domain with a smooth boundary, and let $\rho \in C_0^\infty(\mathbb{R}^2)$ be such that $\int_{\mathbb{R}^2} \rho(y)\, dy = 1$. Then, for $j \in \mathbb{N}$, we have*

$$\int_{SO(2)} \sup_{2^{j-1}\leqslant t\leqslant 2^{j+2}} t^{4/3+\zeta_2} \Big| \sum_{k\in\mathbb{Z}^2_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon(j,\infty)k) \Big|\, d\theta \lesssim 1,$$

*where $d\theta$ is the normalized Haar measure on $SO(2)$ and $\varepsilon(j,\infty) = 2^{-318j/953}$. Furthermore, if the boundary is of finite type $\omega$ then*

$$\int_{SO(2)} \sup_{2^{j-1}\leqslant t\leqslant 2^{j+2}} t^{4/3+\zeta_2+\sigma(2,\omega)} \Big| \sum_{k\in\mathbb{Z}^2_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk) \widehat{\rho}(\varepsilon(j,\omega)k) \Big|\, d\theta \lesssim 1,$$

*where* $\varepsilon(j,\omega) = 2^{-j\alpha(2,\omega)}$,

$$\alpha(2,\omega) = \frac{318\omega - 616}{953\omega - 1848} \quad \textit{and} \quad \sigma(2,\omega) = \frac{616}{953(953\omega - 1848)}.$$

*The implicit constants depend only on* $\mathcal{B}$.

Since the proof is essentially the same as the proof of Lemma 6.1 we will not provide every detail but only a few key estimates (see also the proof of Lemma 6.1 in [5]).

As before we first decompose $\sum_{k\in\mathbb{Z}^2_*} \widehat{\chi}_{\mathcal{B}_\theta}(tk)\widehat{\rho}(\varepsilon k)$ into two parts: Sum I and II.

By Lemma 2.2 and the fact that $\Phi \in L^{2,\infty}(S^1)$ (see Theorem 0.3 in [2]) we get

$$\int_{SO(2)} \sup_{2^{j-1}\leqslant t\leqslant 2^{j+2}} t^{4/3+\zeta_2} \big|\,\text{Sum II}\,\big|\, d\theta \lesssim (2^j)^{4/3-3/2+\zeta_2}\delta^{1/2}\varepsilon^{-1/2} \lesssim 1,$$

where $\varepsilon = \varepsilon(j,\infty)$ and $\delta = \delta(j,\infty) = 2^{-j/953}$.

If $\partial\mathcal{B}$ is of finite type $\omega$, then

$$\int_{SO(2)} \sup_{2^{j-1}\leqslant t\leqslant 2^{j+2}} t^{4/3+\zeta_2+\sigma(2,\omega)} \big|\,\text{Sum II}\,\big|\, d\theta$$
$$\lesssim (2^j)^{4/3-3/2+\zeta_2+\sigma(2,\omega)}\delta^{1/2+1/(\omega-2)}\varepsilon^{-1/2} \lesssim 1,$$

where $\varepsilon = \varepsilon(j,\omega)$ and $\delta = \delta(j,\omega) = 2^{-j\beta(2,\omega)}$ with

$$\beta(2,\omega) = \frac{\omega - 2}{953\omega - 1848}.$$

For Sum I we now use Lemma 3.5 and get

$$\text{Sum I} \lesssim \delta^{-14} t^{-3/2+1/22}\varepsilon^{-7/22} + t^{-3/2}|R_2|.$$

Combining this estimate with the above two of Sum II yields Lemma 7.1.

## A. Several lemmas

Here is a quantitative version of the inverse function theorem (see the appendix in [4]).

**Lemma A.1.** *Suppose* $f$ *is a* $C^{(k)}$ $(k \geqslant 2)$ *mapping from an open set* $\Omega \subset \mathbb{R}^d$ *into* $\mathbb{R}^d$ *and* $b = f(a)$ *for some* $a \in \Omega$. *Assume* $|\det(\nabla f(a))| \geqslant c$ *and that, for any* $x \in \Omega$,

$$|D^\nu f_i(x)| \leqslant C \qquad \textit{for} \quad |\nu| \leqslant 2, \quad 1 \leqslant i \leqslant d.$$

*If* $r_0 \leqslant \sup\{r > 0 : B(a,r) \subset \Omega\}$, *then* $f$ *is bijective from* $B(a,r_1)$ *to an open set containing* $B(b,r_2)$, *where*

$$r_1 = \min\left\{\frac{c}{2d^2 d!\, C^d}, r_0\right\} \quad \textit{and} \quad r_2 = \frac{c}{4d!\, C^{d-1}} r_1.$$

*The inverse mapping* $f^{-1}$ *is in* $C^{(k)}(V)$.

Hörmander's Theorem 7.7.1 in [7] gives the following estimate obtained by integration by parts.

**Lemma A.2.** *Let $K \subset \mathbb{R}^d$ be a compact set, $X$ an open neighborhood of $K$ and $k$ a nonnegative integer. If $u \in C_0^k(K)$, real $f \in C^{k+1}(X)$, then*

$$\Big|\int u(x)\, e^{i\lambda f(x)}\, dx\Big| \leqslant C|K|\lambda^{-k} \sum_{|\nu|\leqslant k} \sup |D^\nu u|\, |\nabla f|^{|\nu|-2k}, \quad \lambda > 0.$$

*Here $C$ is bounded when $f$ stays in a bounded set in $C^{k+1}(X)$.*

The following lemmas are various results of the method of stationary phase. The first one follows from Hörmander's Lemma 7.7.3 in [7]. The second one is Sogge and Stein's Lemma 2 in [26].

**Lemma A.3.** *Let $A$ be a real symmetric non-degenerate matrix. Then we have, for every integer $k > 0$ and integer $s > d/2$,*

$$\Big|\int u(x)\, e^{i\lambda\langle Ax,x\rangle/2}\, dx - (2\pi)^{d/2}\lambda^{-d/2}|\det A|^{-1/2} e^{i\pi\, sgn(A)/4} T_k(\lambda)\Big|$$
$$\leqslant C_k(\|A^{-1}\|/\lambda)^{d/2+k} \sum_{|\alpha|\leqslant 2k+s} \|D^\alpha u\|_{L^2}, \quad u \in \mathscr{S}, \lambda > 0,$$
$$T_k(\lambda) = \sum_0^{k-1} (2i\lambda)^{-j} \langle A^{-1}D, D\rangle^j u(0)/j!\,.$$

**Lemma A.4.** *Suppose $\phi$ and $\psi$ are smooth functions in $B(0,\delta) \subset \mathbb{R}^d$ with $\phi$ real-valued. Assume that $|(\partial/\partial x)^\nu \phi| \leqslant C_1$, $|\nu| \leqslant d+2$ and $|(\partial/\partial x)^\nu \psi| \leqslant C_2 \delta^{-|\nu|}$, $|\nu| \leqslant d$. We also suppose that $(\nabla\phi)(0) = 0$, but $|\det \nabla^2\phi(0)| \geqslant \delta$. Then there exists a positive constant $c_1$ (independent of $\delta$), which is sufficiently small, so that if $\psi$ is supported in $B(0, c_1\delta)$ we can assert that*

$$\Big|\int_{\mathbb{R}^d} \psi e^{i\lambda\phi}\, dx\Big| \leqslant C\, \lambda^{-d/2}\, \delta^{-1/2}.$$

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