Differential calculus on topological spaces with weak Markov structure I

Alexander Brudnyi

Dedicated to the memory of Professor Mikael Passare.

Abstract. The concept of a weak Markov set takes its origin from Whitney problems for differentiable functions on $\mathbb{R}^n$. In the present paper we study a version of the first Whitney problem of *characterizing families of continuous functions satisfying certain differential relations* on weak Markov sets. To this end we develop differential calculus on weak Markov sets similar to that on open subsets of $\mathbb{R}^n$. Then we show that some classical results for smooth functions and differential forms (such as Poincaré lemma, de Rham and Hartogs theorems, Künneth formulas, etc.) are valid also on certain weak Markov sets and more generally certain topological spaces with weak Markov structures. The class of such spaces includes, in particular, C^∞ manifolds with boundaries and some Lipschitz and fractal topological manifolds. Thus the paper offers yet another approach to analysis on fractals, a developing area of modern mathematics that focuses on geometric and dynamical aspects of fractals.

1. Introduction

Motivated by boundary value problems for partial differential equations, classical trace and extension theorems characterize traces of spaces of generalized smoothness (e.g., Sobolev, Besov, etc.) to smooth submanifolds of a Euclidean space. But in many cases one needs similar results for subsets of a more complicated geometric structure (for instance, after the change of variables initial data may be situated on a Lipschitz surface). The subject is originated from the seminal 1934 Hassler Whitney papers [36] and [37], the first of which deals with the following problem: *characterize collections of m-jets on a closed subset of $\mathbb{R}^n$ that are traces of C^m jets, i.e., sets of Taylor polynomials of degree m on $\mathbb{R}^n$ generated by C^m functions.* Whitney developed important analytic and geometric techniques which allowed

him to solve this problem; moreover, he constructed a linear bounded extension operator acting from the trace space to $C^m(\mathbb{R}^n)$. In the second paper, marked by number I, he used his extension method to solve for univariate functions an essentially more difficult (known as the first Whitney) problem of *characterizing continuous functions on an arbitrary closed subset of $\mathbb{R}^n$ that are traces of $C^m(\mathbb{R}^n)$ functions.*

In accordance with the one-dimensional case, it was natural to formulate the second (Whitney) problem: *does there exist a linear bounded extension operator from $C^m(\mathbb{R}^n)$ trace to a closed subset of $\mathbb{R}^n$ to $C^m(\mathbb{R}^n)$?*

Although in the second half of the 20th century some important partial results were obtained, the problems remained unsolved almost 70 years. In the last decade a significant breakthrough in the area has been made by Charles Fefferman (2003-2009) (see, in particular, his survey [13] and references therein) who solved the second Whitney problem and made an important contribution towards the solution of the first Whitney problem justifying the, so-called, Yu. Brudnyi–Shvartsman finiteness principle for traces of C^m functions to closed subsets of $\mathbb{R}^n$. Fefferman's results are based on multileveled, voluminous constructions that can be turned into algorithms for finite sets.

Whitney theory may be viewed as an important step towards differential calculus on closed subsets of Euclidean spaces. (For C^∞ functions on subanalytic sets there are substantially more advanced theories developed by Len Bos and Pierre Milman [7] and by Edward Bierstone and Pierre Milman [5].) However, up to now it was not completely clear what property is responsible for a "differential calculus". It turns out that this is a "weak Markov property" introduced by the author and Yuri Brudnyi in [3] and shared by many classes of sets that arise in applications. For its definition, here and below, by $\mathcal{P}_{k,n}$ we denote the space of real polynomials on $\mathbb{R}^n$ of degree k and by $Q_r(x) \subset \mathbb{R}^n$ the closed cube centered at x of sidelength $2r$.

Definition 1.1. A point x of a subset $S \subset \mathbb{R}^n$ is said to be weak k-Markov if

$$(1.1) \qquad \varliminf_{r \to 0} \left\{ \sup_{p \in \mathcal{P}_{k,n} \setminus \{0\}} \left(\frac{\sup_{Q_r(x)} |p|}{\sup_{S \cap Q_r(x)} |p|} \right) \right\} < \infty.$$

A closed set $S \subset \mathbb{R}^n$ is said to be weak k-Markov if it contains a dense subset of weak k-Markov points.

The term *weak Markov set* is derived from that of *Markov set* introduced by Alf Jonsson and Hans Wallin in [22] by means of local Markov polynomial inequalities in connection with extension and trace problems for functions in Besov spaces, see Section 2 below for an equivalent definition. The class of weak k-Markov sets denoted by $\mathrm{Mar}_k^*(\mathbb{R}^n)$, contains, in particular, the closure of any open set, the Ahlfors p-regular compact subsets of $\mathbb{R}^n$ with $p > n - 1$, a wide class of fractals of arbitrary positive Hausdorff measure and closures of unions of any combination of such sets. The class $\mathrm{Mar}_0^*(\mathbb{R}^n)$ (consisting of all nonempty closed subsets of $\mathbb{R}^n$) admits a filtration

$$\mathrm{Mar}_0^*(\mathbb{R}^n) \supset \mathrm{Mar}_1^*(\mathbb{R}^n) \supset \mathrm{Mar}_2^*(\mathbb{R}^n) \supset \cdots \supset \mathrm{Mar}_k^*(\mathbb{R}^n) \supset \cdots .$$

We set

$$\mathrm{Mar}^*_\infty(\mathbb{R}^n) := \bigcap_{k \geq 0} \mathrm{Mar}^*_k(\mathbb{R}^n)$$

and refer to elements of this class as weak ∞-Markov sets.

The purpose of the present paper is to study a version of the first Whitney problem of *characterizing families of continuous functions satisfying certain differential relations* on weak Markov sets. To this end we develop differential calculus on weak Markov sets first introducing (intrinsically) the notion of derivative of a function on such a set and then establishing its basic properties similar to those for derivatives of functions on $\mathbb{R}^n$. Next, we consider examples of problems for functions satisfying some differential relations and show, in particular, that many classical results for smooth functions and differential forms (such as Poincaré lemma, de Rham and Hartogs theorems, Künneth formulas, etc.) are valid also on certain weak Markov sets and more generally certain topological spaces with weak Markov structures. The class of such spaces includes, in particular, C^∞ manifolds with boundaries and some Lipschitz and fractal topological manifolds. Thus the paper offers yet another approach to analysis on fractals, a developing area of modern mathematics that focuses on geometric and dynamical aspects of fractals. (For known approaches and major developments in the area, see, e.g., survey [33] and references therein.)

The paper is organized as follows. In Section 2 we formulate some basic properties of weak k-Markov sets accompanied by examples. In particular, we show that (a) unlike Markov sets, $\mathrm{Mar}^*_{k+1}(\mathbb{R}^n) \subsetneq \mathrm{Mar}^*_k(\mathbb{R}^n)$ for all $k \in \mathbb{Z}_+$; (b) direct products, closures of unions or of images under surjective C^1 maps of weak k-Markov sets are also weak k-Markov sets (in the corresponding Euclidean spaces); however, classes $\mathrm{Mar}^*_k(\mathbb{R}^n)$, $n \geq 2$, are not invariant under bi-Lipschitz maps; and (c) for a fixed $k \geq 1$, each closed subset of $\mathbb{R}^n$ is disjoint union of a subset in $\mathrm{Mar}^*_k(\mathbb{R}^n) \cup \{\emptyset\}$ and a set of Hausdorff dimension at most $n-1$ whose closure does not contain subsets in $\mathrm{Mar}^*_k(\mathbb{R}^n)$.

In Section 3 we give the (intrinsic) definition of the derivative of a function $f \colon S \to \mathbb{R}$ at a weak 1-Markov point $x \in S \subset \mathbb{R}^n$ similar to the classical one: f has derivative at x if and only if there exists a vector $D_S f(x) \in \mathbb{R}^n$ such that

$$f(y) = f(x) + \langle D_S f(x), y - x \rangle + o(\|y - x\|), \quad y \in S$$

(here $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the standard inner product and the Euclidean norm on $\mathbb{R}^n$). Coordinates of $D_S f(x)$ are called partial derivatives of f at x. We show that if a function f has derivative at x, then it is restriction to S of a differentiable at x (in the usual sense) function $\tilde{f} \colon \mathbb{R}^n \to \mathbb{R}$ whose derivative at x coincides with $D_S f(x)$ (see Proposition 3.2). Moreover, there are analogs of divided differences $\Delta^k_{x,i}$, $1 \leq k \leq n$, $i \in \mathbb{N}$, on S with supports of cardinality $n+1$ converging to x as $i \to \infty$ such that $D_S f(x) = \lim_{i \to \infty} \left(\Delta^1_{x,i}(f), \ldots, \Delta^n_{x,i}(f) \right)$ (see Proposition 3.3).

Similarly, one defines derivatives of order $m \leq k$ of a function $f \colon S \to \mathbb{R}$ at a weak k-Markov point $x \in S \subset \mathbb{R}^n$. If $S \in \mathrm{Mar}^*_k(\mathbb{R}^n)$ and $f \in C(S)$ has derivatives of order $m \leq k$ at all weak k-Markov points of S, then m-jets of f consisting of

all partial derivatives of order $\leq m$ of f at these points are well defined. We say that such f is m-times continuously differentiable (i.e., of class C^m), if the set of m-jets of f satisfies, in each open relatively compact subset of S, the classical Whitney–Glaeser condition for some modulus of continuity ω (see Theorem 3.4). Then it follows that f is restriction to S of a m-times continuously differentiable function $\tilde{f}$ on $\mathbb{R}^n$.

We complete this section with formulations of the results related to the Whitney problems on weak Markov sets, that is, of finiteness principle and finite depth simultaneous extension for traces of differentiable functions to weak k-Markov sets, see Theorem 3.5 and Remark 3.6.

In Section 4, using intrinsic definitions of derivatives at weak k-Markov points of a set $S \in \mathrm{Mar}_k^*(\mathbb{R}^n)$, we define differential operators with continuous coefficients on S of order $m \leq k$ whose domains consist of spaces of m-times continuously differentiable functions on S. Further, we present examples of solving certain homogeneous differential equations on S. In particular, we show that for some Cantor-type weak k-Markov sets traces of polynomials of degree $m \leq k-1$ to them are characterized as $(m+1)$-times continuously differentiable functions on S whose partial derivatives of order m are equal to zero (Theorem 4.3), and describe solutions of analogs of the Laplace equation (Theorem 4.5) and of the Cauchy–Riemann equations (Theorem 4.4) on such sets.

In Section 5 we introduce the class Mar_k^* of paracompact spaces with weak k-Markov structure, analogously to the definition of the class of manifolds with C^k structure in differential topology, i.e., gluing together relatively open subsets of sets in $\mathrm{Mar}_k^*(\mathbb{R}^n)$ by means of local C^m homeomorphisms. (In particular, every C^k manifold and every subset of $\mathrm{Mar}_k^*(\mathbb{R}^n)$, $n \in \mathbb{N}$, belong to Mar_k^*.) We prove that any open cover of the space in Mar_k^* admits a C^k partition of unity subordinate to it (Proposition 5.2), introduce notions of "tangent bundle", "differential" and "C^k diffeomorphism" for spaces in Mar_k^* (generalizing similar notions for C^k manifolds) and prove an analog of the Whitney embedding theorem for them (Proposition 5.4).

Next, we introduce the subclass $\mathrm{Mar}_{k,\Gamma}^* \subset \mathrm{Mar}_k^*$ (containing C^k manifolds as well) consisting of spaces locally C^k diffeomorphic to direct products of simple Γ sets, that is, weak k-Markov sets situated between graphs (in $\mathbb{R}^{n+1}$) of continuous functions $f_1, f_2 : \mathbb{K}^n \to \mathbb{R}$, $f_1 \leq f_2$, where $\mathbb{K}^n := (0,1)^n \subset \mathbb{R}^n$ is the open unit cube. (If, in this definition, $f_1 = f_2$, then, in general, the graph of f_1 is a fractal in $\mathbb{R}^{n+1}$.) We show that fractal topological manifolds in $\mathrm{Mar}_{k,\Gamma}^*$ can be obtained by isotopic perturbations of C^1 manifolds (Theorem 5.5). Further, we introduce and study differential forms on spaces in $\mathrm{Mar}_{\infty,\Gamma}^*$. In particular, we prove that an analog of the classical Poincaré d-lemma for differential forms on C^∞ manifolds is valid also for d-closed differential forms on spaces in $\mathrm{Mar}_{\infty,\Gamma}^*$. Thus the de Rham complex of differential forms on such a space is well defined and, as in the case of C^∞ manifolds, the corresponding de Rham cohomology groups are isomorphic to the Čech cohomology groups of the space (Theorem 5.8). We also establish an analog of the Künneth formula for the de Rham cohomology groups of spaces in $\mathrm{Mar}_{\infty,\Gamma}^*$ (Theorem 5.13).

Sections 6–12 contain proofs of the main results of the paper.

In a forthcoming paper, following ideas of H. Whitney [39], we introduce integration of differential forms over special "fractal" chains in spaces from $\mathrm{Mar}^*_{\infty,\Gamma}$ and establish an analog of the Stokes formula for them.

2. Properties of weak Markov sets

All properties except for (6), (10) and (11) proved in Section 6 below either follow directly from Definition 1.1 or have been established in [3] (see also Volume II, Chapter 10 in [2]).

(1) $\mathrm{Mar}^*_0(\mathbb{R}^n)$ *consists of all nonempty closed sets in* $\mathbb{R}^n$.

(2) $\mathrm{Mar}^*_1(\mathbb{R})$ *consists of all infinite closed subsets of* $\mathbb{R}$ *without isolated points.*

(3) *If* $S \in \mathrm{Mar}^*_k(\mathbb{R}^n)$ *and* $U \subset S$ *is relatively open, then the closure* $\bar{U} \in \mathrm{Mar}^*_k(\mathbb{R}^n)$.

(4) *If* $\{S_i\}_{i\in I} \subset \mathrm{Mar}^*_k(\mathbb{R}^n)$, *then* $\overline{\cup_{i\in I} S_i} \in \mathrm{Mar}^*_k(\mathbb{R}^n)$.

(5) *For closed sets* $S_i \subset \mathbb{R}^{n_i}$, $i = 1, 2$, *their product* $S_1 \times S_2 \in \mathrm{Mar}^*_k(\mathbb{R}^{n_1+n_2})$ *if and only if* $S_i \in \mathrm{Mar}^*_k(\mathbb{R}^{n_i})$, $i = 1, 2$.

(6) *Let* x *be a weak* k-*Markov point of* $S \subset \mathbb{R}^n$. *Suppose a map* $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, $m \leq n$, *is such that the derivative* $D\varphi(x)$ *of* φ *at* x *exists and has rank* m. *Then* $\varphi(x)$ *is a weak* k-*Markov point of* $\varphi(S) \subset \mathbb{R}^m$. *In particular, if the set of such points* x *is dense in* S, *then* $\overline{\varphi(S)} \in \mathrm{Mar}^*_k(\mathbb{R}^m)$.

(7) *A closed set* $S \subset \mathbb{R}^n$ *belongs to the class of Markov sets, denoted by* $\mathrm{Mar}(\mathbb{R}^n)$, *if for some constant* $c > 0$ *and every* $x \in S$, $0 < r \leq \mathrm{diam}\, S$ *and* $p \in \mathcal{P}_{1,n}$,

$$\sup_{Q_r(x)} |p| \leq c \cdot \sup_{S \cap Q_r(x)} |p|.$$

If $S \in \mathrm{Mar}(\mathbb{R}^n)$, *then the above condition holds for polynomials of every degree, see Chapter 2 in* [22]. *Hence,* $\mathrm{Mar}(\mathbb{R}^n) \subset \mathrm{Mar}^*_\infty(\mathbb{R}^n)$.

A closed set $S \subset \mathbb{R}^n$ is called (Ahlfors) d-*regular* ($0 \leq d \leq n$) if for every cube $Q_r(x)$ with $x \in S$ and $0 < r < \mathrm{diam}\, S$

$$c_0\, r^d \leq \mathcal{H}_d(S \cap Q_r(x)) \leq c_1\, r^d,$$

where $c_0, c_1 > 0$ are constants independent of x and r and $\mathcal{H}_d$ stands for the Hausdorff d-measure on $\mathbb{R}^n$.

Every d-regular compact subset of $\mathbb{R}^n$ with $d > n-1$ is Markov, see Chapter 2 of [22]. So *the closure of unions of* d_i-*regular compact sets* $S_i \subset \mathbb{R}^n$ *with* $d_i > n-1$, $i \in I$, *is weak* ∞-*Markov.*

Example 2.1. The classical *von Koch snowflake curve* is d-regular with $d :=$ $\log 4/\log 3$, see, e.g., [10] for its construction.

(8) *Fix $k \in \mathbb{N}_+ \cup \{\infty\}$. Every closed subset $S \subset \mathbb{R}^n$ is disjoint union of a (possibly empty) weak k-Markov subset and a set of Hausdorff dimension at most $n - 1$.*

In particular, let a closed set $S \subset \mathbb{R}^n$ have a base (in the relative topology) consisting of sets of Hausdorff dimension greater than $n - 1$. Then S is weak ∞-Markov. (For otherwise, nonempty open set $S \setminus S_{\max}$, where $S_{\max}$ is the maximal weak ∞-Markov subset of S, contains an element of the base with Hausdorff dimension greater than $n - 1$.)

Using this fact one can prove, e.g., that the graph of the *Weierstrass nowhere differentiable function*

$$\sum_{n=0}^{\infty} a^n \cos(b^n x), \quad \text{where} \quad 0 < a < 1, \ b > 1, \ ab > 1,$$

is weak ∞-Markov. (It follows from Theorem 4 in [32] and estimates in [15].)

More generally, it was proved in Theorem 5 of [32] that for a fixed $0 < \alpha < 1$ the graph of $\varphi(x) = \sum_{n=0}^{\infty} \beta^{-\alpha n} q(\beta^n x + \theta_n)$ with sufficiently large $\beta > 1$ and arbitrary $\theta_0, \theta_1, \ldots$, where q is a Lipschitz function on $\mathbb{R}$ of period one monotone and nonconstant on a compact interval, satisfies the above base topology assumption (for $n = 2$). Hence, this graph belongs to $\mathrm{Mar}_{\infty}^*(\mathbb{R}^2)$.

(9) *A point x of a subset $S \subset \mathbb{R}^n$ is weak k-Markov if and only if there exist a convergent to 0 sequence of positive numbers $\{r_i\}_{i \in \mathbb{N}}$ and a sequence of finite subsets $F_i \subset S \cap Q_{r_i}(x)$ with $\operatorname{card} F_i = \dim \mathcal{P}_{k,n}$, $i \in \mathbb{N}$, such that*

$$(2.1) \qquad \sup_{i \in \mathbb{N}} \left\{ \sup_{p \in \mathcal{P}_{k,n} \setminus \{0\}} \left(\frac{\sup_{Q_{r_i}(x)} |p|}{\sup_{F_i} |p|} \right) \right\} < \infty.$$

The proof is a consequence of the following fact (cf. Proposition 3.5 in [3]):

Let V be an n-dimensional vector space of bounded real functions on a set X (in particular, $\operatorname{card} X \geq n$). There exist a finite subset $F \subset X$ of $\operatorname{card} F = n$ and a constant $c = c(n) > 0$ such that for every $v \in V$,

$$\sup_X |v| \leq c \cdot \max_F |v|.$$

(10) *Unlike Markov sets, $\mathrm{Mar}_{k+1}^*(\mathbb{R}^n) \setminus \mathrm{Mar}_k^*(\mathbb{R}^n) \neq \emptyset$ for all $k \in \mathbb{Z}_+$.*

In view of (5) it suffices to establish this property for $n = 1$ only. It is done in Subsection 6.2 by constructing, for each $k \in \mathbb{Z}_+$, a Cantor-type subset of $\mathbb{R}$ of Hausdorff dimension zero belonging to $\mathrm{Mar}_k^*(\mathbb{R}) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R})$.

(11) *Classes $\mathrm{Mar}_k^*(\mathbb{R})$, $k \in \mathbb{Z}_+ \cup \{\infty\}$, are invariant under bi-Lipschitz maps (i.e., if $S \in \mathrm{Mar}_k^*(\mathbb{R})$ and $\varphi : S \to S' \subset \mathbb{R}$ is a bi-Lipschitz map, then $S' \in \mathrm{Mar}_k^*(\mathbb{R})$), while $\mathrm{Mar}_k^*(\mathbb{R}^n)$, $n \geq 2$, $k \in \mathbb{N} \cup \{\infty\}$, are not.*

To prove the second part of the statement for $k \in \mathbb{N}$ we construct a Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ whose graph $\Gamma_\varphi \in \mathrm{Mar}_k^*(\mathbb{R}^2) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^2)$ (see Subsection 6.3).

3. Traces of differentiable functions to weak Markov sets

Results of this section are proved in Section 7.

Definition 3.1. A function $f\colon S \to \mathbb{R}$ is said to have derivatives of order $\leq m$ $(\leq k)$ at a weak k-Markov point $x \in S \subset \mathbb{R}^n$ if there exists a polynomial $T_x^m(f) \in \mathcal{P}_{m,n}$ such that

$$\lim_{y \to x} \frac{|f(y) - T_x^m(f)(y)|}{\|y - x\|_\infty^m} = 0.$$

If

$$T_x^m(f)(z) := \sum_{|\alpha| \leq m} \frac{c_\alpha}{\alpha!} (z - x)^\alpha, \quad \alpha \in \mathbb{Z}_+^n, |\alpha| := \|\alpha\|_1,$$

then c_α is called the partial derivative of order α at x and is denoted as $D_S^\alpha f(x)$.

(Here $\|\cdot\|_\infty$ and $\|\cdot\|_1$ stand for ℓ^∞-norm on $\mathbb{R}^n$ and ℓ^1-norm on $\mathbb{Z}^n$, respectively.)

Proposition 3.2. *A function $f\colon S \to \mathbb{R}$ has derivatives of order $\leq m\,(\leq k)$ at a weak k-Markov point $x \in S \subset \mathbb{R}^n$ if and only if there exists a function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ having derivatives of order $\leq m$ at x such that $\tilde{f}|_S = f$. Moreover, the Taylor polynomial $T_x^m(\tilde{f})$ of degree m at x of any such extension $\tilde{f}$ coincides with $T_x^m(f)$.*

Next, we express derivatives of $f\colon S \to \mathbb{R}$ in intrinsic terms using some analogs of divided differences.

Proposition 3.3. *For a weak k-Markov point $x \in S \subset \mathbb{R}^n$, and a nonnegative integer number $m \leq k$ there exist a convergent to 0 sequence of positive numbers $\{r_i\}_{i \in \mathbb{N}}$, a sequence of finite subsets $F_i^m \subset S \cap Q_{r_i}(x) \backslash \{x\}$ with $\operatorname{card} F_i = \dim \mathcal{P}_{m,n}$, a sequence of signed measures μ_i^α, $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \leq m$, on F_i^m, $i \in \mathbb{N}$, such that if a function $f\colon S \to \mathbb{R}$ has derivatives of order m at x, then*

$$(3.1) \qquad D_S^\alpha f(x) = \lim_{i \to \infty} \int_{F_i^m} f \, d\mu_i^\alpha.$$

Recall that the space $C^{m,\omega}(\mathbb{R}^n)$ is defined by the norm

$$\|f\|_{C^{m,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + \max_{|\alpha|=m} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x - y\|_\infty)};$$

here $\omega\colon (0, \infty) \to (0, \infty)$ is a nondecreasing concave function with $\omega(0+) = 0$. For a subset $S \subset \mathbb{R}^n$ by $C^{m,\omega}(S)$ we denote the trace space of continuous functions on S equipped with the trace norm

$$\|f\|_{C^{m,\omega}(S)} := \inf \left\{ \|g\|_{C^{m,\omega}(\mathbb{R}^n)}\,;\, g|_S = f \right\}.$$

The following result characterizing functions in $C^{m,\omega}(S)$ is the analog of the classical Whitney–Glaeser theorem (see, e.g., Volume I, Section 2.2 in [2]).

Theorem 3.4. *Let $S \subset \mathbb{R}^n$ be weak k-Markov, $k \in \mathbb{N}$. A function $f \in C(S)$ belongs to $C^{m,\omega}(S)$, $m \leq k$, if and only if it has derivatives of order $\leq m\,(\leq k)$ at each weak k-Markov point $x \in S$ and there exists a constant $\lambda > 0$ such that for all*

weak k-Markov points $x, y \in S$, $z \in \{x, y\}$,

(3.2)
$$\max_{|\alpha| \leq m} |D_S^\alpha f(x)| \leq \lambda \qquad and$$

$$\max_{|\alpha| \leq m} \frac{|D^\alpha (T_x^m(f) - T_y^m(f))(z)|}{\|x - y\|_\infty^{m-|\alpha|}} \leq \lambda \cdot \omega(\|x - y\|_\infty).$$

Moreover,

$$\|f\|_{C^{m,\omega}(S)} \approx \inf \lambda$$

with constants of equivalence depending only on m and n.

Finally, we formulate a result proved in [3] that solves Whitney problems for the trace of space $C^{m,\omega}(\mathbb{R}^n)$ to weak k-Markov sets ($k \geq m$).

Theorem 3.5. *Let $S \subset \mathbb{R}^n$ be weak k-Markov, $k \in \mathbb{N}$, and $m \leq k$.*

(a) **Finiteness principle.** *Suppose that for $f \in C(S)$ its restriction to each subset S' of S of cardinality at most $2\binom{n+m}{n}$ satisfies $\|f|_{S'}\|_{C^{m,\omega}(S')} \leq 1$. Then $f \in C^{m,\omega}(S)$ and there exists $c = c(m, n) > 0$ such that $\|f\|_{C^{m,\omega}(S)} \leq c$.*

(b) **Finite depth simultaneous extension.** *There exists a linear bounded extension operator $E : C^{m,\omega}(S) \to C^{m,\omega}(\mathbb{R}^n)$,*

$$(Ef)(x) := \begin{cases} \displaystyle\sum_{i=1}^\infty \lambda_i(x) f(x_i) & \text{if } x \in \mathbb{R}^n \setminus S, \\ f(x) & \text{if } x \in S; \end{cases}$$

here all $\lambda_i \in C^\infty(\mathbb{R}^n)$ and have compact supports in $\mathbb{R}^n \setminus S$, all $x_i \in S$ and for each $x \in \mathbb{R}^n \setminus S$ the number of nonzero terms in the above sum is at most $\binom{n+m}{n} \cdot w$, where w is the order of the Whitney cover of $\mathbb{R}^n \setminus S$.

Remark 3.6. If $S \subset \mathbb{R}^n$ is an arbitrary closed set, then, as it was discovered by Fefferman [11], a statement similar to (a) is valid with the finiteness constant bounded by $(1 + \dim \mathcal{P}_{m,n})^{3 \cdot \dim \mathcal{P}_{m,n}}$. Later this bound was replaced by $2^{\dim \mathcal{P}_{m,n}}$ due to Bierstone–Milman [6] and, independently and by a different method, Shvartsman [34]. A linear bounded extension operator $C^{m,\omega}(S) \to C^{m,\omega}(\mathbb{R}^n)$ in this case was constructed by Fefferman [12]. Using a modification of his method Luli [26] constructed an extension operator $C^{m,\omega}(S) \to C^{m,\omega}(\mathbb{R}^n)$ of finite depth, cf. (b).

4. Examples of extension problems

Definition 4.1. A function f on an open subset U of a weak k-Markov set $S \subset \mathbb{R}^n$ is said to belong to the space $C^m(U)$, $m \leq k$, if for each $x \in U$ there is its open neighbourhood $O_x \subset U$ such that $f|_{O_x}$ satisfies conditions of Theorem 3.4 with a modulus of continuity ω_x. Here $C^\infty(U) := \cap_{\ell=1}^\infty C^\ell(U)$.

We define $C^m(U, \mathbb{R}^p)$ as the space of maps $U \to \mathbb{R}^p$ with coordinates in $C^m(U)$.

Using Theorem 3.4 and a smooth partition of unity subordinate to a suitable cover of U by open subsets of $\mathbb{R}^n$, one easily shows that $f \in C^m(U)$ iff it is restriction to U of a C^m function defined in an open neighbourhood of U in $\mathbb{R}^n$. In the case $m = \infty$ this follows from a result of Hestens [18].

Let $S \in \operatorname{Mar}_k^*(\mathbb{R}^n)$ and $S_0 \subset S$ be a dense subset of weak k-Markov points of S. For each $f \in C^m(S)$, $m \le k$, the function $D_S^\alpha f \in C(S)$, $|\alpha| \le m$, is defined as the extension by continuity to S of the function $D_S^\alpha(f|_{S_0})$ (which is locally uniformly continuous on S_0 due to Theorem 3.4). Next, for a differential operator $P_D := \sum_{|\alpha| \le m} a_\alpha D^\alpha$, $a_\alpha \in C(W)$, where $W \subset \mathbb{R}^n$ is an open neighbourhood of S, we define

$$(4.1) \qquad P_{D_S} f := \sum_{|\alpha| \le m} (a_\alpha|_S) D_S^\alpha f.$$

If f is restriction to S of a C^m function $\hat{f}$ defined in an open neighbourhood of S in W, then (see Proposition 3.2)

$$(4.2) \qquad P_{D_S} f = (P_D \hat{f})|_S.$$

In what follows for a subset O of a Hausdorff topological space X, $\partial O := \bar{O} \setminus O^o$ and $O^c := X \setminus O$ stand for its boundary and complement in X.

In the formulated examples we deal with Cantor-type sets of the form

$$(4.3) \qquad S := \bar{\Omega} \setminus \Big(\bigcup_{j \ge 1} D_j \Big),$$

where Ω is a bounded open subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ and $\{D_j\}_{j \ge 1}$ is a family of mutually disjoint domains in Ω such that $\partial D_j = \partial(\bar{D}_j)^c$ for all j.

Example 4.2. The *Sierpiński gasket* is a d-regular set, $d := \log 3 / \log 2$, of the form (4.3) obtained by repeatedly removing open equilateral triangles from an initial equilateral triangle, see, e.g., [10].

Our first example describes traces of polynomials to a class of Cantor-type weak Markov sets.

Theorem 4.3. *Let $S \subset \mathbb{R}^n$, $n \ge 2$, be a weak k-Markov set, $k \ge 2$, of the form (4.3) such that $\Omega \Subset \mathbb{R}^n$ and all $(\bar{D}_j)^c$ are domains. Suppose $f \in C^{m+1}(S)$, $m \le k-1$, satisfies $D_S^\alpha f = 0$ for all $|\alpha| = m$. Then there exists a unique polynomial $p_f \in \mathcal{P}_{m-1,n}$ such that $p_f|_S = f$. In particular, S is connected.*

Next we present a Hartogs-type theorem describing traces of holomorphic functions to a class of Cantor-type weak Markov sets.

Theorem 4.4. *Let $S \subset \mathbb{C}^n$, $n \ge 2$, be a weak k-Markov set, $k \ge 2$, of the form (4.3) such that $\Omega \Subset \mathbb{C}^n$ and all $(\bar{D}_j)^c$ are domains. Let $f \in C_{\mathbb{C}}^2(S) := C^2(S) \otimes \mathbb{C}$ satisfy $\bar{\partial}_S f = 0$. Then there exists a unique holomorphic function $F \in \mathcal{O}(\Omega) \cap C_{\mathbb{C}}^1(\bar{\Omega})$ such that $F|_S = f$ and $\sup_{\bar{\Omega}} |F| = \sup_S |f|$.*

Our third example describes solutions of Laplace-type equations on a class of Cantor-type weak Markov sets. In its formulation $C^{2,\alpha}(\mathbb{R}^n)$ stands for the space of bounded C^2 functions on $\mathbb{R}^n$ whose second-order derivatives satisfy the Hölder condition with exponent α.

Theorem 4.5. *Let $S \subset \mathbb{R}^n$, $n \geq 3$, be a weak k-Markov set, $k \geq 2$, of the form* (4.3). *Let $f \in C^{2,\alpha}(S)$, $0 < \alpha < 1$, satisfy $\Delta_S f = 0$. Then there exist functions $F_1, F_2 \in C^{2,\alpha}(\mathbb{R}^n)$ harmonic in $\cup_{j \geq 1} D_j$ and $\left(\overline{\cup_{j \geq 1} D_j}\right)^c$, respectively, such that $\lim_{x \to \infty} F_i(x) = 0$, $i = 1, 2$, and $(F_1 + F_2)|_S = f$. Moreover, if $F_1', F_2' \in C^{2,\alpha}(\mathbb{R}^n)$ is any other pair of such functions, then $F_1' = F_1$ on $\cup_{j \geq 1} D_j$ and $F_2' = F_2$ on $\left(\overline{\cup_{j \geq 1} D_j}\right)^c$.*

5. Spaces with weak Markov structure

5.1. Basic definitions and properties

Most definitions of this subsection are similar to those of the theory of standard C^k manifolds, see, e.g., [30].

Definition 5.1. A weak k-Markov structure on a second countable Hausdorff space S is a collection of pairs $\mathcal{D} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$, called charts, such that

(1) $(U_\alpha)_{\alpha \in \Lambda}$ is an open cover of S;

(2) Each φ_α is a homeomorphism of U_α onto a relatively open subset of a set in $\mathrm{Mar}_k^*(\mathbb{R}^n)$ such that for all $\alpha, \beta \in \Lambda$ maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \, (\subset \mathbb{R}^n)$ are of class C^k;

(3) $\mathcal{D}$ is maximal with respect to property (2).

A function f on an open subset $U \subset S$ is of class C^k if $f \circ \varphi_\alpha \in C^k(\varphi_\alpha(U \cap U_\alpha))$ for all $(U_\alpha, \varphi_\alpha) \in \mathcal{D}$.

The definition implies that S is paracompact and of covering dimension at most n. Moreover, it has a base of topology consisting of relatively compact open subsets.

The class of spaces with weak k-Markov structure will be denoted Mar_k^*. If M is a C^k manifold, then by $\mathrm{Mar}_k^*(M) \subset \mathrm{Mar}_k^*$ we denote the family of all weak k-Markov closed subsets $S \subset M$, i.e., such that in Definition 5.1 each $(U_\alpha, \varphi_\alpha)$ has the form $U_\alpha := V_\alpha \cap S$ and $\varphi_\alpha := \psi_\alpha|_{U_\alpha}$, where (V_α, ψ_α) is a C^k chart on M. In particular, every C^k manifold and weak k-Markov subset of $\mathbb{R}^n$ belong to Mar_k^*.

Proposition 5.2. *Let $S \in \mathrm{Mar}_k^*$. For any open cover of S there exists a C^k partition of unity subordinate to it.*

For $S_1, S_2 \in \mathrm{Mar}_k^*$ a continuous map $f \colon S_1 \to S_2$ is said to be of class C^k if it is of class C^k in all suitable charts $(U_{\alpha 1}, \varphi_{\alpha 1}) \in \mathcal{D}_1$ and $(U_{\alpha 2}, \varphi_{\alpha 2}) \in \mathcal{D}_2$ on S_1 and S_2, respectively (i.e., $\varphi_{\alpha 2} \circ f \circ \varphi_{\beta 1}^{-1} \in C^k\big((\varphi_{\beta 1} \circ f^{-1})(U_{\alpha 2}), \varphi_{\alpha 2}(U_{\alpha 2})\big)$).

Every C^k map $f\colon S_1 \to S_2$ determines an injective morphism of algebras $f^*\colon C^k(U) \to C^k(f^{-1}(U))$, $U \subset S_2$ is open (here f^* is the pullback by f).

For $S \subset \mathrm{Mar}_k^*(\mathbb{R}^n)$, $k \geq 1$, the *tangent bundle* $TS \to S$ on S is defined as the restriction to S of the tangent bundle $T\mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$. Clearly, $TS \in \mathrm{Mar}_k(T\mathbb{R}^n)$. Now, for a C^k map $f\colon S \to \mathbb{R}^p$ the *differential* $Ff\colon TS \to T\mathbb{R}^p$ is defined as the restriction of the differential $D\hat{f}\colon TU \to T\mathbb{R}^p$, where $\hat{f}$ is a C^k extension of f to an open neighbourhood $U \subset \mathbb{R}^n$ of S. According to Propositions 3.2, 3.3, the definition of Df does not depend on the choice of the C^k extension $\hat{f}$ and, in intrinsic terms, $Df := \big(f, (D_S^j f_i)_{1 \leq i \leq n,\, 1 \leq j \leq p}\big)$, where $f := (f_1, \ldots, f_p)$.

Similarly, if $S \in \mathrm{Mar}_k^*$, then the *tangent bundle* $TS \to S$ is defined as the quotient space of disjoint union $\bigsqcup_{\alpha \in \Lambda} TU_\alpha$, where $TU_\alpha \to U_\alpha$ is the pullback by φ_α of the bundle $T\overline{\varphi_\alpha(U_\alpha)}$, by means of the following equivalence relation:

$$TU_\alpha \ni v \sim w \in TU_\beta \iff (\varphi_\beta)_*(w) = D(\varphi_\beta \circ \varphi_\alpha^{-1})\big((\varphi_\alpha)_*(v)\big);$$

here $(U_\alpha, \varphi_\alpha) \in \mathcal{D}$ and $(\varphi_\alpha)_* : TU_\alpha \to T\overline{\varphi_\alpha(U_\alpha)}|_{\varphi_\alpha(U_\alpha)}$ is the isomorphism of bundles generated by φ_α. The weak k-Markov structure on TS is given by the collection of charts $\{\,(TU_\alpha, (\varphi_\alpha)_*)\}_{\alpha \in \Lambda}$.

Next, if $f\colon S_1 \to S_2$ is a C^k map between $S_1, S_2 \in \mathrm{Mar}_k^*$, then the *differential* is a morphism of bundles $Ff\colon TS_1 \to TS_2$ defined in charts $(U_{\alpha 1}, \varphi_{\alpha 1}) \in \mathcal{D}_1$ and $(U_{\beta 2}, \varphi_{\alpha 2}) \in \mathcal{D}_2$ such that $f(U_{\alpha 1}) \subset U_{\beta 2}$ by the formula

$$Df(v) := (\varphi_{\beta 2})_*^{-1}\big(D(\varphi_{\beta 2} \circ f \circ \varphi_{\alpha 1}^{-1})\big((\varphi_{\alpha 1})_*(v)\big)\big).$$

We say that $f\colon S_1 \to S_2$ is a C^k *embedding* if f is an injection and Df is injective on each fibre of TS_1. If, in addition, f is a homeomorphism, then it is called a C^k *diffeomorphism*.

Proposition 5.3. *Suppose* $S_1, S_2 \in \mathrm{Mar}_k^*$, $k \geq 1$, *and* $f\colon S_1 \to S_2$ *is a* C^k *diffeomorphism. Then* $f^{-1}\colon S_2 \to S_1$ *is a* C^k *diffeomorphism as well.*

Let $S \subset \mathbb{R}^p$ be a closed subset and $S \in \mathrm{Mar}_k^*$. We say that the weak k-Markov structure $\mathcal{D}$ on S is induced from $\mathbb{R}^p$ if each φ_α in Definition 5.1 for S is restriction of a C^k map into $\mathbb{R}^n$ defined in an open neighbourhood of U_α in $\mathbb{R}^p$.

Our next result is an analog of the Whitney embedding theorem [38].

Proposition 5.4. *Suppose* $S \in \mathrm{Mar}_k^*$, $k \geq 1$, *and the rank of* TS *is* n. *There exists a proper* C^k *embedding* $f\colon S \to \mathbb{R}^N$, *where* $N := 2n + 1$ *if* $k \geq 2$, *such that* $f(S) \in \mathrm{Mar}_k^*$ *and has weak* k-*Markov structure induced from* $\mathbb{R}^N$. *Moreover,* $f\colon S \to f(S)$ *is a* C^k *diffeomorphism.*

(In case $S \in \mathrm{Mar}_k^*(M)$, $k \geq 2$, where $\dim M = n$, one can replace $2n + 1$ by $2n$ applying the strong Whitney embedding theorem to M.)

5.2. Calculus of differential forms

In what follows $\mathbb{K}^n := (0, 1)^n \subset \mathbb{R}^n$ stands for the open unit cube.

Let $f_{ij} : \mathbb{K}^{n_i} \to \mathbb{R}$, $1 \leq i \leq \ell$, $j = 1, 2$, be continuous functions such that $f_{i1} \leq f_{i2}$ and closures of sets $\Gamma_{f_{i1}, f_{i2}}^{n_i} := \{(x, tf_{i1}(x) + (1-t)f_{i2}(x))\,;\, x \in \mathbb{K}^{n_i},\, t \in [0, 1]\}$

are weak k-Markov subsets of $\mathbb{R}^{n_i+1}$ for all $1 \leq i \leq m$. A set of the form $\Gamma_{f_{11},f_{12}}^{n_1} \times \cdots \times \Gamma_{f_{\ell 1},f_{\ell 2}}^{n_\ell} \subset \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_\ell+1}$ $(=: \mathbb{R}^n, n := \sum_{i=1}^{\ell}(n_i+1))$ will be called a Γ *set*. Each component $\Gamma_{f_{i1},f_{i2}}^{n_i} \subset \mathbb{R}^{n_i+1}$ will be called a *simple Γ set*.

By $\mathrm{Mar}_{k,\Gamma}^* \subset \mathrm{Mar}_k^*$ we denote the subclass of sets S such that for each $x \in S$ there exists a chart $(U_\alpha, \varphi_\alpha)$ with $x \in U_\alpha$ and $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ a Γ set.

If M is a C^k manifold, then by $\mathrm{Mar}_{k,\Gamma}^*(M) \subset \mathrm{Mar}_{k,\Gamma}^*$ we denote the family of closed subsets $S \subset M$ such that in the above definition each $(U_\alpha, \varphi_\alpha)$ has the form $U_\alpha := V_\alpha \cap S$ and $\varphi_\alpha := \psi_\alpha|_{U_\alpha}$, where (V_α, ψ_α) is a C^k chart on M.

In particular, every C^k manifold with a boundary, $k \geq 1$, belongs to $\mathrm{Mar}_{k,\Gamma}^*$. However, one can show that the von Koch snowflake curve with the weak ∞-Markov structure induced from $\mathbb{R}^2$ does not belong to $\mathrm{Mar}_{\infty,\Gamma}^*$. The next result shows that this class contains sufficiently many fractal topological manifolds.

Theorem 5.5. *Let M be a C^k submanifold, $k \geq 1$, of an n-dimensional C^∞ Riemannian manifold X of dimension $m \geq \lfloor n/2 \rfloor \geq 1$ and let $U \subset X$ be an open neighbourhood of M. There is an open neighbourhood $N \subset U$ of M such that*

(A) *For each $d \in [m,n]$ there exist a topological submanifold $M_d \subset \mathrm{Mar}_{\infty,\Gamma}^*(N)$ having a base of topology of sets of Hausdorff dimension d and an isotopy $h_{d,t} : M_d \to N$, $t \in [0,1]$, such that $h_{d,0} = \mathrm{id}$, $h_{d,1}(M_d) = M$, each $h_{d,t}(M_d) \in \mathrm{Mar}_{\infty,\Gamma}^*(N)$, $0 \leq t < 1$, and $h_{d,t} : M_d \to h_{d,t}(M_d)$ is a C^k diffeomorphism for all such t. Moreover, M_m is a Lipschitz manifold and $h_{m,1} : M_m \to M$ is locally bi-Lipschitz.*

(B) *For each $l \in \mathbb{N}$ there exist a Lipschitz submanifold $M_l \in \mathrm{Mar}_{l,\Gamma}^*(N) \setminus \mathrm{Mar}_{l+1,\Gamma}^*(N)$ and an isotopy $h_{l,t} : M_l \to N$, $t \in [0,1]$, such that $h_{l,0} = \mathrm{id}$, $h_{l,1}(M_l) = M$, each $h_{l,t}(M_l) \in \mathrm{Mar}_{l,\Gamma}^*(U) \setminus \mathrm{Mar}_{l+1,\Gamma}^*(N)$, $0 \leq t < 1$ and $h_{l,t} : M_l \to h_{l,t}(M_l)$, is a C^p diffeomorphism, $p := \min\{l,k\}$, for all such t. Moreover, $h_{l,1} : M_l \to M$ is locally bi-Lipschitz.*

For $S_1, S_2 \in \mathrm{Mar}_k^*$, $k \geq 1$, a continuous map $f : S_1 \to S_2$ is said to be *locally Lipschitz* if it is locally Lipschitz in all suitable charts $(U_{\alpha 1}, \varphi_{\alpha 1}) \in \mathcal{D}_1$ and $(U_{\alpha 2}, \varphi_{\alpha 2}) \in \mathcal{D}_2$ on S_1 and S_2, respectively (that is, $\varphi_{\alpha 2} \circ f \circ \varphi_{\beta 1}^{-1} \in \mathrm{Lip}_{\mathrm{loc}}\big((\varphi_{\beta 1} \circ f^{-1})(U_{\alpha 2}), \varphi_{\alpha 2}(U_{\alpha 2})\big)$, where the metric structures of sets in the brackets are induced from those of the corresponding Euclidean spaces). Clearly, the definition does not depend on the choice of charts and each C^k map $S_1 \to S_2$ is locally Lipschitz as well. We say that $f : S_1 \to S_2$ is *locally bi-Lipschitz* if f is a homeomorphism and f, f^{-1} are locally Lipschitz maps. Since the class of locally bi-Lipschitz homeomorphisms preserves Hausdorff dimension and each m-dimensional C^k manifold M, $m \geq 1$, $k \geq 1$, admits a proper C^k embedding into $\mathbb{R}^{2m}$, Theorem 5.5 shows that there is a family of the cardinality of the continuum of locally bi-Lipschitz nonequivalent weak ∞-Markov structures on M.

For $S \in \mathrm{Mar}_k^*$, $k \geq 1$, the *cotangent bundle* $T^*S \to S$ is a bundle dual to the tangent bundle $TS \to S$ and equipped with a weak k-Markov structure induced by that on TS. A *differential form* ω of class C^k on S of degree $\deg \omega = p$ is a C^k section of the bundle $\wedge^p T^*S$. If rank of TS is n and $x = (x_1, \ldots, x_n)$ are

coordinates in a chart on S, then ω can be written locally in these coordinates as

$$\omega := \sum_{|I|=p} c_I(x) dx_I,$$

where c_I are of class C^k and for $I = (i_1, \ldots, i_s)$, $1 \leq i_\ell \leq n$, $1 \leq \ell \leq s$, we define $|I| := s$ and $dx_I := dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_s}$; here $dx_1, \ldots, dx_n$ are local coordinates on T^*S in the chart, dual to the coordinates $\partial/\partial x_1, \ldots, \partial/\partial x_n$ on TS there (pulled back from $T\mathbb{R}^n$ by the map defining this chart).

In what follows we assume that $S \in \mathrm{Mar}^*_\infty$. By $\Omega^p(U)$ we denote the vector space of C^∞ differential forms of degree p on an open set $U \subset S$. The differential $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ is defined exactly as in the case of differential forms on C^∞ manifolds (e.g., if $S \in \mathrm{Mar}^*_\infty(\mathbb{R}^n)$ and $\omega = \sum_{|I|=p} c_I \, dx_I \in \Omega^p(S)$, where $x_1, \ldots, x_n$ are coordinates on $\mathbb{R}^n$, then $d\omega := \sum_{|I|=p} \left(\sum_{i=1}^n D_S^i c_I \, dx_i\right) \wedge dx_I$).

The following result generalizes the classical Poincaré d-lemma for differential forms on C^∞ manifolds.

Theorem 5.6. *If $S \in \mathrm{Mar}^*_{\infty,\Gamma}$ and $\omega \in \Omega^p(S)$ is d-closed, i.e., $d\omega = 0$, then for each $x \in S$ there exists an open neighbourhood U_x of x and, for $p \geq 1$, a form $\eta_x \in \Omega^{p-1}(U_x)$ such that $d\eta_x = \omega$. If $p = 0$, then ω is locally constant.*

By Ω_S^p we denote the sheaf of germs of C^∞ differential forms of degree p on $S \in \mathrm{Mar}^*_\infty$. Then from Proposition 5.2 we get (see, e.g., [19] for basic definitions and results of algebraic topology used below):

Proposition 5.7. *Ω_S^p is a fine sheaf.*

Combining Theorem 5.6 with Proposition 5.7 we obtain straightforwardly:

Theorem 5.8. *Let $S \in \mathrm{Mar}^*_{\infty,\Gamma}$ and $\mathrm{rank}\, TS = n$. Then we have the following resolution of the constant sheaf $\mathbb{R}$ on S by the sheaves of differential forms:*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ d\ } \Omega_S^0 \xrightarrow{\ d\ } \Omega_S^1 \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega_S^n \xrightarrow{\ d\ } 0.$$

In particular, $\check{H}^i(S,\mathbb{R}) = H_{dR}^i(S,\mathbb{R})$, where $\check{H}^i(S,\mathbb{R})$ is the Čech cohomology of sheaf $\mathbb{R}$ on S and

$$H_{dR}^i(S,\mathbb{R}) := \frac{\mathrm{Ker}\big(\Omega^i(S) \xrightarrow{\ d\ } \Omega^{i+1}(S)\big)}{d\big(\Omega^{i-1}(S)\big)}.$$

Since $\check{H}^i(S,\mathbb{R}) = 0$ for $i > \dim_c S$, where $\dim_c$ stands for the covering dimension, Theorem 5.8 yields

Corollary 5.9. *For $i > \dim_c S$ each d-closed form in $\Omega^i(S)$ is d-exact.*

The statement of Theorem 5.8 is not valid if $S \notin \mathrm{Mar}^*_{\infty,\Gamma}$ as the following simple example shows.

Example 5.10. Let $S \subset \mathrm{Mar}_\infty^*(\mathbb{R})$ be the standard Cantor set in $[0, 1]$. Then $\check{H}^0(S, \mathbb{R})$ consists of continuous locally constant functions on S. In particular, each such function assigns finitely many values. Let $[0, 1] \setminus S := \cup_{j=1}^\infty (a_j, b_j)$ and φ be a positive C^∞ function on $(0, 1)$ having zeros of infinite order at 0 and 1. For a uniformly bounded sequence $\{c_j\} \subset \mathbb{R}_+$ consider a sequence of positive functions $\varphi_j \in C^\infty\big((a_j, b_j)\big)$, $\varphi_j(x) := c_j \, e^{-(b_j - a_j)} \varphi\big(\frac{x - a_j}{b_j - a_j}\big)$, $j \in \mathbb{N}$. Then the function $\tilde{\varphi}$ equals φ_j on (a_j, b_j), $j \in \mathbb{N}$, and 0 on $S \cup (\mathbb{R} \setminus [0, 1])$ is C^∞ on $\mathbb{R}$. We set $\psi(x) := \int_{-\infty}^x \tilde{\varphi}(t) \, dt$, $x \in \mathbb{R}$. By the definition $\psi \in C^\infty(\mathbb{R})$ and $d(\psi|_S) = 0$. However, the range of $\psi|_S$ is not finite. This shows that $\check{H}^0(S, \mathbb{R})$ is a proper subset of $H_{dR}^0(S, \mathbb{R})$.

In general, for $S \in \mathrm{Mar}_\infty^*$ as in the case of C^∞ manifolds (using Proposition 5.7) we obtain canonical homomorphisms of groups $h_S^i : \check{H}^i(S, \mathbb{R}) \to H_{dR}^i(S, \mathbb{R})$ such that if $f : S_1 \to S_2$ is a C^∞ map of spaces in Mar_∞^*, then for all i the following diagrams:

$$(5.1) \qquad
\begin{array}{ccc}
\check{H}^i(S_2, \mathbb{R}) & \xrightarrow{\;h_{S_2}^i\;} & H_{dR}^i(S_2, \mathbb{R}) \\
{\scriptstyle f_C^*}\big\downarrow & & {\scriptstyle f_{dR}^*}\big\downarrow \\
\check{H}^i(S_1, \mathbb{R}) & \xrightarrow{\;h_{S_1}^i\;} & H_{dR}^i(S_1, \mathbb{R})
\end{array}
$$

are commutative. Here f_C^* and f_{dR}^* are homomorphisms induced by f.

As another consequence of Theorem 5.8 we obtain the following interpolation result. Let $S \in \mathrm{Mar}_{\infty, \Gamma}^*$ be a closed subset of a space $X \in \mathrm{Mar}_\infty^*$ and the weak ∞-Markov structure on S is induced by that on X, i.e., the embedding $S \hookrightarrow X$ induces surjective morphism $\Omega_X^0 \to \Omega_S^0$ of sheaves of germs of C^∞ functions.

Corollary 5.11. *There exists an open neighbourhood $N \subset X$ of S such that for each d-closed form $\omega \in \Omega^p(S)$ there exists a d-closed form $\widetilde{\omega} \in \Omega^p(N)$ such that $\widetilde{\omega}|_S = \omega$.*

By $H^*(S, \mathbb{R})$ we denote the ring of singular cohomology of $S \in \mathrm{Mar}_\infty^*$ endowed with $\smile$ product (see, e.g., Appendix A in [29] for definitions). Also we consider the ring $H_{dR}^*(S, \mathbb{R})$ endowed with $\wedge$ product. If S is a C^∞ manifold, then the classical de Rham theorem states that these two rings are isomorphic. This result can be extended:

Theorem 5.12. *For $S \in \mathrm{Mar}_{\infty, \Gamma}^*$, there exists an isomorphism of rings $\Phi_S : (H_{dR}^*(S, \mathbb{R}), \wedge) \to (H^*(S, \mathbb{R}), \smile)$ such that if $f : S_1 \to S_2$ is a C^∞ map of spaces in $\mathrm{Mar}_{\infty, \Gamma}^*$, then the following diagram is commutative:*

$$(5.2) \qquad
\begin{array}{ccc}
H_{dR}^*(S_2, \mathbb{R}) & \xrightarrow{\;\Phi_{S_2}\;} & H^*(S_2, \mathbb{R}) \\
{\scriptstyle f_{dR}^*}\big\downarrow & & {\scriptstyle f^*}\big\downarrow \\
H_{dR}^*(S_1, \mathbb{R}) & \xrightarrow{\;\Phi_{S_1}\;} & H^*(S_1, \mathbb{R}).
\end{array}
$$

Here f^* is the ring homomorphism induced by f.

Moreover, if S is a C^∞ manifold, then Φ_S coincides with the classical de Rham isomorphism defined by the pairing of differential forms and chains via integration.

Let $S_i \in \mathrm{Mar}_{\infty,\Gamma}$, $i = 1, 2$. Then $S_1 \times S_2 \in \mathrm{Mar}_{\infty,\Gamma}$ as well. By $\pi_i : S_1 \times S_2 \to S_i$, $i = 1, 2$, we denote the natural projections. The map

$$(\omega_1, \omega_2) \longmapsto \pi_1^*(\omega_1) \wedge \pi_2^*(\omega_2), \qquad \omega_i \in \Omega^*(S_i), \quad i = 1, 2,$$

determines a bilinear map $\wedge : H_{dR}^*(S_1, \mathbb{R}) \otimes H_{dR}^*(S_2, \mathbb{R}) \to H_{dR}^*(S_1 \times S_2, \mathbb{R})$ sending each $\oplus_{p+q=k} H_{dR}^p(S_1, \mathbb{R}) \otimes H_{dR}^q(S_2, \mathbb{R})$ into $H_{dR}^k(S_1 \times S_2, \mathbb{R})$.

As a corollary of Theorem 5.12 we have the following generalization of the classical Künneth formula.

Theorem 5.13. *Suppose S_2 is homotopy equivalent to a CW complex having only finitely many cells in each dimension. Then the map $\wedge \colon H_{dR}^*(S_1, \mathbb{R}) \otimes H_{dR}^*(S_2, \mathbb{R}) \to H_{dR}^*(S_1 \times S_2, \mathbb{R})$ is an isomorphism.*

In case both S_i admit locally finite acyclic open covers, the result was first established by Vladimir Goldshtein by the method analogous to the one described in Section 5 of [8] (by means of the Mayer–Vietoris sequences which clearly exist in the category of spaces in Mar_∞^*, cf. Section 2 of [8] for similar arguments). The question whether a general space in $\mathrm{Mar}_{\infty,\Gamma}$ has such a cover is open.

Theorem 5.13 can be applied, e.g., if $S_2 \in \mathrm{Mar}_{\infty,\Gamma}^*$ is a compact topological manifold (in this case it has homotopy type of a finite CW complex, see [24]).

In a forthcoming paper we present some results, different from those formulated above, on the de Rham cohomology of certain topological manifolds in $\mathrm{Mar}_\infty^* \setminus \mathrm{Mar}_{\infty,\Gamma}^*$ (including von Koch snowflake-type curves) and of Cantor-type sets in $\mathbb{R}^n$ (having similar to the Sierpiński gasket structures).

6. Proofs of Properties (6), (10) and (11)

6.1. Proof of Property (6)

We use the Remez-type inequality of [4]:

Let $V \subset \mathbb{R}^n$ be a bounded convex body and $\omega \subset V$ be a measurable subset. By $|\cdot|$ we denote the Lebesgue measure on $\mathbb{R}^n$. For each $p \in \mathcal{P}_{k,n}$,

$$(6.1) \qquad \sup_V |p| \le T_k \Big(\frac{1 + \beta_n(\lambda)}{1 - \beta_n(\lambda)} \Big) \sup_\omega |p|,$$

where $\lambda := |\omega|/|V|$, $\beta_n(\lambda) := (1 - \lambda)^{1/n}$ and $T_k(x) := \frac{1}{2}((x + \sqrt{x^2 - 1})^k + (x - \sqrt{x^2 - 1})^k)$ is the Chebyshev polynomial of degree k.

(Using inequality (6.1) one easily shows that cubes $Q_r(x)$ in Definition 1.1 of a weak Markov point can be replaced by closed balls $\bar{B}_r(x)$ with respect to a norm on $\mathbb{R}^n$.)

We also use the Markov inequality asserting that

$$(6.2) \qquad \sup_{[a,b]} |p'| \le \frac{k^2}{b - a} \cdot \sup_{[a,b]} |p| \quad \text{for each } p \in \mathcal{P}_{k,1}.$$

Proof. First, we prove the result for $m = n$.

Let $x \in S$ be a weak k-Markov point such that $D\varphi(x)$ is invertible. Since the required statement is clearly true for invertible affine maps, making suitable affine transformations of $\mathbb{R}^n$, we may assume that $x = \varphi(x)$ and $D\varphi(x)$ is the identity map. Then in an open neighbourhood U of x we have

$$(6.3) \qquad \varphi(y) = y + s(y) \cdot \|y - x\|_\infty, \quad \text{where} \ \lim_{y \to x} s(y) = 0.$$

Here $\| \cdot \|_\infty$ is the ℓ_∞ norm on $\mathbb{R}^n$.

Suppose that $\{r_i\}_{i \in \mathbb{N}}$ is a convergent to zero sequence of positive numbers such that all $Q_{r_i}(x) \subset U$ and

$$(6.4) \qquad \sup_{i \in \mathbb{N}} \left\{ \sup_{p \in \mathcal{P}_{k,n} \setminus \{0\}} \left(\frac{\sup_{Q_{r_i}(x)} |p|}{\sup_{S \cap Q_{r_i}(x)} |p|} \right) \right\} =: c < \infty.$$

From (6.3) we get

$$(6.5) \qquad \|\varphi(y) - y\|_\infty \leq s_i \cdot r_i \quad \text{for all} \ y \in Q_{r_i}(x),$$

where $s_i := \sup_{y \in Q_{r_i}(x)} \|s(y)\| \to 0$ as $i \to \infty$. This implies that

$$(6.6) \qquad \varphi(S \cap Q_{r_i}(x)) \subset Q_{r_i(1+s_i)}(x).$$

Now, from (6.3)–(6.6) using inequality (6.1), the intermediate value theorem and inequality (6.2) we deduce that, for each $p \in \mathcal{P}_{k,n}$,

$$(6.7) \qquad \sup_{Q_{r_i(1+s_i)}(x)} |p| \leq T_k \left(\frac{1 + \left((1+s_i)^n - 1\right)^{1/n}}{1 - \left((1+s_i)^n - 1\right)^{1/n}} \right) \sup_{Q_{r_i}(x)} |p|$$

$$\leq c \cdot T_k \left(\frac{1 + \left((1+s_i)^n - 1\right)^{1/n}}{1 - \left((1+s_i)^n - 1\right)^{1/n}} \right) \sup_{S \cap Q_{r_i}(x)} |p|$$

$$\leq c \cdot T_k \left(\frac{1 + \left((1+s_i)^n - 1\right)^{1/n}}{1 - \left((1+s_i)^n - 1\right)^{1/n}} \right) \left(\sup_{\varphi(S \cap Q_{r_i}(x))} |p| + \frac{k^2 \sqrt{n} \cdot s_i \cdot r_i}{r_i(1+s_i)} \sup_{Q_{r_i(1+s_i)}(x)} |p| \right).$$

Due to (6.6) inequality (6.7) implies that, for all sufficiently large i,

$$\sup_{Q_{r_i(1+s_i)}(x)} |p| \leq \frac{c \cdot T_k \left(\frac{1 + ((1+s_i)^n - 1)^{1/n}}{1 - ((1+s_i)^n - 1)^{1/n}} \right)}{1 - c \cdot T_k \left(\frac{1 + ((1+s_i)^n - 1)^{1/n}}{1 - ((1+s_i)^n - 1)^{1/n}} \right) \cdot \frac{k^2 \sqrt{n} \cdot s_i}{1+s_i}} \cdot \sup_{\varphi(S) \cap Q_{r_i(1+s_i)}(x)} |p|.$$

The first factor in the right-hand side of the last inequality tends to c as i tends to ∞. This shows that $x := \varphi(x)$ is a weak k-Markov point of $\varphi(S)$.

The case of $m < n$ is reduced to the previous one. Indeed, the statement is clearly true for linear projections $\mathbb{R}^n \to \mathbb{R}^m$. Making suitable affine transformations of $\mathbb{R}^n$ and $\mathbb{R}^m$ without loss of generality we may assume that $x = \varphi(x)$

and $D\varphi(x)\colon \mathbb{R}^n \to \mathbb{R}^m$ is the linear projection sending $(y_1,\ldots,y_n) \in \mathbb{R}^n$ to $(y_1,\ldots,y_m) \in \mathbb{R}^m$. Let us define $\widetilde{\varphi}\colon \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$\widetilde{\varphi}(y_1,\ldots,y_n) := \big(\varphi(y_1,\ldots,y_n), y_{m+1},\ldots,y_n\big).$$

By the definition $D\widetilde{\varphi}(x)$ exists and is the identity map. Thus the previous case implies that $\widetilde{\varphi}(x)$ is a weak k-Markov point of $\widetilde{\varphi}(S) \subset \mathbb{R}^n$. Applying now the linear projection $D\varphi(x)$ to $\widetilde{\varphi}(x)$ and $\widetilde{\varphi}(S)$ we obtain the required result.

The second part of the statement of the property follows directly from the first one and the definition of weak k-Markov sets. $\qquad\square$

6.2. Proof of Property (10)

For each $k \in \mathbb{Z}_+$ we construct a compact set $S \in \mathrm{Mar}_k^*(\mathbb{R}) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R})$.

For $k = 0$ we define $S := \{0\} \subset \mathbb{R}$. Clearly, $S \in \mathrm{Mar}_0^*(\mathbb{R}) \setminus \mathrm{Mar}_1^*(\mathbb{R})$, see properties (1), (2) of Section 2.

Assume now that $k \geq 1$. We set $S_1 := \{i/k,\ 0 \leq i \leq k\}$. Next, choose $\lambda_1 \in (0, 1/(3k)]$ and define $S_2 := S_1 + \lambda_1 \cdot S_1$. Then choose $\lambda_2 \in (0, \lambda_1/(3k)]$ and define $S_3 := S_2 + \lambda_2 \cdot S_1$. Proceeding similarly at the nth step we define $S_n := S_{n-1} + \lambda_{n-1} \cdot S_1$ for some $\lambda_{n-1} \in (0, \lambda_{n-2}/(3k)]$, etc. (Here $X + Y := \{x + y\,;\, x \in X,\, y \in Y\}$ for $X, Y \subset \mathbb{R}$.)

The required set S is defined as the closure of $\cup_{n=1}^\infty S_n$.

By the definition, each element $x \in S$ is written as a (convergent) series

$$(6.8) \qquad x = \sum_{m=0}^\infty \frac{j_m \cdot \lambda_m}{k},$$

where j_m are integers between 0 and k, and $\lambda_0 := 1$. Moreover, $x \in S_n$ if and only if $j_m = 0$ for all $m \geq n$.

Let us show that presentation (6.8) is unique. Indeed, suppose that $x = \sum_{m=0}^\infty j_m' \cdot \lambda_m/k$ is another such presentation. By $M \in \mathbb{Z}_+$ we denote the minimal number such that $j_m = j_m'$ for $0 \leq m \leq M$ but $j_{M+1} \neq j_{M+1}'$. Then we have

$$0 = \Big| \sum_{m=0}^\infty \frac{(j_m - j_m') \cdot \lambda_m}{k} \Big| \geq \frac{|j_{M+1} - j_{M+1}'| \cdot \lambda_{M+1}}{k} - \sum_{m=M+2}^\infty \frac{|j_m - j_m'| \cdot \lambda_m}{k}$$

$$(6.9) \quad \geq \frac{\lambda_{M+1}}{k} - \sum_{m=M+2}^\infty \lambda_m \geq \frac{\lambda_{M+1}}{k} - \sum_{m=1}^\infty \frac{\lambda_{M+1}}{(3k)^m} \geq \frac{\lambda_{M+1}}{2k} > 0,$$

a contradiction proving the claim.

Proposition 6.1. *Suppose that* $\lim_{i\to\infty} \frac{\lambda_{i+1}}{\lambda_i} = 0$. *Then* $S \in \mathrm{Mar}_k^*(\mathbb{R}) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R})$.

Proof. First, we reformulate property (9) of Section 2 for the case of weak Markov subsets of $\mathbb{R}$.

Lemma 6.2. *A point x of a subset $S \subset \mathbb{R}$ is weak k-Markov if and only if there exist a convergent to 0 sequence of positive numbers $\{r_i\}_{i \in \mathbb{N}}$, a sequence of finite subsets $F_i := \{x_{i1}, \dots, x_{ik+1}\} \subset S \cap Q_{r_i}(x)$, $i \in \mathbb{N}$, and a positive number $\delta(x)$ such that for all $i \in \mathbb{N}$*

$$(6.10) \qquad \min_{m \neq n} |x_{im} - x_{in}| \geq \delta(x) r_i.$$

Proof of Lemma 6.2. According to property (9) it suffices to show that the hypothesis of the lemma is equivalent to (2.1). So assuming that (2.1) is valid, let us show that (6.10) is also valid for some $\delta(x) > 0$.

If, on the contrary, such $\delta(x)$ does not exist, then passing if necessary to a subsequence of $\{r_i\}_{i \in \mathbb{N}}$ without loss of generality we may assume that

(a) $\displaystyle \lim_{i \to \infty} \frac{x_{im} - x_{in}}{r_i} = 0$ for some fixed $m \neq n$;

(b) $\displaystyle \lim_{i \to \infty} \frac{x_{ij} - x}{r_i} =: x_j \in [-1, 1]$ for all $1 \leq j \leq k + 1$.

Due to (a) the set $F := \{x_1, \dots, x_{k+1}\}$ consists of at most k distinct points. Thus there exists a polynomial p of degree k on $\mathbb{R}$ having zeros at points of F and such that $\max_{[-1,1]} |p| = 1$. This and (b) imply for $p_i(t) := p\left(\frac{t-x}{r_i}\right) \in \mathcal{P}_{k,1} \setminus \{0\}$, $i \in \mathbb{N}$, that

$$\sup_{Q_{r_i}(x)} |p_i| = 1 \quad \text{and} \quad \lim_{i \to \infty} \sup_{F_i} |p| = 0,$$

a contradiction with (2.1).

Conversely, if (6.10) is valid then according to an inequality following from the classical Cartan lemma, see page 21 in [25], condition (2.1) is also valid for the same $\{r_i\}_{i \in \mathbb{N}}$ and $F_i \subset Q_{r_i}(x)$, $i \in \mathbb{N}$. □

Returning to the proof of Proposition 6.1, first, let us show that $S \notin \mathrm{Mar}^*_{k+1}(\mathbb{R})$. Assuming the contrary, we find, by Lemma 6.2, a point $x \in S$, a convergent to 0 sequence of positive numbers $\{r_i\}_{i \in \mathbb{N}}$, a sequence of subsets $F_i := \{x_{i1}, \dots, x_{ik+2}\} \subset S \cap Q_{r_i}(x)$, $i \in \mathbb{N}$, and a positive number $\delta(x)$ such that for all $i \in \mathbb{N}$

$$(6.11) \qquad \min_{m \neq n} |x_{im} - x_{in}| \geq \delta(x) \, r_i.$$

By $j(i)$, $i \in \mathbb{N}$, we denote a natural number such that

$$(6.12) \qquad \lambda_{j(i)+1} \leq r_i < \lambda_{j(i)}.$$

Case I. $\qquad \varliminf_{i \to \infty} \dfrac{r_i}{\lambda_{j(i)+1}} < \infty \qquad \text{or} \qquad \varliminf_{i \to \infty} \dfrac{\lambda_{j(i)}}{r_i} < \infty.$

In each of these cases passing to a subsequence of $\{r_i\}_{i \in \mathbb{N}}$ we may assume without loss of generality that there exist constants $c_1, c_2 > 0$ and a sequence $\{k(i)\}_{i \in \mathbb{N}} \subset \mathbb{N}$ such that for all $i \in \mathbb{N}$

$$(6.13) \qquad c_1 \cdot \lambda_{k(i)} \leq r_i \leq c_2 \cdot \lambda_{k(i)}.$$

Next, according to (6.8), there exists a unique nondecreasing sequence $\{x_i\}_{i \in \mathbb{N}}$ such that $x_i \in S_i$ and $|x_{i+1} - x_i| \leq \lambda_i$ for all $i \in \mathbb{N}$, and $\lim_{i \to \infty} x_i = x$.

Lemma 6.3. *For all sufficiently large i, $S \cap Q_{r_i}(x)$ belongs to the set*

$$S(x_{k(i)}) := \Big\{ z \in S \,;\, z := x_{k(i)} + \sum_{m=k(i)}^{\infty} \frac{j_m \cdot \lambda_m}{k}, \quad j_m \in \mathbb{Z}_+ \cap [0,k], \ m \geq k(i) \Big\}.$$

Proof of Lemma 6.3. If $z \notin S(x_{k(i)})$, then one of the first $k(i)$ terms in (6.8) for z differs from the corresponding term in (6.8) for $x_{k(i)}$. Thus, as in (6.9) we have

$$|z - x| \geq |z - x_{k(i)}| - |x_{k(i)} - x| \geq \frac{\lambda_{k(i)-1}}{2k} - \sum_{m=k(i)}^{\infty} \lambda_m \geq \frac{\lambda_{k(i)-1}}{2k} - \frac{3\lambda_{k(i)}}{k}.$$

The latter, due to the hypothesis of the proposition and (6.13), is greater than r_i for all sufficiently large i. This implies that $z \notin S \cap Q_{r_i}(x)$. $\qquad\square$

Passing to a subsequence of $\{r_i\}_{i \in \mathbb{N}}$, if necessary, we may assume without loss of generality that $S \cap Q_{r_i}(x) \subset S(x_{k(i)})$ for all i.

According to the lemma, numbers in (6.11) can be written as

$$x_{ip} := x_{k(i)} + \sum_{m=k(i)}^{\infty} \frac{j_{mp} \cdot \lambda_m}{k}, \quad j_{mp} \in \mathbb{Z}_+ \cap [0,k], \ m \geq k(i), \ 1 \leq p \leq k+2.$$

In particular, there exist integers $1 \leq p_1(i) < p_2(i) \leq k+2$ such that $j_{k(i)p_1(i)} = j_{k(i)p_2(i)}$. This, the hypothesis of the proposition and (6.13) imply that for all sufficiently large i

$$|x_{ip_1(i)} - x_{ip_2(i)}| \leq \sum_{m=k(i)+1}^{\infty} \lambda_m \leq \frac{3\lambda_{k(i)+1}}{k} < \delta(x) \cdot r_i$$

which contradicts (6.11).

Case II. $\qquad \displaystyle\lim_{i \to \infty} \frac{r_i}{\lambda_{j(i)+1}} = \infty \qquad$ and $\qquad \displaystyle\lim_{i \to \infty} \frac{\lambda_{j(i)}}{r_i} = \infty.$

Using the second of these conditions and arguing as in the proof of Lemma 6.3 we obtain that for all sufficiently large i, $S \cap Q_{r_i}(x) \subset S(x_{j(i)+1})$ (see (6.12)). Then for the linear polynomial $p(t) := t - x$, $t \in \mathbb{R}$, we have

$$\sup_{S \cap Q_{r_i}(x)} |p| \leq \sup_{S(x_{j(i)+1})} |p| \leq \sum_{m := j(i)+1}^{\infty} \lambda_m \leq \frac{3\lambda_{j(i)+1}}{k}.$$

On the other hand,
$$\sup_{Q_{r_i}(x)} |p| = r_i.$$

Therefore
$$\varliminf_{i \to \infty} \frac{\sup_{Q_{r_i}(x)} |p|}{\sup_{S \cap Q_{r_i}(x)} |p|} \geq \varliminf_{i \to \infty} \frac{k \cdot r_i}{3\lambda_{j(i)+1}} = \infty,$$

contradicting (6.11) (see the proof of Lemma 6.2).

Thus we have proved that S is not weak $(k+1)$-Markov.

To show that $S \in \mathrm{Mar}_k^*(\mathbb{R})$ it suffices to check the condition of Lemma 6.2 for points $x \in \cup_{i=1}^{\infty} S_i$ (a dense subset of S). In this case if $x \in S_m$ and $i \geq m$, then $S \cap Q_{\lambda_i}(x)$ contains the set $\{x\} + \lambda_i \cdot S_1$ of cardinality $k+1$ with distances between its distinct points being at least λ_i/k. Thus according to Lemma 6.2, each $x \in \cup_{i=1}^{\infty} S_i$ is weak k-Markov. This gives the required statement and completes the proof of Proposition 6.1. $\square$

Remark 6.4. The constructed set S is of Hausdorff dimension zero. For otherwise, due to property (8), see Section 2, S contains a nonempty weak $(k+1)$-Markov subset. However, each point of S is at most weak k-Markov (see the proof of Proposition 6.1), a contradiction.

Taking direct products of n copies of S we obtain, according to property (5) of Section 2, a compact subset of $\mathbb{R}^n$ belonging to $\mathrm{Mar}_k^*(\mathbb{R}^n) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^n)$. This completes the proof of property (10).

6.3. Proof of Property (11)

First, we prove invariance of $\mathrm{Mar}_k^*(\mathbb{R})$ under bi-Lipschitz maps.

Theorem 6.5. *If $x \in S \subset \mathbb{R}$ is a weak k-Markov point and $\varphi : S \to S' \subset \mathbb{R}$ is a bi-Lipschitz map, than $\varphi(x) \in S'$ is weak k-Markov. In particular, if S is a weak k-Markov set, then S' is weak k-Markov as well.*

Proof. Using the McShane extension theorem [27], without loss of generality we may assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz and its restriction $\varphi|_S : S \to S'$ is bi-Lipschitz. Let $\{r_i\}_{i \in \mathbb{N}}$, $F_i := \{x_{i1}, \ldots, x_{ik+1}\} \subset S \cap Q_{r_i}(x)$, $i \in \mathbb{N}$, and $\delta(x) > 0$ be the same as in Lemma 6.2 for the point $x \in S$. Then we have $\varphi(Q_{r_i}(x)) \subset Q_{c_1 r_i}(\varphi(x))$, where c_1 is the Lipschitz constant for φ, and for all $i \in \mathbb{N}$

$$\min_{m \neq n} \left| \varphi(x_{im}) - \varphi(x_{in}) \right| \geq \frac{\delta(x)}{c_2} \cdot r_i,$$

where c_2 is the Lipschitz constant for $\varphi^{-1} : S' \to S$. Thus $\varphi(F_i) \subset S' \cap Q_{c_1 r_i}(\varphi(x))$, $i \in \mathbb{N}$, satisfy conditions of Lemma 6.2 for the point $\varphi(x) \in S'$, sequence $\{c_1 r_i\}_{i \in \mathbb{N}}$ and $\delta(\varphi(x)) := \delta(x)/(c_1 c_2)$. In particular, $\varphi(x) \in S'$ is weak k-Markov.

The second statement follows from the first one because φ is continuous and so sends dense subsets of S to dense subsets of S'. $\square$

To show that $\mathrm{Mar}_k^*(\mathbb{R}^n)$, $n \geq 2$, $k \in \mathbb{N}$, is not invariant under bi-Lipschitz maps, we construct Lipschitz functions $\varphi : \mathbb{R} \to \mathbb{R}$ whose graphs $\Gamma_\varphi \in \mathrm{Mar}_k^*(\mathbb{R}^2) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^2)$.

($\star$) *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function such that*

(a) *a point $(x_0, f(x_0)) \in \Gamma_f$ is weak k- but not weak $(k+1)$-Markov, $1 \leq k < \infty$;*

(b) *$f'|_{\mathbb{R} \setminus \{x_0\}}$ has discontinuities of the first kind at some points of a set $\{x_i\}_{i \in \mathbb{N}}$;*

(c) *f is differentiable at all points of the set $\mathbb{R} \setminus \left\{ \{x_i\}_{i \in \mathbb{Z}_+} \right\}$.*

Existence of such f is established in Theorem 6.8.

Suppose $\{c_i\}_{i\in\mathbb{N}}$ is a countable dense subset of an open interval $(-t, t) \subset \mathbb{R}$ such that

$$(6.14) \qquad \{x_0 + c_i\}_{i\in\mathbb{N}} \bigcap \{x_i + c_j\}_{i,j\in\mathbb{N}} = \emptyset.$$

(For instance, $\{x_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{Q}$ and $\{c_i\}_{i\in\mathbb{N}}$ is a linearly independent family over $\mathbb{Q}$.)

Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence of nonzero numbers such that

$$\sum_{i=1}^{\infty} |a_i| < \infty.$$

We define a bounded Lipschitz function

$$(6.15) \qquad \varphi(x) := \sum_{i=1}^{\infty} a_i f(x - c_i), \quad x \in [x_0 - t, x_0 + t].$$

Theorem 6.6. $\Gamma_\varphi \in \mathrm{Mar}_k^*(\mathbb{R}^2) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^2)$.

Proof. We start with:

Lemma 6.7. (1) *If $f'(x_0)$ exists, then the point $(x_0, f(x_0)) \in \Gamma_f$ is weak 0- but not weak 1-Markov.*

(2) *If $f'(x_0+)$ and $f'(x_0-)$ exist and distinct, then the point $(x_0, f(x_0)) \in \Gamma_f$ is weak 1- but not weak 2-Markov.*

Proof of Lemma 6.7. (1) Assume that $f'(x_0)$ exists. Consider a map $F : \mathbb{R}^2 \to \mathbb{R}^2$,

$$F(x, y) := (x, y - f(x)).$$

By our assumption the derivative $DF(x_0, f(x_0))$ exists and has rank 2. Thus according to property (6) of Section 2, $(x_0, f(x_0)) \in \Gamma_f$ is weak k-Markov if and only if $(x_0, 0) := F(x_0, f(x_0)) \in F(\Gamma_f) := \mathbb{R} \times \{0\}$ is weak k-Markov (with respect to $\mathcal{P}_{k,2}$). Since $F(\Gamma_f)$ belongs to the zero set of the linear polynomial y, the point $F(x_0, f(x_0))$ is 0-Markov only. This gives the required statement for $(x_0, f(x_0)) \in \Gamma_f$.

(2) Suppose $f'(x_0-) \neq f'(x_0+)$ exist. Consider the function

$$g(x) := \begin{cases} f(x_0) + f'(x_0-)(x - x_0) & \text{for } x \leq x_0 \\ f(x_0) + f'(x_0+)(x - x_0) & \text{for } x > x_0. \end{cases}$$

By the definition $\tilde{f} := f - g$ has derivative 0 at x_0. Consider a map $\widetilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\widetilde{F}(x, y) := (x, y - \tilde{f}(x)).$$

Then $\widetilde{F}$ has derivative of rank 2 at each point (x_0, y). In particular, $(x_0, f(x_0)) \in \Gamma_f$ is weak k-Markov iff the point $(x_0, 0) := \widetilde{F}(x_0, f(x_0)) \in \widetilde{F}(\Gamma_f) := \Gamma_g$ is weak k-Markov. The last point is obviously 1-Markov, because

(a) the convex hull of the intersection of a square Q centered at $(x_0, 0)$ with Γ_g is a triangle $T(Q)$ such that the ratio of its area to the area of Q is a positive constant independent of the size of the square;

(b) maximum of modulus of a linear polynomial on $Q \cap \Gamma_g$ is equal to maximum of modulus of this polynomial on $T(Q)$;

(c) from (a) and Remez-type inequality (6.1) follow that maximum of modulus of a linear polynomial on Q is bounded from above by an absolute constant times maximum of modulus of this polynomial on $T(Q)$.

To show that $(x_0, 0) \in \Gamma_g$ is not weak 2-Markov note that the nonzero quadratic polynomial $p(x, y) := \big(y - f(x_0) - f'(x_0-)(x - x_0)\big)\big(y - f(x_0) - f'(x_0+)(x - x_0)\big)$ is equal to zero on Γ_g. This gives the required statement. $\qquad \square$

To prove the theorem, first, let us show that each point $(x_0 + c_i, \varphi(x_0 + c_i)) \in \Gamma_\varphi$ is weak k- but not weak $(k+1)$-Markov. Indeed, according to condition (6.14) all functions $a_j f(x - c_j)$, $j \neq i$, are differentiable at $x_0 + c_i$. Therefore the bounded Lipschitz function

$$\varphi_i(x) := \sum_{j \neq i} a_j f(x - c_j), \quad x \in [x_0 - t, x_0 + t],$$

is differentiable at $x_0 + c_i$ as well.

Next, consider an invertible map $F : (x_0 - t, x_0 + t) \times \mathbb{R} \to (x_0 - t, x_0 + t) \times \mathbb{R}$ defined by

$$F(x, y) := (x, y - \varphi_i(x)).$$

Clearly, the derivative $DF(x_0 + c_i, \varphi(x_0 + c_i))$ exists and has rank 2. Thus according to property (6) of Section 2, the point $(x_0 + c_i, \varphi(x_0 + c_i)) \in \Gamma_\varphi$ is weak m-Markov iff $F(x_0 + c_i, \varphi(x_0 + c_i)) \in F(\Gamma_\varphi)$ is weak m-Markov. By the definition $F(\Gamma_\varphi)$ is the graph of the function $a_i f(x - c_i)$, $x \in (x_0 - t, x_0 + t)$, and $F(x_0 + c_i, \varphi(x_0 + c_i)) = (x_0 + c_i, a_i f(x_0))$. Applying to $F(\Gamma_\varphi)$ the invertible linear transformation of $\mathbb{R}^2$ sending each (x, y) to $(x - c_i, y/a_i)$ we obtain that $(x_0 + c_i, \varphi(x_0 + c_i)) \in \Gamma_\varphi$ is weak m-Markov iff $(x_0, f(x_0)) \in \Gamma_f$ is weak m-Markov. The above argument and the hypothesis of the theorem imply that each $(x_0 + c_i, \varphi(x_0 + c_i)) \in \Gamma_\varphi$ is weak k- but not weak $(k+1)$-Markov.

Now, consider points $(x_i + c_j, \varphi(x_i + c_j)) \in \Gamma_\varphi$, $i, j \in \mathbb{N}$. Due to condition (6.14), for a fixed $n \in \mathbb{N}$ either $a_n f'(x_i + c_j - c_n)$ or $a_n f'(x_i + c_j - c_n+) \neq a_n f'(x_i + c_j - c_n-)$ exist. Hence, the same holds for φ, i.e., either $\varphi'(x_i + c_j)$ or $\varphi'(x_i + c_j-) \neq \varphi'(x_i + c_j+)$ exist. Then Lemma 6.7 implies that points $(x_i + c_j, \varphi(x_i + c_j)) \in \Gamma_\varphi$, $i, j \in \mathbb{N}$, are at most weak 1-Markov.

Finally, by the hypothesis, φ is differentiable on the set $[x_0 - t, x_0 + t] \setminus \{\{x_i + c_j\}_{i \in \mathbb{Z}_+ ; j \in \mathbb{N}}\}$. Thus image of this set in Γ_φ consists of weak 0-Markov points.

Combining these facts and using that $\{x_0 + c_i\}_{i \in \mathbb{N}}$ is dense in $[x_0 - t, x_0 + t]$ and $k \geq 1$, we obtain that $\Gamma_\varphi \in \mathrm{Mar}_k^*(\mathbb{R}^2) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^2)$ as required. $\qquad \square$

Theorem 6.8. *For each $k \in \mathbb{N}$ there exists a bounded Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ satisfying assumptions* (a)–(c) *of condition* ($\star$).

Proof. For $k = 1$ we define the function f by the formula

$$f(x) := \begin{cases} |x - x_0| & \text{for } |x - x_0| \leq 1, \\ 1 & \text{for } |x - x_0| > 1. \end{cases}$$

Then f' has discontinuities of the first kind at points $x_0 \pm 1$ and x_0 and according to Lemma 6.7, $(x_0, f(x_0)) \in \Gamma_f$ is weak 1- but not weak 2-Markov.

Now, we consider the case of $k \geq 2$. We set $t_n := 2^{-n^2}$, $n \in \mathbb{N}_+$. Let p be a univariate polynomial of degree k equals zero at $t_0 := 1$ and $t_1 := 1/2$ and such that $\sup_{[-1,1]} |p| \leq 1$. (If $k \geq 4$, then we can choose p so that its derivatives at t_0 and t_1 are zeros as well.) We define the function f by the formula

$$f(x) := \left\{ \begin{array}{ll} 0 & \text{if } x \in \big(\bigcup_{j \in \mathbb{Z}_+} [t_{2j}/2, t_{2j+2}] \big) \bigcup (\mathbb{R} \setminus (0,1)), \\ t_{2j} \cdot p\big(t_{2j}^{-1} \cdot x\big) & \text{if } x \in [t_{2j}, t_{2j}/2]; \ j \in \mathbb{Z}_+. \end{array} \right.$$

According to this formula, f is Lipschitz and bounded on $\mathbb{R}$ and f' has at most first kind discontinuities at points t_{2j} and $t_{2j}/2$, $j \in \mathbb{Z}_+$. (If $k \geq 4$ and p' is zero at t_0 and t_1, then f' is continuous on $\mathbb{R} \setminus \{0\}$.) Let us show that $(0, f(0)) = (0,0) \in \Gamma_f$ is weak k-Markov.

Lemma 6.9. *Let $\Gamma_h \subset Q_1(0)$ be the graph of $h := f|_{[1/3,1]}$. Then*

$$\sup_{q \in \mathcal{P}_{k,2} \setminus \{0\}} \Big(\frac{\sup_{Q_1(0)} |q|}{\sup_{\Gamma_h} |q|} \Big) =: C < \infty.$$

Proof of Lemma 6.9. Assuming, on the contrary, that the statement is false, using a compactness argument we find a polynomial $q \in \mathcal{P}_{k,2} \setminus \{0\}$ such that $q|_{\Gamma_h \cap Q_1(0)} = 0$. Therefore considering p and q in complex variables $(z, w) \in \mathbb{C}^2$ we obtain that $q(z, w)$ equals zero on the complex irreducible curves $\{(z, w) \in \mathbb{C}^2;\ w = p(z)\}$ and $\{(z, w) \in \mathbb{C}^2;\ w = 0\}$. Since p is of degree k, for each w from a Zariski open subset U of $\mathbb{C}$ the equation $p(z) = w$ has k distinct solutions $z_1(w), \ldots, z_k(w) \in \mathbb{C}$. Thus the univariate holomorphic polynomial $q(\cdot, w)$ being of degree at most k has roots $z_1(w), \ldots, z_k(w)$. This implies that $q(z, w) = r(w)(p(z) - w)$ for all $z \in \mathbb{C}$ and $w \in U$. Since $d^k q/dz^k = const$, we get that r is constant on the (connected) open set U. Also, $q(z, 0) = 0$ for all $z \in \mathbb{C}$. Hence, $r = 0$ (because $p \not\equiv 0$). We conclude that $q \equiv 0$, a contradiction. $\qquad \square$

Let us consider squares $Q_{t_{2j}}(0) \subset \mathbb{R}^2$, $j \in \mathbb{Z}_+$. Observe that, by the definition of f, the graph of $t_{2j}^{-1} \cdot f(t_{2j}x)$, $x \in [1/3, 1]$, coincides with Γ_h for all $j \in \mathbb{Z}_+$. Hence, by Lemma 6.9 for each $q \in \mathcal{P}_{k,2} \setminus \{0\}$ we have

$$\frac{\sup_{Q_{t_{2j}}(0)} |q|}{\sup_{\Gamma_f \cap Q_{t_{2j}}(0)} |q|} \leq \frac{\sup_{(x,y) \in Q_1(0)} |q(t_{2j}x, t_{2j}y)|}{\sup_{(x,y) \in \Gamma_h \cap Q_1(0)} |q(t_{2j}x, t_{2j}y)|} \leq C.$$

This and Definition 1.1 imply that $(0,0) \in \Gamma_f$ is weak k-Markov.

Now, let us show that $(0,0) \in \Gamma_f$ is not weak $(k+1)$-Markov. Suppose that this is wrong. Then there exists a convergent to zero sequence of positive numbers $\{r_i\}_{i \in \mathbb{N}}$ such that

(6.16)
$$\sup_{i \in \mathbb{N}} \left\{ \sup_{q \in \mathcal{P}_{k+1,2} \setminus \{0\}} \Big(\frac{\sup_{Q_{r_i}(0)} |q|}{\sup_{\Gamma_f \cap Q_{r_i}(0)} |q|} \Big) \right\} =: c < \infty.$$

For each i by $t_{2j(i)} \in \{t_{2j}\}_{j \in \mathbb{Z}_+}$ we denote the maximal number not exceeding r_i.

Applying (6.16) to $q(x, y) := y$ we obtain

$$\sup_{i \in \mathbb{N}} \left\{ \frac{\sup_{(x,y) \in Q_{r_i}(0)} |y|}{\sup_{(x,y) \in \Gamma_f \cap Q_{r_i}(0)} |y|} \right\} = \sup_{i \in \mathbb{N}} \left\{ \frac{r_i}{t_{2j(i)}} \right\} \leq c.$$

Thus

(6.17) $$t_{2j(i)} \leq r_i \leq c\, t_{2j(i)} \quad \text{for all } i \in \mathbb{N}.$$

Consider the polynomial $q_i(x, y) := y \big(y - t_{2j(i)} \cdot p\,(t_{2j(i)}^{-1} \cdot x) \big) \in \mathcal{P}_{k+1,2}$. Then $\sup_{Q_{r_i}(0)} |q_i| \geq |q_i(r_i, t_{2j(i)})| = r_i^2$. Also, inequality (6.17) and definition of the sequence $\{t_n\}$ imply that for all sufficiently large i,

$$\frac{1}{2} \cdot t_{2(j(i)-1)} \geq c\, t_{2j(i)} \geq r_i.$$

By the definition of f for such i we have

$$\Gamma_f \cap \big\{ [t_{2j(i)}, r_i] \times [-r_i, r_i] \big\} \subset \big\{ (x, y) \in \mathbb{R}^2 \,;\, y = 0 \big\}.$$

In turn, this implies that (recall that $\sup_{[-1,1]} |p| \leq 1$)

$$\sup_{\Gamma_f \cap Q_{r_i}(0)} |q_i| = \sup_{\Gamma_f \cap \big\{ [0, t_{2(j(i)+1)}] \times [-r_i, r_i] \big\}} |q_i|$$

$$= \sup_{\Gamma_f \cap Q_{t_{2(j(i)+1)}}(0)} |q_i| \leq \sup_{Q_{t_{2(j(i)+1)}}(0)} |q_i| \leq 2 \cdot t_{2j(i)} \cdot t_{2(j(i)+1)}.$$

From these estimates we obtain

$$\varlimsup_{i \to \infty} \left(\frac{\sup_{Q_{r_i}(0)} |q_i|}{\sup_{\Gamma_f \cap Q_{r_i}(0)} |q_i|} \right) \geq \varlimsup_{i \to \infty} \frac{r_i^2}{2 \cdot t_{2j(i)} \cdot t_{2(j(i)+1)}} \geq \varlimsup_{i \to \infty} \frac{t_{2j(i)}}{2 \cdot t_{2(j(i)+1)}} = \infty.$$

This contradicts (6.16) and shows that $(0,0) \in \Gamma_f$ is not weak $(k+1)$-Markov. $\square$

Proposition 6.10. *There exists a bounded Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ whose graph $\Gamma_\varphi \in \mathrm{Mar}_\infty^*(\mathbb{R}^2)$.*

Proof. According to Theorem 6.6, there exists a uniformly bounded sequence of Lipschitz functions $\{\varphi_i\}_{i \in \mathbb{N}}$ on $\mathbb{R}$ with uniformly bounded Lipschitz constants[1] such that φ_i is differentiable outside a countable dense subset $S_i \subset \mathbb{R}$ and $\Gamma_{\varphi_i} \subset \mathbb{R}^2$ is weak i-Markov, $i \in \mathbb{N}$. We choose a sequence of numbers $\{c_i\}_{i \in \mathbb{N}}$ such that $\{c_i + S_i\} \cap \{c_j + S_j\} = \emptyset$ for all $i \neq j$ (e.g., choose $\{c_i\}_{i \in \mathbb{N}}$ to be a linearly independent family over a vector space over $\mathbb{Q}$ generated by all S_i, $i \in \mathbb{N}$), and define a bounded Lipschitz function

$$\varphi(x) := \sum_{i=1}^{\infty} \frac{\varphi_i(x - c_i)}{i^2}.$$

Then φ is differentiable outside the countable set $\cup_{i=1}^{\infty} \{S_i + c_i\}$. As in the proof of Theorem 6.6 one shows that Γ_φ is weak i-Markov for all $i \in \mathbb{N}$ because each $\{S_i + c_i\}$ is dense in $\mathbb{R}$. Thus $\Gamma_\varphi \in \mathrm{Mar}_\infty^*(\mathbb{R}^2)$. $\square$

---

[1] For a Lipschitz function $f \colon \mathbb{R} \to \mathbb{R}$ its Lipschitz constant is $L(f) := \sup_{x \neq y} |f(x) - f(y)|/|x - y|$.

Example 6.11. Point $(0,0) \in \mathbb{R}^2$ is weak ∞-Markov for the graph of $f(x) = x \sin(1/x)$ (*exercise for the reader*). However, it is weak 0-Markov only for graphs of functions $g(x) = h(x) \sin(1/x)$, where $h \in C(\mathbb{R}_+)$ and $h(x) = o(x)$ as $x \to 0^+$.

Let us prove now that classes $\mathrm{Mar}_k^*(\mathbb{R}^n)$, $n \geq 2$, $k \in \mathbb{N} \cup \{\infty\}$, are not invariant under bi-Lipschitz maps.

Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $\Gamma_\varphi \in \mathrm{Mar}_k^*(\mathbb{R}^2)$ for some $k \in \mathbb{N} \cup \{\infty\}$. We set $S := \Gamma_\varphi \times \mathbb{R}^{n-2} \subset \mathbb{R}^n$ (if $n = 2$, then $S := \Gamma_\varphi$). Due to property (5) of Section 2, $S \in \mathrm{Mar}_k^*(\mathbb{R}^n)$. We define a bi-Lipschitz map $F \colon \mathbb{R}^n \to \mathbb{R}^n$,

$$F(x,y,z) := (x, y - \varphi(x), z), \quad (x,y) \in \mathbb{R}^2, \ z \in \mathbb{R}^{n-2}.$$

Then $F(S) := \{(x, 0, z) \in \mathbb{R}^n; x \in \mathbb{R}, \ z \in \mathbb{R}^{n-2}\}$. If $\mathrm{Mar}_k^*(\mathbb{R}^n)$ is invariant under F, then $F(S) \in \mathrm{Mar}_k^*(\mathbb{R}^n)$ as well. However, $F(S) \notin \mathrm{Mar}_1^*(\mathbb{R}^n)$ as the set of zeros of the linear polynomial $y \in \mathcal{P}_{1,n}$. This proves the second part of property (11).

7. Proofs of Propositions 3.2, 3.3 and Theorem 3.4

Proof of Proposition 3.2. First, let us show that the polynomial $T_x^m(f)$ in Definition 3.1 is unique. Suppose $\tilde{T}_x^m(f)$ and $T_x^m(f)$ are polynomials subject to this definition. Then $|\tilde{T}_x^m(f)(y) - T_x^m(f)(y)| = o(\|y - x\|_\infty^m)$. By the definition of a weak k-Markov point, there exists a convergent to zero sequence of positive numbers $\{r_i\}_{i \in \mathbb{N}}$ and a constant $c > 0$ such that, for all i,

$$\sup_{Q_{r_i}(x)} |\tilde{T}_x^m(f) - T_x^m(f)| \leq c \cdot \sup_{S \cap Q_{r_i}(x)} |\tilde{T}_x^m(f) - T_x^m(f)|.$$

On the other hand, by the one-dimensional Remez inequality (cf. (6.1)),

$$\sup_{Q_1(x)} |\tilde{T}_x^m(f) - T_x^m(f)| \leq \left(\frac{4\sqrt{n}}{r_i}\right)^m \cdot \sup_{Q_{r_i}(x)} |\tilde{T}_x^m(f) - T_x^m(f)|.$$

Combining these inequalities we get

$$\sup_{Q_1(x)} |\tilde{T}_x^m(f) - T_x^m(f)| \leq c \cdot \left(\frac{4\sqrt{n}}{r_i}\right)^m \cdot o(r_i^m) \longrightarrow 0 \quad \text{as} \quad i \to \infty.$$

Thus $\tilde{T}_x^m(f) = T_x^m(f)$.

Now, if $\tilde{f} \colon \mathbb{R}^n \to \mathbb{R}$ has derivatives of order $\leq m$ at x and $T_x^m(\tilde{f})$ is its Taylor polynomial of degree m at x, then due to Definition 3.1, the function $f := \tilde{f}|_S$ has derivatives of order $\leq m$ at x, and $T_x^m(f) = T_x^m(\tilde{f})$ by the uniqueness just proved. So let us prove the converse statement. To this end choose a retraction map $\psi \colon \mathbb{R}^n \to S$ such that $\|y - \psi(y)\|_\infty \leq 2 \cdot \mathrm{dist}_\infty(y, S)$ for all $y \in \mathbb{R}^n$. We define

$$\hat{f}(y) := \frac{f(y) - T_x^m(f)(y)}{\|y - x\|_\infty^m}, \quad y \in S \setminus \{x\}, \quad \hat{f}(x) := 0, \quad \text{and}$$

$$\tilde{f}(y) := T_x^m(f)(y) + \|y - x\|_\infty^m \cdot (\hat{f} \circ \psi)(y), \quad y \in \mathbb{R}^n.$$

Then $\tilde{f}$ is the required extension of f. To prove this we must check that $\hat{f} \circ \psi$ is continuous at x. (Since $(\hat{f} \circ \psi)(x) = 0$, this will imply the required statement.)

For otherwise, there exists a sequence $\{y_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^n$ tending to x such that $\overline{\lim}_{i \to \infty} |(\hat{f} \circ \psi)(y_i)| \neq 0$. By definition, $\|y_i - \psi(y_i)\|_\infty \leq 2 \cdot \mathrm{dist}_\infty(y_i, S) \leq 2 \cdot \|y_i - x\|_\infty$. Hence, $\{\psi(y_i)\}_{i \in \mathbb{N}} \subset S$ tends to x as well. But $\hat{f}$ is continuous (and equals zero) at x by Definition 3.1. This implies that $\overline{\lim}_{i \to \infty} |(\hat{f} \circ \psi)(y_i)| = \lim_{i \to \infty} |\hat{f}(\psi(y_i))| = 0$, a contradiction showing that $\hat{f} \circ \psi$ is continuous and equals zero at x. $\square$

Proof of Proposition 3.3. Let $\{r_i\}_{i \in \mathbb{N}}$ and $F_i \subset S \cap Q_{r_i}(x)$, $\mathrm{card}\, F_i = \dim \mathcal{P}_{k,n}$, $i \in \mathbb{N}$, be the same as in property (9) for x, see (2.1). Perturbing each F_i, without loss of generality we may assume that $x \notin F_i$ for all $i \in \mathbb{N}$. Since inequalities of property (9) are also valid for polynomials in $\mathcal{P}_{m,n}$, by Proposition 3.5 in [3] there exist subsets $F_i^m \subset F_i$, $\mathrm{card}\, F_i = \dim \mathcal{P}_{m,n} (=: d_{m,n})$, $i \in \mathbb{N}$, such that

$$(7.1) \qquad \sup_{i \in \mathbb{N}} \left\{ \sup_{p \in \mathcal{P}_{m,n} \setminus \{0\}} \left(\frac{\sup_{Q_{r_i}(x)} |p|}{\sup_{F_i^m} |p|} \right) \right\} =: c < \infty.$$

We set $\{r_i\}_{i \in \mathbb{N}}$ and $\{F_i^m\}_{i \in \mathbb{N}}$ to be the required objects of the proposition.

Next, from (7.1) follows that each F_i^m is a minimal interpolating set for polynomials of degree m on $\mathbb{R}^n$. If $F_i^m := \{x_{i1}, \ldots, x_{id_{m,n}}\}$ by $\ell_{ij} \in \mathcal{P}_{m,n}$ we denote a polynomial equal to 1 at x_{ij} and 0 at all other points x_{is}, $s \neq j$. Then we define

$$(7.2) \qquad \mu_i^\alpha(\{x_{ij}\}) := \frac{1}{\alpha!} \cdot D^\alpha \ell_{ij}(x), \quad i \in \mathbb{N}, \ 1 \leq j \leq d_{m,n}, \ |\alpha| \leq m.$$

Let us prove (3.1). By definition,

$$\int_{F_i^m} f \, d\mu_i^\alpha = \sum_{j=1}^{d_{m,n}} f(x_{ij}) \cdot \frac{D^\alpha \ell_{ij}(x)}{\alpha!} = \frac{1}{\alpha!} \cdot D^\alpha \left(\sum_{j=1}^{d_{m,n}} f(x_{ij}) \cdot \ell_{ij} \right)(x)$$

$$= \frac{D^\alpha(P_i(f))(x)}{\alpha!},$$

where $P_i(f) \in \mathcal{P}_{m,n}$ is such that $P_i(f)(x_{ij}) := f(x_{ij})$ for all $1 \leq j \leq d_{m,n}$. Thus to prove the required statement it suffices to show that the sequence of polynomials $\{P_i(f)\}_{i \in \mathbb{N}}$ converges (uniformly on a fixed cube) to T_x^m. To this end consider polynomials $P_i(f) - T_x^m$, $i \in \mathbb{N}$. Since $P_i(f)|_{F_i^m} = f$, there exists a convergent to zero sequence of positive numbers $\{c_i\}_{i \in \mathbb{N}}$ such that, for all i,

$$\max_{y \in F_i^m} \frac{|P_i(f)(y) - T_x^m(y)|}{\|y - x\|_\infty^m} \leq c_i.$$

From these inequalities arguing as in the proof of Proposition 3.2 we obtain that

$$\sup_{Q_1(x)} |P_i(f) - T_x^m| \leq \tilde{c}_i$$

for some convergent to zero sequence $\{\tilde{c}_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_+$. This shows that $\{P_i(f)\}_{i \in \mathbb{N}}$ converges uniformly on $Q_1(x)$ to T_x^m and completes the proof. $\square$

Proof of Theorem 3.4. Inequalities (3.2) and the Whitney–Glaeser theorem (see, e.g., Volume I, Section 2.2 in [2]) imply that there exists a function $F \in C^{m,\omega}(\mathbb{R}^n)$ of norm at most $c(m,n)\lambda$ such that $T_x^m(F) = T_x^m(f)$ for all weak k-Markov points $x \in S$. In particular, $F = f$ at weak k-Markov points of S. Since the set of such points is dense in S and f is continuous, $F|_S = f$, i.e., $f \in C^{m,\omega}(S)$. Conversely, if $f \in C^{m,\omega}(S)$, then by the Whitney–Glaeser theorem it satisfies inequalities (3.2).

Also, due to Proposition 3.2, $T_x^m(f)$ coincides with $T_x^m(F)$ for any $C^{m,\omega}$ extension F of f. Therefore by the Whitney–Glaeser theorem applied to an almost optimal $C^{m,\omega}$ extension of f, $\inf \lambda \approx \|f\|_{C^{m,\omega}(S)}$ with constants of equivalence depending on m and n only. □

8. Proofs of Theorems 4.3, 4.4 and 4.5

Proof of Theorem 4.3. Since $f \in C^{m+1}(S)$ and S is compact, there exists a function $\hat{f} \in C^{m+1}(\mathbb{R}^n)$ with compact support such that $\hat{f}|_S = f$. We prove the theorem by induction on m.

Assume first that $m = 1$, i.e., $D_S^\alpha f = 0$ for all $|\alpha| = 1$. For each $D_j \subset \Omega$ consider the d-closed 1-form $\lambda_j := d(\hat{f}|_{D_j})$. We extend λ_j to $(D_j)^c$ by zero denoting the extended form $\hat{\lambda}_j$.

Lemma 8.1. *There exist a modulus of continuity ω and a constant $C > 0$ such that if $\hat{\lambda}_j = \sum_{i=1}^n a_{ij}\,dx_i$, then all $a_{ij} \in C^{1,\omega}(\mathbb{R}^n)$ and $\sup_{\ell,\,1 \le i \le n} \|\sum_{j=1}^\ell a_{ij}\|_{C^{1,\omega}(\mathbb{R}^n)} \le C$, and all $\hat{\lambda}_j$ are d-closed.*

Proof of Lemma 8.1. Since $\hat{f} \in C_c^2(\mathbb{R}^n)$, there exists a modulus of continuity ω such that $\hat{f} \in C^{2,\omega}(\mathbb{R}^n)$. Let us prove the statement for such ω and $C := \|\hat{f}\|_{C^{2,\omega}(\mathbb{R}^n)}$.

Let $d\hat{f} := \sum_{i=1}^n c_i\,dx_i$. By the hypothesis, all c_i and their partial derivatives $\partial c_i/\partial x_j$ vanish on S. Moreover, $a_{ij}|_{D_j} := c_i|_{D_j}$. Hence, for all admissible $\ell \in \mathbb{N}$, $\sum_{j=1}^\ell a_{ij}(x)$ and $\sum_{j=1}^\ell \nabla a_{ij}(x)$ tend to 0 uniformly in x as $D_\ell' := \cup_{j=1}^\ell D_j \ni x \to \cup_{j=1}^\ell (\bar{D}_j \setminus D_j)\ (\subset S)$. This and the definition of a_{ij} imply that $a_{i\ell}' := \sum_{j=1}^\ell a_{ij} \in C_c^1(\mathbb{R}^n)$. Moreover, $\|a_{i\ell}'\|_{C^{1,\omega}(D_\ell')} = \|c_i|_{D_\ell'}\|_{C^{1,\omega}(D_\ell')} \le C$ and if $x_1 \in D_\ell'$ and $x_2 \in (D_\ell')^c$, then

$$
\sup_{\mathbb{R}^n} |a_{i\ell}'| + \max_{|\alpha|=1} \frac{|D^\alpha a_{i\ell}'(x_1) - D^\alpha a_{i\ell}'(x_2)|}{\omega(\|x_2 - x_1\|_\infty)} \le \sup_{\mathbb{R}^n} |c_i| + \max_{|\alpha|=1} \frac{|D^\alpha c_i(x_1) - D^\alpha c_i(z)|}{\omega(\|x_2 - x_1\|_\infty)}
$$
$$
\le \left(1 - \frac{\omega(\|z - x_1\|_\infty)}{\omega(\|x_2 - x_1\|_\infty)}\right) \sup_{\mathbb{R}^n} |c_i|
$$
$$
+ \frac{\omega(\|z - x_1\|_\infty)}{\omega(\|x_2 - x_1\|_\infty)} \left(\sup_{\mathbb{R}^n} |c_i| + \max_{|\alpha|=1} \frac{|D^\alpha c_i(x_1) - D^\alpha c_i(z)|}{\omega(\|z - x_1\|_\infty)} \right) \le C
$$

for some $z \in [x_1, x_2] \cap \overline{D_\ell'} \setminus D_\ell'\ (\ne \emptyset)$.

These show that $a_{ij} \in C^{1,\omega}(\mathbb{R}^n)$ and $\sup_{\ell,\,1 \le i \le n} \|a_{i\ell}'\|_{C^{1,\omega}(\mathbb{R}^n)} \le C$.

The closedness of $\hat{\lambda}_j$ follows from its definition. □

Let us fix a point $o \notin \bar{\Omega}$ and define

$$g_j(x) := \int_o^x \hat{\lambda}_j, \quad x \in \mathbb{R}^n.$$

Since $\hat{\lambda}_j$ is d-closed, the integral is independent of the path joining o and x. In particular, since by our assumption $(\bar{D}_j)^c$ is connected and $\partial D_j = \partial(\bar{D}_j)^c$, the continuous function g_j is equal to zero on $(D_j)^c$ and $\sup_{\mathbb{R}^n} |g_j| \le C'$ for some $C' > 0$ independent of j. Next, due to Lemma 8.1 the function $g'_\ell := \sum_{j=1}^\ell g_j$ equals zero on $(D'_\ell)^c$ and satisfies $\|g'_\ell\|_{C^{2,\omega}(\mathbb{R}^n)} \le C''$ for some $C'' > 0$ independent of ℓ. Also, by our construction, $d(\hat{f}|_{D'_\ell} - g'_\ell) = 0$ for all ℓ.

Further, according to a compactness result for bounded sequences in $C^{2,\omega}(\mathbb{R}^n)$, see, e.g. Volume II, Proposition 5.4 in [2], there exists a subsequence $\{g'_{\ell_p}\}_{p \in P}$ of $\{g'_\ell\}$ and a function $g \in C^{2,\omega}(\mathbb{R}^n)$ such that for every closed cube $Q \subset \mathbb{R}^n$,

$$\lim_{p \in P} \|g - g'_{\ell_p}\|_{C^2(Q)} = 0.$$

In particular, g equals zero on $(\cup_j D_j)^c$ and $d(\hat{f}|_\Omega - g|_\Omega) = 0$ (since $\bar{\Omega} := (\cup_j D_j) \sqcup S$ and $d_S(\hat{f}|_S) = 0$). Connectedness of Ω implies that $\hat{f}|_{\bar{\Omega}} - g|_{\bar{\Omega}} = const$. On the other hand, $\hat{f}|_S = f$ and $g|_S = 0$. This shows that f is the restriction to S of a (unique) polynomial of degree zero, as required. As a corollary, we get immediately that S is connected. This proves the case $m = 1$.

Now, assuming that the theorem is valid for $m - 1 \ge 1$ we prove it for m.

To this aim we consider functions $D_S^\alpha f$, $|\alpha| = m - 1$. Then $d_S(D_S^\alpha f) = 0$ and by the induction hypothesis, $D_S^\alpha f = const$ for all α. Since $D_S^\alpha f = D_S^{\pi(\alpha)} f$ for any permutation π defined on the set of coordinates of α, there exists a unique homogeneous polynomial $p \in \mathcal{P}_{m-1,n}$ such that $D^\alpha p = D_S^\alpha f$ for all $|\alpha| = m - 1$. In particular, $D^\alpha(f - p|_S) = 0$ for all $|\alpha| = m - 1$ and $f - p|_S \in C^{m+1}(S)$. Applying to this function the induction hypothesis we obtain that $f - p|_S = q|_S$ for a unique $q \in \mathcal{P}_{m-2,n}$. Thus f is the restriction to S of the (unique) polynomial $p_f := p + q \in \mathcal{P}_{m-1,n}$. $\square$

Proof of Theorem 4.4. Since $f \in C_\mathbb{C}^2(S)$ and S is compact, there exist a modulus of continuity ω and a function $\hat{f} \in C_\mathbb{C}^{2,\omega}(\mathbb{C}^n) := C^{2,\omega}(\mathbb{R}^{2n}) \otimes \mathbb{C}$ with compact support such that $\hat{f}|_S = f$. We set $\eta := \bar{\partial}(\hat{f}|_{\bar{\Omega}})$ and by $\hat{\eta}$ denote its extension to $\mathbb{C}^n$ by zero. Then repeating literally the proof of Lemma 8.1 we obtain that if $\hat{\eta} := \sum_{j=1}^n a_j \, d\bar{z}_j$, then all $a_j \in C_\mathbb{C}^{1,\omega}(\mathbb{C}^n)$ and $\hat{\eta}$ is a $\bar{\partial}$-closed $(0,1)$-form on $\mathbb{C}^n$ with support in $\overline{(\cup_{j \ge 1} D_j)}$.

Lemma 8.2. *There exists a function $g \in C_\mathbb{C}^1(\mathbb{C}^n)$ equals zero on $S \cup \bar{\Omega}^c$ such that $\bar{\partial}g = \hat{\eta}$.*

Proof of Lemma 8.2. We set

$$\delta(z) := dz_1 \wedge \cdots \wedge dz_n \quad \text{and} \quad \delta'(z) := \sum_{k=1}^n (-1)^{k-1} dz_1 \wedge \cdots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \cdots \wedge dz_n.$$

Without loss of generality we may assume that $\bar{\Omega} \subset B$, the open unit Euclidean ball in $\mathbb{C}^n$. Then the function

$$(8.1) \qquad g(z) := \int_{\xi \in B} \hat{\eta}(\xi) \wedge \delta'\Big(\frac{\bar{\xi} - \bar{z}}{\|\xi - z\|^2}\Big) \wedge \delta(\xi), \quad z \in \mathbb{C}^n,$$

satisfies $\overline{\partial}(g|_B) = \hat{\eta}|_B$, see Section 6.4, Theorem 6.9 in [17]; here $\| \cdot \|$ is the the ℓ_2-norm on $\mathbb{C}^n$. Since $\mathrm{supp}\,\hat{\eta} \Subset B$, $g \in C^1_{\mathbb{C}}(\mathbb{C}^n)$, and moreover, is holomorphic on $B \setminus \mathrm{supp}\,\hat{\eta}$. Further, (8.1) shows that g is complex analytic on $(\mathrm{supp}\,\hat{\eta})^c$ and $\lim_{z \to \infty} g(z) = 0$. Therefore by the classical Hartogs theorem in $\mathbb{C}^n$, $n \geq 2$, the function g is holomorphic on $(\mathrm{supp}\,\hat{\eta})^c$ and equals zero in the connected component of this set containing $(\bar{B})^c$.

On the other hand, we can write $\hat{\eta} = \sum_{j \geq 1} \hat{\eta}_j$, where $\hat{\eta}_j$ is extension by zero of $\eta|_{D_j}$. Then we rewrite (8.1) as

$$g(z) = \int_{\xi \in B} \sum_{j \geq 1} \hat{\eta}_j(\xi) \wedge \delta'\Big(\frac{\bar{\xi} - \bar{z}}{\|\xi - z\|^2}\Big) \wedge \delta(\xi)$$

$$= \sum_{j \geq 1} \int_{\xi \in B} \hat{\eta}_j(\xi) \wedge \delta'\Big(\frac{\bar{\xi} - \bar{z}}{\|\xi - z\|^2}\Big) \wedge \delta(\xi).$$

(The change of the order is possible because the series on the right is absolutely convergent.) Since each $\hat{\eta}_j$ is of class C^1 and $\overline{\partial}$-closed on $\mathbb{C}^n$ (cf. the proof of Lemma 8.1) and $(\bar{D}_j)^c$ is connected and $\partial D_j = \partial(\bar{D}_j)^c$, each term, say g_j, on the right-hand side of the above identity is zero on $(D_j)^c$ and satisfies $\overline{\partial} g_j = \hat{\eta}_j$ on $\mathbb{C}^n$ (cf. a similar argument above for g). Hence, $g = \sum_{j \geq 1} g_j$ equals zero on $\cup_{j \geq 1}(D_j)^c = S \sqcup \bar{\Omega}^c$ as required. $\qquad\square$

Now, consider the function $F := \hat{f}|_{\bar{\Omega}} - g|_{\bar{\Omega}}$. Then by Lemma 8.2, $F \in C^1_{\mathbb{C}}(\bar{\Omega})$, $F|_S = f$ and $\overline{\partial} F|_\Omega = 0$. Thus $F \in \mathcal{O}(\Omega) \cap C^1_{\mathbb{C}}(\bar{\Omega})$. Further, by the maximum modulus principle, $\sup_{D_j} |F| = \sup_{\bar{D}_j \setminus D_j} |F|$ for all j. Moreover, $\cup_{j \geq 1}(\bar{D}_j \setminus D_j) \subset S$ and $\bar{\Omega} := (\cup_{j \geq 1} D_j) \cup S$. Therefore $\sup_{\bar{\Omega}} |F| = \sup_S |f|$. This implies that such an extension F is unique. $\qquad\square$

Proof of Theorem 4.5. Let $\hat{f} \in C^{2,\alpha}(\mathbb{R}^n)$ be a function with compact support such that $\hat{f}|_S = f$. Consider the functions $\tilde{f}_1$, obtained as extension of $(\Delta \hat{f})|_{\overline{(\cup_j D_j)}^c}$ by zero to $\mathbb{R}^n$, and $\tilde{f}_2$, obtained as extension of $(\Delta \hat{f})|_{\overline{\cup_j D_j}}$ by zero to $\mathbb{R}^n$. Then from the hypotheses of the theorem as in the proof of Theorem 4.3 one obtains that $\tilde{f}_i \in C^{0,\alpha}(\mathbb{R}^n)$, $i = 1, 2$, and have compact supports. By the definition, $\Delta \hat{f} = \tilde{f}_1 + \tilde{f}_2$ (because $(\Delta \hat{f})|_S = 0$).

We set

$$F_i(x) := \frac{1}{n(2-n)\omega_n} \int_{\mathbb{R}^n} \frac{\tilde{f}_i(y)}{\|x - y\|^{n-2}} \, dy_1 \wedge \cdots \wedge dy_n,$$

where ω_n is the volume of the unit Euclidean ball in $\mathbb{R}^n$ and $\| \cdot \|$ is the ℓ_2-norm on $\mathbb{R}^n$. Then according to the Schauder theorem, see, e.g., Chapter 4 in [14],

$F_i \in C^{2,\alpha}(\mathbb{R}^n)$ and $\Delta F_i = \tilde{f}_i$, $i = 1, 2$. Moreover, $\lim_{x \to \infty} F_i = 0$ (because $n \geq 3$ and $\tilde{f}_i$ have compact supports) and F_1, F_2 are harmonic in $\cup_{j \geq 1} D_j$ and $(\overline{\cup_{j \geq 1} D_j})^c$, respectively. By our construction $\Delta(F_1 + F_2 - \hat{f}) = 0$, i.e., $F_1 + F_2 - \hat{f}$ is a harmonic function having limit zero at infinity. Thus by the maximum principle $F_1 + F_2 - \hat{f} = 0$. In particular, $(F_1 + F_2)|_S = f$, as required.

If $F_1', F_2' \in C^{2,\alpha}(\mathbb{R}^n)$ is another pair of functions satisfying the conclusion of the theorem, then $F_1 - F_1' = F_2 - F_2'$ on S and $F_1 - F_1'$, $F_2 - F_2'$ are harmonic in $\cup_{j \geq 1} D_j$ and $(\overline{\cup_{j \geq 1} D_j})^c$, respectively. Since $F_i - F_i' \in C^{2,\alpha}(\mathbb{R}^n)$ and S is weak k-Markov with $k \geq 2$, the latter implies that the function $H \in C(\mathbb{R}^n)$ equals $F_1 - F_1'$ on $\overline{\cup_{j \geq 1} D_j}$ and $F_2 - F_2'$ on $(\overline{\cup_{j \geq 1} D_j})^c$ is harmonic and has limit zero at infinity. Thus $H = 0$. This proves the uniqueness part of the theorem. $\square$

9. Proofs of results of Subsection 5.1

Proof of Proposition 5.2. Let $V = (V_\beta)_{\beta \in B}$ be an open cover of S. Consider the open cover $W = (W_\gamma)_{\gamma \in \Gamma}$ consisting of all possible sets $V_\beta \cap U_\alpha$ where $(U_\alpha, \varphi_\alpha) \in \mathcal{D}$ is a chart. Since S is paracompact of covering dimension $\leq n$ having a base of relatively compact open subsets, there exist locally finite open refinements $W' = (W_i')_{i \in I}$ of W and $W'' = (W_i'')_{i \in I}$ of W' such that the order of W' is $\leq n + 1$ and $W_i'' \Subset W_i'$ for each $i \in I$, see, e.g. arguments in the proof of Lemma 1.4 in [30].

To prove the result it suffices for each $i \in I$ to construct a nonnegative C^k function g_i on S such that $g_i > 0$ on $\bar{W}_i''$ and $\operatorname{supp} g_i \subset W_i'$. Then

$$\rho_i := \frac{g_i}{\sum_{s \in I} g_s}, \quad i \in I,$$

is the required C^k partition of unity.

To this end for each $\bar{W}_i'' \subset W_i'$ choose $(U_{\alpha(i)}, \varphi_{\alpha(i)}) \in \mathcal{D}$ such that $W_i' \subset U_{\alpha(i)}$. Then $\varphi_{\alpha(i)}(\bar{W}_i'')$ is a compact subset of the relatively open subset $\varphi_{\alpha(i)}(W_i')$ of a weak k-Markov set $S_{\alpha(i)}$ of $\mathbb{R}^n$. In a standard way we construct a nonnegative C^k function $\tilde{g}_i$ on $\mathbb{R}^n$ positive on $\varphi_{\alpha(i)}(\bar{W}_i'')$ such that $(\operatorname{supp} \tilde{g}_i) \cap S_{\alpha(i)} \Subset \varphi_{\alpha(i)}(W_i')$, see, e.g., [30]. Finally, we determine $g_i := \varphi_{\alpha(i)}^* \tilde{g}_i$. $\square$

Proof of Proposition 5.3. By definitions of the objects involved, it suffices to prove the following version of the proposition:

If $S_1, S_2 \in \operatorname{Mar}_k^*(\mathbb{R}^n)$, $k \geq 1$, *and* $f : S_1 \to S_2$ *is a* C^k *diffeomorphism, then* $f^{-1} : S_2 \to S_1$ *is a* C^k *diffeomorphism as well.*

In this case the definition of the C^k diffeomorphism and the (classical) inverse function theorem in $\mathbb{R}^n$ imply that there exist open neighbourhoods U_1 of S_1 and U_2 of S_2 and a C^k extension $\hat{f} : U_1 \to U_2$ of f which is a C^k diffeomorphism. Therefore $\hat{f}^{-1} : U_2 \to U_1$ is a C^k diffeomorphism of manifolds, see, e.g., [30]. Hence, $f^{-1} := \hat{f}^{-1}|_{S_2} : S_2 \to S_1$ is a C^k diffeomorphism by definition. $\square$

Proof of Proposition 5.4. Since the covering dimension of S is at most n and S is separable, it can be covered by at most $n + 1$ charts $(U_i, \varphi_i)_{1 \leq i \leq m}$ so that each U_i is disjoint union of at most countable family of mutually disjoint open sets

$V_{ij} \Subset S$ such that $\varphi_i(V_{ij}) \Subset \mathbb{R}^n$ is of diameter (measured in the Euclidean norm) at most $1/4$. Choose vectors $v_{ij} \in \mathbb{R}^n$ such that $v_{ij} + \varphi_i(V_{ij})$ contain $0 \in \mathbb{R}^n$. Let $L \subset \mathbb{R}^n$ be a one-dimensional subspace containing a unit vector e. Choose a sequence of points $\{x_{ij}\} \subset L$ enumerated by the same indices as the family $\{V_{ij}\}$ such that $x_{ij} := n_{ij}e$, $n_{ij} \in \mathbb{N}$. In particular, distances between distinct points of the sequence are greater than or equal to 1.

Let $\{\rho_{ij}\}$ be a C^k partition of unity subordinate to the open cover (V_{ij}) of S. Then $\rho_i := \sum_j \rho_{ij}$, $1 \leq i \leq m$, is a C^k partition of unity subordinate to the open cover (U_i). We define maps $\tilde{\varphi}_i : S \to \mathbb{R}^n$,

$$\tilde{\varphi}_i(x) := \rho_i(x)\big(x_i(x) + v_i(x) + \varphi_i(x)\big), \quad x \in S.$$

Here $x_i := \sum_j x_{ij}\chi_{ij}$ and $v_i := \sum_j v_{ij}\chi_{ij}$, where χ_{ij} is the characteristic function of V_{ij}.

Then we define a map $\Phi : S \to \mathbb{R}^{m(n+1)}$ by the formula

$$\Phi(x) := \big(\tilde{\varphi}_1(x), \ldots, \tilde{\varphi}_m(x), \rho_1(x), \ldots, \rho_m(x)\big).$$

Note that $\Phi(x) = \Phi(x')$ implies that $\rho_i(x) = \rho_i(x') \neq 0$ for some i and hence $x, x' \in U_i$. Assume that $x \in V_{ij}$ and $x' \in V_{ij'}$. Then $\rho_{ij}(x) =: \rho_i(x) = \rho_i(x') := \rho_{ij'}(x')$. In turn, this implies that $x_{ij} + v_{ij} + \varphi_i(x) = x_{ij'} + v_{ij'} + \varphi_i(x')$. Hence, if $j \neq j'$, then $1 \leq \|x_{ij} - x_{ij'}\|_2 = \|(v_{ij'} + \varphi_i(x')) - (v_{ij} + \varphi_i(x))\|_2 \leq 1/2$, a contradiction. Thus $j = j'$ and $\varphi_i(x) = \varphi_i(x')$, i.e., $x = x'$.

Now we check that $D\Phi$ is injective on each fibre of TS. At each point $x \in S$ the differential $D\Phi(x)$ sends a vector $v \in T_xS \subset TS$ of the tangent space at x to the following vector in $\mathbb{R}^n \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \cdots \times \mathbb{R} \subset T_{\Phi(x)}\mathbb{R}^{m(n+1)}$:

$$
\begin{aligned}
(D\Phi(x))(v) := \big(&(D\rho_1(x))(v)(x_1(x) + v_1(x) + \varphi_1(x)) + \rho_1(x)(D\varphi_1(x))(v), \ldots, \\
&(D\rho_m(x))(v)(x_m(x) + v_m(x) + \varphi_m(x)) + \rho_m(x)(D\varphi_m(x))(v), \\
&(D\rho_1(x))(v), \ldots, (D\rho_m(x))(v)\big).
\end{aligned}
$$

Let $1 \leq s \leq m$ be such that $\rho_s(x) \neq 0$. Assuming that $(D\Phi(x))(v) = 0$ we get $(D\varphi_s(x))(v) = 0$, a contradiction.

Finally, let us show that Φ is proper, i.e., preimage under Φ of a compact subset of $\mathbb{R}^{m(n+1)}$ is compact in S. Assuming, on the contrary, that this is false we find a sequence $\{s_n\}_{n\in\mathbb{N}} \subset S$ without limit points such that $\{\Phi(s_n)\}_{n\geq 1} \subset \mathbb{R}^{m(n+1)}$ is bounded. This implies that the sequence $\big\{\sum_{i=1}^m \rho_i(s_n)x_i(s_n)\big\}_{n\geq 1} \subset L$ is bounded. We get from here that $\sup_{n\geq 1}\min_{i\in J_n}\|x_i(s_n)\|_2 < \infty$, where J_n is the set of indices $i \in \{1,\ldots,m\}$ for which $\rho_i(s_n) \neq 0$. Since $\|x_i(s_n)\|_2 \in \mathbb{N}$ for all $i \in J_n$ and $n \in \mathbb{N}$, the latter inequality implies that there exist a number i_0 and an infinite subsequence $\{s_{n_p}\}_{p\in\mathbb{N}}$ of $\{s_n\}$ such that $x_{i_0}(s_{n_p}) \neq 0$ and are equal for all $p \in \mathbb{N}$. This is possible only if all s_{n_p} belong to some $V_{i_0 j_0}$ contradicting our assumption (because $V_{i_0 j_0} \subset S$ is relatively compact but $\{s_{n_p}\}$ does not have limit points in S). Thus Φ is proper.

Lemma 9.1. *Let $S \in \mathrm{Mar}_k^*$, $k \geq 1$, and the rank of TS be n. Suppose $f : S \to M$ is a C^k map into a smooth Riemannian manifold M of $\dim M > n$. Then $f(S)$ has measure zero.*

Proof. Since S can be covered by at most countable family of charts it suffices to prove the result for $S \in \mathrm{Mar}_k^*(\mathbb{R}^n)$. Next, since f is C^k and M is a smooth manifold, there is an open neighbourhood $U \subset \mathbb{R}^n$ of S and a C^k map $\hat{f} : U \to M$ such that $\hat{f}|_S = f$. Then by the classical Sard theorem $\hat{f}(U)$ is of measure zero in M. Since $f(S) \subset \hat{f}(U)$ is union of a countable family of compact subsets, it is measurable and so has measure zero as well. $\square$

If $S \in \mathrm{Mar}_k^*$ with $k \geq 2$, starting with the proper embedding $\Phi \colon S \to \mathbb{R}^N$, $N := m(n+1) > 2n+1$, as above, we then show that by projecting onto a hyperplane it is possible to obtain an embedding into $\mathbb{R}^{N-1}$.

A vector $v \in \mathbb{S}^{N-1} \subset \mathbb{R}^N$ defines a hyperplane (the orthogonal complement) and let $P_v \colon \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the orthogonal projection onto this hyperplane. We show that the set of v for which $\Phi_v = P_v \circ \Phi$ fails to be a proper embedding is of measure zero, hence that it is possible to choose v for which Φ_v is a proper embedding.

The map Φ_v fails to be an embedding exactly when it is not injective or $D\Phi_v$ is not injective at some point. Let us consider the two failures separately:

If v is in the image of the map $\beta \colon (S \times S) \setminus \Delta_S \to \mathbb{S}^{n-1}$, where $\Delta_S \subset S \times S$ is the diagonal, given by

$$\beta(x_1, x_2) := \frac{\Phi(x_1) - \Phi(x_2)}{\|\Phi(x_1) - \Phi(x_2)\|_2},$$

then Φ_v will fail to be injective. Note however that β maps a space in Mar_k^*, $k \geq 2$, with rank of tangent bundle $2n$ into $(N-1)$-dimensional Riemannian manifold, and if $N > 2n+1$, then Lemma 9.1 implies that the image has measure zero.

The immersion condition is a local one, which we may analyze in a chart (U, φ). The map Φ_v will fail to be an immersion in U precisely when v coincides with a vector in the normalized image of $D(\Phi \circ \varphi^{-1})$, where

$$\Phi \circ \varphi^{-1} \colon \varphi(U) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^N.$$

Hence, we have a map (letting $N(w) = \|w\|_2$)

$$\frac{D(\Phi \circ \varphi^{-1})}{N \circ D(\Phi \circ \varphi^{-1})} \colon \varphi(U) \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{N-1}.$$

Note that since $k \geq 2$, this map is at least of class C^1. Thus, by Lemma 9.1, its image has measure zero as long as $2n - 1 < N - 1$, which is certainly true since $2n < N - 1$. Taking union over at most countably many charts, we see that immersion fails on a set of measure zero in $\mathbb{S}^{N-1}$.

In addition, Φ_v is not proper if there exists a sequence $\{s_n\} \subset S$ such that $\{\Phi(s_n)\} \subset \mathbb{R}^N$ converges to ∞ and $\{\Phi_v(s_n)\} \subset \mathbb{R}^{N-1}$ is bounded. This implies that limit points of $\{\Phi(s_n)/\|\Phi(s_n)\|_2\}$ are either v or $-v$. However, by the definition of Φ limit points of the latter sequence coincide with limit points of the sequence

$$\left\{ \frac{(\rho_1(s_n)x_1(s_n), \ldots, \rho_m(s_n)x_m(s_n), 0, \ldots, 0)}{\|\rho_1(s_n)x_1(s_n), \ldots, \rho_m(s_n)x_m(s_n), 0, \ldots, 0)\|_2} \right\}_{n \geq 1}$$

$$\subset \mathbb{S}^{N-1} \cap \left(L \times \cdots \times L \times \{0\} \times \cdots \times \{0\} \right) =: M \cong \mathbb{S}^{m-1}.$$

Since $m - 1 \leq n < N - 1$, M is a set of measure zero in $\mathbb{S}^{N-1}$. Thus for Φ_v to be proper it suffice to choose $v \notin M$.

We see that Φ_v fails to be a proper embedding for a set of $v \in \mathbb{S}^{N-1}$ of measure zero. Hence repeating the above arguments (with Φ replaced by a suitable Φ_v) we may reduce N all the way to $N = 2n + 1$.

Next, let us show that if $f : S \to \mathbb{R}^N$ is a proper embedding, then $f(S) \in \mathrm{Mar}_k^*$ and has weak k-Markov structure induced from $\mathbb{R}^N$.

Let (U, φ) be a chart on S. By definition the map $f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \mathbb{R}^N$ is a C^k embedding. In particular, there exist an open neighbourhood $O \subset \mathbb{R}^n$ of $\varphi(U)$ and a C^k embedding $\hat{f} : O \to \mathbb{R}^N$ which coincides with $f \circ \varphi^{-1}$ on $\varphi(U)$. Since f is a proper embedding, $f(U) \subset f(S)$ is open in the induced from $\mathbb{R}^N$ topology and for each point $x \in \varphi(U)$ there exists an open ball $B_x \Subset O$ centered at x such that $\hat{f}(B_x) \cap f(S) = (f \circ \varphi^{-1})(B_x \cap \varphi(U))$. Then $\hat{f}(B_x)$ is a C^k submanifold of the C^k manifold

$$M_x := \mathbb{R}^N \setminus \left(\hat{f}(\partial B_x) \cup (f(S) \setminus (f \circ \varphi^{-1})(B_x \cap \varphi(U))) \right);$$

here ∂B_x is the boundary of B_x. By the tubular neighbourhood theorem, there exists an open neighbourhood $N_x \subset M_x$ of $\hat{f}(B_x)$ and a C^k retraction $r_x : N_x \to \hat{f}(B_x)$. In particular, $N_x \cap f(S) = (f \circ \varphi^{-1})(B_x \cap \varphi(U))$.

By the definition $\varphi \circ f^{-1}|_{(f \circ \varphi^{-1})(B_x \cap \varphi(U))}$ is restriction to $f(U)$ of the C^k map $\hat{f}^{-1} \circ r_x : N_x \to \mathbb{R}^n$. Using the open cover $(N_x)_{x \in \varphi(U)}$ of $f(U)$ we find its locally finite open refinement $(W_i)_{i \in I}$ and a subordinate C^∞ partition of unity $\{\rho_i\}_{i \in I}$ with $\mathrm{supp}\, \rho_i \Subset W_i$. If $\tau : I \to \varphi(U)$ is a refinement map such that $W_i \subset N_{\tau(i)}$, then we define the map $g : W := \cup_{i \in I} W_i \to \mathbb{R}^n$ by the formula

$$g(x) := \sum_{i \in I} \rho_i(x) \cdot \left(\hat{f}^{-1} \circ r_{\tau(i)} \right)(x), \quad x \in W.$$

For $x \in \hat{f}(\varphi(U))$, let $i_1, \ldots, i_q$ be all indices for which $\rho_{i_m}(x) \neq 0$, $1 \leq m \leq q$. Then

$$x \in \left(\bigcap_{m=1}^q N_{\tau(i_m)} \right) \bigcap f(S) = \bigcap_{m=1}^q \left(f \circ \varphi^{-1} \right) \left(B_{\tau(i_m)} \cap \varphi(U) \right),$$

so that

$$g(x) = \sum_{m=1}^q \rho_{i_m}(x)(\varphi \circ f^{-1})(x) = (\varphi \circ f^{-1})(x).$$

Hence,

$$g|_{f(U)} = \varphi \circ f^{-1}.$$

Finally, we equip $f(S)$ with the weak k-Markov structure consisting of all charts of the form $(f(U), \varphi \circ f^{-1})$. Since f is proper, $f(U)$ is open in the induced topology so that the above argument shows that $f(S) \in \mathrm{Mar}_k^*$ and has the weak k-Markov structure induced from $\mathbb{R}^N$ and $f : S \to f(S)$ is a C^k diffeomorphism. $\qquad \square$

10. Proof of Theorem 5.5

10.1. Hausdorff dimension of graphs of continuous functions

In this subsection we formulate and prove auxiliary results used in the proof of the theorem.

In what follows $\dim_H$ and $\underline{\dim}_B$ and $\overline{\dim}_B$ will denote the Hausdorff and lower and upper box-counting dimensions of a bounded subset of a Euclidean space. If $\underline{\dim}_B(F) = \overline{\dim}_B(F)$ we refer to this value as the box-counting dimension of F (denoted $\dim_B(F)$). We will use the following product formulas, see, e.g., page 94 in [10]:

If $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ are any Borel sets, then

$$(10.1) \qquad \dim_H E + \dim_H F \leq \dim_H(E \times F) \leq \dim_H E + \overline{\dim}_B(F).$$

Let $f \colon I \subset \mathbb{R} \to \mathbb{R}$ be bounded continuous on an interval I and $\Gamma_f \subset \mathbb{R}^2$ be its graph. In many cases Γ_f may be fractal. We formulate several results of this type (for other results, see, e.g., [9], [10], [21], [31], [32] and references therein).

If

$$(10.2) \qquad w_\Theta(x) := \sum_{n=0}^{\infty} a^n \cos(2\pi(b^n x + \theta_n)), \quad 0 < a < 1 < b, \quad ab > 1,$$

is the Weierstrass function with a random phase added to each term, and if each θ_n is chosen independently with respect to the uniform probability measure in $[0, 1]$, then with probability one the Hausdorff dimension of the graph of w_Θ over each (nontrivial) subinterval $I \subset \mathbb{R}$ is $D := 2 + (\ln a)/(\ln b)$, see Theorem 1 and pages 796–798 in [21]. In the case of all $\theta_n = 0$, i.e., of the classical Weierstrass function w, it remains still unknown whether the Hausdorff dimension of Γ_w has the same value although it is proved in [23] that $\dim_B \Gamma_w = D$.

Also, it was shown on page 796 of [21] that on each nontrivial subinterval $I \subset \mathbb{R}$, the function w_Θ satisfies the Hölder condition with exponent $2 - D$. Thus according to Corollary 11.2 of [10], for almost all sequences $\Theta = \{\theta_n\}$ chosen independently with respect to the uniform probability measure in $[0, 1]$, the graph of w_Θ over each (nontrivial) subinterval $I \subset \mathbb{R}$ satisfies

$$(10.3) \qquad \dim_H(\Gamma_{w_\Theta}|_I) = \overline{\dim}_B(\Gamma_{w_\Theta}|_I) = D.$$

Now, if $\Gamma_{w_{\Theta_1}}, \ldots, \Gamma_{w_{\Theta_n}} \subset \mathbb{R}^2$ are graphs of functions satisfying (10.3), then according to the product formula (10.1)

$$(10.4) \qquad \dim_H\big(\Gamma_{w_{\Theta_1}}|_{I_1} \times \cdots \times \Gamma_{w_{\Theta_n}}|_{I_n} \times \bar{\mathbb{K}}^s\big) = nD + s$$

for all nontrivial closed intervals $I_1, \ldots, I_n \subset \mathbb{R}$.

Observe that D attains any value in open interval $(1, 2)$. Thus nD attains any value in the interval $(n + s, 2n + s)$. Moreover, $\Gamma_{w_{\Theta_1}}|_{I_1} \times \cdots \times \Gamma_{w_{\Theta_n}}|_{I_n} \times \bar{\mathbb{K}}^s \in \mathrm{Mar}_\infty^*(\mathbb{R}^{2n+s})$ ($\bar{\mathbb{K}}^s := [0, 1]^s$) for all nontrivial closed intervals $I_1, \ldots, I_n \subset \mathbb{R}$, see properties (5) and (8) of Section 2.

Example 10.1. The graph of the Weierstrass function

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{7}{10}\right)^n \cos(9^n \pi x)$$

has a base of (induced) topology of sets of Hausdorff dimension more than one, see, e.g., [10] and [32].

Further, it was proved in [40] by relatively simple methods that for the function

$$(10.5) \qquad \Phi(x) := \sum_{k=1}^{\infty} \lambda^{-k^\alpha} \phi(\lambda^{k^\beta} x), \quad x \in [0, 1],$$

where $\lambda > 1$, $\beta > \alpha > 1$, and $\phi(x) = 2x$ for $0 \le x \le 1/2$, $\phi(-x) = \phi(x)$ and $\phi(x + 1) = \phi(x)$, its graph over any nontrivial closed subinterval of $[0, 1]$ has Hausdorff dimension 2. Taking the direct product of n copies of graphs of such functions and of $\bar{\mathbb{K}}^s$ we obtain the graph of a continuous map $\bar{\mathbb{K}}^{n+s} \to \bar{\mathbb{K}}^n$ having Hausdorff dimension $2n + s$ over each nondegenerate closed cube in $\bar{\mathbb{K}}^{n+s}$ (cf. (10.1)). Also, this graph belongs to $\mathrm{Mar}_\infty^*(\mathbb{R}^{2n+s})$ due to properties (5) and (8) of Section 2.

Finally, in the proofs of Theorem 6.6 and Proposition 6.10 we constructed bounded Lipschitz functions $\mathbb{R} \to \mathbb{R}$ whose graphs either belong to $\mathrm{Mar}_k^*(\mathbb{R}^2) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^2)$ for a fixed $k \in \mathbb{N}$ or to $\mathrm{Mar}_\infty^*(\mathbb{R}^2)$. In turn, taking direct products of n copies of such graphs with $\mathbb{R}^s$ we obtain graphs of Lipschitz maps $\mathbb{R}^{n+s} \to \mathbb{R}^n$ either belonging to $\mathrm{Mar}_k^*(\mathbb{R}^{2n+s}) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^{2n+s})$ for a fixed $k \in \mathbb{N}$ or to $\mathrm{Mar}_\infty^*(\mathbb{R}^{2n+s})$, see property (5) of Section 2.

In the proof of the theorem we use the following result.

Suppose $\Gamma_F := \{(x, F(x)) \,;\, x \in Q\} \subset \mathbb{R}^{2n}$ is the graph of a continuous map $F : Q \subset \mathbb{R}^n \to \mathbb{R}^n$ defined over a closed nondegenerate cube Q. Assume that

(P) Γ_F is either belongs to $\mathrm{Mar}_k^*(\mathbb{R}^{2n}) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^{2n})$ for a fixed $k \in \mathbb{N}$, or to $\mathrm{Mar}_\infty^*(\mathbb{R}^{2n})$, or has a base of (induced) topology of sets of Hausdorff dimension $n \le d \le 2n$.

Let $T \Subset Q$ be a convex open subsets and let ρ be a C^∞ function on $\mathbb{R}^n$ with all derivatives equal to zero on $T^c := \mathbb{R}^n \setminus T$ and nonzero at each point of T (e.g., as such a ρ one may take a regularized distance to the complement of T, see [35]).

Lemma 10.2. *The graph $\Gamma_{\rho F} := \{(x, \rho(x)F(x)) \,;\, x \in \bar{T}\}$ of ρF over $\bar{T}$ has the same properties as Γ_F, see* (P).

Proof. Let us define a C^∞ map $\tilde{\rho} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by the formula $\tilde{\rho}(x, y) := (x, \rho(x)y)$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$. Then by the hypothesis $\tilde{\rho}$ has a C^∞ inverse on $T \times \mathbb{R}^n$. Now implications $\Gamma_{\rho F} \in \mathrm{Mar}_k^*(\mathbb{R}^{2n}) \setminus \mathrm{Mar}_{k+1}^*(\mathbb{R}^{2n})$ for a fixed $k \in \mathbb{N}$, or $\Gamma_{\rho F} \in \mathrm{Mar}_\infty^*(\mathbb{R}^{2n})$ follow from property (6) of Section 2 provided that the same is true for Γ_F. The remaining property follows from the fact that $\tilde{\rho}$ is locally bi-Lipschitz over $T \times \mathbb{R}^n$ and so preserves Hausdorff dimension of any Borel measurable subset of this set. $\qquad\square$

10.2. Proof of Theorem 5.5

Proof. According to the assumptions of the theorem, there exists an open neighbourhood $U' \subset U$ of M and a C^∞ submanifold M' of U' such that M and M' are C^k isotopic inside U', see, e.g., Section 4 in [30]. Proving the result for M' and taking suitable compositions of the isotopies between M'_r and M', $r \in \{d, l\}$, and M' and M, we obtain the required result for M. Thus without loss of generality we may assume in the proof that M is of class C^∞.

Let $N_M \to M$ be the normal bundle on M. There exists a convex open neighbourhood W of the zero section $M \hookrightarrow N_M$ and a C^∞ diffeomorphism $g : W \to X$ whose image N is a tubular neighbourhood of M in U, see, e.g., Section 5 in [30]. Thus it suffices to construct the required objects in W and then embed them into X by g.

To this end recall that since M is of class C^∞, it admits a C^∞ triangulation (see, e.g., Section 10 in [30]). Shrinking this triangulation, if necessary, we may assume that N_M is C^∞ trivial in an open neighbourhood of the closure of each simplex of the triangulation. Now each nondegenerate simplex σ of the triangulation is C^∞ diffeomorphic to the standard m-dimensional simplex $\Delta \subset \bar{\mathbb{K}}^m$. Thus we may identify (by means of C^∞ diffeomorphisms) σ with Δ and N_M on σ with $\Delta \times \mathbb{R}^{n-m}$. Using results of the previous subsection we construct graphs $\Gamma_\sigma \subset \mathbb{R}^n$ of continuous maps $F_\sigma : \Delta \to \mathbb{R}^{n-m}$ equal zero on the boundary of Δ such that Γ_σ either belong to $\mathrm{Mar}^*_l(\mathbb{R}^n) \setminus \mathrm{Mar}^*_{l+1}(\mathbb{R}^n)$ for a fixed $\ell \in \mathbb{N}$ (in this case the corresponding maps F_σ are Lipschitz), or to $\mathrm{Mar}^*_\infty(\mathbb{R}^n)$ and have bases of (induced) topology of sets of Hausdorff dimension $m \leq d \leq n$ (in case $d = m$ maps F_σ are Lipschitz as well). Using diffeomorphisms identifying σ with Δ and N_M on σ with $\Delta \times \mathbb{R}^{n-m}$, we may regard each F_σ as a continuous section of N_M over σ equals zero on the boundary of σ. Choosing F_σ so that $\max_\sigma \|F_\sigma\|_2$ are sufficiently small, we may assume that images of all F_σ belong to W. Let us define a continuous section $F : M \to W$ by the formula $F|_\sigma := F_\sigma$ for all σ of the triangulation of M. We set $M_d := (g \circ F)(M)$ if all Γ_σ have bases of (induced) topology of sets of Hausdorff dimension $m \leq d \leq n$ and $M_l := (g \circ F)(M)$ if all $\Gamma_\sigma \in \mathrm{Mar}^*_l(\mathbb{R}^n) \setminus \mathrm{Mar}^*_{l+1}(\mathbb{R}^n)$. Moreover, since W is convex, $(1 - t)F(M) \subset W$ for all $t \in [0, 1]$. We define the required homotopy $h_{r,t}(x) := g\big((1 - t)g^{-1}(x)\big)$, $x \in M_r$, $t \in [0, 1]$; here $r \in \{d, l\}$. It is a matter of definitions to check that the introduced objects satisfy the required conditions. $\square$

11. Proof of Theorem 5.6

Due to the definitions of $\mathrm{Mar}^*_{\infty,\Gamma}$ and of $\Omega^p(S)$ (see Subsection 5.2), it suffices to prove the following version of the theorem.

Theorem 11.1. *Let $S = \Gamma^{n_1}_{f_{11},f_{12}} \times \cdots \times \Gamma^{n_\ell}_{f_{\ell 1},f_{\ell 2}} \Subset \mathbb{R}^n := \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_\ell+1}$ be a Γ set such that $\bar{S} \in \mathrm{Mar}^*_\infty(\mathbb{R}^n)$. Let $\omega := \sum_{|I|=p} c_I dx_I$ be a d-closed C^∞ p-form on $\bar{S}$. Then for $p \geq 1$ and an open subset $U \Subset S$ there exists a C^∞ $(p-1)$-form η_U on U such that $d\eta_U = \omega|_U$. If $p = 0$, then ω is a constant function on S.*

Proof. We will prove the result by induction on ℓ. For $\ell = 1$ we prove the following more general statement.

Let $S_1 \times S_2 \Subset \mathbb{R}^{n+1} \times \mathbb{R}^m$ be such that $S_1 \Subset \mathbb{R}^{n+1}$, $\bar{S}_1 \in \mathrm{Mar}_\infty^*(\mathbb{R}^{n+1})$, is a simple Γ set and $S_2 \Subset \mathbb{R}^m$, $\bar{S}_2 \in \mathrm{Mar}_\infty^*(\mathbb{R}^m)$, is a Γ set. Let $x = (x_1, \ldots, x_{n+1})$ and $y = (y_1, \ldots, y_m)$ be the standard coordinates on $\mathbb{R}^{n+1}$ and $\mathbb{R}^m$, respectively. By d_x we denote the differential on $\Omega^*(\bar{S}_1 \times \bar{S}_2)$ with respect to the coordinate x for y fixed. For a differential form $\omega = \sum c_{IJ} dx_I \wedge dy_J$ on $\bar{S}_1 \times \bar{S}_2$ we set $\deg_x \omega := \max\{|I| \,;\, c_{IJ} \not\equiv 0\}$.

Lemma 11.2. *Suppose that ω is a d_x-closed C^∞ differential form on $\bar{S}_1 \times \bar{S}_2$. If $\deg_x \omega \geq 1$, then there exists a C^∞ differential form η on $S_1 \times \bar{S}_2$ such that $d_x \eta = \omega|_{S_1 \times \bar{S}_2}$. If $\deg_x \omega = 0$, then ω is independent of x.*

Proof of Lemma 11.2. Without loss of generality we may assume that

$$S_1 = \left\{ \big(x_1, \ldots, x_n, t f_1(x_1, \ldots, x_n) + (1-t) f_2(x_1, \ldots, x_n)\big) \,;\, x \in \mathbb{K}^n, \, t \in [0,1] \right\}$$

for some continuous $f_1, f_2 : \mathbb{K}^n \to \mathbb{R}$, $\mathbb{K}^n := (0,1)^n$, such that $f_1 \leq f_2$. By the definition of C^∞ functions on weak ∞-Markov sets (see Section 4), ω is restriction to $\bar{S}_1 \times \bar{S}_2$ of a C^∞ differential form ω' (of the same degree as ω) defined in an open neighbourhood of $\bar{S}_1 \times \bar{S}_2$ in $\mathbb{R}^{n+1} \times \mathbb{R}^m$. Since $\bar{S}_1 \times \bar{S}_2$ is compact, multiplying ω' by a suitable C^∞ cut-off function, we obtain a C^∞ differential form $\widetilde{\omega}$ on $\mathbb{R}^{n+1} \times \mathbb{R}^m$ such that $\widetilde{\omega}|_{\bar{S}_1 \times \bar{S}_2} = \omega$ and $\deg \widetilde{\omega} = \deg \omega$, $\deg_x \widetilde{\omega} = \deg_x \omega$.

Since $d_x \widetilde{\omega} = 0$ on $\bar{S}_1 \times \bar{S}_2$ and $\bar{S}_1 \in \mathrm{Mar}_\infty^*(\mathbb{R}^{n+1})$,

$$d_x \big(\widetilde{\omega}|_{(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2} \big) = \beta_- + \beta_+,$$

where each $\beta_-(\cdot, y)$ coincides with $d_x \widetilde{\omega}(\cdot, y)$ on the connected component of $\big((\mathbb{K}^n \times \mathbb{R}) \setminus S_1 \big) \times \{y\}$ containing $\mathbb{K}^n \times \{-\infty\} \times \{y\}$ and on the other connected component of $\big((\mathbb{K}^n \times \mathbb{R}) \setminus S_1 \big) \times \{y\}$ equals zero. Similarly, each $\beta_+(\cdot, y)$ coincides with $d_x \widetilde{\omega}(\cdot, y)$ on the connected component of $\big((\mathbb{K}^n \times \mathbb{R}) \setminus S_1 \big) \times \{y\}$ containing $\mathbb{K}^n \times \{+\infty\} \times \{y\}$ and on the other connected component of $\big((\mathbb{K}^n \times \mathbb{R}) \setminus S_1 \big) \times \{y\}$ equals zero; here $y \in \bar{S}_2$ (see Section 8 for similar arguments). By the definition of the d_x operator on weak ∞-Markov sets, the differential forms $\beta_\pm$ are of class C^∞ on $(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2$. We write $\beta_\pm = (\widetilde{\omega}_{n+1})_\pm + \lambda_\pm$, where $(\widetilde{\omega}_{n+1})_\pm := dx_{n+1} \wedge \tilde{\lambda}_\pm$ and $\lambda_\pm$ do not contain dx_{n+1}. Since $d_x \omega = 0$, the forms $\beta_\pm$ are d_x-closed on $(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2$.

We set

$$\theta_\pm(x, y) := \int_{\pm\infty}^{x_{n+1}} \tilde{\lambda}_\pm(x_1, \ldots, x_n, t_{n+1}, y) \, dt_{n+1}.$$

Clearly, $\theta_\pm$ are of class C^∞ on $(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2$. Now, $\beta_\pm - d_x \theta_\pm$ are d_x-closed C^∞ differential forms on $(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2$ not containing terms with dx_{n+1}. Hence, partial derivatives of their coefficients with respect to x_{n+1} are zeros. In particular, these coefficients are zeros, because at each $y \in \bar{S}_2$ the forms $(\beta_\pm - d_x \theta_\pm)(\cdot, y)$ are equal to zero on the corresponding connected components containing $\mathbb{K}^n \times \{\pm\infty\} \times \{y\}$.

Thus we have

$$\beta_\pm = (\widetilde{\omega}_{n+1})_\pm = d_x \theta_\pm.$$

In turn, equality $d_x(\widetilde{\omega}|_{(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2}) = \beta_+ + \beta_- = d_x(\theta_+ + \theta_-)$ and the classical Poincaré lemma (for d-closed differential forms on $\mathbb{K}^n \times \mathbb{R}$) imply that if $\deg_x \widetilde{\omega} = \deg_x \omega \geq 1$, then there exists a C^∞ differential form η on $(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2$ such that $\widetilde{\omega}|_{(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2} - \theta_+ - \theta_- = d_x \eta$. Moreover, by our construction forms $\theta_\pm$ equal zero on $S_1 \times \bar{S}_2$ so that the previous equality implies that

$$d_x\big(\eta|_{S_1 \times \bar{S}_2}\big) = \omega$$

completing the proof of the lemma in this case.

Next, if $\deg_x \widetilde{\omega} = \deg_x \omega = 0$, then $\widetilde{\omega}|_{(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2} - \theta_+ - \theta_- = \sum c_J dy_J$ and $d_x c_J = 0$ for all J on $(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2$. Hence, all c_J do not depend on $x \in \mathbb{K}^n \times \mathbb{R}$, i.e., $\widetilde{\omega}|_{(\mathbb{K}^n \times \mathbb{R}) \times \bar{S}_2} - \theta_+ - \theta_-$ is a form in y. Since $\theta_\pm$ equal zero on $S_1 \times \bar{S}_2$, the differential form $\omega := \widetilde{\omega}|_{\bar{S}_1 \times \bar{S}_2}$ depends on y only. □

Lemma 11.2 shows that the theorem is valid for $\ell = 1$ and any open $U \Subset S$. Assuming that it is valid for $\ell - 1$ let us establish it for ℓ.

To this end by $x = (x_1, \ldots, x_{n_\ell + 1})$ and $y := (y_1, \ldots, y_{n-n_\ell-1})$ we denote the standard coordinates on $\mathbb{R}^{n_\ell + 1}$ and $\mathbb{R}^{n-n_\ell-1} := \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_{\ell-1}+1}$, respectively. We will prove the statement by induction on $\deg_x \omega$.

For $\deg_x \omega = 0$ from d-closedness of ω and Lemma 11.2 we obtain that ω is a d-closed form depending on y only. Thus it is pullback to S (under the natural projection $\mathbb{R}^n \to \mathbb{R}^{n_{\ell-1}+1}$) of a d-closed differential form on $\bar{\Gamma}^{n_1}_{f_{11}, f_{12}} \times \cdots \times \bar{\Gamma}^{n_{\ell-1}}_{f_{\ell-11}, f_{\ell-12}} \Subset \mathbb{R}^{n-n_\ell-1}$ and, since the number of terms in the direct product is $\ell - 1$, the required statement (of the theorem) follows from the first induction hypothesis.

Suppose that we have proved the statement for $\deg_x \omega := s - 1 \geq 0$; let us prove it for $\deg_x \omega := s$.

We write $\omega = \omega_1 + \omega_2$, where ω_1 is the sum of all terms of ω of $\deg_x \omega := s$ and $\deg_x \omega_2 \leq s - 1$. Then d-closedness of ω implies that $d_x \omega_1 = 0$. Therefore from Lemma 11.2 we obtain that $\omega_1 = d_x \eta_1$ for some C^∞ differential form on S with $\deg_x \eta_1 \leq s - 1$. Consider the form $\omega - d\eta_1 := \omega_2 - d_y \eta_1$. It is d-closed on S with $\deg_x$ at most $s - 1$. Thus by the induction hypotheses applied to a Γ set $U' \Subset S$ containing $\bar{U}$, $\omega|_{U'} = d\eta_{U'}$ for a C^∞ differential form $\eta_{U'}$ on U'. This proves the step of the second induction and therefore the step of the first induction completing the proof of the theorem. □

12. Proofs of Corollary 5.11 and Theorems 5.12 and 5.13

12.1. Proof of Corollary 5.11

Lemma 12.1. *Any set of the form*

$$(12.1) \quad Z := \big\{\big(x, t f_1(x) + (1-t) f_2(x)\big) \in \mathbb{R}^{n+1} \,;\, x \in \bar{\mathbb{K}}^n, \, t \in [0,1]\big\}$$

for some continuous $f_1, f_2 : \bar{\mathbb{K}}^n \to \mathbb{R}$ such that $f_1 \leq f_2$, is a strong deformation retract of $\mathbb{R}^{n+1}$.

Proof of Lemma 12.1. We refer to [20] for basic definitions of homotopy theory.

Clearly $\bar{\mathbb{K}}^n \times \mathbb{R}$ is a strong deformation retract of $\mathbb{R}^{n+1}$. Thus it suffices to check that Z is a strong deformation retract of $\bar{\mathbb{K}}^n \times \mathbb{R}$. For $(x, x_{n+1}, s) \in (\bar{\mathbb{K}}^n \times \mathbb{R}) \times [0,1]$ we define the required retraction $F : (\bar{\mathbb{K}}^n \times \mathbb{R}) \times [0,1] \to Z$ as follows:

$$F(x, x_{n+1}, s) := \begin{cases} (x, f_2(x) + s(x_{n+1} - f_2(x))) & \text{if} \quad x_{n+1} \geq f_2(x), \\ (x, x_{n+1}) & \text{if} \quad f_1(x) \leq x_{n+1} \leq f_2(x), \\ (x, f_1(x) + s(x_{n+1} - f_1(x))) & \text{if} \quad x_{n+1} \leq f_1(x). \end{cases}$$

$\square$

The lemma implies, in particular, that finite direct products of sets of the form (12.1) are absolute retracts. Also, any $S \in \mathrm{Mar}^*_{\infty, \Gamma}$ is countable union of such finite direct products, see Subsection 5.2; hence, by a result of Hanner [16], S is an absolute neighbourhood retract. In particular, in the setting of the theorem, there exists an open neighbourhood $N \subset X$ of S and a continuous retraction $r : N \to S$. We embed X as a closed subset into some $\mathbb{R}^n$ by Proposition 5.4 so that the weak ∞-Markov structure on X is induced from that on $\mathbb{R}^n$. Thus without loss of generality we may assume in the proof that X (and therefore N) belongs to $\mathrm{Mar}^*_{\infty, \Gamma}$.

By $i : S \hookrightarrow N$ we denote the (C^∞) embedding map. Then the pullbacks of r and i define the following homomorphisms of Čech cohomology groups:

$$r^*_C : \check{H}^p(S, \mathbb{R}) \longrightarrow \check{H}^p(N, \mathbb{R}), \qquad i^*_C : \check{H}^p(N, \mathbb{R}) \longrightarrow \check{H}^p(S, \mathbb{R})$$

such that $i^*_C \circ r^*_C = \mathrm{id}$. Thus

$$\check{H}^p(N, \mathbb{R}) = r^*_C(\check{H}^p(S, \mathbb{R})) \oplus \mathrm{Ker}\, i^*_C.$$

For a d-closed form $\omega \in \Omega^p(S)$, we denote by $[\omega] \in H^p_{dR}(S, \mathbb{R})$ its de Rham cohomology class. Consider

$$(r^*_C \circ (h^p_S)^{-1})([\omega]) \in \check{H}^p(N, \mathbb{R}),$$

where $h^p_S : \check{H}^p(S, \mathbb{R}) \to H^p_{dR}(S, \mathbb{R})$ is the isomorphism of Theorem 5.8 for S. Since $N \in \mathrm{Mar}^*_{\infty, \Gamma}$, the de Rham theorem is valid on N, i.e., $h^p_N : \check{H}^p(N, \mathbb{R}) \to H^p_{dR}(N, \mathbb{R})$ is an isomorphism (see (5.1)). Let $\widehat{\omega} \in \Omega^p(N)$ be a d-closed form whose de Rham cohomology class coincides with $(h^p_N \circ r^*_C \circ (h^p_S)^{-1})([\omega])$. Using the commutative diagram (5.1) for the map i and identity $i^*_C \circ r^*_C = \mathrm{id}$ we obtain

$$(i^*_{dR} \circ h^p_N \circ r^*_C \circ (h^p_S)^{-1})([\omega]) = (h^p_S \circ i^*_C \circ r^*_C \circ (h^p_S)^{-1})([\omega]) = [\omega].$$

Hence, the restriction of $\widehat{\omega}$ to S belongs to the same de Rham cohomology class as ω. Therefore $\omega - \widehat{\omega}|_S = d\eta$ with $\eta \in \Omega^{p-1}(S)$ (here we assume that $p \geq 1$, for otherwise, the statement of the theorem is trivial). As η admits locally C^∞ extensions to open subsets of N, using a suitable open cover of the closed set S and a subordinate to it C^∞ partition of unity, we patch such extensions together to find a C^∞ $(p-1)$-form $\widehat{\eta}$ on N such that $\widehat{\eta}|_S = \eta$. We set $\widetilde{\omega} := \widehat{\omega} - d\widehat{\eta}$. Then $\widetilde{\omega}|_S = \omega$.

12.2. Proof of Theorem 5.12

Assume first that $S \in \mathrm{Mar}^*_{\infty,\Gamma}$ is a closed subset of some $\mathbb{R}^n$ with the induced weak ∞-Markov structure (S admits such an embedding by Proposition 5.4). By $i \colon S \hookrightarrow \mathbb{R}^n$ we denote the C^∞ embedding and by $r \colon N \to S$ a continuous retraction of an open neighbourhood $N \subset \mathbb{R}^n$ of S onto S, cf. Theorem 5.11. Then the induced homomorphisms

$$r_* \colon H_k(N, \mathbb{R}) \to H_k(S, \mathbb{R}) \quad \text{and} \quad i_* \colon H_k(S, \mathbb{R}) \to H_k(N, \mathbb{R})$$

of singular homology groups satisfy $r_* \circ i_* = \mathrm{id}$. In particular, i_* is an injection and $H_k(N, \mathbb{R}) = i_*(H_k(S, \mathbb{R})) \oplus \operatorname{Ker} r_*$.

Let ω be a d-closed k-form on S and $[\omega] \in H^k_{dR}(S, \mathbb{R})$ be its cohomology class. We choose a d-closed k-form $\widetilde{\omega}$ on N such that $\widetilde{\omega}|_S = \omega$ (existing by Theorem 5.11). Let σ be a singular k-cycle in S and let $[\sigma] \in H_k(S, \mathbb{R})$ stand for its homology class. We define a map $\Phi_S \colon H^k_{dR}(S, \mathbb{R}) \to H^k(S, \mathbb{R})$ by the formula

$$(12.2) \qquad \big\langle \Phi_S([\omega]), [\sigma] \big\rangle := \int_{\sigma'} \widetilde{\omega}, \qquad [\sigma] \in H_k(S, \mathbb{R}),$$

where σ' is a piecewise smooth singular chain in N with homology class $i_*([\sigma])$.

Identifying $[\widetilde{\omega}] \in H^k_{dR}(N, \mathbb{R})$ with an element of $H^k(N, \mathbb{R})$ by the classical de Rham theorem, we obtain that *the expression on the right in* (12.2) *can be written as* $\langle i^*([\widetilde{\omega}]), [\sigma] \rangle$, *where* $i^* \colon H^k(N, \mathbb{R}) \to H^k(S, \mathbb{R})$ *is the map transpose to* i_*.

We must check that Φ_S is correctly defined, independent of embedding into $\mathbb{R}^n$ and determines the required isomorphism of cohomology rings.

To check that Φ_S is correctly defined, suppose that $\widetilde{\omega}'$ is another d-closed k form on N satisfying $\widetilde{\omega}'|_S = \omega$. Using the isomorphism h^*_N between Čech and de Rham cohomology on N expressed in Theorem 5.8, consider $(h^k_N)^{-1}\big([\widetilde{\omega} - \widetilde{\omega}']\big) \in \check{H}^k(N, \mathbb{R})$. Then due to (5.1)

$$\big(i^*_C \circ (h^k_N)^{-1}\big)\big([\widetilde{\omega} - \widetilde{\omega}']\big) = \big((h^k_S)^{-1} \circ i^*_{dR}\big)\big([\widetilde{\omega} - \widetilde{\omega}']\big) = (h^k_S)^{-1}(0) = 0.$$

Let $K \subset S$ be compact such that $\sigma \in Z_k(K, \mathbb{R})$, the abelian group of singular cycles in K. Since K is projective limit of the system of its open relatively compact neighbourhoods U in N with the partial order defined by inclusion, $\check{H}^k(K, \mathbb{R})$ is isomorphic to inductive limit $\varinjlim \check{H}^k(U, \mathbb{R})$ taking along this system (see, e.g., [1] for basic results of sheaf theory). Since $\omega - \widetilde{\omega} = 0$ on S,

$$\big((i_{K \hookrightarrow U})^*_C \circ (h^k_U)^{-1}\big)\big([(\widetilde{\omega} - \widetilde{\omega}')|_U]\big) = 0,$$

where $i_{K \hookrightarrow U} \colon K \to U$ is the embedding. As $\check{H}^k(K, \mathbb{R})$ is the inductive limit of groups $\check{H}^k(U, \mathbb{R})$, the latter implies that there exists open $U_0 \Subset N$ containing K such that $(h^k_{U_0})^{-1}\big([(\widetilde{\omega} - \widetilde{\omega}')|_{U_0}]\big) = 0$ in $\check{H}^k(U_0, \mathbb{R})$, that is $(\widetilde{\omega} - \widetilde{\omega}')|_{U_0}$ is d-exact. Let $\sigma' \in Z_k(U_0, \mathbb{R})$ be a piecewise smooth singular k-cycle in U_0 homologous to σ. Then the d-exactness of $(\widetilde{\omega} - \widetilde{\omega}')|_{U_0}$ implies that

$$\int_{\sigma'} \widetilde{\omega} = \int_{\sigma'} \widetilde{\omega}'.$$

Since (12.2) does not depend on the choice of $\sigma' \in Z_k(N, \mathbb{R})$, the above equality shows that (12.2) does not depend on the choice of the extension $\widetilde{\omega}$ of ω; similarly it is independent of the choice of the form representing the class $[\omega]$ as well. Thus Φ_S is correctly defined.

Next, if $\Phi_S([\omega]) = 0$, then $[\widetilde{\omega}]_s \in \operatorname{Ker} i^*$ (here $[\widetilde{\omega}]_s$ stands for the singular cohomology class determined by $\widetilde{\omega}$ as above via the de Rham theorem on N). Since S (being an absolute neighbourhood retract) is homotopy equivalent to a CW complex, see, e.g., Theorem 1 in [28], singular cohomology on S is *naturally isomorphic* to Čech cohomology. Thus

$$(h_N^k)^{-1})\big([\widetilde{\omega}]\big) \in \operatorname{Ker} i_C^* \subset \check{H}^k(N, \mathbb{R}) \cong H_{dR}(N, \mathbb{R}).$$

Therefore,

$$0 = \big(i_C^* \circ (h_N^k)^{-1}\big)\big([\widetilde{\omega}]\big) = \big((h_S^k)^{-1} \circ i_{dR}^*\big)\big([\widetilde{\omega}]\big) = (h_S^k)^{-1}\big([\omega]\big).$$

Thus, by Theorem 5.8, $[\omega] = 0 \in H_{dR}^k(S, \mathbb{R})$ showing that Φ_S is an injection.

Further, for $\lambda \in H^k(S, \mathbb{R})$ consider $r^*(\lambda) \in H^k(N, \mathbb{R})$. Let $[\widetilde{\omega}] \in H_{dR}^k(N, \mathbb{R})$ be the de Rham cohomology class representing $r^*(\lambda)$ (existing by the de Rham theorem for C^∞ manifolds). Since $(i^* \circ r^*)(\lambda) = \lambda$, by (12.2) we have $\Phi_S([\omega]) = i^*([\widetilde{\omega}]) = \lambda$ for $\omega := \widetilde{\omega}|_S$. This shows that Φ_S is a surjection.

The fact that $\Phi_S(\lambda_1 \wedge \lambda_2) = \Phi_S(\lambda_1) \smile \Phi_S(\lambda_2)$, $\lambda_1, \lambda_2 \in H_{dR}^*(S, \mathbb{R})$, follows from (12.2) and the similar identity for cohomology on N (valid by the classical de Rham theorem for C^∞ manifolds).

Thus $\Phi_S : H_{dR}^*(S, \mathbb{R}) \to H^*(S, \mathbb{R})$ is a ring isomorphism.

Next, we show that Φ_S does not depend on embedding $i : S \hookrightarrow \mathbb{R}^n$.

Suppose that $i_p : S \hookrightarrow \mathbb{R}^{n_p}$, $p = 1, 2$, are proper C^∞ embeddings. Consider the proper C^∞ embedding $i := (i_1, i_2) : S \hookrightarrow \mathbb{R}^n := \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. By $\pi_p : \mathbb{R}^n \to \mathbb{R}^{n_p}$, $p = 1, 2$, we denote the natural projections. Clearly, $\pi_p \circ i = i_p$, $p = 1, 2$. Due to the preceding discussion, the map $\Phi_{i(S)} : H_{dR}^*(i(S), \mathbb{R}) \to H^*(i(S), \mathbb{R})$ is well defined. Moreover, due to (12.2) for $[\omega] \in H_{dR}^k(S, \mathbb{R})$ and $\sigma \in H_k(S, \mathbb{R})$ we have

$$\begin{aligned}
&\big\langle \big(i_p^* \circ \Phi_{i_p(S)} \circ (i_p^*)^{-1}\big)([\omega]),\, \sigma \big\rangle \\
&\qquad = \big\langle \big(\Phi_{i_p(S)} \circ ((\pi_p|_{i(S)})^*)^{-1} \circ (i^*)^{-1}\big)([\omega]),\, (\pi_p \circ i)_*(\sigma) \big\rangle \\
&\qquad = \big\langle \big(\Phi_{i(S)} \circ (i^*)^{-1}\big)([\omega]),\, ((\pi_p|_{i(S)})^{-1} \circ \pi_p \circ i)_*(\sigma) \big\rangle \\
&\qquad = \big\langle \big(\Phi_{i(S)} \circ (i^*)^{-1}\big)([\omega]),\, (i)_*(\sigma) \big\rangle.
\end{aligned}$$

Hence, $i_p^* \circ \Phi_{i_p(S)} \circ (i_p^*)^{-1} = i^* \circ \Phi_{i(S)} \circ (i^*)^{-1}$, $p = 1, 2$. This shows that Φ_S does not depend on embedding into $\mathbb{R}^n$.

Let us prove now the commutativity of diagram (5.2). So assume that $f : S_1 \to S_2$ is a C^∞ map of spaces in $\operatorname{Mar}_{\infty, \Gamma}^*$. Without loss of generality we may assume that $S_i \subset \mathbb{R}^{n_i}$, $i = 1, 2$, are closed and equipped with the induced weak ∞-Markov structures. We embed S_1 into $S_1 \times S_2$ as the closed subset $\Gamma_f := \{(s_1, f(s_1)) ; s_1 \in S_1\}$ and by $\pi_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ we denote the natural projection. Thus we may assume that $S_1 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ (equipped with the induced weak ∞-Markov structure,

see arguments in the proof of Proposition 5.4) and $f := \pi_2|_{S_1}$. Since $S_2 \subset \mathbb{R}^{n_2}$, by definition (12.2) for $[\omega] \in H^k_{dR}(S_2, \mathbb{R})$ and $\sigma \in H_k(S_1, \mathbb{R})$ we have

$$\left\langle \left(f^* \circ \Phi_{S_2}\right)([\omega]), \lambda \right\rangle = \left\langle \Phi_{S_2}([\omega]), f_*(\lambda) \right\rangle = \left\langle \left(\Phi_{S_1} \circ f^*_{dR}\right)([\omega]), \lambda \right\rangle,$$

as required.

The statement asserting that in the case of S being a C^∞ manifold Φ_S coincides with the classical de Rham isomorphism follows from the fact that for $S \subset \mathbb{R}^n$ being a C^∞ submanifold the singular chain σ' in (12.2) can be chosen from $Z^k(S, \mathbb{R})$.

The proof of the theorem is complete.

12.3. Proof of Theorem 5.13

We define a map $\times : H^*(S_1, \mathbb{R}) \otimes H^*(S_2, \mathbb{R}) \to H^*(S_1 \times S_2, \mathbb{R})$ of singular cohomology rings by the formula

$$\lambda_1 \times \lambda_2 := \pi_1^*(\lambda_1) \smile \pi_2^*(\lambda_2), \qquad \lambda_1 \otimes \lambda_2 \in H^*(S_1, \mathbb{R}) \otimes H^*(S_2, \mathbb{R}).$$

Due to Theorem 5.12 we have

$$\Phi_{S_1 \times S_2}(c_1 \wedge c_2) = \Phi_{S_1}(c_1) \times \Phi_{S_2}(c_2), \qquad c_1 \otimes c_2 \in H^*_{dR}(S_1, \mathbb{R}) \otimes H^*_{dR}(S_2, \mathbb{R}).$$

In turn, it is known, see, e.g., Appendix A in [29], that as spaces S_1 and S_2 are homotopy equivalent to CW complexes, under the assumptions of the theorem,

$$\times : H^*(S_1, \mathbb{R}) \otimes H^*(S_2, \mathbb{R}) \to H^*(S_1 \times S_2, \mathbb{R})$$

is an isomorphism. Since $\Phi_{S_1 \times S_2}$, Φ_{S_1} and Φ_{S_2} are ring isomorphisms, this fact and the previous identity imply the required.

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Alexander Brudnyi: Department of Mathematics and Statistics, University of Calgary, T2N 1N4 Calgary, Canada.
E-mail: albru@math.ucalgary.ca