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A note on theta divisors of stable bundles

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Abstract. Let C be a smooth complex irreducible projective curve of genus $g \geq 3$. We show that if C is a Petri curve with $g \geq 4$, a general stable vector bundle E on C, with integer slope, admits an irreducible and reduced theta divisor Θ_E, whose singular locus has dimension $g-4$. If C is non-hyperelliptic of genus 3, then actually Θ_E is smooth and irreducible for a general stable vector bundle E with integer slope on C.

1. Introduction

Let C be a smooth, irreducible, complex projective curve of genus $g \geq 3$, and let ω_C be the canonical line bundle on C. We recall that C is said a *Petri curve* if for any line bundle L on the curve, the Petri map, given by multiplication of sections,

$$\mu_L\colon H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C), \tag{1.1}$$

is injective, see [1].

Let $\mathcal{U}(r,d)$ denote the moduli space of S-equivalence classes of semistable vector bundles of rank $r \geq 2$ and degree d on C. It is a normal irreducible projective variety of dimension $r^2(g-1)+1$. Except when $r=g=2$ or r and d are coprime, $\mathcal{U}(r,d)$ is singular and the open subset $\mathcal{U}(r,d)^s \subset \mathcal{U}(r,d)$ of smooth points corresponds to isomorphism classes of stable bundles. Moreover, $\mathcal{U}(r,d) \simeq \mathcal{U}(r,d')$ whenever $d'-d = kr, \quad k \in \mathbb{Z}$. In particular, if $d = r(g-1)$ a natural Brill–Noether locus is defined as follows:

$$\Theta_r = \big\{ [E] \in \mathcal{U}(r, r(g-1)) \mid h^0(\mathrm{gr}(E)) \geq 1 \big\}, \tag{1.2}$$

where $[E]$ denotes the S-equivalence class of E and $\mathrm{gr}(E)$ is the polystable bundle defined by a Jordan–Hölder filtration of E, see [13]. Actually, Θ_r is an integral Cartier divisor, see [6], which is called *theta divisor* of $\mathcal{U}(r, r(g-1))$.

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For semistable vector bundles with integer slope we can introduce the notion of theta divisors, see [2]. Let E be a semistable vector bundle on C with integer slope $m = d/r \in \mathbb{Z}$. We set $h = g - 1 - m$. The tensor product defines a map:

$$\tau : \mathcal{U}(r, rm) \times \mathrm{Pic}^h(C) \longrightarrow \mathcal{U}(r, r(g-1)), \tag{1.3}$$

sending $([E], N) \to [E \otimes N]$. We can consider the pull-back $\tau^*\Theta_r$ of Θ_r. When the intersection $\tau^*\Theta_r \cdot [E] \times \mathrm{Pic}^h(C)$ is proper, it defines an effective divisor Θ_E on $\mathrm{Pic}^h(C)$ which is called the *theta divisor* of E, see [12], [11], and [3], which is set theoretically:

$$\Theta_E = \left\{ N \in \mathrm{Pic}^h(C) \mid h^0(\mathrm{gr}(E) \otimes N) \geq 1 \right\}. \tag{1.4}$$

If $\det E \simeq M^{\otimes r}$, with $M \in \mathrm{Pic}^m(C)$, then it is well known that

$$\Theta_E \in |r\Theta_M|, \tag{1.5}$$

where $\Theta_M = \{N \in \mathrm{Pic}^h(C) \mid h^0(M \otimes N) \geq 1\}$, is a translate of the canonical theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$. Our result is the following:

Theorem 1.1. *Let $r \geq 2$ and $m \in \mathcal{Z}$.*

1. *Let C be a Petri curve of genus $g \geq 4$. For a general stable vector bundle $[E] \in \mathcal{U}(r, rm)$, Θ_E is an irreducible and reduced divisor, whose singular locus*

$$\mathrm{Sing}(\Theta_E) = \{N \in \mathrm{Pic}^h(C) \mid h^0(E \otimes N) \geq 2\}, \quad h = g - 1 - m,$$

   has dimension $g - 4$.

2. *Let C be a non-hyperelliptic curve of genus 3. A general stable vector bundle $[E] \in \mathcal{U}(r, rm)$ admits a smooth irreducible and reduced theta divisor Θ_E.*

The above description of $\mathrm{Sing}(\Theta_E)$ actually holds for a general stable bundle E on any smooth curve, this is also proved with different arguments in [14], see also [5] for a generalization. Petri condition is required to prove the dimensional formula.

2. Preliminary results

Before proving our result we will recall some facts on the theta divisor Θ_r of the moduli space $\mathcal{U}(r, r(g-1))$. For any $k \geq 1$, we can define the following Brill–Noether loci:

$$B(r, r(g-1), k) = \left\{ [F] \in \mathcal{U}(r, r(g-1))^s \mid h^0(F) \geq k \right\}, \tag{2.1}$$

$$\tilde{B}(r, r(g-1), k) = \left\{ [F] \in \mathcal{U}(r, r(g-1)) \mid h^0(gr(F)) \geq k \right\}, \tag{2.2}$$

which are closed subschemes of their moduli spaces. Note that

$$\tilde{B}(r, r(g-1), 1) = \Theta_r. \tag{2.3}$$

Moreover, we recall Laszlo's singularity theorem, see [8]:

Theorem 2.1. *The multiplicity of Θ_r at a stable point $[F] \in \Theta_r$ is $h^0(F)$.*

This implies that

$$B(r, r(g-1), 2) = \{[F] \in \mathcal{U}(r, r(g-1))^s \mid [F] \in \mathrm{Sing}(\Theta_r)\}. \tag{2.4}$$

Brill–Noether loci have a determinantal description, which gives the following general results, see for instance [4] and [9].

Lemma 2.2. *For $(r, r(g-1), k)$, the Brill–Noether number is the following:*

$$\beta(r, r(g-1), k) = r^2(g-1) + 1 - k^2. \tag{2.5}$$

If $B(r, r(g-1), k)$ is not empty and $B(r, r(g-1), k) \neq \mathcal{U}(r, r(g-1))^s$, then we have the following properties:

(1) *Every irreducible component of $B(r, r(g-1), k)$ has dimension $\geq \beta(r, r(g-1), k)$;*

(2) *$B(r, r(g-1), k+1) \subset \mathrm{Sing}(r, r(g-1), k)$;*

(3) *The tangent space of $B(r, r(g-1), k)$ at a point $[F]$ with $h^0(F) = k$ can be identified with the dual of the cokernel of the Petri map, given by multiplication of sections:*

$$\mu_F\colon H^0(F) \otimes H^0(\omega_C \otimes F^*) \longrightarrow H^0(F \otimes F^* \otimes \omega_C);$$

(4) *$B(r, r(g-1), k)$ is smooth of dimension $\beta(r, r(g-1), k)$ at $[F]$ if and only if the Petri map μ_F is injective.*

It is easy to produce semistable vector bundles $[F] \in \tilde{B}(r, r(g-1), k)$, however the non-emptiness of $B(r, r(g-1), k)$ is a more delicate question. For $k = 2$ we have the following result:

Theorem 2.3. *Let C be a smooth curve of genus $g \geq 3$. Then for any $r \geq 2$, the Brill–Noether locus $B(r, r(g-1), 2)$ is non-empty.*

This follows from a result of [10], which extends to arbitrary smooth curves the result of [15].

As an application of the study of moduli spaces of coherent systems we have the following:

Theorem 2.4. *Let C be a Petri curve of genus $g \geq 3$ and $r \geq 2$. Then the Brill–Noether locus $B(r, r(g-1), 2)$ is irreducible of dimension*

$$\beta(r, r(g-1), 2) = r^2(g-1) - 3 = \dim\, \mathcal{U}(r, r(g-1))^s - 4.$$

For the proof see [4], Theorem 11.11.

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let C be a smooth curve of genus $g \geq 3$. We will consider the map defined in (1.3):

$$\tau\colon \mathcal{U}(r, rm) \times \mathrm{Pic}^h(C) \longrightarrow \mathcal{U}(r, r(g-1))$$

sending $([E], N) \to [E \otimes N]$. Note that the restriction

$$\tau_N = \tau_{|\mathcal{U}(r,rm)\times N}\colon \mathcal{U}(r, rm) \longrightarrow \mathcal{U}(r, r(g-1))$$

is an isomorphism for any $N \in \mathrm{Pic}^h(C)$. We will consider the restriction of τ to stable bundles and we will denote it by τ_s:

$$\tau_s\colon \mathcal{U}(r, rm)^s \times \mathrm{Pic}^h(C) \longrightarrow \mathcal{U}(r, r(g-1))^s. \tag{3.1}$$

Note that τ_s is a smooth morphism whose fibers are biregular to $\mathrm{Pic}^h(C)$. Let p_i, $i = 1, 2$, denote the projections of $\mathcal{U}(r, rm)^s \times \mathrm{Pic}^h(C)$ onto factors. Let us consider the pull-back $\tau_s^*\Theta_r$ of Θ_r and the restriction of p_2 to $\tau_s^*\Theta_r$:

$$p_{2|\tau_s^*\Theta_r}\colon \tau_s^*\Theta_r \longrightarrow \mathrm{Pic}^h(C). \tag{3.2}$$

It is a surjective map whose fibers are all isomorphic to $\Theta_{r|\mathcal{U}(r,r(g-1))^s}$, since the restriction $\tau_{s|\mathcal{U}(r,rm)^s\times N}$ is an isomorphism for any N. So all fibers are irreducible of the same dimension. This implies that $\tau_s^*\Theta_r$ is an irreducible subscheme of codimension 1 of $\mathcal{U}(r, rm)^s \times \mathrm{Pic}^h(C)$. Moreover, since τ_s is smooth, by (2.4) we have:

$$\mathrm{Sing}(\tau_s^*\Theta_r) = \tau_s^*\big\{[F] \in \mathcal{U}(r, r(g-1))^s \mid [F] \in \mathrm{Sing}(\Theta_r)\big\} = \tau_s^* B(r, r(g-1), 2).$$

By Theorem 2.4, $B(r, r(g-1), 2)$ is an irreducible subscheme of $\mathcal{U}(r, r(g-1))^s$ of codimension 4. By the same argument used above, we can conclude that $\tau_s^* B(r, r(g-1), 2)$ is also an irreducible subscheme of $\mathcal{U}(r, rm)^s \times \mathrm{Pic}^h(C)$ of codimension 4.

Let us consider the restriction of p_1 to $\tau_s^*(\Theta_r)$:

$$p_{1|\tau_s^*(\Theta_r)}\colon \tau_s^*(\Theta_r) \longrightarrow \mathcal{U}(r, rm)^s, \tag{3.3}$$

for general $[E] \in \mathcal{U}(r, rm)^s$ the fiber at $[E]$ is actually the theta divisor Θ_E. Let us consider the open subset of smooth points of $\tau_s^*(\Theta_r)$:

$$X = \tau_s^*(\Theta_r) \setminus \tau_s^* B(r, r(g-1), 2), \tag{3.4}$$

and look at the restriction of p_1 to X:

$$p_{1|X}\colon X \longrightarrow \mathcal{U}(r, rm)^s.$$

It is a dominant map, moreover since X and $\mathcal{U}(r, rm)^s$ are smooth and irreducible, by generic smoothness, see [7], there exists an open subset

$$V \subset \mathcal{U}(r, rm)^s$$

such that $p_{1|\ X\cap p_1{}^{-1}(V)}\colon X\cap p_1{}^{-1}(V)\to V$ is a smooth morphism. This implies that for $[E]\in V$ the fiber $\Theta_E\cap X$ is smooth, that is,

$$\operatorname{Sing}(\Theta_E)=\Theta_E\cdot\tau_s^*B(r,r(g-1),2)=\big\{N\in\operatorname{Pic}^h(C)\mid h^0(E\otimes N)\geq 2\big\}. \tag{3.5}$$

By the above considerations, we can conclude that for a general vector bundle $[E]\in\mathcal{U}(r,rm)^s$, the divisor Θ_E is irreducible and reduced, and its singular locus is described in (3.5). So Theorem 1.1 follows from the next two lemmas.

Lemma 3.1. *Let C be a non-hyperelliptic curve of genus 3. Then, for a general stable $[E]\in\mathcal{U}(r,rm)^s$, the intersection $\Theta_E\cdot\tau_s^*B(r,2r,2)$ is empty.*

Proof. If C is non-hyperelliptic of genus 3 then

$$\dim\ \tau_s^*B(r,2r,2)=2r^2<\dim\ \mathcal{U}(r,rm)^s=2r^2+1.$$

Let us consider the restriction of p_1 to $\tau_s^*B(r,2r,2)$:

$$p_{1|\tau_s^*B(r,2r,2)}\colon\tau_s^*B(r,2r,2)\longrightarrow\mathcal{U}(r,rm)^s. \tag{3.6}$$

The image of $p_{1|\tau_s^*B(r,2r,2)}$ is a closed subvariety of $\mathcal{U}(r,rm)^s$, so we can conclude for general $[E]\in\mathcal{U}(r,rm)^s$ the intersection $\Theta_E\cdot\tau_s^*B(r,2r,2)$ is empty. □

Lemma 3.2. *Let C be a Petri curve of genus $g\geq 4$. Then, for a general stable $[E]\in\mathcal{U}(r,rm)^s$, the intersection $\Theta_E\cdot\tau_s^*B(r,r(g-1),2)$ has dimension $g-4$.*

Proof. Let us consider the restriction of p_1 to $\tau_s^*B(r,r(g-1),2)$:

$$p_{1|\tau_s^*B(r,r(g-1),2)}\colon\tau_s^*B(r,r(g-1),2)\longrightarrow\mathcal{U}(r,rm)^s. \tag{3.7}$$

We prove that the map (3.7) is dominant, hence a general fiber has dimension $g-4$.

For $([E],N)\in\mathcal{U}(r,rm)^s\times\operatorname{Pic}^h(C)$ let $d(p_1)_{[E],N}$ be the induced map on tangent spaces:

$$d(p_1)_{[E],N}\colon T_{[E]}(\mathcal{U}(r,rm))\times T_N(\operatorname{Pic}^h(C))\longrightarrow T_{[E]}(\mathcal{U}(r,rm)),$$

whose kernel is the tangent space $T_{[E],N}([E]\times\operatorname{Pic}^h(C))$ of the fiber of p_1 at $[E]$. For a general $([E],N)\in\tau_s^*B(r,r(g-1),2)$, let us consider the restriction

$$d(p_1)_{[E],N|T_{[E],N}(\tau_s^*B(r,r(g-1),2))}\colon T_{[E],N}\big(\tau_s^*B(r,r(g-1),2)\big)\longrightarrow T_{[E]}\big(\mathcal{U}(r,rm)\big),$$

it is a surjective map if and only we have:

$$\mathit{dim}\ T_{[E],N}\big(\tau_s^*B(r,r(g-1),2)\big)\cap T_{[E],N}\big([E]\times\operatorname{Pic}^h(C)\big)=g-4. \tag{3.8}$$

We recall that for any $([E],N)\in\mathcal{U}(r,rm)^s\times\operatorname{Pic}^h(C)$ we have:

$$T_{[E],N}\big([E]\times\operatorname{Pic}^h(C)\big)\simeq H^1(O_C),$$
$$T_{[E],N}\big(\tau_s^*B(r,r(g-1),2)\big)=\tau_s^*\big(T_{[E\otimes N]}(B(r,r(g-1),2))\big).$$

Finally if $h^0(E\otimes N)=2$, by Lemma 2.2, we have:

$$T_{[E\otimes N]}\big(B(r,r(g-1),2)\big)\simeq (\operatorname{coker}\mu_{E\otimes N})^*\subset H^1(E\otimes E^*),$$

where $\mu_{E\otimes N}$ is the Petri map of $E\otimes N$, and

$$H^1(E\otimes E^*)\simeq T_{[E]}\big(\mathcal{U}(r,rm)\big)\simeq T_{[E\otimes N]}\big(\mathcal{U}(r,r(g-1))\big).$$

We also recall that $E\otimes E^*$ is the sheaf of endomorphisms of E and the trace homomorphism tr defines the subsheaf of tracenull endomorphisms:

$$0\to \operatorname{End}_0(E)\to E\otimes E^*\to O_C\to 0. \tag{3.9}$$

If E is stable then $H^0(E\otimes E^*)=\{\lambda\cdot \mathrm{id}_E|\lambda\in\mathcal{C}\}\simeq H^0(O_C)$, so that we have

$$H^1(E\otimes E^*)=H^1(\operatorname{End}_0(E))\oplus H^1(O_C); \tag{3.10}$$

moreover, $H^1(O_C)$ is the image of the tangent space of $[E]\times \operatorname{Pic}^h(C)$:

$$d(\tau_E)_N\colon H^1(O_C)\longrightarrow H^1(E\otimes E^*). \tag{3.11}$$

So to prove (3.8), it is enough to prove that for a general stable vector bundle $E\otimes N\in B(r,r(g-1),2)$ we have that

$$\dim\big(H^1(O_C)\cap(\operatorname{coker}\mu_{E\otimes N})^*\big)=g-4. \tag{3.12}$$

By taking dual spaces of (3.10), let

$$\pi_E\colon H^0(E\otimes E^*\otimes\omega_C)\longrightarrow H^0(\omega_C) \tag{3.13}$$

be the natural projection map. Then condition (3.12) is satisfied if and only if we require that the composition of the Petri map $\mu_{E\otimes N}$ with π_E has maximal rank:

$$\begin{array}{ccc} H^0(E\otimes N)\otimes H^0(\omega_C\otimes E^*\otimes N^*) & \xrightarrow{\mu_{E\otimes N}} & H^0(E\otimes E^*\otimes\omega_C)\\ & \searrow^{\pi_E\cdot\mu_{E\otimes N}} & \downarrow{\scriptstyle \pi_E}\\ & & H^0(\omega_C). \end{array}$$

Note that actually π_E can be identified with the map induced on global sections from the exact sequence (3.9) tensored with the canonical line bundle ω_C.

Finally, we prove by induction on r that the map $\pi_{grF}\cdot\mu_{grF}$ has maximal rank for a general $[F]\in\tilde{B}(r,r(g-1),2)$.

Let $r=1$: for a general line bundle $L\in B(1,g-1,2)=W^1_{g-1}$, the Petri map is injective

$$\mu_L\colon H^0(L)\otimes H^0(\omega_C\otimes L^*)\longrightarrow H^0(\omega_C)$$

since C is Petri, and the map π_L is the identity.

Let $G\in B(r-1,(r-1)(g-1),2)$ be a general stable bundle satisfying the claim. Let us consider the following semistable vector bundle:

$$F=G\oplus L,\quad L\in\operatorname{Pic}^{g-1}(C),\quad h^0(L)=0. \tag{3.14}$$

Then $H^0(F) \simeq H^0(G)$ and $H^0(\omega_C \otimes F^*) \simeq H^0(\omega_C \otimes G^*)$, so we can conclude that $[F] \in \tilde{B}(r, r(g-1), 2)$. Let $i\colon H^0(G \otimes G^* \otimes \omega_C) \hookrightarrow H^0(F \otimes F^* \otimes \omega_C)$ be the natural inclusion. We have the following commutative diagram:

$$\begin{array}{ccccc} H^0(G) \otimes H^0(\omega_C \otimes G^*) & \xrightarrow{\mu_G} & H^0(G \otimes G^* \otimes \omega_C) & \xrightarrow{\pi_G} & H^0(\omega_C) \\ \simeq \downarrow & & i \downarrow & & \| \\ H^0(F) \otimes H^0(\omega_C \otimes F^*) & \xrightarrow{\mu_F} & H^0(F \otimes F^* \otimes \omega_C) & \xrightarrow{\pi_F} & H^0(\omega_C). \end{array}$$

By induction hypothesis the composition map $\pi_G \cdot \mu_G$ has maximal rank, so $\pi_F \cdot \mu_F$ has maximal rank too. Let $U \subset \tilde{B}(r, r(g-1), 2)$ be the subset corresponding to classes $[F]$ such that the map $\pi_{grF} \cdot \mu_{grF}$ has maximal rank, U is not empty. Since this condition is open on each family of vector bundles of $\tilde{B}(r, r(g-1), 2)$, it follows that U is a non empty open subset of $\tilde{B}(r, r(g-1), 2)$. This implies that a general $[F] \in \tilde{B}(r, r(g-1), 2)$ satisfies the property too. □

This concludes the proof of Theorem 1.1.

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