



Multilinear paraproducts revisited

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In memory of Nigel Kalton

Abstract. We prove that multilinear paraproducts are bounded from products of Lebesgue spaces $L^{p_1} \times \cdots \times L^{p_{m+1}}$ to $L^{p, \infty}$, when $1 \leq p_1, \dots, p_m, p_{m+1} < \infty$, $1/p_1 + \cdots + 1/p_{m+1} = 1/p$. We focus on the endpoint case when some indices p_j are equal to 1, in particular we obtain a new proof of the estimate $L^1 \times \cdots \times L^1 \rightarrow L^{1/(m+1), \infty}$.

1. Introduction

Paraproducts have become tools of great use in analysis and PDEs. They are traditionally built by Littlewood–Paley square functions and may appear in different forms. Paraproducts first emerged in Bony’s theory of paradifferential operators [5], which has taken a step further the pseudodifferential operator theory of Coifman and Meyer [6]. They provide important examples of operators with specific properties and have been used in significant applications, such as the proof of the $T1$ theorem by David and Journé [7]. The relationship of paraproducts with Carleson measures and BMO is so intimate that the former have been on the forefront of research in harmonic analysis through almost a quarter century. The boundedness of paraproducts on L^p spaces for $p > 1$ is easily achieved via duality, but the extension to indices $p \leq 1$ is more delicate and was proved independently by Grafakos and Kalton [9] and by Auscher, Hofmann, Muscalu, Thiele, and Tao [1]; a different proof was given by Bényi, Maldonado, Nahmod, and Torres [2]. Hundreds of references exist on paraproducts today; of these the articles [4], [9], [13] and [14] focus on delicate boundedness properties of them. The expository article of Bényi, Maldonado, Naibo [3] presents a well-motivated introduction to paraproducts.

Multilinear paraproducts may have first appeared explicitly in the work of Yabuta [16] and later resurfaced in the work of Sato and Yabuta [15] who obtained their L^p boundedness for $p \geq 1$. Although paraproducts fit into the class

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of multilinear Calderón–Zygmund theory, one may wonder if there are insightful direct proofs of their L^p (resp. weak L^p) boundedness, especially in the difficult case $p < 1$. Such proofs would take into account the specific form of paraproducts and would reflect the interplay of their intrinsic orthogonality with the orthogonality of L^p (resp. weak L^p). In this work we undertake this task and we include the endpoint cases when at least one index is 1. Our work is based on a weak type square function inequality (Lemma 1.2) recently obtained in [11], which is valid for all $0 < p < \infty$. Another type of m -linear paraproducts built by sums of wave packets associated with dyadic intervals on the line has been studied by Lacey and Metcalfe [12] who obtained similar endpoint estimates to the ones in this article for the paraproducts built by the *Littlewood–Paley operators*.

We will be working on \mathbb{R}^d for some natural number d . For a Schwartz function Φ we denote by Δ_j^Φ the Littlewood–Paley operator given by convolution with the function $\Phi_{2^{-j}}(x) = 2^{jd}\Phi(2^jx)$. We denote by $S_j^\Phi = \sum_{k \leq j} \Delta_k^\Phi$ the partial sum operator of the Δ_k^Φ 's. For fixed smooth bumps Φ and Θ whose Fourier transforms have compact supports that do not contain the origin, we define the paraproduct operator

$$P_2(f, g) = \sum_{j \in \mathbb{Z}} \sum_{k \leq j} \Delta_j^\Theta(f) \Delta_k^\Phi(g) = \sum_{j \in \mathbb{Z}} \Delta_j^\Theta(f) S_j^\Phi(g),$$

for Schwartz functions f, g . This operator and its $(m+1)$ -linear version is the main object of study of this paper. This is defined by

$$P_{m+1}(f_0, f_1, \dots, f_m) = \sum_{j \in \mathbb{Z}} \Delta_j^\Theta(f_0) S_j^{\Theta_1}(f_1) \cdots S_j^{\Theta_m}(f_m),$$

for Schwartz functions f_0, f_1, \dots, f_m and smooth bumps $\Theta, \Theta_1, \dots, \Theta_m$.

For $0 < p < \infty$, we denote by L^p the space of all measurable functions on \mathbb{R}^d whose p th power is integrable over \mathbb{R}^d and by $L^{p,\infty}$ the space of all measurable functions h that satisfy

$$\|h\|_{L^{p,\infty}} = \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^d : |h(x)| > \lambda\}|^{1/p} < \infty.$$

Given a bump Ψ , we define the square function associated with Ψ by

$$\mathbf{S}^\Psi(f) = \left(\sum_{\ell \in \mathbb{Z}} |\Delta_\ell^\Psi(f)|^2 \right)^{1/2}.$$

We will also work with the “lacunary” square function

$$\mathbf{S}_q^\Psi(f) = \left(\sum_{\ell \in \mathbb{Z}} |\Delta_{q\ell}^\Psi(f)|^2 \right)^{1/2}.$$

defined for a positive integer q . (Notice that $\mathbf{S}_1^\Psi = \mathbf{S}^\Psi$.) Under very mild assumptions on Ψ (such as $|\Psi(x)| + |\nabla\Psi(x)| \leq A(1 + |x|)^{-d-\varepsilon}$ and $\int_{\mathbb{R}^d} \Psi(x) dx = 0$), it is known that \mathbf{S}^Ψ (also \mathbf{S}_q^Ψ) maps $L^r(\mathbb{R}^d)$ to $L^{r,\infty}(\mathbb{R}^d)$ for all $1 \leq r < \infty$ (see [8]).

Finally, we denote by \mathbf{M} the Hardy–Littlewood maximal operator. We recall that

$$\sup_{j \in \mathbb{Z}} |\Delta_j^\Theta(f)| + \sup_{j \in \mathbb{Z}} |S_j^\Theta(f)| \leq C_\Theta \mathbf{M}(f),$$

for all Schwartz functions f , for some constant C_Θ .

The main goal of this paper is to indicate how to obtain boundedness for P_{m+1} from the product of Lebesgue spaces $L^{p_0} \times L^{p_1} \times \dots \times L^{p_m}$ to $L^{p,\infty}$ whenever $1 \leq p_0, p_1, \dots, p_m < \infty$ and $p = (p_0^{-1} + p_1^{-1} + \dots + p_m^{-1})^{-1}$. The case $p \geq 1$ is quite easy to deal with via duality and Hölder’s inequality, but the case $p < 1$ is more delicate and we will focus on it. In particular, we show paraproducts map $L^1 \times \dots \times L^1 \rightarrow L^{1/(m+1),\infty}$ which is the strongest endpoint estimate concerning them.

When $m = 1$ this result is known, see for instance [9], [1], [12], but the contribution of this paper is to provide a simple proof of it that does not rely on deep technical machinery (tiles, Carleson measures) and which also works for all $m \geq 1$. The following is our main result.

Theorem 1.1. *Fix an integer $m \geq 1$ and smooth bumps $\Theta, \Theta_1, \dots, \Theta_m$ whose Fourier transforms are compactly supported in $\mathbb{R}^d \setminus \{0\}$. For each $0 \leq k \leq m - 1$ and functions f_j in the Schwartz class of \mathbb{R}^d define the $(m + 1)$ -linear paraproduct*

$$(1.1) \quad P_{m+1}^{(k)}(f_0, f_1, \dots, f_m) = \sum_{j \in \mathbb{Z}} \left[\Delta_j^\Theta(f_0) \prod_{s=1}^k \Delta_j^{\Theta_s}(f_s) \prod_{s=k+1}^m S_j^{\Theta_s}(f_s) \right],$$

with the understanding that when $k = 0$, the first product is missing. Let p be defined by $p^{-1} = p_0^{-1} + p_1^{-1} + \dots + p_m^{-1}$. Then $P_{m+1}^{(k)}$ is bounded from $L^{p_0}(\mathbb{R}^d) \times L^{p_1}(\mathbb{R}^d) \times \dots \times L^{p_m}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ when $1 \leq p_j < \infty$ and into $L^p(\mathbb{R}^d)$ when $1 < p_j < \infty$ for all j .

We will need the following lemma, which is Corollary 4 in [11].

Lemma 1.2. *Let Ψ be a smooth bump whose Fourier transform is supported in an annulus that does not contain the origin and satisfies, for some positive integer q ,*

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-jq}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Then for any $0 < p < \infty$ there is a constant $C_{p,d}$ (that also depends on Ψ) such that for all functions g in L^2 we have

$$\|g\|_{L^{p,\infty}} \leq C_{p,d} \|\mathbf{S}_q^\Psi(g)\|_{L^{p,\infty}}.$$

2. The proof of the Theorem 1.1

When all $p_j > 1$, the fact $P_{m+1}^{(k)}: L^{p_0} \times L^{p_1} \times \dots \times L^{p_m} \rightarrow L^p$ is a consequence of the corresponding weak type estimate via multilinear interpolation, see [10]. It will therefore suffice to prove that $P_{m+1}^{(k)}$ maps $L^{p_0} \times L^{p_1} \times \dots \times L^{p_m}$ to $L^{p,\infty}$ when $1/(m + 1) \leq p < \infty$.

We suppose that the Fourier transform of Θ is supported in the annulus $a_0 < |\xi| < b_0$ for some $0 < a_0 < b_0 < \infty$, of Θ_j is supported in the annulus $a_j < |\xi| < b_j$ for some $0 < a_j < b_j < \infty$, $1 \leq j \leq m$.

Case 1: $m \geq 1$ and $k = m - 1$.

Subcase 1.a: $m \geq 2$.

When $k = m - 1$ only one partial sum operator S_j appears in the product in (1.1). Then, for $m \geq 2$, $P_m^{(m-1)}(f, f_1, \dots, f_m)$ is pointwise bounded by

$$\left(\sum_{j \in \mathbb{Z}} |\Delta_j^{\Theta_1}(f_1) \cdots \Delta_j^{\Theta_{m-1}}(f_{m-1})|^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} |\Delta_j^\Theta(f_0) S_j^{\Theta_m}(f_m)|^2 \right)^{1/2}.$$

This is in turn controlled by

$$(2.1) \quad \mathbf{S}^{\Theta_1}(f_1) [\mathbf{M}(f_2) \cdots \mathbf{M}(f_{m-1})] [\mathbf{S}^\Theta(f_0) \mathbf{M}(f_m)]$$

(with the understanding that the middle factor does not appear when $m = 2$) which is easily shown to satisfy the claimed conclusion, by applying Hölder’s inequality on weak L^p spaces (i.e., $\|g_0 g_1 \cdots g_m\|_{L^{p,\infty}} \leq \|g_0\|_{L^{p_0,\infty}} \|g_1\|_{L^{p_1,\infty}} \cdots \|g_m\|_{L^{p_m,\infty}}$) and using the boundedness of the maximal and square functions from L^r to $L^{r,\infty}$ for $1 \leq r < \infty$.

Subcase 1.b: $m = 1$.

In this case we write

$$S_j^{\Theta_1} = S_{j+r_0}^{\Theta_1} + \sum_{i=j+r_0+1}^j \Delta_i^{\Theta_1},$$

for some $r_0 < 0$ chosen so that the spectra of $S_{j+r_0}^{\Theta_1}$ and Δ_j^Θ are disjoint; picking r_0 so that $b_1 2^{r_0+j} < a_0 2^j$ suffices. Then the function $\Delta_j^\Theta(f) S_{j+r_0}^{\Theta_1}(f_1)$ is supported in the annulus

$$(a_0 - b_1 2^{r_0}) 2^j < |\xi| < (b_0 + b_1 2^{r_0}) 2^j.$$

We pick integers $n_0 < m_0$ such that

$$2^{n_0} < a_0 - b_1 2^{r_0} < b_0 + b_1 2^{r_0} < 2^{m_0}$$

and we choose a function Ω whose Fourier transform equals 1 on the annulus $2^{n_0} < |\xi| < 2^{m_0}$, vanishes off the annulus $2^{n_0-1} < |\xi| < 2^{m_0+1}$, and satisfies

$$(2.2) \quad \sum_{\ell \in \mathbb{Z}} \widehat{\Omega}(2^{(m_0-n_0+1)\ell} \xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

It follows from (2.2) that

$$(2.3) \quad \sum_{\ell \in \mathbb{Z}} \widehat{\Omega}(2^\ell \xi) = m_0 - n_0 + 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Then we write

$$(2.4) \quad P_2^{(0)}(f_0, f_1) = \sum_{j \in \mathbb{Z}} \Delta_j^\Omega (\Delta_j^\Theta (f_0) S_{j+r_0}^{\Theta_1} (f_1)) + E,$$

where E is a finite sum of terms of the form $\sum_j \Delta_j^\Theta (f_0) \Delta_{j+c}^{\Theta_1} (f_1)$. Since E is pointwise bounded by a constant multiple of $\mathbf{S}^\Theta (f_0) \mathbf{S}^{\Theta_1} (f_1)$, the required conclusion follows for E via an application of Hölder’s inequality for weak type spaces.

We need to argue a bit more to handle the first term on the right in (2.4). We pick a function Ψ whose Fourier transform is equal to 1 on the annulus $2^{n_0-2} < |\xi| < 2^{m_0+2}$ and vanishes off the annulus $2^{n_0-3} < |\xi| < 2^{m_0+3}$. Set $q = m_0 - n_0 + 5$. We split \mathbb{Z} as a disjoint union of sets $I_s = \{\ell q + s, \ell \in \mathbb{Z}\}$, $0 \leq s \leq q - 1$. Next we split the sum in (2.4) as a finite sum over $s \in \{0, 1, \dots, q - 1\}$ of the sums

$$(2.5) \quad \Sigma_s = \sum_{j \in I_s} \Delta_j^\Omega [\Delta_j^\Theta (f_0) S_{j+r_0}^{\Theta_1} (f_1)].$$

We also define a function Ψ_s by setting $\widehat{\Psi}_s(\xi) = \widehat{\Psi}(2^{-s}\xi)$ and we note that $\sum_\ell \widehat{\Psi}_s(2^{-\ell q}\xi) = 1$ for ξ in $\mathbb{R}^d \setminus \{0\}$.

We make the following crucial observation: for $j \in I_s$ and $\ell \in \mathbb{Z}$ the supports of the functions $\xi \rightarrow \widehat{\Psi}_s(2^{-\ell q}\xi)$ and $\xi \rightarrow \widehat{\Omega}(2^{-j}\xi)$ intersect exactly when $j = \ell q + s$ and this case $\Delta_j^\Omega \Delta_{\ell q}^{\Psi_s} = \Delta_j^\Omega$ as the first function equals 1 on the support of the second. We deduce that for $j \in I_s$ and $\ell \in \mathbb{Z}$ we have

$$\Delta_{\ell q}^{\Psi_s} \left[\sum_{j \in I_s} \Delta_j^\Omega [\Delta_j^\Theta (f_0) S_{j+r}^{\Theta_1} (f_1)] \right] = \Delta_{\ell q+s}^\Omega [\Delta_{\ell q+s}^\Theta (f_0) S_{\ell q+s+r_0}^{\Theta_1} (f_1)],$$

and this exactly equals $\Delta_{\ell q+s}^\Theta (f) S_{\ell q+s+r_0}^{\Theta_1} (f_1)$. It follows that

$$\mathbf{S}_q^{\Psi_s}(\Sigma_s) = \left(\sum_{\ell \in \mathbb{Z}} |\Delta_{\ell q}^{\Psi_s}(\Sigma_s)|^2 \right)^{1/2} = \left(\sum_{\ell \in \mathbb{Z}} |\Delta_{\ell q+s}^\Theta (f_0) S_{\ell q+s+r_0}^{\Theta_1} (f_1)|^2 \right)^{1/2},$$

which is pointwise controlled by a constant multiple of $\mathbf{S}_q^\Theta (f_0) \mathbf{M}(f_1)$. To apply Lemma 1.2 we need to show that Σ_s defined in (2.5) lies in L^2 . By the orthogonality of L^2 -norms, we have

$$\begin{aligned} \left\| \sum_{j \in I_s} \Delta_j^\Omega [\Delta_j^\Theta (f_0) S_{j+r_0}^{\Theta_1} (f_1)] \right\|_{L^2}^2 &= \sum_{j \in I_s} \int_{\mathbf{R}^n} |\Delta_j^\Omega [\Delta_j^\Theta (f_0) S_{j+r_0}^{\Theta_1} (f_1)](x)|^2 dx \\ &\leq C \|M(f_1)\|_{L^\infty} \sum_{j \in I_s} \int_{\mathbf{R}^n} |\Delta_j^\Theta (f_0)(x)|^2 dx \\ &\leq C \|f_1\|_{L^\infty} \|f_0\|_{L^2}^2 < \infty. \end{aligned}$$

Using Lemma 1.2, for each $s \in \{0, 1, \dots, q - 1\}$ we obtain that

$$\|\Sigma_s\|_{L^{p,\infty}} \leq C_p \|\mathbf{S}_q^{\Psi_s}(\Sigma_s)\|_{L^{p,\infty}},$$

and by the previous discussion this expression at most a constant multiple of $\|\mathbf{S}_q^\ominus(f_0)\mathbf{M}(f_1)\|_{L^{p,\infty}}$. The required conclusion is an easy consequence of Hölder’s inequality and of the boundedness of the maximal and square functions from L^r to $L^{r,\infty}$ for $1 \leq r < \infty$.

Case 2: $m \geq 2$ and $k < m - 1$.

Having established the case $k = m - 1$, we continue the proof by reverse induction on k . Fix a $k \in \{0, 1, \dots, m - 2\}$ and assume that the conclusion is valid for all $k' > k$ (and $k' \leq m - 1$.) We need to prove the same conclusion for k .

We begin by writing for all $s \in \{k + 1, \dots, m\}$

$$S_j^{\ominus_s} = S_{j+r_s}^{\ominus_s} + \sum_{i=j+r_s+1}^j \Delta_i^{\ominus_s}$$

for some $r_s < 0$ that satisfy

$$(2.6) \quad b_{k+1}2^{rk+1} + \dots + b_m2^{rm} < a_0$$

so that the spectra of $S_{j+r_{k+1}}^{\ominus_{k+1}}(f_{k+1}) \cdots S_{j+r_m}^{\ominus_m}(f_m)$ and $\Delta_j^\ominus(f_0)$ are disjoint.

Then we express $P_{m+1}^{(k)}$ as a finite sum of operators of the form $P_{m+1}^{(k+1)}, P_{m+1}^{(k+2)}, \dots, P_{m+1}^{(m-1)}$ plus

$$(2.7) \quad \sum_{j \in \mathbb{Z}} \left[\Delta_j^\ominus(f_0) \prod_{s=k+1}^m S_{j+r_s}^{\ominus_s}(f_s) \right] \left[\prod_{s=1}^k \Delta_j^{\ominus_s}(f_s) \right],$$

with the understanding that if $k = 0$, the last product does not appear. The induction hypothesis on k yields the boundedness of $P_{m+1}^{(k+1)}, P_{m+1}^{(k+2)}, \dots, P_{m+1}^{(m-1)}$, while the boundedness of (2.7) is discussed below considering two subcases.

Subcase 2.a: $k \geq 1$.

In this subcase things are straightforward. We apply the Cauchy–Schwarz inequality to control (2.7) by the product of the ℓ^2 norms of the expressions inside the square brackets and therefore by the product

$$\mathbf{S}^\ominus(f_0) \mathbf{S}^{\ominus_1}(f_1) \left[\prod_{s=2}^m \mathbf{M}(f_s) \right].$$

Obviously, this expression is bounded from $L^{p_0} \times \dots \times L^{p_m}$ to $L^{p,\infty}$.

Subcase 2.b: $k = 0$.

Condition (2.6) implies that the function $\Delta_j^\ominus(f_0) S_{j+r_1}^{\ominus_1}(f_1) \cdots S_{j+r_m}^{\ominus_m}(f_m)$ is supported in the annulus $2^{n_0}2^j < |\xi| < 2^{m_0}2^j$ where $n_0 < m_0$ are integers chosen so that

$$2^{n_0} < (a_0 - (b_12^{r_1} + \dots + b_m2^{r_m})) < (b_0 + b_12^{r_1} + \dots + b_m2^{r_m}) < 2^{m_0}.$$

We choose a smooth function Ω which is equal to 1 on the annulus $2^{n_0} < |\xi| < 2^{m_0}$ and vanishes off the annulus $2^{n_0-1} < |\xi| < 2^{m_0+1}$. Then we write the expression in (2.7) as follows:

$$(2.8) \quad \sum_{j \in \mathbb{Z}} \Delta_j^\Omega \left[\Delta^\Theta(f_0) \prod_{s=1}^m S_{j+r_s}^{\Theta_s}(f_s) \right].$$

We now pick a function Ψ whose Fourier transform is equal to 1 on the annulus $2^{n_0-2} < |\xi| < 2^{m_0+2}$ and vanishes outside the annulus $2^{n_0-3} < |\xi| < 2^{m_0+3}$. Set $q = m_0 - n_0 + 5$. We split \mathbb{Z} as a disjoint union of sets $I_s = \{\ell q + s, \ell \in \mathbb{Z}\}$, $0 \leq s < q$. Next we split the sum in (2.8) as a finite sum over $s \in \{0, 1, \dots, q-1\}$ of the sums Σ_s where the indices j in (2.8) run over the set I_s . We also define a function $\widehat{\Psi}_s$ by setting $\widehat{\Psi}_s(\xi) = \widehat{\Psi}(2^{-s}\xi)$ and we note that $\sum_\ell \widehat{\Psi}_s(2^{-\ell q}\xi) = 1$ for ξ in $\mathbb{R}^d \setminus \{0\}$.

We observe that for $j \in I_s$ and $\ell \in \mathbb{Z}$ the supports of the functions $\xi \rightarrow \widehat{\Psi}_s(2^{-\ell q}\xi)$ and $\xi \rightarrow \widehat{\Omega}(2^{-j}\xi)$ intersect nontrivially exactly when $j = \ell q + s$ and this case $\Delta_j^\Omega \Delta_{\ell q}^{\widehat{\Psi}_s} = \Delta_j^\Omega$. We are therefore in a position to use Lemma 1.2, since again we can control the L^2 -norm of $\sum_{j \in I_s} \Delta_j^\Omega [\Delta^\Theta(f_0) \prod_{s=1}^m S_{j+r_s}^{\Theta_s}(f_s)]$ by $C \prod_{s=1}^m \|f_s\|_{L^\infty} \|f_0\|_{L^2} < \infty$, and argue as in Subcase 2.2 to complete the proof.

Remark 2.1. The exponent p_j can be taken to be equal to infinity whenever the maximal function $\mathbf{M}(f_j)$ appears in the estimate controlling $P_{m+1}^{(k)}$ (pointwise or in norm). For instance, when $m \geq 2$ and $k = m - 1$, we may take $p_2 = \dots = p_m = \infty$; see (2.1).

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