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An Application of Orthoisomorphisms to Non-Commutative L^p-Isometries

By

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Abstract

We prove that if there exists an into linear isometry between non-commutative L^p-spaces then there exists an into Jordan $*$-isomorphism between underlying von Neumann algebras, as an application of Araki-Bunce-Wright's theorem concerning the characterization of orthogonality preserving positive maps between preduals. Moreover, we determine the structure of a linear non-commutative L^p-isometry when it is surjective and $*$-preserving.

§ 0. Introduction

In this paper, we consider the following problems. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras. Let $1<p<\infty$, $p\neq 2$ and let $L^p(\mathcal{M}_1)$ and $L^p(\mathcal{M}_2)$ be associated non-commutative L^p-spaces. Suppose that there exists a linear isometry T from $L^p(\mathcal{M}_1)$ to $L^p(\mathcal{M}_2)$. Then, at first, can we find a Jordan $*$-isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$? Secondly, can we describe the structure of T in terms of the induced Jordan $*$-isomorphism?

These problems have the origin in Banach [B]. Several authors had developed the theory, and there is a complete description of isometries for the case of semifinite von Neumann algebras in Yeadon [Y].

On the other hand, after the development of the modular theory, one can construct non-commutative L^p-spaces associated with von Neumann algebras which are not necessarily semifinite. Although there are different methods of construction, those are by Haagerup [H3] (see also [T1]), Araki-Masuda [AM], Hilsum [Hi], Kosaki [Ko2], Terp [T2] etc., it is known that for a fixed von Neumann algebra those L^p-spaces are canonically isometrically isomorphic each other.

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Some difficulties to deal with non-commutative L^p-spaces associated with arbitrary von Neumann algebras come from the following facts. Though one can embed the original von Neumann algebra to its L^p-spaces in the σ-finite case, no one knows which embedding is most canonical. In other words, there appear highly non-commutative obstructions such as Radon-Nikodym derivatives, which turn to be central elements in the semifinite case. So it does not seem easy to obtain a common area between the L^p-spaces and the original von Neumann algebra, and it seems that many techniques used in semifinite case are no more available.

We work on Haagerup's L^p-spaces, since those elements are (unbounded) operators, and their polar decompositions give us informations related to the original von Neumann algebra or its predual.

In [W1], the existence of a surjective Jordan $*$-isomorphism was shown when $\mathcal{M}_1$, $\mathcal{M}_2$ are σ-finite and T is surjective $*$-preserving.

In Section 2, we will prove the existence of a Jordan $*$-isomorphism without any restrictions on $\mathcal{M}_1$, $\mathcal{M}_2$ and T, making use of Araki-Bunce-Wright's theorem which characterizes orthogonality preserving positive maps between preduals of von Neumann algebras.

In [W2], the structure of T was described when $\mathcal{M}_1$, $\mathcal{M}_2$ are σ-finite and T is surjective positive.

In Section 3, we will prove that if T is surjective $*$-preserving then T is the composition of the induced Jordan $*$-isomorphism and the canonical $*$-isomorphism arised from the change of weights followed by multiplication by a fixed central symmetry.

§1. Preliminaries

We begin with some basic definitions concerning Haagerup's non-commutative L^p-spaces associated with arbitrary von Neumann algebras. For details and proofs we refer to [H3] and [T1]. Let φ_0 be a fixed faithful normal semifinite weight on $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$. Let $\{\sigma_t^{\varphi_0}\}_{t\in\mathbb{R}}$ be the modular automorphism group with respect to φ_0. We denote by $\mathcal{N}$ the crossed product $\mathcal{M}\rtimes_{\sigma\phi_0}\mathbb{R}$, which is a von Neumann algebra generated by $\pi(x)$, $x\in\mathcal{M}$ and λ_s, $s\in\mathbb{R}$, defined by

$$(\pi(x)\xi)(t)=\sigma_{-t}^{\phi_0}(x)\xi(t),\ \xi\in L^2(\mathbb{R},\mathcal{H}),\ t\in\mathbb{R},$$
$$(\lambda_s\xi)(t)=\xi(t-s),\ \xi\in L^2(\mathbb{R},\mathcal{H}),\ t\in\mathbb{R}.$$

The dual actions, θ_s, $s\in\mathbb{R}$, naturally extend to automorphisms on $\widehat{\mathcal{N}}_+$, which is the extended positive part of $\mathcal{N}$ (cf. [H1; Section 1]). For each normal weight

φ on $\mathcal{M}$, we denote by $\widetilde{\varphi}$ the dual weight of φ on $\mathcal{N}$. It is well-known that there exists a unique faithful normal semifinite trace τ on $\mathcal{N}$ characterized by the Connes' cocycle $(D\widetilde{\varphi}_0 : D\tau)_t = \lambda_t$, $t \in \mathbb{R}$, and τ satisfies $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$ (cf. [H2]; Lemma 5.2]).

Haagerup's L^p-spaces are realized as subspaces consist of measurable operators with respect to this trace τ. A densely defined closed operator a affiliated with $\mathcal{N}$, with its domain $\mathcal{D}(a)$, is said to be τ-measurable if there is, for each $\delta > 0$, a projection $e \in \mathcal{N}$ such that $eL^2(\mathbb{R}, \mathcal{H}) \subset \mathcal{D}(a)$ and $\tau(1-e) \leq \delta$. We denote by $\widetilde{\mathcal{N}}$ the set of all τ-measurable operators, which becomes a complete Hausdorff topological $*$-algebra under the strong operations in the measure topology. For any subset $\mathcal{S}$ of $\widetilde{\mathcal{N}}$, the set of all selfadjoint (resp. positive selfadjoint) operators in $\mathcal{S}$ shall be denoted by $\mathcal{S}_{sa}$ (resp. $\mathcal{S}_+$).

Now the dual actions θ_s, $s \in \mathbb{R}$, are extended to continuous $*$-automorphisms of $\widetilde{\mathcal{N}}$. For $0 < p < \infty$, the Haagerup's L^p-space is defined by

$$L^p(\mathcal{M}; \varphi_0) = \left\{ a \in \widetilde{\mathcal{N}};\ \theta_s(a) = e^{-s/p}a, \quad s \in \mathbb{R} \right\},$$

and simply denoted $L^p(\mathcal{M})$ whenever it is not necessary to indicate the weight φ_0. For each normal weight φ on $\mathcal{M}$, we simply denote by

$$h_\varphi = \frac{d\widetilde{\varphi}}{d\tau}$$

the non-commutative Radon-Nikodym derivative of $\widetilde{\varphi}$ with respect to τ. It is well-known that $\varphi \in \mathcal{M}_{*,+}$, which is the set of all normal positive linear functionals on $\mathcal{M}$, if and only if h_φ is τ-measurable. The mapping $\varphi \to h_\varphi$ is extended to a linear order isomorphism from $\mathcal{M}_*$ onto $L^1(\mathcal{M})$, and so the positive linear functional tr on $L^1(\mathcal{M})$ is defined by

$$tr(h_\varphi) = \varphi(1),\ \varphi \in \mathcal{M}_*.$$

For $0 < p < \infty$, the (quasi-)norm of $L^p(\mathcal{M})$ is defined by $\|a\|_p = tr(|a|^p)^{1/p}$, $a \in L^p(\mathcal{M})$. When $1 \leq p < \infty$, $L^p(\mathcal{M})$ is a Banach space, and its dual Banach space is $L^q(\mathcal{M})$ with $1/p + 1/q = 1$ by the following duality ;

$$\langle a,b \rangle = tr(ab) = tr(ba),\ a \in L^p(\mathcal{M}),\ b \in L^q(\mathcal{M}).$$

Note that for any $a = u|a| \in L^p(\mathcal{M})$ with its polar decomposition, u belongs to $\mathcal{M}$ and $|a|$ belongs to $L^p(\mathcal{M})_+$. Also for any $a = a_+ - a_- \in L^p(\mathcal{M})_{sa}$ with its Jordan decomposition, one has $a_+, a_- \in L^p(\mathcal{M})_+$.

§2. Existence of a Jordan $*$-Isomorphism

In this section, we prove that if there exists an into linear isometry between non-commutative L^p-spaces then there exists an into Jordan $*$-isomorphism between underlying von Neumann algebras. Araki-Bunce-Wright's theorem allows us to prove our result without σ-finiteness of von Neumann algebras and surjectivity of L^p-isometry.

In an interesting article, Araki [A] initiated the study of orthogonal decomposition preserving positive linear maps (o. d. homomorphisms) between preduals of von Neumann algebras. Bunce and Wright [BW] solved a problem in [A] and characterized those maps in a general setting.

Now we state the Bunce-Wright theorem for injective case only, which is just we need here. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be arbitrary von Neumann algebras. Let β : $(\mathcal{M}_1)_* \longrightarrow (\mathcal{M}_2)_*$ be an o.d. homomorphism (that is, β is a continuous linear map which preserves both order and orthogonal decomposition). Moreover, we assume that β is injective. We define $(\mathcal{M}_2)_\beta$ to be the σ-weak closed $*$-subalgebra of $\mathcal{M}_2$ generated by $\{s(\beta(\varphi))\,;\ \varphi \in (\mathcal{M}_1)_{*,+}\}$, where $s(\psi)$ denotes the support projection of $\psi \in (\mathcal{M}_2)_{*,sa}$.

Theorem 1 (Bunce and Wright [BW; Theorem 2.6]). *There is a weak$*$ continuous and surjective Jordan $*$-isomorphism J : $\mathcal{M}_1 \longrightarrow (\mathcal{M}_2)_\beta$ such that $\beta^*(J(x)) = \beta^*(1)x$, for all x* in $\mathcal{M}_1$.

Theorem 2. *Let $1 < p < \infty$ and $p \neq 2$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be arbitrary von Neumann algebras. Let φ_0 (resp. ψ_0) be a faithful normal semifinite weight on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$). Let T be a linear isometry from $L^p(\mathcal{M}_1;\varphi_0)$ to $L^p(\mathcal{M}_2;\psi_0)$. Then there exists a Jordan $*$-isomorphism J from $\mathcal{M}_1$ to $\mathcal{M}_2$ satisfying.*

$$|T(h_\varphi^{1/p})| = h_{\varphi\circ J^{-1}}^{1/p},\ \varphi \in (\mathcal{M}_1)_{*,+}.$$

Proof. For each $\varphi \in (\mathcal{M}_1)_{*,+}$, $|T(h_\varphi^{1/p})|$ belongs to $L^p(\mathcal{M}_2;\ \psi_0)_+$. Hence we can define a map β from $(\mathcal{M}_1)_{*,+}$, to $(\mathcal{M}_2)_{*,+}$, by $h_{\beta(\varphi)}^{1/p} = |T(h_\varphi^{1/p})|$, $\varphi \in (\mathcal{M}_1)_{*,+}$.

Then β satisfies the following conditions;

(1) $\beta(\alpha\varphi) = \alpha\beta(\varphi)$, $\alpha \geq 0$, $\varphi \in (\mathcal{M}_1)_{*,+}$

(2) $\beta(\sum\varphi_n) = \sum\beta(\varphi_n)$, whenever $\{\varphi_n\}$ is a countable family in $(\mathcal{M}_1)_{*,+}$ whose supports are orthogonal each other and the sum $\sum\varphi_n$ exists in $(\mathcal{M}_1)_{*,+}$

(3) $\|\beta(\varphi)\| = \|\varphi\|$, $\varphi \in (\mathcal{M}_1)_{*,+}$

(4) $\beta(\varphi_n) \to \beta(\varphi)$, whenever $\{\varphi_n\}$ is a family in $(\mathcal{M}_1)_{*,+}$ and $\|\varphi_n - \varphi\| \to 0$.

Indeed, it is immediate to see (1) and (3). For the condition (4), if $\varphi_n \to \varphi$, then we have $h_{\varphi_n} \to h_\varphi$. It follows from [Ko1; Theorem 4.2] that $h_{\varphi_n}^{1/p} \to h_\varphi^{1/p}$, so $T(h_{\varphi_n}^{1/p}) \to T(h_\varphi^{1/p})$. It follows from [Ko1; Theorem 4.4] that $|T(h_{\varphi_n}^{1/p})| \to |T(h_\varphi^{1/p})|$, hence we have $\beta(\varphi_n) \to \beta(\varphi)$ again by [Ko1; Theorem 4.2].

For the condition (2), if φ_1 and φ_2 are orthogonal in $(\mathcal{M}_1)_{*,+}$, by the equality condition for the Clarkson's inequality, we have $T(h_{\varphi_1}^{1/p})^* T(h_{\varphi_2}^{1/p}) = T(h_{\varphi_1}^{1/p}) T(h_{\varphi_2}^{1/p})^* = 0$. Therefore

$$\begin{aligned}|T(h_{\varphi_1}^{1/p}+h_{\varphi_2}^{1/p})|^2 &= T(h_{\varphi_1}^{1/p})^* T(h_{\varphi_1}^{1/p}) + T(h_{\varphi_2}^{1/p})^* T(h_{\varphi_2}^{1/p}) \\ &= |T(h_{\varphi_1}^{1/p})|^2 + |T(h_{\varphi_2}^{1/p})|^2 = (|T(h_{\varphi_1}^{1/p})| + |T(h_{\varphi_2}^{1/p})|)^2.\end{aligned}$$

Hence we have

$$\begin{aligned}h_{\beta(\varphi_1+\varphi_2)}^{1/p} &= |T(h_{\varphi_1+\varphi_2}^{1/p})| = |T(h_{\varphi_1}^{1/p}+h_{\varphi_2}^{1/p})| = |T(h_{\varphi_1}^{1/p})| + |T(h_{\varphi_2}^{1/p})| \\ &= h_{\beta(\varphi_1)}^{1/p} + h_{\beta(\varphi_2)}^{1/p} = h_{\beta(\varphi_1)+\beta(\varphi_2)}^{1/p}.\end{aligned}$$

This implies the condition (2), since we have already checked (4).

Thus the map β induces a continuous finite measure on the predual in the sense of [W2; Definition 2], so β is additive as in the proof of [W2; Theorem 5]. We extend β to a positive linear map from $(\mathcal{M}_1)_*$ to $(\mathcal{M}_2)_*$, and denote by β also. It is obvious that β is orthogonal decomposition preserving. Then we can conclude by Bunce-Wright theorem that there exists a weak* continuous Jordan $*$-isomorphism $J: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ such that $\beta^*(J(x)) = \beta^*(1)x$, $x \in \mathcal{M}_1$.

For each $\varphi \in (\mathcal{M}_1)$, we have $\beta(\varphi)(J(x)) = \varphi(\beta^*(J(x))) = \varphi(\beta^*(1)x)$. Hence $\beta(\varphi) = (\varphi \circ J^{-1})(J(\beta^*(1))\cdot)$ on $J(\mathcal{M}_1)$. Since $\|\beta(\varphi)\| = \|\varphi\|$, $\varphi \in (\mathcal{M}_1)_{*,+}$, we have $\varphi(\beta^*(1)) = \beta(\varphi)(J(1)) = \varphi(1)$, $\varphi \in (\mathcal{M}_1)_{*,+}$. Thus $\beta^*(1) = 1$ or $\beta(\varphi) = \varphi \circ J^{-1}$. This completes the proof. ∎

§3. The Structure of Surjective $*$-Preserving Linear L^p-Isometry

In this section, we shall prove the implementation of surjective $*$-preserving linear L^p-isometries.

Let $1 < p < \infty$ and $p \neq 2$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be arbitrary von Neumann algebras. Let φ_0 (resp. ψ_0) be a fixed faithful normal semifinite weight on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$). Let T be a linear isometry from $L^p(\mathcal{M}_1;\varphi_0)$ to $L^p(\mathcal{M}_2;\psi_0)$. Let J be the Jordan $*$-isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ induced by T due to Theorem 2.

At first, it J is $*$-isomorphic, then we have

$$\sigma_t^{\varphi_0 \circ J^{-1}} = J \circ \sigma_t^{\varphi_0} \circ J^{-1},\ t \in \mathbb{R}$$

by the uniqueness of the modular automorphism group.

Secondly, if J is $*$-antiisomorphic, then we have

$$\sigma_t^{\varphi_0\circ J^{-1}} = J\circ\sigma_{-t}^{\varphi_0}\circ J^{-1},\ t\in\mathbb{R}.$$

Actually we compute

$$\begin{aligned} f(t) &= (\varphi_0\circ J^{-1})(x(J\circ\sigma_{-t}^{\varphi_0}\circ J^{-1})(y)) \\ &= \varphi_0(\sigma_{-t}^{\varphi_0}(J^{-1}(y))J^{-1}(x)) \\ &= \varphi_0(J^{-1}(y)\sigma_t^{\varphi_0}(J^{-1}(x))) \end{aligned}$$

by the invariance. By using the KMS condition for $(\varphi_0, \sigma^{\varphi_0})$, we have

$$\begin{aligned} f(-i+t) &= (\varphi_0\circ J^{-1})(x(J\circ\sigma_{-(-i+t)}^{\varphi_0}\circ J^{-1})(y)) \\ &= \varphi_0(J^{-1}(y)\sigma_{-i+t}^{\varphi_0}(J^{-1}(x))) \\ &= \varphi_0(\sigma_t^{\varphi_0}(J^{-1}(x))J^{-1}(y)) \\ &= \varphi_0(J^{-1}(x)\sigma_{-t}^{\varphi_0}(J^{-1}(y))) \\ &\qquad \text{(by the invariance again)} \\ &= (\varphi_0\circ J^{-1})((J\circ\sigma_{-t}^{\varphi_0}\circ J^{-1})(y)\,x). \end{aligned}$$

In the general case, we have a central projection q in $\mathcal{M}_1$ such that J is $*$-isomorphic on $\mathcal{M}_1q$ and $*$-antiisomorphic on $\mathcal{M}_1q^\perp$. Note that $\pi_{\sigma\varphi_0}(q)$ is central in $\mathcal{M}_1 \rtimes_{\sigma\varphi_0} \mathbb{R}$ by $\sigma_t^{\varphi_0}(q) = q$ (cf. [S; 2.21]).
Now we have from the above arguments

$$\begin{aligned} \sigma_t^{\varphi_0\circ J^{-1}} &= J\circ\sigma_t^{\varphi_0}\circ J^{-1} && \text{on } J(\mathcal{M}_1q) \\ \sigma_t^{\varphi_0\circ J^{-1}} &= J\circ\sigma_{-t}^{\varphi_0}\circ J^{-1} && \text{on } J(\mathcal{M}_1q^\perp). \end{aligned}$$

Therefore, using the notation $\check{\sigma}_t = \sigma_{-t}, t\in\mathbb{R}$,

$$\begin{aligned} \mathcal{M}_1q \rtimes_{\sigma\varphi_0} \mathbb{R} &\cong J(\mathcal{M}_1q) \rtimes_{\sigma\varphi_0\circ J^{-1}} \mathbb{R} && (*\text{-isomorphic}) \\ \mathcal{M}_1q^\perp \rtimes_{\sigma\varphi_0} \mathbb{R} &\cong J(\mathcal{M}_1q^\perp) \rtimes_{\check{\sigma}\varphi_0\circ J^{-1}} \mathbb{R} && (*\text{-antiisomorphic}), \end{aligned}$$

where the latter $*$-antiisomorphism is given by $\pi_{\sigma\varphi_0}(x) \mapsto \pi_{\check{\sigma}\varphi_0\circ J^{-1}}(J(x))$ and $\lambda_t \mapsto \lambda_{-t}$. So there exists a Jordan $*$-isomorphism from $\mathcal{M}_1 \rtimes_{\sigma\varphi_0} \mathbb{R}$ onto $J(\mathcal{M}_1q) \rtimes_{\sigma\varphi_0\circ J^{-1}} \mathbb{R} \oplus J(\mathcal{M}_1q^\perp) \rtimes_{\check{\sigma}\varphi_0\circ J^{-1}} \mathbb{R}$, extending J. However, there exists a canonical $*$-isomorphism j from $J(\mathcal{M}_1q^\perp) \rtimes_{\check{\sigma}} \mathbb{R}$ onto $J(\mathcal{M}_1q^\perp) \rtimes_{\sigma} \mathbb{R}$ defined by $j(\pi_{\check{\sigma}}(x)) = \pi_\sigma(x)$ and $j(\lambda_t) = \lambda_{-t}$. Consequently, we have a canonical Jordan $*$-isomorphism $\widetilde{J}$ from $\mathcal{M}_1 \rtimes_{\sigma\varphi_0} \mathbb{R}$ onto $J(\mathcal{M}_1) \rtimes_{\sigma\varphi_0\circ J^{-1}} \mathbb{R}$ satisfying that $\widetilde{J}(\pi_{\sigma\varphi_0}(x)) = \pi_{\sigma\varphi_0\circ J^{-1}}(J(x))$ and $\widetilde{J}(\lambda_t) = \lambda_t$. Moreover, we can extend $\widetilde{J}$ to a

Jordan $*$-isomorphism between the $*$-algebras of measurable operators, which is a homeomorphism with respect to their measure topologies, and the restriction of $\widetilde{J}$ to $L^p(\mathcal{M}_1; \varphi_0)$ is a canonical positive linear isometry from $L^p(\mathcal{M}_1; \varphi_0)$ onto $L^p(J(\mathcal{M}_1); \varphi_0 \circ J^{-1})$ (cf. [W1; Section 4]).

Now we assume that there exists a faithful normal semifinite operator valued weight $E: \mathcal{M}_1 \longrightarrow J(\mathcal{M}_1)$.

Put $\psi_2 = \varphi_0 \circ J^{-1} \circ E$. Then there are two faithful normal semifinite weights on $\mathcal{M}_2$, ψ_0 and ψ_2. We denote the crossed product with respect to ψ_0 (resp. ψ_2) by $\mathcal{N}_{\psi_0} = \mathcal{M}_2 \rtimes_{\sigma\psi_0} \mathbb{R}$ (resp. $\mathcal{N}_{\psi_2} = \mathcal{M}_2 \rtimes_{\sigma\psi_2} \mathbb{R}$). Let $\widetilde{\mathcal{N}}_{\psi_0}$ (resp. $\widetilde{\mathcal{N}}_{\psi_2}$) be the $*$-algebra of all measurable operators (with respect to the canonical trace) on $L^2(\mathbb{R}, \mathcal{H})$.

Define a unitary operator u on $L^2(\mathbb{R}; \mathcal{H})$ by

$$(u\xi)(t) = (D\psi_2; D\psi_0)_{-t}\, \xi(t), \; \xi \in L^2(\mathbb{R}, \mathcal{H}), \; t \in \mathbb{R}.$$

Put $\kappa(a) = uau^*$, $a \in \mathcal{N}_{\psi_0}$. Then κ is the canonical $*$-isomorphism from $\mathcal{N}_{\psi_0}$ onto $\mathcal{N}_{\psi_2}$, which is related to change of weights from ψ_0 to ψ_2. Moreover, κ extends to a $*$-isomorphism $\widetilde{\kappa}$ from $\widetilde{\mathcal{N}}_{\psi_0}$ onto $\widetilde{\mathcal{N}}_{\psi_2}$, and the restriction of $\widetilde{\kappa}$ is a positive linear isometry from $L^p(\mathcal{M}_2; \psi_0)$ to $L^p(\mathcal{M}_2; \psi_2)$ (cf. [W1; Lemma 2.1, Lemma 2.2]).

Moreover, we obtain a canonical inclusion $\iota: J(\mathcal{M}_1) \rtimes_{\sigma\varphi_0 \circ J^{-1}} \mathbb{R} \longrightarrow \mathcal{M}_2 \rtimes_{\sigma\psi_2} \mathbb{R}$, since $\sigma_t^{\psi_2} = \sigma_t^{\varphi_0 \circ J^{-1} \circ E} = \sigma_t^{\varphi_0 \circ J^{-1}}, t \in \mathbb{R}$ on $J(\mathcal{M}_1)$ (cf. [S; Theorem 11.9]). ι is extended to the inclusion between the $*$-algebras of measurable operators, still denoted by ι.

Thus we have a canonical positive linear isometry $\widetilde{\kappa}^{-1} \circ \iota \circ \widetilde{J}$ from $L^p(\mathcal{M}_1; \varphi_0)$ to $L^p(\mathcal{M}_2; \psi_0)$.

Proposition 3. *Keep the situation as above. Assume that T is positive and that there exists a faithful normal semifinite operator valued weight $E: \mathcal{M}_2 \longrightarrow J(\mathcal{M}_1)$. Then T equals to the restriction of $\widetilde{\kappa}^{-1} \circ \iota \circ \widetilde{J}$ to $L^p(\mathcal{M}_1; \varphi_0)$.*

Proof. The existence of an operator valued weight E guarantees the canonical positive linear isometry mentioned above. Since T is positive, $T(h_\varphi^{1/p}) = h_{\varphi \circ J^{-1}}^{1/p}$, $\varphi \in (\mathcal{M}_1)_{*,+}$ by Theorem 2. Therefore, we easily compute the Radon-Nikodym derivative as in the proof of [W2; Theorem 5] to have $T(h_\varphi{}^{1/p})^p = \widetilde{\kappa}^{-1} \circ \iota \circ \widetilde{J}(h_\varphi)$. This completes the proof. ∎

Question 4. *When T is an L^p-isometry and J is the induced Jorden $*$-isomorphism, does there always exist a faithful normal semifinite operator valued weight $E: \mathcal{M}_2 \longrightarrow J(\mathcal{M}_1)$?*

Corollary 5. *Assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are semifinite von Neumann algebras and T is positive. Then T equals to the restrietion of $\widetilde{\kappa}^{-1}\circ\iota\circ\widetilde{J}$* to $L^p(\mathcal{M}_1;\ \varphi_0)$.

Theorem 6. *Let $1<p<\infty$ and $p\neq 2$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be arbitrary von Neumann algebras. Let φ_0 (resp. ψ_0) be a fixed faithful normal semifinite weight on $\mathcal{M}_1$ (resp. $\mathcal{M}_2$). Let T be a surjective and $*$-preserving linear isometry from $L^p(\mathcal{M}_1;\ \varphi_0)$ to $L^p(\mathcal{M}_2;\ \psi_0)$. Let J be the Jordan $*$-isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ induced by T due to Theorem 2, and let κ be the canonical isomorphism associated with the change of weights ψ_0 and $\varphi_0\circ J^{-1}$. Then there exists a central symmetry z in $\mathcal{M}_2$ and T equals to the restriction of $z\cdot\widetilde{\kappa}^{-1}\circ\widetilde{J}$ to $L^p(\mathcal{M}_1;\ \varphi_0)$.*

Proof. For each $\varphi\in(\mathcal{M}_1)_{*,+}$, there exists a unique pair ψ_+ and ψ_- in $(\mathcal{M}_2)_{*,+}$ satisfying $T(h_\varphi^{1/p}) = h_{\psi+}^{1/p}-h_{\psi-}^{1/p}$. Hence we can define maps β_+ (resp. β_-) from $(\mathcal{M}_1)_{*,+}$ to $(\mathcal{M}_2)_{*,+}$ by $h_{\beta+(\varphi)}^{1/p} = h_{\psi+}^{1/p}$ (resp. $h_{\beta-(\varphi)}^{1/p} = h_{\psi-}^{1/p}$). It follows from the equality condition of the Clarkson's inequality than β_+ and β_- preserves orthogonality. Though $\|\beta_+(\varphi)\|\leq\|\varphi\|$, $\varphi\in(\mathcal{M}_1)_{*,+}$ instead of $\|\beta(\varphi)\| = \|\varphi\|$, β_+ and β_- turns to be additive and extended to o.d. homomorphisms. Hence $\beta_+^*(1)$ and $\beta_-^*(1)$ are central elements. Define a map β_0 by $\beta_0(\varphi) = \beta_+(\varphi)-\beta_-(\varphi)$, $\varphi\in(\mathcal{M}_1)_{*,+}$. Then β_0 can be extended to an $\mathbb{R}$-linear map from $(\mathcal{M}_1)_{*,sa}$ to $(\mathcal{M}_2)_{*,sa}$.

For each $\varphi\in(\mathcal{M}_1)_{*,sa}$, let $\varphi = \varphi_+-\varphi_-$ be the Jordan decomposition. Then we have

$$\begin{aligned}\|\beta_0(\varphi)\| &= \|\beta_0(\varphi_+)-\beta_0(\varphi_-)\|\\ &= \|\beta_+(\varphi_+)+\beta_-(\varphi_-)\| + \|\beta_+(\varphi_-)+\beta_-(\varphi_+)\|\\ &= \|\varphi_+\| + \|\varphi_-\| = \|\varphi\|,\end{aligned}$$

since $\beta_+(\varphi_+)$, $\beta_+(\varphi_-)$, $\beta_-(\varphi_+)$ and $\beta_-(\varphi_-)$, are orthogonal each other in $(\mathcal{M}_2)_{*,+}$. Thus β_0 is isometric on $(\mathcal{M}_1)_{*,sa}$.

It follows from the surjectivity of T that β_0 is also surjective. Put $\beta(\varphi) = \beta_0(\varphi(\beta_0^*(1)\cdot))$, $\varphi\in(\mathcal{M}_1)_{*,sa}$. We claim β is positive. It suffices to show that β^* is positive. It is easy to see that $\beta^*(y) = \beta_0^*(1)\beta_0^*(y)$. In particular, $\beta^*(1) = \beta_0^*(1)^2$. Obviously we have $\|\beta^*\|\leq 1$. Since any unital linear contraction between C*-algebras is positive, it is enough to show that $\beta_0^*(1)^2 = 1$. However, a surjective $\mathbb{R}$-linear isometry β_0^* maps extreme points of the closed unit ball of $(\mathcal{M}_1)_{sa}$ to those of $(\mathcal{M}_2)_{sa}$. Since they are symmetries, we conclude that $\beta_0^*(1)^2 = 1$.

Finally, since $\beta_0^*(1)$ is central and since β_0 preserves orthogonality, β is an o.d. homomorphism. By the Bunce-Wright theorem, there exists a weak $*$ continuous Jordan $*$-isomorphism J such that $\beta^*(J(x)) = \beta^*(1)x$, $x\in\mathcal{M}_1$ or $\beta(\varphi) = \varphi\circ J^{-1}$, $\varphi\in(\mathcal{M}_1)_*$. Thus we have $\beta_0(\varphi) = z_0\cdot\varphi\circ J^{-1}$, where $z_0 = \beta_0^*(1)$

is a fixed symmetry.

There exists a suitable central projection $e_0 \in \mathcal{M}_2$ such that $z_0 = 2e_0 - 1$. For each $\varphi \in (\mathcal{M}_1)_{*,+}$,

$$\beta_0(\varphi) = e_0 \cdot \varphi \circ J^{-1} - (1-e_0) \cdot \varphi \circ J^{-1}$$

is the Jordan decomposition. By the uniqueness, we have

$$\beta_+(\varphi) = e_0 \cdot \varphi \circ J^{-1} \quad \text{and} \quad \beta_-(\varphi) = (1-e_0) \cdot \varphi \circ J^{-1}.$$

Thus we have

$$h_{\beta+(\varphi)} = \frac{d(e_0 \cdot \varphi \circ J^{-1})^{\sim \phi_0}}{d\tau_{\phi_0}} = \tilde{\kappa}^{-1}\left(\frac{d(e_0 \cdot \varphi \circ J^{-1})^{\sim \phi_2}}{d\tau_{\phi_2}}\right)$$

$$= \tilde{\kappa}^{-1} \circ \tilde{J}\left(\frac{d(e \cdot \varphi)^{\sim \varphi_0}}{d\tau_{\varphi_0}}\right) = \tilde{\kappa}^{-1} \circ \tilde{J}(e \cdot h_\varphi) \quad \text{with} \quad e = J^{-1}(e_0).$$

This implies that $h^{1/p}_{\beta+(\varphi)} = \tilde{\kappa}^{-1} \circ \tilde{J}\left(e \cdot h^{1/p}_\varphi\right)$, $\varphi \in (\mathcal{M}_1)_{*,+}$. So we can conclude that $T(h^{1/p}_\varphi) = z \cdot \tilde{\kappa}^{-1} \circ \tilde{J}(h^{1/p}_\varphi)$, $\varphi \in (\mathcal{M}_1)_{*,+}$, where $z = 2\tilde{\kappa}^{-1}(e_0) - 1$. This completes the proof. ∎

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