Rev. Mat. Iberoam. **31** (2015), no. 3, 935–976
DOI 10.4171/RMI/859

Brownian motion on treebolic space: escape to infinity

Alexander Bendikov, Laurent Saloff-Coste,
Maura Salvatori and Wolfgang Woess

Abstract. Treebolic space is an analog of the Sol geometry, namely, it is the horocylic product of the hyperbolic upper half plane $\mathbb{H}$ and the homogeneous tree $\mathbb{T} = \mathbb{T}_{\mathsf{p}}$ with degree $\mathsf{p}+1 \geq 3$, the latter seen as a one-complex. Let $\mathfrak{h}$ be the Busemann function of $\mathbb{T}$ with respect to a fixed boundary point. Then for real $\mathsf{q} > 1$ and integer $\mathsf{p} \geq 2$, treebolic space $\mathsf{HT}(\mathsf{q},\mathsf{p})$ consists of all pairs $(z = x + \mathfrak{i}\, y, w) \in \mathbb{H} \times \mathbb{T}$ with $\mathfrak{h}(w) = \log_{\mathsf{q}} y$. It can also be obtained by glueing together horizontal strips of $\mathbb{H}$ in a tree-like fashion. We explain the geometry and metric of HT and exhibit a locally compact group of isometries (a horocyclic product of affine groups) that acts with compact quotient. When $\mathsf{q} = \mathsf{p}$, that group contains the amenable Baumslag–Solitar group $\mathsf{BS}(\mathsf{p})$ as a co-compact lattice, while when $\mathsf{q} \neq \mathsf{p}$, it is amenable, but non-unimodular. $\mathsf{HT}(\mathsf{q},\mathsf{p})$ is a key example of a strip complex in the sense of [4].

Relying on the analysis of strip complexes developed by the same authors in [4], we consider a family of natural Laplacians with "vertical drift" and describe the associated Brownian motion. The main difficulties come from the singularities which treebolic space (as any strip complex) has along its bifurcation lines. In this first part, we obtain the rate of escape and a central limit theorem, and describe how Brownian motion converges to the natural geometric boundary at infinity. Forthcoming work will be dedicated to positive harmonic functions.

1. Introduction

Let $\mathbb{H} = \{x + \mathfrak{i}\, y : x \in \mathbb{R}\,,\ y > 0\}$ be hyperbolic upper half space, and $\mathbb{T} = \mathbb{T}_{\mathsf{p}}$ be the homogeneous tree, drawn in such a way that every vertex of $\mathbb{T}$ has one predecessor and p successors. *Treebolic space* is a Riemannian 2-complex, a *horocyclic product* of $\mathbb{H}$ and $\mathbb{T}$. Let us start with a picture and an informal description.

Mathematics Subject Classification (2010): Primary 60J25; Secondary 60J65, 53C23, 20F65, 05C05.
Keywords: tree, hyperbolic plane, horocyclic product, Laplacian, Brownian motion, rate of escape, central limit theorem, boundary convergence.

Let $1 < \mathsf{q} \in \mathbb{R}$. For $k \in \mathbb{Z}$, subdivide $\mathbb{H}$ into the strips $\mathsf{S}_k = \{x + \mathfrak{i}\,y : x \in \mathbb{R}, \mathsf{q}^{k-1} \le y \le \mathsf{q}^k\}$ (see Figure 3 further below). Each strip is bounded by two horizontal lines of the form $\mathsf{L}_k = \{x + \mathfrak{i}\,\mathsf{q}^k : x \in \mathbb{R}\}$, which in hyperbolic geometry are horocycles with respect to the "upper" boundary point $\boldsymbol{\infty}$ (or rather $\mathfrak{i}\,\infty$). In treebolic space $\mathsf{HT}(\mathsf{q},\mathsf{p})$, infinitely many copies of those strips are glued together in a tree-like fashion: for each $k \in \mathbb{Z}$, the bottom lines of p copies of S_k are identified with each other and with the top line of a copy of S_{k-1}. Thus, every copy of any of the L_k becomes a *bifurcation line* whose "side view" is a vertex v of $\mathbb{T}$ that can be used to identify the line as L_v (instead of L_k). In the same way, we write S_v for the strip sitting below L_v in our picture. Each strip is equipped with the standard hyperbolic length element, and combining this with the tree metric, one obtains a natural metric on $\mathsf{HT}(\mathsf{q},\mathsf{p})$. A more formal description will be given in §2.

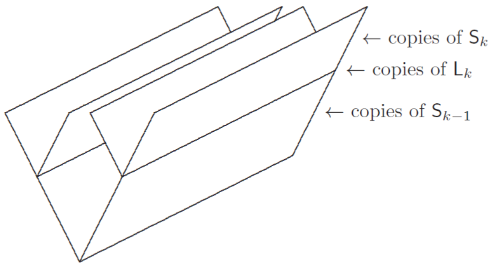


FIGURE 1. A finite section of treebolic space, with $\mathsf{p} = 2$.[1]

Why is this space interesting? First of all, it is a key example of a *strip complex* in the sense of [4]. Strip complexes are a class of Riemannian complexes. Laplacians and the associated potential theory on Riemannian complexes appear in the book of Eells and Fuglede [18]. A study of Brownian motion and harmonic functions on Euclidean complexes was undertaken by Brin and Kifer [9]. In [4], the theory of Laplacians and diffusion on strip complexes, properties of the heat kernel, etc., were studied in a careful and rigorous way. In this spirit, the present paper is the first detailed case study of what can be achieved on the basis of that theory, which provides a highly non-trivial extension of the very popular subject of analysis and probability on "quantum graphs" (metric graphs) to what one might also call "quantum complexes".

Second, treebolic space is a horosphere in the product space $\mathbb{H} \times \mathbb{T}$, where the tree $\mathbb{T}$ is viewed as a one-dimensional complex in which each edge is a copy of a suitable compact interval. In other words, it is the *horocyclic product* of $\mathbb{H}$ and $\mathbb{T}$. A first appearance of such a horocyclic product was that of two trees with (integer) branching numbers p and $\mathsf{q} \ge 2$, respectively. This is the Diestel–Leader graph $\mathsf{DL}(\mathsf{p},\mathsf{q})$, which for $\mathsf{p} \ne \mathsf{q}$ was proposed by Diestel and Leader [17] as a candidate example to answer the following question of Woess [30]: is there a vertex-transitive graph which is not quasi-isometric with a Cayley graph? It was only

[1] This figure also appears in [4].

quite recently that Eskin, Fisher and Whyte finally showed, as part of impressive work [19], [20], [21], that $\mathsf{DL}(\mathsf{p},\mathsf{q})$ is indeed such an example. On the other hand, in case of equal branching numbers of the two trees, $\mathsf{DL}(\mathsf{p},\mathsf{p})$ is a Cayley graph of the lamplighter group $\mathbb{Z}(\mathsf{p}) \wr \mathbb{Z}$. This geometric realisation of the latter lead to a good understanding of random walks, spectra and boundary theory of those groups, the DL-graphs, and of horocyclic products of more than 2 trees, see the work of Bertacchi, Bartholdi, Brofferio, Neuhauser, Woess [6], [33], [1], [12], [13], [2].

Besides $\mathsf{DL}(\mathsf{p},\mathsf{q})$, treebolic space has another, more classical sister structure. This is $\mathsf{Sol}(\mathsf{p},\mathsf{q})$, the horocyclic product of two hyperbolic planes with curvatures $-\mathsf{p}^2$ and $-\mathsf{q}^2$, respectively, where $\mathsf{p},\mathsf{q} > 0$. Besides being a 3-dimensional Riemannian manifold, $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ can be seen as a Lie group, which is the semidirect product of $\mathbb{R}$ with $\mathbb{R}^2$ induced by the action $(x,y) \mapsto (e^{\mathsf{p}z}x, e^{-\mathsf{q}z}y)$, $z \in \mathbb{R}$. $\mathsf{Sol}(1,1)$ is one of Thurston's eight model geometries in dimension 3. The Brownian motion generated by the Laplace–Beltrami operator on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ is studied in detail in a sister paper to the present one, by Brofferio, Salvatori and Woess [11].

The analogy between $\mathsf{DL}(\mathsf{p},\mathsf{q})$ and $\mathsf{Sol}(\mathsf{p},\mathsf{q})$ becomes also apparent in [19], [20], and [21], where the quasi-isometry classes of these graphs, resp. manifolds are determined. Coming back to treebolic space, we mention that like the above sister structures, it is neither Gromov hyperbolic nor Cat(0). We shall explain below that the amenable Baumslag–Solitar group $\mathsf{BS}(\mathsf{p}) = \langle a, b \mid ab = b^{\mathsf{p}}a \rangle$ acts on $\mathsf{HT}(\mathsf{p},\mathsf{p})$ by isometries and with compact quotient. This fact has been exploited by Farb and Mosher [22] (without describing the space as a horocyclic product) in order to determine the quasi-isometry types of the Baumslag–Solitar groups. On the other hand, we shall see that for $\mathsf{p} \neq \mathsf{q}$, no discrete group can act in such a way on our space.

In the present paper, in §2 we first exhibit more details about the geometry of treebolic space and its metric $\mathsf{d}(\cdot,\cdot) = \mathsf{d}_{\mathsf{HT}}(\cdot,\cdot)$ and explain its isometry group, which is (up to the obvious reflections with respect to vertical hyperplanes) obtained as a "horocyclic" product of the group $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ of all affine mappings $z \mapsto \mathsf{q}^k z + b$ ($z \in \mathbb{H}$, $k \in \mathbb{Z}$, $b \in \mathbb{R}$) and the affine group of the tree $\mathbb{T}_{\mathsf{p}}$, that is, the group of all automorphisms (self-isometries) of the tree that fix a given boundary point.

We next, in §3, turn our attention to the Laplace operator on HT, whose rigorous construction as an essentially self-adjoint diffusion operator bears a serious challenge in view of the singularities which our structure has along the bifurcation lines. This challenge was faced in the general setting of strip complexes in [4]. As a matter of fact, we consider a family of Laplacians $\Delta_{\alpha,\beta}$ with two "vertical drift" parameters α and β. When looking at Bownian motion (BM), that is, the diffusion $(X_t)_{t\geq 0}$ on HT whose infinitesimal generator is $\Delta_{\alpha,\beta}$, it is hyperbolic BM with linear drift parameter α in the interior of each strip. On the other hand, β is responsible for the random choice of the strip into which BM should make its next infinitesimal step when the current position is on one of the bifurcation lines. The overall drift relies on both in terms of the number $\mathsf{a} = \beta\,\mathsf{p}\,\mathsf{q}^{\alpha-1}$. The drift is 0 if and only if $\mathsf{a} = 1$, while BM has an "upwards" (resp. "downwards") drift when $\mathsf{a} > 1$ (resp. < 1). The Laplacian and Brownian motion admit natural projections on $\mathbb{T}$ and $\mathbb{H}$, as well as on $\mathbb{R}$. The projection onto $\mathbb{R}$ associates with each point its height: it is the Busemann function with respect to the boundary point at infinity of $\mathbb{H}$ (as well as of the tree). The projected Brownian motion $(Z_t)_{t\geq 0}$ on $\mathbb{H}$ is in general not ordinary hyberbolic BM with drift parameter α, except when $\beta = 1/\mathsf{p}$. That is, it evolves like hyperbolic BM with drift in the interior of each of the strips S_k into which $\mathbb{H}$ has been "sliced", while it receives an additional vertical "kick" (absent only when $\beta = 1/\mathsf{p}$) on each of the lines L_k.

The projection $(W_t)_{t\geq 0}$ on $\mathbb{T}$ is a typical example of BM on a metric graph (the tree). The study of the corresponding Laplace operators is by now well established, and more straightforward than the higher dimensional version on strip complexes that we are dealing with here. See e.g. Cattaneo [14], Keller and Lenz [29], Bendikov and Saloff-Coste [3]. Analogously, the projection $(Y_t)_{t\geq 0}$ on $\mathbb{R}$ evolves like ordinary BM with drift as long as it does not visit any integer. When it visits an integer, BM receives an additional random "kick" in the positive or negative direction.

The main goal of this paper is to describe how Brownian motion on HT evolves spatially. Main tool for this study is the sequence $\big(\tau(n)\big)$ of the stopping times of the successive visits of (X_t) in the bifurcation lines L_v, $v \in V(\mathbb{T})$ (the vertex set of the tree). The increments $\tau(n) - \tau(n-1)$ are i.i.d. for $n \geq 2$, have exponential moments and an explicitly computable Laplace transform, see §4. That section contains further basic preliminary results. In particular, we study the distribution of the location of the process at time $\tau(1)$, which is the law governing the process $(X_{\tau(n)})$. This is quite subtle, because the singularities of our structure require care when trying to implement methods that appear to be "obvious" in the classical smooth setting.

The state space of the induced Markov process $(X_{\tau(n)})_{n\geq 0}$ is the disjoint union of all bifurcation lines. The projection $(Z_{\tau(n)})_{n\geq 0}$ of that process on $\mathbb{H}$ can be interpreted as a random walk on the group $\mathsf{Aff}(\mathbb{H},\mathsf{q})$. It can be treated via the methods of the work of Grincevicius [24], [25]. At the same time, the projection $(W_{\tau(n)})_{n\geq 0}$ is a nearest neighbour random walk on the (vertex set of the) tree whose transition probabilities are invariant under the action of the affine group of $\mathbb{T}$. It can also be considered as a random walk on that group. Random walks of this type were studied in detail by Cartwright, Kaimanovich and Woess [15]. The synthesis of those results on the two affine groups of $\mathbb{H}$ and of $\mathbb{T}$ is crucial for our study.

In §5, we consider the natural geometric boundary at infinity of HT. Since HT is naturally embedded in the direct product $\mathbb{H} \times \mathbb{T}$, its natural compactification is its closure in $\widehat{\mathbb{H}} \times \widehat{\mathbb{T}}$. The boundary of HT is the set of points added in this way. Here, $\widehat{\mathbb{T}}$ is the well-known end compactification of the tree, while $\widehat{\mathbb{H}}$ is the classical compactification of hyperbolic plane (the closed unit disk in the disk model of $\mathbb{H}$, or equivalently – in the upper half plane situation – the upper half plane together with its bottom line $\mathbb{R}$ and the "upper" boundary point at infinity). We show that in the topology of that compactification, Brownian motion on HT converges almost surely to a limit random variable that lives on the boundary. In general, we can get quite good information about the law of that limit random variable, but it can be given explicitly only in special cases regarding the choice of the parameters α and β. Convergence to the boundary goes hand in hand with computation of the linear *rate of escape* $\ell(\alpha,\beta)$, that is,

$$\mathsf{d}_{\mathsf{HT}}(X_t,X_0)/t \to \ell(\alpha,\beta) \quad \text{almost surely, as } t\to\infty\,.$$

It is the same as the rate of escape of (Y_t) on $\mathbb{R}$. A basic tool for boundary convergence and rate of escape is the the notion of *regular sequences* of Kaimanovich [28].

Next, in §6, we derive a central limit theorem, concerning convergence in law of

$$\big(\mathsf{d}_{\mathsf{HT}}(X_t,X_0) - t\,\ell(\alpha,\beta)\big)/\sqrt{t}\,.$$

When $\ell(\alpha,\beta)>0$, the limit law is centred normal distribution, and we also explain how to compute its variance $\sigma^2(\alpha,\beta)>0$. When $\ell(\alpha,\beta)=0$, the result as well as the limit distribution are somewhat more complicated.

The interplay of BM with the boundary provides the bridge to the potential theoretic part of our work, that will be laid out in forthcoming work [5].

In concluding the introduction, we want to underline how similar the geometric features as well as the properties of Brownian motion (resp. random walks) and the associated harmonic functions are on DL-graphs and lamplighter groups, the Sol-manifold (resp. -group) and treebolic space. In spite of the different techniques needed for each of the three, the realisation of those analogies, as well as the detailed study undertaken here, have become possible via the geometric interpretation of those structures as horocyclic products.

On the other hand, as already indicated, the elaboration and use of the analytic and probabilistic tools for this study are quite subtle in view of the singularities of HT at the bifurcation lines, thus providing a first concrete implementation of the analysis on strip complexes developed in [4].

2. Geometry and isometries of treebolic space

We start by describing the relevant features of the homogeneous tree $\mathbb{T}=\mathbb{T}_{\mathsf{p}}$. Here, we consider $\mathbb{T}$ as a one-complex, where each edge is a copy of the unit interval $[0\,,\,1]$. The discrete graph metric $\mathsf{d}_{\mathbb{T}}(v_1,v_2)$ on the vertex set (0-skeleton) $V(\mathbb{T})$ of $\mathbb{T}$ is the length (number of edges) on the shortest path between v_1 and v_2. This metric has an obvious "linear" extension to the one-skeleton.

We partition the vertex set into countably many sets H_k, $k\in\mathbb{Z}$, such that each H_k is countably infinite, and every vertex $v\in H_k$ has precisely one neighbour v^- (the *predecessor* of v) in H_{k-1} and p neighbours in H_{k+1} (the *successors* of v), each of with has v as its predecessor. See Figure 2. The sets H_k are called *horocycles*. For $v\in H_k$, we define $\mathfrak{h}(v)=k$. There is also a horocycle H_t for any real t: if $k=\lceil t\rceil$ and $v\in V(\mathbb{T})$ with $\mathfrak{h}(v)=k$, then the metric edge $[v^-,v]$ meets H_t precisely in the point w which is at distance $k-t$ from v, and we set $\mathfrak{h}(w)=t$.

In addition to this basic description, we shall need further details. A *geodesic path*, resp. *geodesic ray*, resp. *infinite geodesic* in $\mathbb{T}$ is the image of an isometric embedding $t \mapsto w_t \in \mathbb{T}$ of a finite interval $[a\,,b]$, resp. one-sided infinite interval $[a\,,\infty)$, resp. $\mathbb{R}$, that is, $\mathsf{d}(w_s,w_t) = |t-s|$ for all s,t. An *end* of $\mathbb{T}$ is an equivalence class of rays, where two rays (w_t) and $(\bar{w}_t)$ are equivalent if they coincide up to finite initial pieces, i.e., there are $s,t_0 \in \mathbb{R}$ such that $\bar{w}_t = w_{s+t}$ for all $t \ge t_0$. We write $\partial\mathbb{T}$ for the space of ends, and $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$. For all $\eta,\zeta \in \widehat{\mathbb{T}}$ there is a unique geodesic $\overline{\eta\,\zeta}$ that connects the two. In particular, if $w \in \mathbb{T}$ and $\xi \in \partial\mathbb{T}$ then $\overline{w\,\xi}$ is the ray that starts at w and represents ξ. Furthermore, if $\xi,\eta \in \partial\mathbb{T}$ $(\xi \ne \eta)$ then $\overline{\eta\,\xi}$ is the infinite geodesic whose two halves (split at any of its points) are rays that represent η and ξ, respectively. For $v,w \in \mathbb{T}$, $v \ne w$, we define the cone $\widehat{\mathbb{T}}(v,w) = \{\zeta \in \widehat{\mathbb{T}} : w \in \overline{v\,\zeta}\}$. For $\xi \in \partial\mathbb{T}$, the collection of all cones containing ξ is a neighbourhood basis of ξ, while a neighbourhood basis of $w \in \mathbb{T}$ is given by all open balls in the tree metric. Thus, we obtain a topology which makes $\widehat{\mathbb{T}}$ a compact Hausdorff space with the vertex set $V(\mathbb{T})$ as a discrete subset and $\partial\mathbb{T}$

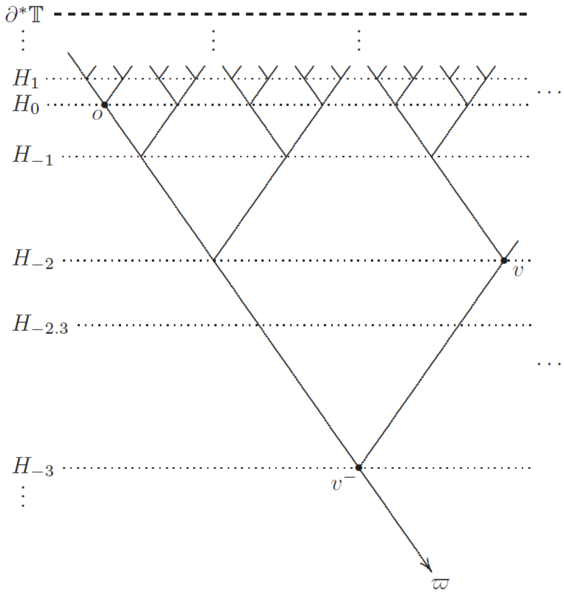


FIGURE 2. The "upper half plane" drawing of $\mathbb{T}_2$ (top down).[2]

We choose and fix a reference vertex (root) $o \in H_0$. The geodesic ray whose vertices consist of the root and all its ancestors (= iterated predecessors) defines a reference end $\varpi \in \partial\mathbb{T}$, the lower boundary point in Figure 2. We set $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\varpi\}$, the upper boundary in Figure 2. For $w_1,w_2 \in \widehat{\mathbb{T}} \setminus \{\varpi\}$, their confluent (or maximal common ancestor) $b = w_1 \curlywedge w_2$ with respect to ϖ is defined by $\overline{w_1\,\varpi} \cap \overline{w_2\,\varpi} = \overline{b\,\varpi}$. The function $\mathfrak{h} : \mathbb{T} \to \mathbb{R}$ defined above is the *Busemann function* of $\mathbb{T}$ with respect

[2]This figure also appears in [4].

to ϖ, which can be written as

$$\mathfrak{h}(w) = \mathsf{d}(w, w \curlywedge o) - \mathsf{d}(o, w \curlywedge o). \tag{2.1}$$

In addition, we note that

$$\mathsf{d}_{\mathbb{T}}(v, w) = \mathsf{d}_{\mathbb{T}}(v\,, v \curlywedge w) + \mathsf{d}_{\mathbb{T}}(v \curlywedge w, w) = \mathfrak{h}(v) + \mathfrak{h}(w) - 2\mathfrak{h}(v \curlywedge w). \tag{2.2}$$

There is a natural Lebesgue measure dw on $\mathbb{T}$, which on each edge is a copy of standard Lebesgue measure on the unit interval.

The natural compactification $\widehat{\mathbb{H}}$ of the hyperbolic plane $\mathbb{H}$ is the closed unit disk, when we use the Poincaré disk model. In our upper half plane model, $\widehat{\mathbb{H}}$ is the closed upper half plane together with the the point at infinity ∞. The corresponding boundary $\partial\mathbb{H}$ consists of ∞ together with the lower boundary line $\partial^*\mathbb{H} = \mathbb{R}$. The Busemann function on $\mathbb{H}$ with respect to ∞ is $z \mapsto \log(\mathit{Im}\, z)$, where $\mathit{Im}\, z$ is the imaginary part of z. For $z, z' \in \widehat{\mathbb{H}} \setminus \{\infty\}$, we can define the hyperbolic analogue $z \wedge z'$ of the confluent: $z \wedge z = z$, and when $z \neq z'$, then $z \wedge z'$ is the point on the (hyperbolic) geodesic $\overline{z\,z'}$ with maximal imaginary part. Recall that $\overline{z\,z'}$ is part of a circle centred on $\mathbb{R}$ which is orthogonal to that boundary line. The function $(z, z') \mapsto z \wedge z'$ is continuous from $(\widehat{\mathbb{H}} \setminus \{\infty\}) \times (\widehat{\mathbb{H}} \setminus \{\infty\})$ to $\widehat{\mathbb{H}} \setminus \{\infty\}$. (The analogous property holds for the tree.) Similarly to (2.2), we have for $z, z' \in \mathbb{H}$

$$\begin{aligned} &\mathsf{d}_{\mathbb{H}}(z, z') = \mathsf{d}_{\mathbb{H}}(z\,, z \wedge z') + \mathsf{d}_{\mathbb{H}}(z \wedge z', z') \quad \text{and} \\ &\big|\mathsf{d}_{\mathbb{H}}(z, z') \;-\; \big(2\log(\mathit{Im}\; z \wedge z') - \log(\mathit{Im}\, z) - \log(\mathit{Im}\, z')\big)\big| \le \log 4. \end{aligned} \tag{2.3}$$

We are not sure whether the last inequality appears in the literature very often; its proof is an amusing exercise of handling the hyperbolic metric.

In the same way as $\mathbb{T}$ is subdivided horizontally by the horocycles H_k, $k \in \mathbb{Z}$, we subdivide $\mathbb{H}$ into the horizontal strips S_k delimited by the lines L_k consisting of all $x + \mathfrak{i}\, y \in \mathbb{H}$ with $y = \mathsf{q}^k$, $k \in \mathbb{Z}$, see Figure 3. Note that all S_k are hyperbolically isometric.

As outlined in introduction and abstract, treebolic space with parameters q and p is

$$\mathsf{HT}(\mathsf{q}, \mathsf{p}) = \{\mathfrak{z} = (z, w) \in \mathbb{H} \times \mathbb{T}_{\mathsf{p}} : \mathfrak{h}(w) = \log_{\mathsf{q}}(\mathit{Im}\, z)\}\,. \tag{2.4}$$

Thus, Figures 2 and 3 are the "side" and "front" views of HT, that is, the images of HT under the projections $\pi_{\mathbb{H}} : (z, w) \mapsto z$ and $\pi_{\mathbb{T}} : (z, w) \mapsto w$, respectively. For a vertex $v \in V(\mathbb{T})$, let

$$\mathsf{L}_v = \{\mathfrak{z} = (z, v) \in \mathsf{HT} : \mathit{Im}\, z = \mathsf{q}^{\mathfrak{h}(v)}\} = L_{\mathfrak{h}(v)} \times \{v\}\,. \tag{2.5}$$

Then L_v and L_{v^-} are the upper and lower lines (respectively) in HT that delimit the strip

$$\mathsf{S}_v = \{\mathfrak{z} = (z, w) \in \mathsf{HT} : w \in [v^-, v]\}\,. \tag{2.6}$$

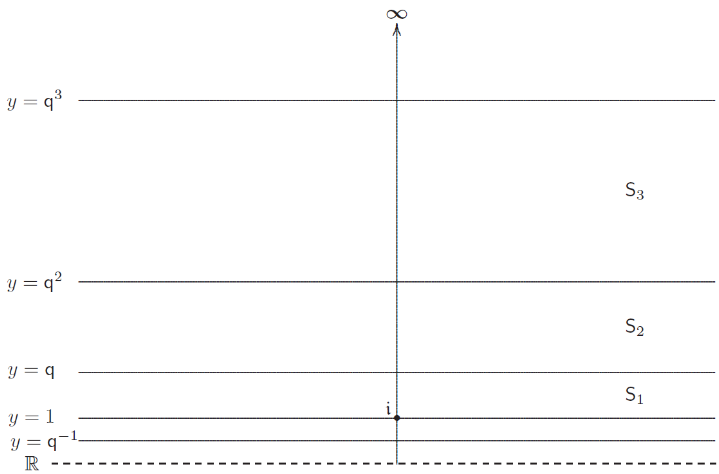


FIGURE 3. Hyperbolic upper half plane $\mathbb{H}$ subdivided in isometric strips.[3]

Here, $w \in [v^-, v]$ is an element of the edge $[v^-, v]$ of $\mathbb{T}$, which is (recall) a copy of the unit interval. For $\mathfrak{z} = (z, w) \in \mathsf{HT}$, we shall sometimes write $Re\,\mathfrak{z} = Re\,z$ for the real part of z.

For each end $\xi \in \partial^*\mathbb{T}$, treebolic space contains the isometric copy

$$\mathbb{H}_\xi = \{\mathfrak{z} = (z, w) \in \mathsf{HT} : w \in \overline{\xi\,\varpi}\}$$

of $\mathbb{H}$, and if $\xi, \eta \in \partial^*\mathbb{T}$ are distinct and $v = \xi \curlywedge \eta$ (a vertex), then $\mathbb{H}_\xi$ and $\mathbb{H}_\eta$ ramify along the line L_v, that is, $\mathbb{H}_\xi \cap \mathbb{H}_\eta = \{(z, w) \in \mathsf{HT} : w \in \overline{v\,\varpi}\}$.

The metric of HT is given by the hyperbolic length element in the interior of each strip. Its natural geodesic continuation is given as follows: consider two points $\mathfrak{z}_1 = (z_1, w_1), \mathfrak{z}_2 = (z_2, w_2) \in \mathsf{HT}$. Let $d_{\mathbb{H}}(z_1, z_2)$ by the hyperbolic distance between z_1 and z_2, and let $v = w_1 \curlywedge w_2$. Then

$$\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_1, \mathfrak{z}_2) = \begin{cases} \mathsf{d}_{\mathbb{H}}(z_1, z_2)\,, & \text{if } v \in \{w_1, w_2\}, \\ \min\{\mathsf{d}_{\mathbb{H}}(z_1, z) + \mathsf{d}_{\mathbb{H}}(z, z_2) : z \in L_{\mathfrak{h}(v)}\}\,, & \text{if } v \notin \{w_1, w_2\}. \end{cases} \tag{2.7}$$

Indeed, in the first case, $\mathfrak{z}_1$ and $\mathfrak{z}_2$ belong to a common copy $\mathbb{H}_\xi$ of $\mathbb{H}$. In the second case, $v \in V(\mathbb{T})$, and there are $\xi_1, \xi_2 \in \partial^*\mathbb{T}$ such that $\xi_1 \curlywedge \xi_2 = v$ and $\mathfrak{z}_i \in \mathbb{H}_{\xi_i}$ lie above the line L_v, so that it is necessary to pass through some point $\mathfrak{z} = (z, v) \in \mathsf{L}_v$ on the way from $\mathfrak{z}_1$ to $\mathfrak{z}_2$. See Figure 4.

Proposition 2.8. *For* $\mathfrak{z}_1 = (z_1, w_1)$, $\mathfrak{z}_2 = (z_2, w_2) \in \mathsf{HT}$, *with* $\delta = \log(1 + \sqrt{2})$,

$$\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_1, \mathfrak{z}_2) \le \mathsf{d}_{\mathbb{H}}(z_1, z_2) + (\log \mathsf{q})\, \mathsf{d}_{\mathbb{T}}(w_1, w_2) - \underbrace{(\log \mathsf{q})\, |\mathfrak{h}(w_1) - \mathfrak{h}(w_2)|}_{|\log Im\, z_1 - \log Im\, z_2|} \le \mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_1, \mathfrak{z}_2) + 2\delta\,.$$

[3]This figure also appears in [4].

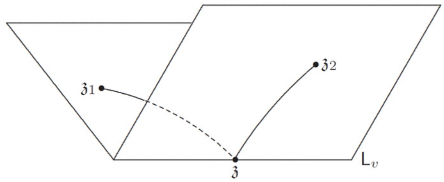


FIGURE 4. Geodesic connecting $\mathfrak{z}_1$ and $\mathfrak{z}_2$ in HT.

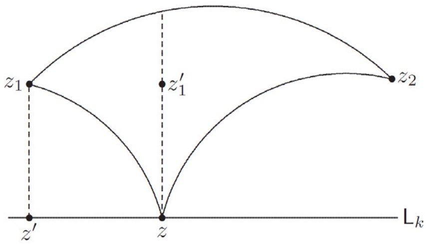


FIGURE 5. The geodesic triangle in $\mathbb{H}$ formed by the projections of $\mathfrak{z}_1$, $\mathfrak{z}_2$ and $\mathfrak{z}$.

Proof. In the first case of (2.7), we have $\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_1, \mathfrak{z}_2) = \mathsf{d}_{\mathbb{H}}(z_1, z_2)$ and $\mathsf{d}_{\mathbb{T}}(w_1, w_2) = |\mathfrak{h}(w_1) - \mathfrak{h}(w_2)|$. Therefore, the left-hand side inequality of the proposition is indeed an equality.

We consider the second case of (2.7). We suppose without loss of generality that $Im\, z_1 \le Im\, z_2$ and $Re\, z_1 \le Re\, z_2$. Set $k = \mathfrak{h}(v)$, and let $z \in \mathsf{L}_k$ be a point that realizes the minimum in (2.7), corresponding to $\mathfrak{z} = (z, v)$ in Figure 4. On the vertical ray in $\mathbb{H}$ going upwards from z to ∞, let z_1' be the point with $Im\, z_1' = Im\, z_1$. Also, we let z' be the point on L_k with $Re\, z' = Re\, z_1$. See Figure 5, showing the respective points and geodesic arcs in $\mathbb{H}$.

We start with the left-hand one of the two proposed inequalities, and use the minimising property of z: the distance $\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_1, \mathfrak{z}_2)$ is bounded above by the length of any path in $\mathbb{H}$ that starts at z_1 , ends at z_2 and visits L_k in between. We choose the following path: we first move vertically from z_1 down to z', then back up to z_1, and then along the geodesic arc from z_1 to z_2 . The length of this path is

$$2(\log Im\, z_1 - \log Im\, z') + \mathsf{d}_{\mathbb{H}}(z_1, z_2)\,,$$

which coincides with the middle term of our double inequality.

We now consider the right-hand one of the two proposed inequalities. By (2.7), $\mathsf{d}_{\mathsf{HT}}(\mathfrak{z}_1, \mathfrak{z}_2) = \mathsf{d}_{\mathbb{H}}(z_1, z) + \mathsf{d}_{\mathbb{H}}(z_2, z)$. On the other hand,

$$(\log \mathsf{q})\, \mathsf{d}_{\mathbb{T}}(w_1, w_2) = (\log Im\, z_1 - \log Im\, z) + (\log Im\, z_2 - \log Im\, z) \quad \text{and}$$
$$(\log \mathsf{q})\, |\mathfrak{h}(w_1) - \mathfrak{h}(w_2)| = \log Im\, z_2 - \log Im\, z_1\,,$$

because the last term was assumed to be ≥ 0. Thus,

$$(\log \mathsf{q})\,\mathsf{d}_{\mathbb{T}}(w_1,w_2) - (\log \mathsf{q})\,|\mathfrak{h}(w_1)-\mathfrak{h}(w_2)| = 2\,\mathsf{d}_{\mathbb{H}}(z_1',z)\,.$$

Thus, the claim of the proposition is equivalent with

$$\mathsf{d}_{\mathbb{H}}(z_1',z) - 2\delta \leq \frac{1}{2}\big(\mathsf{d}_{\mathbb{H}}(z_1,z)+\mathsf{d}_{\mathbb{H}}(z_2,z)-\mathsf{d}_{\mathbb{H}}(z_1,z_2)\big) \leq \mathsf{d}_{\mathbb{H}}(z_1',z)\,,$$

and we still need to proof the left-hand inequality. Now recall that $\mathbb{H}$ is the classical model of a geodesic metric space which is Gromov-hyperbolic: for the given value of $\delta \geq 0$, every geodesic triangle is δ-thin. (That is, for any point on one of the three sides, there is a point at distance at most δ on the union of the other two sides.) We refer to Gromov [26], Coornaert, Delzant and Papadopoulos [16] and/or Ghys and de la Harpe [23] for all details regarding hyperbolic metrics and spaces. Now, $\frac{1}{2}\big(\mathsf{d}(z_1,z)+\mathsf{d}(z_2,z)-\mathsf{d}(z_1,z_2)\big) = (z_1|z_2)_z$ is just the so-called *Gromov product* of z_1 and z_2 with respect to the reference point z. It is well known that the Gromov product on any geodesic hyperbolic metric space satisfies

$$\mathsf{d}(z,\overline{z_1\,z_2}) - 2\delta \leq (z_1|z_2)_z \leq \mathsf{d}(z,\overline{z_1\,z_2})\,,$$

where $\overline{z_1\,z_2}$ is of course the geodesic arc between z_1 and z_2. In our situation, we have $\mathsf{d}_{\mathbb{H}}(z,\overline{z_1\,z_2}) \geq \mathsf{d}_{\mathbb{H}}(z_1',z)$, and the desired inequality follows. □

Proposition 2.8 should be compared with the formula of Proposition 3.1 in [6] for the graph metric of the DL-graphs, which is of the same form (without the δ). Note that the width of a strip S_v is $\mathsf{d}_{\mathsf{HT}}(\mathsf{L}_{v^-},\mathsf{L}_v) = \log \mathsf{q}$, while its image under the projection $\pi_{\mathbb{T}}$ is the (metric) edge $[v^-,v]$, which has length 1. That is, in the construction of HT from $\mathbb{T}$ and $\mathbb{H}$, the tree is stretched by a factor of $\log \mathsf{q}$.

Also note that the coordinates (z,w) of HT used in (2.4) are useful in order to see the nature of HT as a horocylic product and for deducing algebraic–geometric properties. However, by their nature, these coordinates are not independent. The resulting redundancy can be avoided by yet another description, more suitable for analytic purposes; see [4], §2.B. It is not used here in order to avoid abundance of multiple notation.

The *area element* of HT is $d\mathfrak{z} = y^{-2}dx\,dy$ for $\mathfrak{z} = (z,w)$ in the interior of every S_v, where $z = x + \mathfrak{i}\,y$ and dx, dy are Lebesgue measure: this is (a copy of) the standard hyperbolic upper half plane area element. The area of the lines L_v is of course 0.

Definition 2.9. For a real function f on HT, we write f_v for its restriction to the closed strip S_v, where $v \in V(\mathbb{T})$. For its values, we write $f_v(z) = f(z,w)$, where $w \in \mathbb{T}$ is the unique element on the edge $[v^-,v]$ such that $(z,w) \in \mathsf{HT}$ (that is, $\mathfrak{h}(w) = \log_{\mathsf{q}}(Im\,z)$).

Analogous notation is used for the restriction of a function f defined on $\Omega \subset \mathsf{HT}$ to $\Omega \cap \mathsf{S}_v$.

While we think of f_v as a function on S_v, it is formally a function defined for complex $z = x + \mathfrak{i}\, y \in \mathsf{S}_{\mathfrak{h}(v)} \subset \mathbb{H}$. The integral of f with respect to the area element is given by

$$\int_{\mathsf{HT}} f(\mathfrak{z})\, d\mathfrak{z} = \sum_{v \in V(\mathbb{T})} \int_{\mathsf{S}_{\mathfrak{h}(v)}} f_v(x + \mathfrak{i}\, y)\, y^{-2}\, dx\, dy\,, \tag{2.10}$$

whenever this is well defined in the sense of Lebesgue integration.

Next, we determine the isometry group of $\mathsf{HT}(\mathsf{q},\mathsf{p})$ and its modular function. Recall here that the modular function $\boldsymbol{\delta}$ of an arbitrary locally compact group G is the continuous homomorphism from the group into the multiplicative group $\mathbb{R}_+$ with the property that for left Haar measure dg on G, one has

$$\int_G f(gg_0)\, dg = \boldsymbol{\delta}(g_0)^{-1} \int_G f(g)\, dg$$

for every integrable function f on G.

Consider the action on $\mathbb{H}$ of the group of affine transformations

$$\mathsf{Aff}(\mathbb{H},\mathsf{q}) = \left\{ g = \begin{pmatrix} \mathsf{q}^n & b \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\,,\ b \in \mathbb{R} \right\}, \quad \text{where } gz = \mathsf{q}^n z + b\,,\ z \in \mathbb{H}\,. \tag{2.11}$$

Thus, $g_1 g_2 = \begin{pmatrix} \mathsf{q}^{n_1+n_2} & b_1 + \mathsf{q}^{n_1} b_2 \\ 0 & 1 \end{pmatrix}$ for $g_i = \begin{pmatrix} \mathsf{q}^{n_i} & b_i \\ 0 & 1 \end{pmatrix}$, $i = 1, 2$. This group acts by isometries on $\mathbb{H}$ and leaves the set of lines $y = \mathsf{q}^k$, $k \in \mathbb{Z}$, invariant. The full group of isometries of $\mathbb{H}$ with the latter property is generated by $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ and the reflection along the y-axis. Our group is locally compact, and left Haar measure dg and its modular function $\boldsymbol{\delta}_{\mathbb{H}} = \boldsymbol{\delta}_{\mathbb{H},\mathsf{q}}$ are given by

$$dg = \mathsf{q}^{-n}\, dn\, db \quad \text{and} \quad \boldsymbol{\delta}_{\mathbb{H}}(g) = \mathsf{q}^{-n}\,, \quad \text{if} \quad g = \begin{pmatrix} \mathsf{q}^n & b \\ 0 & 1 \end{pmatrix}. \tag{2.12}$$

Here, dn is counting measure on $\mathbb{Z}$ and db is Lebesgue measure on $\mathbb{R}$.

Regarding the tree, first note that every isometry is the natural linear extension of an automorphism, that is, a neighbourhood preserving permutation of the vertex set $V(\mathbb{T})$. Also, note that the action of each isometry extends continuously to $\widehat{\mathbb{T}}$, since isometries send geodesic rays to geodesic rays and preserve their equivalence. Let $\mathsf{Aut}(\mathbb{T}_\mathsf{p})$ denote the full isometry group of $\mathbb{T}_\mathsf{p}$. Following Cartwright, Kaimanovich and Woess [15], the *affine group* of $\mathbb{T}_\mathsf{p}$ is

$$\mathsf{Aff}(\mathbb{T}_\mathsf{p}) = \{\gamma \in \mathsf{Aut}(\mathbb{T}_\mathsf{p}) : \gamma\,\varpi = \varpi\}\,. \tag{2.13}$$

This is a locally compact, totally disconnected and compactly generated group with respect to the topology of pointwise convergence, and it acts transitively on $V(\mathbb{T})$. The name is chosen (1) because of the analogy with the classical affine group which is just the group of (orientation preserving) isometries of $\mathbb{H}$ that fix the boundary point ∞, and (2) because the affine group over any local field whose residual field has order p embeds naturally into $\mathsf{Aff}(\mathbb{T}_\mathsf{p})$, see [15] and below. The elements γ of $\mathsf{Aff}(\mathbb{T})$ are also characterized by the property $\gamma(v^-) = (\gamma v)^-$ for every $v \in V(\mathbb{T})$, or equivalently, by $\gamma(w_1 \curlywedge w_2) = (\gamma w_1) \curlywedge (\gamma w_2)$ for all $w_i \in \mathbb{T}$. Consequently, the mapping $\Phi : \mathsf{Aff}(\mathbb{T}) \to \mathbb{Z}$ defined by $\gamma \mapsto \mathfrak{h}(\gamma w) - \mathfrak{h}(w)$ is independent of $w \in \mathbb{T}$ and a homomorphism. Thus $\gamma(H_t) = H_{t+k}$ if $\mathfrak{h}(\gamma w) - \mathfrak{h}(w) = k$. As a matter of fact, this mapping appears in the modular function $\boldsymbol{\delta}_{\mathbb{T}} = \boldsymbol{\delta}_{\mathbb{T}_\mathsf{p}}$ of $\mathsf{Aff}(\mathbb{T}_\mathsf{p})$, see [15]:

$$(2.14)\quad \boldsymbol{\delta}_{\mathbb{T}}(\gamma) = \mathsf{p}^{\Phi(\gamma)} \quad \text{where} \quad \Phi(\gamma) = \mathfrak{h}(\gamma w) - \mathfrak{h}(w)\,, \quad \text{if} \quad \gamma \in \mathsf{Aff}(\mathbb{T}_\mathsf{p})\,,\ w \in \mathbb{T}\,.$$

In the following theorem, we collect several rather straightforward properties of the isometry group of $\mathsf{HT}(\mathsf{q},\mathsf{p})$.

Theorem 2.15. *The group*

$$\mathcal{A} = \mathcal{A}(\mathsf{q},\mathsf{p}) = \{(g,\gamma) \in \mathsf{Aff}(\mathbb{H},\mathsf{q}) \times \mathsf{Aff}(\mathbb{T}_\mathsf{p}) : \log_\mathsf{q} \boldsymbol{\delta}_{\mathbb{H}}(g) + \log_\mathsf{p} \boldsymbol{\delta}_{\mathbb{T}}(\gamma) = 0\}$$

acts on $\mathsf{HT}(\mathsf{q},\mathsf{p})$ *by isometries* $(g,\gamma)(z,w) = (gz,\gamma w)$. *It is the semidirect product*

$$\mathcal{A} = \mathbb{R} \rtimes \mathsf{Aff}(\mathbb{T}) \quad \textit{with respect to the action} \quad b \mapsto \mathsf{q}^{\Phi(\gamma)}\, b\,,\ \gamma \in \mathsf{Aff}(\mathbb{T})\,,\ b \in \mathbb{R}\,.$$

The full group of isometries of $\mathsf{HT}(\mathsf{q},\mathsf{p})$ *is generated by* $\mathcal{A}(\mathsf{q},\mathsf{p})$ *and the reflection*

$$\mathfrak{s}(x + \mathfrak{i}\, y, w) = (-x + \mathfrak{i}\, y, w)\,.$$

It acts on $\mathsf{HT}(\mathsf{q},\mathsf{p})$ *with compact quotient isomorphic with the circle of length* $\log \mathsf{q}$, *and it leaves the area element of* HT *invariant.*

As a closed subgroup of $\mathsf{Aff}(\mathbb{H},\mathsf{q}) \times \mathsf{Aff}(\mathbb{T}_\mathsf{p})$, *the group* $\mathcal{A}$ *is locally compact, compactly generated and amenable, and its modular function is given by*

$$\boldsymbol{\delta}_{\mathcal{A}}(g,\gamma) = (\mathsf{p}/\mathsf{q})^{\Phi(\gamma)}\,.$$

Proof. (1) Let $(g,\gamma) \in \mathcal{A}$ and $(z,w) \in \mathsf{HT}$, with $g = \begin{pmatrix} \mathsf{q}^n & b \\ 0 & 1 \end{pmatrix}$ and $z = x + \mathfrak{i}\, y$. Then $gz = (b + \mathsf{q}^n x) + \mathfrak{i}\,(\mathsf{q}^n y)$, and $\mathfrak{h}(\gamma w) = \mathfrak{h}(w) + n$. Thus, $\mathfrak{h}(w) = \log_q(Im\, z)$ implies $\mathfrak{h}(\gamma w) = \log_q(Im\, gz)$, whence $(gz,\gamma w) \in \mathsf{HT}$. From (2.7), one sees that this is an isometry. Indeed, let $(z_i,w_i) \in \mathsf{HT}$ $(i = 1,2)$ and $v = w_1 \curlywedge w_2$. Then $\gamma v = \gamma w_1 \curlywedge \gamma w_2$. So, if $v \in \{w_1,w_2\}$ then $\gamma v \in \{\gamma w_1, \gamma w_2\}$ and

$$d\big((gz_1,\gamma w_1),(gz_2,\gamma w_2)\big) = d_{\mathbb{H}}(gz_1,gz_2) = d_{\mathbb{H}}(z_1,z_2) = d\big((z_1,w_1),(z_2,w_2)\big)\,.$$

If $v \notin \{w_1,w_2\}$ then $v \in V(\mathbb{T})$ and $\gamma \mathsf{L}_v = \mathsf{L}_{\gamma v}$. If z_0 minimizes $d_{\mathbb{H}}(z_1,z) + d_{\mathbb{H}}(z,z_2)$ among all $z \in L_{\mathfrak{h}(v)}$, then gz_0 minimizes $d_{\mathbb{H}}(gz_1,\tilde z) + d_{\mathbb{H}}(\tilde z,z_2)$ among all $\tilde z \in L_{\mathfrak{h}(\gamma v)}$. Thus, $d\big((gz_1,\gamma w_1),(gz_2,\gamma w_2)\big) = d\big((z_1,w_1),(z_2,w_2)\big)$ in this case as well. Thus, $\mathcal{A}$ acts by isometries.

(2) We can identify each element $\mathfrak{g} = (g,\gamma) \in \mathcal{A}$ with the pair $[b,\gamma] \in \mathbb{R} \times \mathsf{Aff}(\mathbb{T})$, where $g = \begin{pmatrix} \mathsf{q}^{\Phi(\gamma)} & b \\ 0 & 1 \end{pmatrix}$ as an affine mapping. It is immediate that with this identification, $\mathcal{A} = \mathbb{R} \rtimes \mathsf{Aff}(\mathbb{T})$ with the proposed action of $\mathsf{Aff}(\mathbb{T})$ on $\mathbb{R}$, namely, the group operation is $[b_1\,,\gamma_1][b_2\,,\gamma_1] = [b_1 + \mathsf{q}^{\Phi(\gamma_1)} b_2\,,\gamma_1\gamma_2]$.

(3) Let $\mathfrak{g}$ be an isometry of HT. Then it is clear that $\mathfrak{g}$ sends each line L_v to some other line $\mathsf{L}_{\tilde v}$; compare with [22]. Thus, there is some $\gamma \in \mathsf{Aut}(\mathbb{T})$ such that $\mathfrak{g}\mathsf{L}_v = \mathsf{L}_{\gamma v}$ for every $v \in V(\mathbb{T})$. We claim that $\gamma \in \mathsf{Aff}(\mathbb{T})$, that is, $\gamma v^- = (\gamma v)^-$ for all $v \in V(\mathbb{T})$.

For $v \in V(\mathbb{T})$, let $\tilde{v} = \gamma v$. Suppose that $\gamma v^- \neq \tilde{v}^-$. Then $\tilde{u}_1 = \gamma v^-$ must be a successor of $\tilde{v}$, that is, $\tilde{u}_1^- = \tilde{v}$. Also, since $\mathsf{p} \geq 2$, there must be some successor u of v ($u^- = v$) such that $\tilde{u}_2 = \gamma u$ is a successor of $\tilde{v}$. Then $\mathfrak{g}$ maps $\mathsf{S}_u \cup \mathsf{S}_v$ isometrically to $\mathsf{S}_{u_1} \cup \mathsf{S}_{u_2}$. Now, writing $k = \mathfrak{h}(v)$ and $\tilde{k} = \mathfrak{h}(\gamma v)$, we have that $\mathsf{S}_u \cup \mathsf{S}_v$ is an isometric copy of $\mathsf{S}_{k-1} \cup \mathsf{S}_k \subset \mathbb{H}$, while $\mathsf{S}_{u_1} \cup \mathsf{S}_{u_2}$ consists of two copies of S_{k-1} glued together along their bottom line. With the metric (2.7), these two pieces are *not* isometric. Thus, it must be $\gamma v^- = (\gamma v)^-$, and $\gamma \in \mathsf{Aff}(\mathbb{T})$. We now also see that $\mathfrak{g}\mathsf{S}_v = \mathsf{S}_{\gamma v}$ for all $v \in V(\mathbb{T})$.

Now consider an end $\xi \in \partial^*\mathbb{T}$. It follows from the above that $\mathfrak{g}\mathbb{H}_\xi = \mathbb{H}_{\gamma\xi}$. Thus, there must be an isometry g_ξ of $\mathbb{H}$ such that for $(z,w) \in \mathbb{H}_\xi$, $\mathfrak{g}(z,w) = (g_\xi z, \gamma w)$. Then g_ξ must be either a Möbius transformation or a Möbius transformation followed by reflection along the y-axis. Since $\mathfrak{g}\mathsf{L}_v = \mathsf{L}_{\gamma v}$ for each $v \in V(\mathbb{T}) \cap \overline{\xi\,\varpi}$, in both of the last cases, that Möbius transformation is in $\mathsf{Aff}(\mathbb{H},\mathsf{q})$. Now let $\eta \in \partial^*\mathbb{T} \setminus \{\xi\}$ and set $v = \xi \curlywedge \eta$. Then $\mathbb{H}_\xi$ and $\mathbb{H}_\eta$ coincide below (and including) the line $\mathsf{L}_v \subset \mathsf{HT}$, whence g_ξ and g_η coincide below the line $L_{\mathfrak{h}(v)}$. But this implies that $g_\xi = g_\eta =: g$ for all $\xi, \eta \in \partial^*\mathbb{T}$. Every $(z,w) \in \mathsf{HT}$ lies in $\mathbb{H}_\xi$ for some $\xi \in \partial^*\mathbb{T}$. Therefore $\mathfrak{g}(z,w) = (gz, \gamma w)$ for all $(z,w) \in \mathsf{HT}$. This means that either $\mathfrak{g} \in \mathcal{A}$ (when g itself is a Möbius transformation), or $\mathfrak{s}\mathfrak{g} \in \mathcal{A}$ (when g is a Möbius transformation followed by reflection along the y-axis).

The statement abut the co-compact action and factor space is obvious, and it is straightforward that the action of $\mathcal{A}$, as well as $\mathfrak{s}$, preserve the area element of HT.

(4) We compute the modular function of $\mathcal{A}$. Let db be Lebesgue measure on $\mathbb{R}$ and $d\gamma$ left Haar measure on $\mathsf{Aff}(\mathbb{T})$. It will be useful to normalise $d\gamma$ such that

$$\int_{\mathsf{Aff}(\mathbb{T})} \mathbf{1}_{\mathsf{Stab}(o)}(\gamma)\, d\gamma = 1\,, \quad \text{where} \quad \mathsf{Stab}(x) = \{\gamma \in \mathsf{Aff}(\mathbb{T}) : \gamma o = o\} \tag{2.16}$$

is the stabiliser of o. It is an open-compact subgroup of $\mathsf{Aff}(\mathbb{T})$. It is a straightforward exercise that in the $[b,\gamma]$-coordinates of the semidirect product, left Haar measure on $\mathcal{A}$ is given by

$$d\mathfrak{g} = \mathsf{q}^{-\Phi(\gamma)}\, db\, d\gamma\,. \tag{2.17}$$

Now let $\mathfrak{g}_0 = [b_0, \gamma_0]$, and let $f \in \mathcal{C}_c(\mathsf{HT})$, the space of continuous, compactly supported functions. Then, using (2.14)

$$\begin{aligned}
\int_{\mathcal{A}} f(\mathfrak{g}\mathfrak{g}_0)\, d\mathfrak{g} &= \int_{\mathsf{Aff}(\mathbb{T})} \int_{\mathbb{R}} \mathsf{q}^{-\Phi(\gamma)} f[b + \mathsf{q}^{\Phi(\gamma)} b_0, \gamma\gamma_0]\, db\, d\gamma \\
&= \mathsf{q}^{\Phi(\gamma_0)} \int_{\mathbb{R}} \int_{\mathsf{Aff}(\mathbb{T})} \mathsf{q}^{-\Phi(\gamma\gamma_0)} f[b, \gamma\gamma_0]\, d\gamma\, db \\
&= \mathsf{q}^{\Phi(\gamma_0)} \int_{\mathbb{R}} \boldsymbol{\delta}_{\mathbb{T}}(\gamma_0)^{-1} \int_{\mathsf{Aff}(\mathbb{T})} \mathsf{q}^{-\Phi(\gamma)} f[b,\gamma]\, d\gamma\, db = (\mathsf{q}/\mathsf{p})^{\Phi(\gamma_0)} \int_{\mathcal{A}} f(\mathfrak{g})\, d\mathfrak{g}\,.
\end{aligned}$$

This yields $\boldsymbol{\delta}_{\mathcal{A}}(\gamma_0) = (\mathsf{p}/\mathsf{q})^{\Phi(\gamma_0)}$, as proposed. □

Corollary 2.18. *When* $\mathsf{p} \neq \mathsf{q}$*, there is no* discrete *group that acts on* $\mathsf{HT}(\mathsf{q},\mathsf{p})$ *with compact quotient.*

Indeed, such a group would be a co-compact lattice in the isometry group of $\mathsf{HT}(\mathsf{q},\mathsf{p})$, which cannot exist, since the latter group is non-unimodular. When $\mathsf{p}=\mathsf{q}$, the situation is different.

Proposition 2.19. *The Baumslag–Solitar group* $\mathsf{BS}(\mathsf{p}) = \langle \mathfrak{a},\mathfrak{b} : \mathfrak{a}\mathfrak{b} = \mathfrak{b}^{\mathsf{p}}\mathfrak{a}\rangle$ *embeds as a co-compact, discrete subgroup into* $\mathcal{A}(\mathsf{p},\mathsf{p})$*.*

Proof. It is well known that

$$\mathsf{BS}(\mathsf{p}) = \left\{ \begin{pmatrix} \mathsf{p}^n & k/\mathsf{p}^l \\ 0 & 1 \end{pmatrix} : k,l,n \in \mathbb{Z} \right\}. \tag{2.20}$$

In this representation, $\mathfrak{a} = \begin{pmatrix} \mathsf{p} & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathfrak{b} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, it is immediate that $\mathsf{BS}(\mathsf{p})$ is a (non-discrete) subgroup of $\mathsf{Aff}(\mathbb{H},\mathsf{p})$.

We now explain how our group acts on $\mathbb{T}=\mathbb{T}_{\mathsf{p}}$, compare e.g. with § 4.A in [15]. The *ring* $\mathbb{Q}_{\mathsf{p}}$ of p-adic numbers consists of all Laurent series in powers of p of the form

$$\mathfrak{u} = \sum_{k=m}^{\infty} a_k\, \mathsf{p}^k\,, \quad m \in \mathbb{Z}\,,\ a_k \in \{0,\dots,\mathsf{p}-1\}\,. \tag{2.21}$$

If $a_m \neq 0$ in (2.21), then we set $|\mathfrak{u}|_{\mathsf{p}} = \mathsf{p}^{-m}$, the p-adic norm of $\mathfrak{u}$. In addition, we get the neutral element 0 of $\mathbb{Q}_{\mathsf{p}}$ when $a_k=0$ for all k in (2.21). Of course, $|0|_{\mathsf{p}}=0$. Addition and multiplication in $\mathbb{Q}_{\mathsf{p}}$ extend the respective operations on those elements (2.21) which are finite sums (i.e., $a_k=0$ for all but finitely many k), performed within the rational numbers. That is, carries to higher positions of coefficients that exceed $\mathsf{p}-1$ have to be taken care of.

We have the following for all $\mathfrak{u},\mathfrak{v}\in\mathbb{Q}_{\mathsf{p}}$ and $m\in\mathbb{Z}$:

$$\begin{aligned} &\text{(i)} && |\mathfrak{u}|_{\mathsf{p}} = 0 \iff \mathfrak{u}=0\,,\\ &\text{(ii)} && |\mathfrak{u}+\mathfrak{v}|_{\mathsf{p}} \le \max\{|\mathfrak{u}|_{\mathsf{p}}, |\mathfrak{v}|_{\mathsf{p}}\}\,,\\ &\text{(iii)} && |\mathfrak{u}\,\mathfrak{v}|_{\mathsf{p}} \le |\mathfrak{u}|_{\mathsf{p}}\,|\mathfrak{v}|_{\mathsf{p}}\,,\\ &\text{(iv)} && |\mathsf{p}^m\,\mathfrak{v}|_{\mathsf{p}} = \mathsf{p}^{-m}|\mathfrak{v}|_{\mathsf{p}}\,. \end{aligned} \tag{2.22}$$

If p is prime, then we always have equality in (iii), and $\mathbb{Q}_{\mathsf{p}}$ is a field. Otherwise, it is only a ring. By (ii), the norm induces an ultrametric. Any metric ball in $\mathbb{Q}_{\mathsf{p}}$ is open and compact, and $\mathbb{Q}_{\mathsf{p}}$ is totally disconnected (a Cantor set). Let $\overline{B}(\mathfrak{u},\mathsf{p}^{-k})$ be the closed ball with radius p^{-k} and centre $\mathfrak{u}$. Each of its points is a centre for that ball. It is the disjoint union of p closed balls with radius p^{-k-1}. Now consider

$$H_m = \{v = \overline{B}(\mathfrak{u},\mathsf{p}^{-m}) : \mathfrak{u}\in\mathbb{Q}_{\mathsf{p}}\}\,.$$

This is going to be the horocycle at level m of our tree, and for $v=\overline{B}(\mathfrak{u},\mathsf{p}^{-m})$ as a vertex of $\mathbb{T}=\mathbb{T}_{\mathsf{p}}$, its predecessor is $v^- = \overline{B}(\mathfrak{u},\mathsf{p}^{-m+1})$. This gives us the tree structure. We find that $\partial^*\mathbb{T}=\mathbb{Q}_{\mathsf{p}}$.

We see that via the matrix representation (2.20), $\mathsf{BS}(\mathsf{p})$ acts on $\partial^*\mathbb{T}$ by affine transformations of the ring $\mathbb{Q}_\mathsf{p}$. This action extends to the tree: if $\gamma = \begin{pmatrix} \mathsf{p}^n & k/\mathsf{p}^l \\ 0 & 1 \end{pmatrix}$ and $v = \overline{B}(\mathfrak{u}, \mathsf{p}^{-m})$ then

$$\gamma v = \big\{ \mathsf{p}^n \mathfrak{v} + k/\mathsf{p}^l : \mathfrak{v} \in \overline{B}(\mathfrak{u}, \mathsf{p}^{-m}) \big\} = \overline{B}(\mathsf{p}^n \mathfrak{u} + k/\mathsf{p}^l, \mathsf{p}^{-m-n})$$

is a closed ball with radius p^{-m-n}, so that it is another vertex of our tree, which lies in H_{m+n}: (iv) of (2.22) is crucial here. In this way, γ defines an element of $\mathsf{Aff}(\mathbb{T}_\mathsf{p})$.

We can now take the diagonal embedding $\gamma \mapsto (\gamma, \gamma)$ of $\mathsf{BS}(\mathsf{p})$ into $\mathsf{Aff}(\mathbb{H}, \mathsf{p}) \times \mathsf{Aff}(\mathbb{T}_\mathsf{p})$. This embedding is compatible with the level structure of both $\mathbb{H}$ and the tree, so that $\mathsf{BS}(\mathsf{p})$ is embedded into $\mathcal{A}(\mathsf{p}, \mathsf{p})$. It is easily seen to be discrete. It is co-compact because the factor space is compact. Indeed, a fundamental domain for the action of $\mathsf{BS}(\mathsf{p}, \mathsf{p})$ is obtained as follows: In $\mathbb{H}$, take the Euclidean (!) rectangle R with vertices $\mathfrak{i}$, $\mathsf{p} + \mathfrak{i}$, $\mathsf{p} + \mathfrak{i}\,\mathsf{p}$ and $\mathfrak{i}\,\mathsf{p}$. Then $\{(z, w) \in \mathsf{HT} : z \in R,\ w \in [o^-, o]\}$ is a fundamental domain. The reader is invited to elaborate these last details as an exercise; compare once more with [22]. □

Remark 2.23. In the analogy between tree and hyperbolic upper half plane, $\partial^*\mathbb{T}$ corresponds to the lower boundary line $\mathbb{R}$ of $\mathbb{H}$. In this spirit, the natural analogue of Lebesgue measure on $\mathbb{R}$ is the measure λ^* on $\partial\mathbb{T}$ which corresponds to (suitably normalised) Haar measure on the Abelian group $\mathbb{Q}_p$ under the identification of $\partial^*\mathbb{T}$ with $\mathbb{Q}_p$. The basic open-closed sets in $\partial^*\mathbb{T}$ (the ultrametric balls) and their measures are

$$\partial_v^*\mathbb{T} = \{\xi \in \partial\mathbb{T} : v \in \overline{\varpi\,\xi}\} \quad \text{and} \quad \lambda^*(\partial_v^*\mathbb{T}) = \mathsf{p}^{-\mathfrak{h}(v)}, \quad v \in V(\mathbb{T}).$$

3. Laplacians with drift

We now explain our family of natural Laplace operators $\Delta^{\mathsf{HT}} = \Delta^{\mathsf{HT}}_{\alpha,\beta}$ on $\mathsf{HT}(\mathsf{q}, \mathsf{p})$ with "vertical drift" parameters $\alpha \in \mathbb{R}$ and $\beta > 0$. Their rigorous construction is carried out in detail in [4]. Here, we reproduce the basic facts.

Definition 3.1. We let $\mathcal{C}^\infty(\mathsf{HT})$ be the set of those continuous functions f on HT such that, for each $v \in V(\mathbb{T})$, the restriction f_v of f to the strip S_v (as in Definition 2.9) has continuous derivatives $\partial_x^m \partial_y^n f_v(z)$ of all orders in the interior S_v^o which satisfy, for all $R > 0$,

$$\sup \big\{ |\partial_x^m \partial_y^n f_v(z)| : z = x + \mathfrak{i}\, y \in \mathsf{S}^o_{\mathfrak{h}(v)}\,,\ |Re\, z| \le R \big\} < \infty\,.$$

Thus, on each strip S_v, each partial derivative has a continuous extension $\partial_x^m \partial_y^n f_v(z)$ to the strip's boundary. Note that when $w^- = v$, it is in general *not* true that $\partial_x^m \partial_y^n f_w = \partial_x^m \partial_y^n f_v$ on $\mathsf{L}_v = \mathsf{S}_v \cap \mathsf{S}_w$, unless $m = n = 0$. We have the (hyperbolic) gradient ∇f given by

$$\nabla f_v(z) = \big(y^2 \partial_x f_v(z)\,,\, y^2 \partial_y f_v(z)\big)$$

which is defined without ambiguity in the interior of each strip. However, on any bifurcation line L_v, we have to distinguish between all the one-sided limits of the gradient, obtaining the family

$$\nabla f_v(z) \quad \text{and} \quad \nabla f_w(z) \quad \text{for all } w \in V(\mathbb{T}) \text{ with } w^- = v\,, \quad (z,v) \in \mathsf{L}_v\,.$$

Let $\mathcal{C}_c^\infty(\Omega)$ be the space of those functions in $\mathcal{C}^\infty(\mathsf{HT})$ that have compact support contained in Ω. We shall write

$$\mathsf{LT} = \bigcup_{v \in V(\mathbb{T})} \mathsf{L}_v \quad \text{and} \quad \mathsf{HT}^o = \bigcup_{v \in V(\mathbb{T})} \mathsf{S}_v^o = \mathsf{HT} \setminus \mathsf{LT}\,. \tag{3.2}$$

For $\alpha \in \mathbb{R}$, $\beta > 0$, we define the measure $\mathbf{m}_{\alpha,\beta}$ on HT by

$$\begin{gathered} d\mathbf{m}_{\alpha,\beta}(\mathfrak{z}) = \phi_{\alpha,\beta}(\mathfrak{z})\, d\mathfrak{z} \quad \text{with} \\ \phi_{\alpha,\beta}(\mathfrak{z}) = \beta^{\mathfrak{h}(v)}\, y^\alpha \quad \text{for} \quad \mathfrak{z} = (x + \mathfrak{i}\, y, w) \in \mathsf{S}_v \setminus \mathsf{L}_{v^-}\,, \quad \text{where} \quad v \in V(\mathbb{T})\,, \end{gathered} \tag{3.3}$$

that is, $w \in (v^-, v]$ and $\log_{\mathsf{q}} y = \mathfrak{h}(w)$.

Definition 3.4. For $f \in \mathcal{C}^\infty(\mathsf{HT})$ and $\mathfrak{z} = (x + \mathfrak{i}\, y, w) \in \mathsf{HT}^o$, we set

$$\Delta_{\alpha,\beta} f(\mathfrak{z}) = y^2(\partial_x^2 + \partial_y^2) f(\mathfrak{z}) + \alpha\, y\, \partial_y f(\mathfrak{z})\,.$$

Let $\mathcal{D}_{\alpha,\beta,c}^\infty$ be the space of all functions $f \in \mathcal{C}_c^\infty(\mathsf{HT})$ with the following properties.

(i) For any k, the k-th iterate $\Delta_{\alpha,\beta}^k f$, originally defined on HT^o, admits a continuous extension to all of HT (which then belongs to $\mathcal{C}_c^\infty(\mathsf{HT})$ and is also denoted $\Delta_{\alpha,\beta}^k f$).

(ii) The function f, as well as each of its iterates $\Delta_{\alpha,\beta}^k f$, satisfies the *bifurcation conditions*

$$\partial_y f_v = \beta \sum_{w\,:\, w^- = v} \partial_y f_w \quad \text{on } L_v \text{ for each } v \in V(\mathbb{T})\,. \tag{3.5}$$

The Laplacian $\Delta_{\alpha,\beta}$ as a differential operator on HT^o apparently depends only on α. Dependence on β is through the domain of functions on which the Laplacian acts, which have to satisfy (3.5). In the following propositions, we present some of the essential properties proved in [4], where additional details can be found.

Proposition 3.6. *The space $\mathcal{D}_{\alpha,\beta,c}^\infty$ is dense in the Hilbert space $\mathcal{L}^2(\mathsf{HT}, \mathbf{m}_{\alpha,\beta})$. The operator $(\Delta_{\alpha,\beta}, \mathcal{D}_{\alpha,\beta,c}^\infty)$ is essentially self-adjoint in $\mathcal{L}^2(\mathsf{HT}, \mathbf{m}_{\alpha,\beta})$.*

With a small abuse of notation, we write $\big(\Delta_{\alpha,\beta}, \mathrm{Dom}(\Delta_{\alpha,\beta})\big)$ for its unique self-adjoint extension. Indeed, at a higher level of rigour in Definition 2.16 of [4], the differential operator of Definition 3.4 (i), defined on $\mathcal{D}_{\alpha,\beta,c}^\infty$, is denoted $\mathfrak{A}_\alpha$, and the notation $\Delta_{\alpha,\beta}$ is reserved for the extension. The detailed construction of the latter in [4] is carried out via Dirichlet form theory.

Proposition 3.7. (a) *The heat semigroup* $e^{t\Delta_{\alpha,\beta}}$, $t > 0$, *acting on* $\mathcal{L}^2(\mathsf{HT}, \mathbf{m}_{\alpha,\beta})$ *admits a continuous positive symmetric transition kernel* $(0,\infty) \times \mathsf{HT} \times \mathsf{HT} \ni (t, \mathfrak{w}, \mathfrak{z}) \mapsto \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})$ *such that for all* $f \in \mathcal{C}_c(\mathsf{HT})$,

$$e^{t\Delta_{\alpha,\beta}} f(\mathfrak{z}) = \int_{\mathsf{HT}} \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})\, f(\mathfrak{z})\, d\mathbf{m}_{\alpha,\beta}(\mathfrak{z})\,.$$

(b) *For each fixed* $(t, \mathfrak{w})$, *the function* $\mathfrak{z} \mapsto \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})$ *is in* $\mathcal{C}^\infty(\mathsf{HT})$ *and satisfies* (3.5).

(c) *The heat semigroup is conservative, that is,* $\int_{\mathsf{HT}} \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \cdot)\, d\mathbf{m}_{\alpha,\beta} = 1$.

(d) *It sends* $\mathcal{L}^\infty(\mathsf{HT})$ *into* $\mathcal{C}^\infty(\mathsf{HT}) \cap \mathcal{L}^\infty(\mathsf{HT})$ *and* $\mathcal{C}_0(\mathsf{HT})$, *the space of continuous functions vanishing at infinity, into itself.*

The general theory of Markov processes tells us that $\Delta_{\alpha,\beta}$ is the infinitesimal generator of a Hunt process $(X_t)_{t\ge 0}$. This is our Brownian motion on HT. It is defined for every starting point $\mathfrak{w} \in \mathsf{HT}$, has infinite life time and continuous sample paths. Its family of distributions $(\mathbb{P}^{\alpha,\beta}_{\mathfrak{w}})_{\mathfrak{w}\in\mathsf{HT}}$ on $\mathbf{\Omega} = \mathcal{C}([0,\infty] \to \mathsf{HT})$ is determined by the one-dimensional distributions

$$\mathbb{P}^{\alpha,\beta}_{\mathfrak{w}}[X_t \in U] = \int_U \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})\, d\mathbf{m}_{\alpha,\beta}(\mathfrak{z}) = \int_U \mathbf{p}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})\, d\mathfrak{z}\,,$$

where U is any Borel subset of HT and

$$\mathbf{p}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z}) = \mathbf{h}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z})\, \phi_{\alpha,\beta}(\mathfrak{z})$$

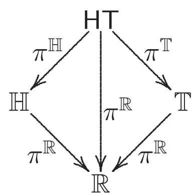
with the function $\phi_{\alpha,\beta}$ as in (3.3). We note that this transition density with respect to $d\mathfrak{z}$ is invariant under the action of the group $\mathcal{A}$ of Theorem 2.15:

$$\mathbf{p}_{\alpha,\beta}(t, \mathfrak{g}\mathfrak{w}, \mathfrak{g}\mathfrak{z}) = \mathbf{p}_{\alpha,\beta}(t, \mathfrak{w}, \mathfrak{z}) \quad \text{for all} \quad t > 0\,,\ \mathfrak{w}, \mathfrak{z} \in \mathsf{HT} \text{ and } \mathfrak{g} \in \mathcal{A}\,. \tag{3.8}$$

We next say a few words about the natural projections of HT. We have

$$\pi^{\mathbb{H}} : \mathsf{HT} \to \mathbb{H}\,,\ \ \mathfrak{z} = (z, w) \mapsto z\,, \quad \pi^{\mathbb{T}} : \mathsf{HT} \to \mathbb{T}\,,\ \ \mathfrak{z} = (z, w) \mapsto w\,, \quad \text{and}$$
$$\pi^{\mathbb{R}} : \mathsf{HT} \to \mathbb{R}\,,\ \ \mathfrak{z} = (z, w) \mapsto \log_{\mathsf{q}} Im(z).$$

We also interpret $\pi^{\mathbb{R}}$ as a projection $\mathbb{H} \to \mathbb{R}$, where $z \mapsto \log_{\mathsf{q}} Im(z)$, and as a projection $\mathbb{T} \to \mathbb{R}$, where $w \mapsto \mathfrak{h}(w)$. Thus, the following diagram commutes.



The "sliced" hyperbolic plane as in Figure 3 can be interpreted as $\mathsf{HT}(\mathsf{q},1)$, that is, the tree is $\mathbb{Z}$, the bi-infinite line graph. Everything that has been said above also applies here, so that we have the operator $\Delta^{\mathbb{H}}_{\alpha,\beta}$ on $\mathbb{H}$.

Analogously, we have a Laplacian $\Delta^{\mathbb{T}}_{\alpha,\beta}$ on the metric tree, introduced in the same way as above. However, we should take care of the slightly different parametrisation, that is, the stretching factor $\log \mathsf{q}$ in the construction of HT, while in $\mathbb{T}$, each edge $[v^-,v]$ corresponds to the real interval $[\mathfrak{h}(v)-1\,,\mathfrak{h}(v)]$. The functions that we consider now depend on one real variable in each open edge. We write f_v for the restriction of $f:\mathbb{T}\to\mathbb{R}$ to $[v^-,v]$. We have to redefine the analogue of the measure of (3.3):

$$
(3.9)\qquad \begin{aligned} d\mathbf{m}^{\mathbb{T}}_{\alpha,\beta}(w) &= \phi^{\mathbb{T}}_{\alpha,\beta}(w)\,dw \quad \text{with}\\ \phi^{\mathbb{T}}_{\alpha,\beta}(w) &= \beta^{\mathfrak{h}(v)}\,\mathsf{q}^{(\alpha-1)\mathfrak{h}(w)}\,\log\mathsf{q} \quad \text{for}\ \ w\in(v^-,v]\,, \end{aligned}
$$

where $v\in V(\mathbb{T})$ and (recall) dw is the standard Lebesgue measure in each edge. The space $\mathcal{C}^\infty(\mathbb{T})$ is defined as in Definition 3.1, considering the edges of $\mathbb{T}$ as the strips. The analogues of the crucial Definition 3.4 plus the bifurcation condition (3.5) now become the following: every $f\in\operatorname{Dom}(\Delta^{\mathbb{T}}_{\alpha,\beta})\cap\mathcal{C}^\infty(\mathbb{T})$ must satisfy for every $v\in V(\mathbb{T})$

$$
(3.10)\qquad \begin{aligned} f_v'(v) &= \beta\sum_{w\,:\,w^-=v} f_w'(v) \quad \text{and}\\ \Delta^{\mathbb{T}}_{\alpha,\beta}f &= \frac{1}{(\log\mathsf{q})^2}f'' + \frac{\alpha-1}{\log\mathsf{q}}f' \quad \text{in the open edge } (v^-,v)\,. \end{aligned}
$$

Finally, the analogue on the real line is comprised in the above by identifying $\mathbb{R}$ with the tree with branching number 1 (degree 2). In this case, the vertices are the integers, the edges are the intervals $[k-1\,,\,k]$, where $k\in\mathbb{Z}$, and the Laplacian becomes $\Delta^{\mathbb{R}}_{\alpha,\beta}$. Its definition as a differential operator in each open interval $(k-1\,,\,k)$ is the same as in (3.10), while the bifurcation condition becomes $f'(k-)=\beta\,f'(k+)$ for all $k\in\mathbb{Z}$.

With these modifications, propositions 3.6 and 3.7 apply to all those Laplacians. We write $\mathbf{h}^{\mathbb{H}}_{\alpha,\beta}$, $\mathbf{h}^{\mathbb{T}}_{\alpha,\beta}$ and $\mathbf{h}^{\mathbb{R}}_{\alpha,\beta}$ for the respective associated transition kernels.

Proposition 3.11. *Let (X_t) be the process on $\mathsf{HT}(\mathsf{q},\mathsf{p})$ whose infinitesimal generator is $\Delta_{\alpha,\beta}$. Set*

$$
Z_t=\pi^{\mathbb{H}}(X_t)\,,\quad W_t=\pi^{\mathbb{T}}(X_t)\,,\quad \textit{and}\quad Y_t=\pi^{\mathbb{R}}(X_t)\,,\quad t\ge 0.
$$

(a) *The process (Z_t) is a Markov process on $\mathbb{H}$ whose infinitesimal generator is $\Delta^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}$. Its transition kernel with respect to the measure $\mathbf{m}^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}$ is $\mathbf{h}^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}$.*

(b) *The process (W_t) is a Markov process on $\mathbb{T}$ whose infinitesimal generator is $\Delta^{\mathbb{T}}_{\alpha,\beta}$. Its transition kernel with respect to the measure $\mathbf{m}^{\mathbb{T}}_{\alpha,\beta}$ is $\mathbf{h}^{\mathbb{T}}_{\alpha,\beta}$.*

(c) *The process (Y_t) is a Markov process on $\mathbb{R}$ whose infinitesimal generator is $\Delta^{\mathbb{R}}_{\alpha,\beta\mathsf{p}}$. Its transition kernel with respect to the measure $\mathbf{m}^{\mathbb{R}}_{\alpha,\beta\mathsf{p}}$ is $\mathbf{h}^{\mathbb{R}}_{\alpha,\beta\mathsf{p}}$.*

Definition 3.12. For any open domain $\Omega \subset \mathsf{HT}$, we let $\tau^\Omega = \inf\{t > 0 : X_t \in \mathsf{HT} \setminus \Omega\}$ be the *first exit time* of (X_t) from Ω, and if $\tau = \tau^\Omega < \infty$ almost surely for the starting point $X_0 = \mathfrak{w} \in \Omega$, then we write $\mu_{\mathfrak{w}}^\Omega$ for the distribution of X_τ .

The probability measure $\mu_{\mathfrak{w}}^\Omega$ is usually supported by $\partial\Omega$ (we do not specify the meaning of "usually"; for the sets that we are going to consider, this will be true). We shall use analogous notation on $\mathbb{H}$, $\mathbb{T}$ and $\mathbb{R}$. We note that

$$\mu_{\mathfrak{w}}^\Omega(B) = \mu_{\mathfrak{g}\mathfrak{w}}^{\mathfrak{g}\Omega}(\mathfrak{g}B) \quad \text{for every } \mathfrak{g} \in \mathcal{A} \text{ and Borel set } B \subset \mathsf{HT}\,. \tag{3.13}$$

Definition 3.14. Let $\Omega \subset \mathsf{HT}$ be open. A continuous function $f : \Omega \to \mathbb{R}$ is called *harmonic on* Ω if for every open, relatively compact set U with $\overline{U} \subset \Omega$,

$$f(\mathfrak{z}) = \int f\, d\mu_{\mathfrak{z}}^U \quad \text{for all } \mathfrak{z} \in U\,.$$

From the classical analytic viewpoint, this definition may be unsatisfactory; "harmonic" should mean "annihilated by the Laplacian" (as a differential operator). However, for general open domains in HT, the correct formulation in these terms is quite subtle in view of the relative location of the bifurcations.

More details will be stated and used in [5].

4. Brownian motion and the induced random walks

Our basic approach is to study BM on HT via the random walk resulting from observing the processes during its successive visits in the set LT of all bifurcation lines.

Thus, we define the stopping times $\tau(n)$, $n \in \mathbb{N}_0$,

$$\tau(0) = 0, \quad \tau(n+1) = \inf\bigl\{t > \tau(n) : Y_t \in \mathbb{Z} \setminus \{Y_{\tau(n)}\}\bigr\}\,. \tag{4.1}$$

They are not only the times of the successive visits of (Y_t) in $\mathbb{Z}$: by Proposition 3.11, if X_0 lies in some open strip S_v^o, then $\tau(1)$ is the exit time from that open strip, that is, the instant when X_t first meets a point on $\mathsf{L}_v \cup \mathsf{L}_{v^-}$. If $X_{\tau(n)} \in \mathsf{L}_v$ for some $v \in V(\mathbb{T})$ (which holds for all $n \geq 1$, and possibly also for $n = 0$), then $\tau(n+1)$ is the first instant $t > \tau(n)$ when X_t meets one of the bifurcation lines L_{v^-} or L_w with $w^- = v$. The $\tau(n)$ are also the times of the successive visits of (Z_t) in the union of all the lines L_k that subdivide $\mathbb{H}$, as well as the times of the successive visits of (W_t) in the vertex set $V(\mathbb{T})$ of $\mathbb{T}$. Later on, we shall also need the integer random variables $\mathbf{n}_t$, defined by

$$\tau(\mathbf{n}_t) \leq t < \tau(\mathbf{n}_t + 1)\,, \quad \text{where } t \geq 0\,, \tag{4.2}$$

as well as the stopping time

$$\sigma = \inf\bigl\{t > 0 : Y_t \in \{-1, 1\}\bigr\}\,, \quad \text{where} \quad Y_0 = y_0 \in [-1\,,\,1]. \tag{4.3}$$

This is the exit time from $[-1\,,\,1]$. Note that $\sigma = \tau(1)$ when $y_0 = 0$, but not when $0 < |y_0| < 1$.

Lemma 4.4. *For any starting point in $\mathbb{R}$, resp. $[-1\,,\,1]$, the stopping times $\tau(1)$ and σ are almost surely finite.*

Proof. We start with σ. Consider the function $g(y) = \mathsf{Pr}_y[\sigma < \infty]$ on $[-1\,,1]$. It is a weak solution of the Dirichlet problem $\Delta^{\mathbb{R}}_{\alpha,\beta\mathsf{p}} g = 0$ on $(-1\,,\,1)$ with boundary values 1 at ± 1. By Theorem 5.9 in [4] (in a simplified version, because here we are dealing with the infinite line as a metric graph), g is a strong solution. Thus, g satisfies the following "broken" differential equation, where we have to use $\Delta^{\mathbb{R}}_{\alpha,\beta\mathsf{p}}$:

$$\frac{1}{(\log \mathsf{q})^2}\, g'' + \frac{\alpha - 1}{\log \mathsf{q}}\, g' = 0\,, \qquad g'(0-) = \beta\, \mathsf{p}\, g'(0+)\,, \tag{4.5}$$

with $g(\pm 1) = 1$. The unique solution is $g \equiv 1$, whence $\mathsf{Pr}_y[\sigma < \infty] = g(y) = 1$.

Now let us consider $\tau(1)$. By (3.8), the transition density of (Y_t) is invariant under translation by integers. Therefore we may suppose that the starting point is in $[0\,,\,1)$. If it is 0 then $\tau(1) = \sigma$, so we restrict to starting points $y \in (0\,,\,1)$. Set $h(y) = \mathsf{Pr}_y[\tau(1) < \infty]$. Then h satisfies the differential equation

$$\frac{1}{(\log \mathsf{q})^2}\, h'' + \frac{\alpha - 1}{\log \mathsf{q}}\, h' = 0$$

on the interval $(0\,,\,1)$, with boundary values $h(0) = h(1) = 1$. Again, the unique solution is $h \equiv 1$, whence $\mathsf{Pr}_y[\tau(1) < \infty] = 1$. $\square$

We shall need detailed computations regarding the two integer random variables

$$\tau = \tau(2) - \tau(1) \quad \text{and} \quad Y = Y_{\tau(2)} - Y_{\tau(1)},$$

in particular their expected values and variances. Note that Y takes the values ± 1.

Proposition 4.6. (a) *The increments $\tau(n) - \tau(n-1)$, $n \geq 1$, are independent and almost surely finite.*

(b) *They are identically distributed for $n \geq 2$, and when $Z_0 \in \mathsf{LT}$, then also $\tau(1)$ has the same distribution.*

(c) *Let $b = (\alpha - 1)\log \mathsf{q}/2$ and consider the real functions $s(\lambda) = b^2 + (\log \mathsf{q})^2\,\lambda$ and*

$$r(\lambda) = (\beta\,\mathsf{p} + 1)\sum_{n=0}^{\infty} \frac{s(\lambda)^n}{(2n)!} + (\beta\,\mathsf{p} - 1)\, b \sum_{n=0}^{\infty} \frac{s(\lambda)^n}{(2n+1)!}\,, \quad \lambda \in \mathbb{R}\,.$$

Then the random variables Y and τ defined above are independent,

$$\mathsf{E}(e^{-\lambda\tau}\mathbf{1}_{[Y=1]}) = \beta\,\mathsf{p}\, e^{b}/r(\lambda) \quad \textit{and} \quad \mathsf{E}(e^{-\lambda\tau}\mathbf{1}_{[Y=-1]}) = e^{-b}/r(\lambda).$$

(d) *In particular, setting $\mathsf{a} = \beta\,\mathsf{p}\,\mathsf{q}^{\alpha-1}$,*

$$\mathsf{Pr}[Y = 1] = \frac{\mathsf{a}}{\mathsf{a}+1}\,, \quad \mathsf{Pr}[Y = -1] = \frac{1}{\mathsf{a}+1}\,,$$

$$\mathsf{E}(Y) = \frac{\mathsf{a}-1}{\mathsf{a}+1}\,, \quad \textit{and} \quad \mathsf{Var}(Y) = \frac{4\mathsf{a}}{(\mathsf{a}+1)^2}\,.$$

(e) *The Laplace transform* $\lambda \mapsto \mathsf{E}(e^{-\lambda\tau}) = (\mathsf{a}+1)e^{-b}/r(\lambda)$ *is analytic in a neighbourhood of* 0*, so that* τ *has finite exponential moment* $\mathsf{E}(e^{\lambda_0\tau})$ *for some* $\lambda_0 > 0$*.*

(f) *The expectation and variance of* τ *are*

$$\mathsf{E}(\tau) = r'(0)e^b/(\mathsf{a}+1) \quad \textit{and} \quad \mathsf{Var}(\tau) = \mathsf{E}(\tau)^2 - r''(0)e^b/(\mathsf{a}+1).$$

Proof. (a) and (b) are clear.

For (c), we fix $\lambda \ge 0$ and consider again the exit time σ of the process (Y_t) from the interval $[-1\,,\,1]$. We let $f_{\pm1}(y) = \mathsf{E}_y(e^{-\lambda\sigma}\mathbf{1}_{[Y_\sigma=\pm1]})$, respectively, defined for the starting point $Y_0 = y \in [-1\,,\,1]$. We note that $\mathsf{E}(e^{-\lambda\tau}\mathbf{1}_{[Y=\pm1]}) = f_{\pm1}(0)$. Each of the two functions $f_{\pm1}$ is a weak, whence strong ([4], Theorem 5.9) solution of the Dirichlet problem $\Delta^{\mathbb{R}}_{\alpha,\beta\mathsf{p}} f_{\pm1} = \lambda \cdot f_{\pm1}$ on the interval $[-1\,,\,1]$ with boundary values 0 and 1, resp. 1 and 0 at the endpoints -1 and 1. Thus, f_{-1} and f_1 satisfy the "broken" differential equation

$$\frac{1}{(\log \mathsf{q})^2}\, f''_{\pm1} + \frac{\alpha-1}{\log \mathsf{q}}\, f'_{\pm1} = \lambda \cdot f_{\pm1}\,, \qquad f'_{\pm1}(0-) = \beta\mathsf{p}\, f'_{\pm1}(0+)\,,$$
$$f_1(-1) = 0\,, f_1(1) = 1\,, \quad \text{resp.} \quad f_{-1}(-1) = 1\,, f_{-1}(1) = 0.$$

The computation of the solutions is a lengthy, but basic exercise that leads to (c); it may be useful here to note that

$$r(\lambda) = \begin{cases} (\beta\,\mathsf{p}+1)\cosh\sqrt{s(\lambda)} \;+\; (\beta\,\mathsf{p}-1)\,b\,\sinh\sqrt{s(\lambda)}\Big/\sqrt{s(\lambda)}\,, & s(\lambda) \ge 0\,,\\ (\beta\,\mathsf{p}+1)\cos\sqrt{-s(\lambda)} \;+\; (\beta\,\mathsf{p}-1)\,b\,\sin\sqrt{-s(\lambda)}\Big/\sqrt{-s(\lambda)}\,, & s(\lambda) \le 0\,. \end{cases}$$

Statement (d) is obtained by setting $\lambda = 0$ in (c).

A short computation now shows that $\mathsf{E}(e^{-\lambda\tau}\mathbf{1}_{[Y=\pm1]}) = \mathsf{E}(e^{-\lambda\tau})\mathsf{Pr}([Y = \pm1]$ for all $\lambda \ge 0$, which yields independence of Y and τ.

Statement (e) is obvious from the form of the Laplace transform.

Statement (f) is obtained by direct computations of the first and second derivatives of the transform. □

We can compute

$$\mathsf{E}(\tau) = \begin{cases} \dfrac{(\log \mathsf{q})^2}{2b^2}\, \dfrac{(\beta\mathsf{p}-1)b\cosh b + [(\beta\mathsf{p}+1)b - (\beta\mathsf{p}-1)]\sinh b}{(\beta\mathsf{p}+1)\cosh b + (\beta\mathsf{p}-1)\sinh b}\,, & \text{if } \alpha \ne 1\,,\\ \dfrac{(\log \mathsf{q})^2}{2}\,, & \text{if } \alpha = 1\,, \end{cases} \tag{4.7}$$

However, we omit the lengthy formula for $\mathsf{Var}(\tau)$, which can be obtained by tedious computation but provides no specific insight.

The following is obtained by completely similar, but simpler computations. (Namely, we have to solve the same differential equation as above for computing $\mathsf{E}_y(e^{-\lambda\tau(1)})$, but it is not "broken".)

Lemma 4.8. *For any $y \in \mathbb{R}$, there is $\lambda = \lambda(y) > 0$ (depending only on the fractional part of y) such that for the process (Y_t) starting at y*

$$\mathsf{E}_y\big(e^{\lambda\tau(1)}\big) < \infty\,.$$

We now clarify the nature of the induced processes on $\mathbb{Z}$ and on $\mathbb{T}$, respectively.

Corollary 4.9. *With $\mathsf{a} = \beta\,\mathsf{p}\,\mathsf{q}^{\alpha-1}$ as in Proposition 4.6 (d),*

(a) *the process $\big(Y_{\tau(n)}\big)_{n\ge 1}$ is a nearest neighbour random walk on $\mathbb{Z}$ with transition probabilities*

$$p_{\mathbb{Z}}(k,l) = \mathsf{Pr}[Y_{\tau(n+1)} = l \mid Y_{\tau(n)} = k] = \begin{cases} \dfrac{\mathsf{a}}{1+\mathsf{a}}\,, & \text{if } l = k+1\,,\\ \dfrac{1}{1+\mathsf{a}}\,, & \text{if } l = k-1\,,\\ 0\,, & \text{otherwise.}\end{cases}$$

(b) *The process $\big(W_{\tau(n)}\big)_{n\ge 1}$ is a transient nearest neighbour random walk on (the vertex set of) $\mathbb{T}$ with transition probabilities*

$$p_{\mathbb{T}}(v,w) = \mathsf{Pr}[W_{\tau(n+1)} = w \mid W_{\tau(n)} = v] = \begin{cases} \dfrac{\mathsf{a}}{(1+\mathsf{a})\mathsf{p}}\,, & \text{if } w^- = v\,,\\ \dfrac{1}{1+\mathsf{a}}\,, & \text{if } w = v^-\,,\\ 0\,, & \text{otherwise,}\end{cases}$$

where $v, w \in V(\mathbb{T})$.

Proof. (a) is immediate from Proposition 4.6 (d).

Part (b) is an immediate consequence of (a), because for any $v \in V(\mathbb{T})$, we must have $p_{\mathbb{T}}(v, v^-) = p_{\mathbb{Z}}(k, k-1)$, while $p_{\mathbb{T}}(v,w)$ must be the same for all successors w of v, with sum $p_{\mathbb{Z}}(k, k+1)$. It is well-known and easy to prove that this random walk on $V(\mathbb{T})$ is transient (visits any finite set only finitely often a.s.), compare with [15] or [33]. □

The transition kernel of the induced processes on HT, resp. $\mathbb{H}$, cannot be computed as explicitly. We need to consider the non-compact set

$$\Omega_v = \{(z,w) \in \mathsf{HT} : w \in N(v)^o\} \subset \mathsf{HT}\,, \tag{4.10}$$

where $N(v)$ is the "neighbourhood star" in $\mathbb{T}$ at $v \in V(\mathbb{T})$. That is, $N(v)$ is the union of all edges ($\equiv$ intervals!) of $\mathbb{T}$ which have v as one endpoint. It is a compact metric subtree of $\mathbb{T}$, whose boundary $\partial N(v)$ consists of all neighbours of v in $V(\mathbb{T})$. We write $\partial^+ N(v) = \partial N(v) \setminus \{v^-\}$ (the forward neighbours of v).

For any starting point $\mathfrak{w} \in \Omega_v$, the exit time τ^{Ω_v} is almost surely finite by (3.8) and Lemma 4.4. Thus, we have the probability measure $\mu_{\mathfrak{w}}^{\Omega_v}$ on the boundary of Ω_v in HT,

$$\partial\Omega_v = \bigcup_{w \in \partial N(v)} \mathsf{L}_w \,.$$

For $\mathfrak{w} \in \mathsf{L}_v$, this is the transition probability of the Markov chain $(X_{\tau(n)})$ on LT: for any $\mathfrak{w} \in \mathsf{L}_v$ $(v \in V(\mathbb{T}))$ and Borel set $B \subset \partial\Omega_v$,

$$\mathsf{Pr}[X_{\tau(n+1)} \in B \mid X_{\tau(n)} = \mathfrak{w}] = \mu_{\mathfrak{w}}^{\Omega_v}(B)\,. \tag{4.11}$$

Lemma 4.12. *For any $\mathfrak{w} \in \Omega_v$, the measure $\mu_{\mathfrak{w}}^{\Omega_v}$ is supported by whole of the boundary of Ω_v in HT,*

$$\partial\Omega_v = \mathsf{L}_{v^-} \cup \bigcup_{w \in V(\mathbb{T}) : w^- = v} \mathsf{L}_w \,.$$

In particular, the process $(X_{\tau(n)})$ is irreducible on LT: for any starting point $\mathfrak{w} \in \mathsf{LT}$ and any non-empty open interval I that lies on one of the bifurcation lines,

$$\mathsf{Pr}_{\mathfrak{w}}[\exists n : X_{\tau(n)} \in I] > 0\,.$$

Proof. The second statement follows from the first one. The first one follows from ellipticity of $\Delta_{\alpha,\beta}$. More specifically, we can also see this as follows. A boundary point $\mathfrak{z}$ of any open domain $\Omega \subset \mathsf{HT}$ is regular for the Dirichlet problem with respect to $\partial\Omega$ if and only if $\mathsf{Pr}_{\mathfrak{z}}[\tau^{\Omega} = 0] = 1$ (a general fact from potential theory).

Every boundary point of Ω_v *is* regular. This follows from the fact that τ^{Ω_v} is the same as the exit time of the process (W_t) on $\mathbb{T}$ from the neighbourhood star $N(v)^o$. But the Dirichlet problem for the latter (with boundary values at the finitely many neighbours of v in $V(\mathbb{T})$) is obviously solvable, as one can verify by direct, elementary computations similar to those used in the proof of Lemma 4.4.

To conclude, recall that every regular point has to be in the support of the first exit measure. □

We choose the point $\mathfrak{o} = (\mathfrak{i}\,, o) \in \mathsf{HT}$ as the *origin* of treebolic space. Let

$$\mu = \mu_{\mathfrak{o}}^{\Omega}\,, \quad \text{where} \quad \Omega = \Omega_o\,.$$

By group invariance (3.13), we have

$$\mu_{\mathfrak{w}}^{\Omega_v} = \delta_{\mathfrak{g}} * \mu\,, \quad \text{when} \quad \mathfrak{g} \in \mathcal{A}\,,\;\; \mathfrak{g}\mathfrak{o} = \mathfrak{w} \in \mathsf{L}_v\,. \tag{4.13}$$

The convolution of the Dirac measure at $\mathfrak{g}$ with μ is defined via the action of the group $\mathcal{A}$, which is transitive on LT. That is, LT is a homogeneous space of $\mathcal{A}$ (the stabilizer of $\mathfrak{o}$ in $\mathcal{A}$ is a non-trivial compact subgroup), and $(X_{\tau(n)})$ is a random walk on that homogeneous space.

The transition kernel of $(Z_{\tau(n)})$ can be obtained analogously. That process evolves on

$$\mathsf{L}\mathbb{H} = \bigcup_{k \in \mathbb{Z}} \mathsf{L}_k \subset \mathbb{H}.$$

We let

$$\widetilde{\Omega}_k = \big(\mathsf{S}_{k-1} \cup \mathsf{S}_k\big)^o = \pi^{\mathbb{H}}(\Omega_v) \quad \text{for any } v \in H_k \subset V(\mathbb{T})\,.$$

Its boundary within $\mathbb{H}$ is $\partial\widetilde{\Omega}_k = \mathsf{L}_{k-1} \cup \mathsf{L}_{k+1}$. For any starting point $z \in \widetilde{\Omega}_k$, we let $\widetilde{\mu}_z^{\widetilde{\Omega}_k}$ be the exit distribution from $\widetilde{\Omega}_k$. In analogy with Lemma 4.12, it is supported by the whole of $\partial\widetilde{\Omega}_k$, and for any $z \in \mathsf{L}_k$ and Borel set $B \subset \partial\widetilde{\Omega}_k$,

$$\mathsf{Pr}[Z_{\tau(n+1)} \in B \mid Z_{\tau(n)} = z] = \widetilde{\mu}_z^{\widetilde{\Omega}_k}(B)\,. \tag{4.14}$$

We set

$$\widetilde{\mu} = \widetilde{\mu}_{\mathfrak{i}}^{\widetilde{\Omega}}\,, \quad \text{where} \quad \widetilde{\Omega} = \widetilde{\Omega}_0\,. \tag{4.15}$$

This is the image of μ under the projection $\pi^{\mathbb{H}}$. Once more by group invariance (3.13), we have

$$\widetilde{\mu}_z^{\widetilde{\Omega}_k} = \delta_g * \widetilde{\mu}\,, \quad \text{when} \quad g \in \mathsf{Aff}(\mathbb{H}, \mathsf{q})\,, \ \ g\mathfrak{i} = z \in \mathsf{L}_k\,.$$

Now we note that the group $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ acts *simply transitively* on $\mathsf{L}\mathbb{H}$. Indeed, $\mathsf{L}\mathbb{H}$ can be identified with $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ via the homeomorphic one-to-one correspondence

$$z = x + \mathfrak{i}\,\mathsf{q}^k \leftrightarrow g = \begin{pmatrix} \mathsf{q}^k & x \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad g\mathfrak{i} = z\,. \tag{4.16}$$

Thus, group invariance tells us that we can consider the process $(Z_{\tau(n)})$ as the right random walk on $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ with law $\widetilde{\mu}$. In other words, the increments $Z_{\tau(n-1)}^{-1} Z_{\tau(n)}$, $n \geq 2$ (resp. $n \geq 1$, when $Z_0 \in \mathsf{HT}$) are i.i.d. random variables with distribution $\widetilde{\mu}$, when we consider inverses in $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ via the identification (4.16).

Corollary 4.17. *The random walk $(Z_{\tau(n)})$ on $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ is transient.*

Proof. The support of the probability measure $\widetilde{\mu}$ on $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ generates that group as a semigroup, that is, the random walk is irreducible (every open set is reached with positive probability). We know from (2.12) that the group $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$ is non-unimodular. Now, any irreducible random walk on a non-unimodular group must be transient, see [27], or, for a shorter proof, [32]. □

The remainder of this section is dedicated to a study of properties of the probability measures μ on $\partial\Omega = \mathsf{L}_{o^-} \cup \bigcup_{v:v^-=o} \mathsf{L}_v \subset \mathsf{LT}$ and $\widetilde{\mu}$ on $\partial\widetilde{\Omega} = \mathsf{L}_{-1} \cup \mathsf{L}_1 \subset \mathsf{L}\mathbb{H}$, respectively.

An important step is to show that in between two successive times $\tau(n)$ and $\tau(n+1)$, the processes (Z_t) on $\mathbb{H}$ and thus also (X_t) on HT cannot escape too far "sideways" within the the current strip (i.e., the strip to which the process is confined between those two times).

Proposition 4.18. *Suppose (Z_t) starts in $\widetilde{\Omega}$. There is $\rho < 1$ such that for every $n \in \mathbb{N}$, we have for the exit time σ from $\widetilde{\Omega}$*

$$\mathsf{Pr}_{z_0}\big[\max\big\{|\,Re\,Z_t - Re\,z_0| : 0 \leq t \leq \sigma\big\} \geq n\big] \leq 2\,\rho^n \quad \textit{for every} \ \ z_0 \in \widetilde{\Omega}.$$

(Recall again that $\sigma = \tau(1)$ when $z_0 \in \mathsf{L}_0$.)

Proof. By invariance under horizontal translations, we may assume that $Re\, z_0 = 0$.
Consider the vertical segments, resp. open sets,

$$J_n = \{n + \mathfrak{i}\, y : \mathsf{q}^{-1} < y < \mathsf{q}\} \subset \widetilde{\Omega} \quad \text{and} \quad \widetilde{\Omega}^{(n)} = \{z \in \widetilde{\Omega} : Re\, z < n\}\,, \quad n \in \mathbb{Z}\,,$$

so that J_n is the right-hand side boundary of $\widetilde{\Omega}^{(n)}$. For any starting point in $(\mathsf{S}_0 \cup \mathsf{S}_1)^o$, the exit time of (Z_t) from $\widetilde{\Omega}$ is the σ from (4.3), and when $Z_0 \in \mathsf{L}_0$ then $\sigma = \tau(1)$. Analogously, we let $\sigma(n)$ be the exit time of (Z_t) from $\widetilde{\Omega}^{(n)}$.

Now our argument will be as follows: if (Z_t) starts at z_0 and there is some $t \le \tau(1)$ such that $Re\, Z_t \ge n$ then $(Z_t)_{t<\sigma}$ must pass through each J_k, $k = 1, \dots, n$.

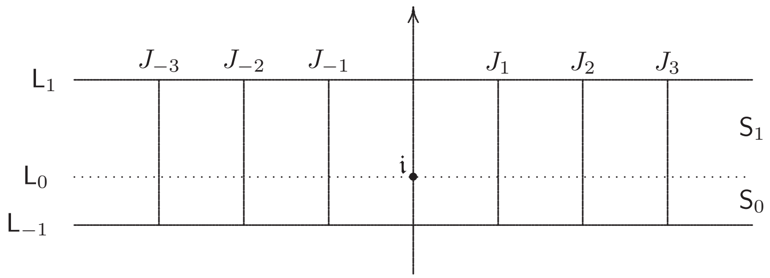


FIGURE 6. The set $\widetilde{\Omega} \subset \mathbb{H}$, subdivided by the bifurcation line L_0 and the vertical segments J_k.

The function $z \mapsto \mathsf{Pr}_z[Z_{\sigma(1)} \in \mathsf{L}_{-1} \cup \mathsf{L}_1]$ is weakly harmonic (harmonic in the sense of distributions) on $\widetilde{\Omega}^{(1)}$, whence strongly harmonic by Theorem 5.9 in [4], and thus continuous. We consider this function on $\bar{J}_0$. At the endpoint of that segment, it is $= 1$, while inside J_0 it is < 1. Thus, there is $z_0 \in J_0$ where our function attains its minimum, and

$$\rho = 1 - \mathsf{Pr}_{z_0}[Z_{\sigma(1)} \in \mathsf{L}_{-1} \cup \mathsf{L}_1] < 1\,.$$

But then

$$\mathsf{Pr}_z[Z_{\sigma(1)} \in J_1] = 1 - \mathsf{Pr}_z[Z_{\sigma(1)} \in \mathsf{L}_{-1} \cup \mathsf{L}_1] \le \rho \quad \text{for every } z \in J_0\,.$$

By invariance under the group $\mathsf{Aff}(\mathbb{H}, \mathsf{q})$, and in particular under translations by reals, we also have for all $k \ge 1$

$$\mathsf{Pr}_z[Z_{\sigma(k)} \in J_k] \le \rho \quad \text{for every } z \in J_{k-1}\,.$$

We now use "balayage" in probabilistic terms. Just for the next lines, consider the measure $\widetilde{\nu}(B) = \mathsf{Pr}_{\mathfrak{i}}[Z_{\sigma(k-1)} \in B]$ for Borel sets $B \subset J_{k-1}$. If $Z_0 = \mathfrak{i}$ and $Z_{\sigma(k)} \in J_k$ then we must have $Z_{\sigma(k-1)} \in J_{k-1}$. Therefore (by the strong Markov property),

$$\begin{aligned}\mathsf{Pr}_{z_0}[Z_{\sigma(k)} \in J_k] = \mathsf{Pr}_{z_0}[Z_{\sigma(k)} \in J_k\,,\ Z_{\sigma(k-1)} \in J_{k-1}] = \int_{J_{k-1}} \mathsf{Pr}_z[Z_{\sigma(k)} \in J_k]\, d\widetilde{\nu}(z)\\ \le \rho \cdot \widetilde{\nu}(J_{k-1}) = \rho \cdot \mathsf{Pr}_{z_0}[Z_{\sigma(k-1)} \in J_{k-1}]\,.\end{aligned}$$

Inductively,

$$\mathsf{Pr}_{z_0}[Z_{\sigma(k)} \in J_k] \le \rho^k \quad \text{for every } k \ge 1\,.$$

If $Z_0 = z_0$ and $Re\, Z_t \ge n$ for some $t \le \sigma$ then a visit to J_n must have occured before time t. That is, $\sigma(n) \le t$, whence

$$\mathsf{Pr}_{z_0}\big[\max\{Re\, Z_t : 0 \le t \le \sigma\} \ge n\big] \le \rho^n\,. \tag{4.19}$$

Now observe that our process is also invariant under the reflection $x + \mathfrak{i}\,y \mapsto -x + \mathfrak{i}\,y$. Therefore

$$\mathsf{Pr}_{\mathfrak{i}}\big[\min\{Re\, Z_t : 0 \le t \le \sigma\} \le -n\big] \le \rho^n\,.$$

The proposed inequality follows. □

Relying again on group invariance (3.8), we deduce the following.

Corollary 4.20. *The random variables*

$$\begin{aligned} D_n &= \max\big\{\mathsf{d}_{\mathsf{HT}}(X_t, X_{\tau(n)}) : \tau(n) \le t \le \tau(n+1)\big\} \\ &= \max\big\{\mathsf{d}_{\mathbb{H}}(Z_t, Z_{\tau(n)}) : \tau(n) \le t \le \tau(n+1)\big\}\,, \quad n \in \mathbb{N}\,, \end{aligned}$$

are i.i.d. and

$$\limsup_{n\to\infty} \frac{D_n}{\log\log n} \le 2 \quad \textit{almost surely.}$$

In particular,

$$\mathsf{E}\big(\exp(\exp(D_n/3))\big) < \infty\,.$$

Proof. It is clear that the D_n are i.i.d. For the purpose of the proofs of this and the next corollary, set

$$M_n = \max\big\{|\, Re\, Z_t - Re\, Z_{\tau(n-1)}| : \tau(n-1) \le t \le \tau(n)\big\}.$$

These random variables are also i.i.d. With ρ as in Proposition 4.18, and for arbitrary $\varepsilon > 0$,

$$\sum_{n=2}^{\infty} \mathsf{Pr}_{\mathfrak{i}}\,[M_n \ge \tfrac{1+\varepsilon}{\log(1/\rho)} \log n] \le 2 \sum_{n=2}^{\infty} \exp\big((\log\rho)\lfloor \tfrac{1+\varepsilon}{\log(1/\rho)} \log n \rfloor\big) \le \frac{2}{\rho} \sum_{n=2}^{\infty} \frac{1}{n^{1+\varepsilon}}\,,$$

which is finite. By the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \frac{M_n}{\log n} \le \frac{1}{\log(1/\rho)} \quad \text{almost surely.}$$

We also see that

$$\mathsf{E}(e^{\lambda_1 M_1}) < \infty \quad \text{for} \quad 0 < \lambda_1 < \log(1/\rho). \tag{4.21}$$

By simple computations with the hyperbolic metric, for any $\mathfrak{z} = (z, w) \in \partial\Omega$, and thus $z \in \partial\widetilde{\Omega} = \mathsf{L}_1 \cup \mathsf{L}_{-1}$, one has

$$\log(1 + |\, Re\, z|^2) - \log \mathsf{q} \le d_{\mathsf{HT}}(\mathfrak{z}, \mathfrak{o}) = d_{\mathbb{H}}(z, \mathfrak{i}\,) \le \log \mathsf{q} + 2\log(1 + |\, Re\, z|). \tag{4.22}$$

Therefore $D_n \le \log \mathsf{q} + 2\log(1+M_n)$, whence as above,

$$\sum_{n=3}^{\infty} \mathsf{Pr}_{\mathfrak{i}}\,[D_n \ge (2+\varepsilon)\log\log n] < \infty$$

for every $\varepsilon > 0$. We get $\limsup D_n/\log\log n \le 2$ a.s. Also, for some $c > 0$, $e^{D_1/3} \le \mathsf{q}^{1/3}(1+M_1)^{2/3} \le c + \lambda_1 M_1$. Now (4.21) yields the doubly exponential moment condition for D_1. □

From the last corollary and (4.22), we also get the following.

Corollary 4.23. *With $\lambda_1 > 0$ as in (4.21),*

$$\int_{\partial\widetilde{\Omega}} \exp\bigl(\lambda_1\, Re\, z\bigr)\, d\widetilde{\mu}(z) < \infty.$$

In particular, μ and $\widetilde{\mu}$ satisfy doubly exponential moment conditions

$$\int_{\partial\widetilde{\Omega}} \exp\bigl(\exp\bigl(d_{\mathbb{H}}(\mathfrak{i}\,, z)/3\bigr)\bigr)\, d\widetilde{\mu}(z) = \int_{\partial\Omega} \exp\bigl(\exp\bigl(d_{\mathsf{HT}}(\mathfrak{o}, \mathfrak{z})/3\bigr)\bigr)\, d\mu(\mathfrak{z}) < \infty.$$

Finally, we anticipate a result from [5] which appears very natural, but whose proof is quite subtle.

Proposition 4.24. *Let $\Omega = \Omega_v$ or $\Omega = \mathsf{S}_v^o \subset \mathsf{HT}$ ($v \in V(\mathbb{T})$). Then for any starting point $\mathfrak{z} \in \Omega$, the exit measure $\mu_{\mathfrak{z}}^{\Omega}$ has a continuous, strictly positive density with respect to Lebesgue measure on the finitely many bifurcation lines that make up $\partial\Omega$.*

The analogous statement holds on "sliced" hyperbolic plane.

5. Rate of escape and convergence to the boundary at infinity

Theorem 5.1. *In the natural metric of HT, the Brownian motion (X_t) on HT generated by $\Delta_{\alpha,\beta}$ has the following rate of escape.*

$$\lim_{t\to\infty} \frac{1}{t}\, \mathsf{d}_{\mathsf{HT}}(X_t, X_0) = |\ell(\alpha,\beta)| \quad \textit{almost surely, where} \quad \ell(\alpha,\beta) = \frac{\log \mathsf{q}}{\mathsf{E}(\tau)}\,\frac{\mathsf{a}-1}{\mathsf{a}+1}\,,$$

with a and $\mathsf{E}(\tau)$ given by Proposition 4.6 (d) and (4.7), respectively.

The proof of this theorem will go hand in hand with the one of Theorem 5.5 below, concering convergence of (X_t) to the boundary.

The tree $\mathbb{T}$ has its natural geometric compactification $\widehat{\mathbb{T}}$ with boundary at infinity $\partial\mathbb{T} = \partial^*\mathbb{T} \cup \{\varpi\}$, see Figure 2. Analogously, the hyperbolic plane $\mathbb{H}$ has its standard hyperbolic compactification $\widehat{\mathbb{H}}$ with boundary $\partial\mathbb{H} = \partial^*\mathbb{H} \cup \{\infty\}$, where $\partial^*\mathbb{H} = \mathbb{R}$, see Figure 3. Since HT is a topological subspace of $\mathbb{H} \times \mathbb{T}$, we can compactify it as follows.

Definition 5.2. The *geometric compactification* $\widehat{\mathsf{HT}}$ of HT is the closure of HT in the compact space $\widehat{\mathbb{H}} \times \widehat{\mathbb{T}}$. The *geometric boundary* at infinity of HT is

$$\partial\,\mathsf{HT} = \widehat{\mathsf{HT}} \setminus \mathsf{HT}\,.$$

The boundary consists of the following five pieces:

$$(5.3)\quad \partial\,\mathsf{HT} = \big(\{\infty\} \times \partial^*\mathbb{T}\big) \cup \big(\partial^*\mathbb{H} \times \{\varpi\}\big) \cup \big(\{\infty\} \times \mathbb{T}\big) \cup \big(\mathbb{H} \times \{\varpi\}\big) \cup \{(\infty, \varpi)\}\,.$$

For a better understanding (and future use), we describe convergence to the boundary.

5.4. Consider a sequence $\mathfrak{z}_n = (z_n, w_n)$ in HT, with $z_n = x_n + \mathfrak{i}\, y_n$.

(a) $\mathfrak{z}_n \to (\infty, \xi) \in \{\infty\} \times \partial^*\mathbb{T}$ if $w_n \to \xi$ in $\widehat{\mathbb{T}}$, in which case necessarily $z_n \to \infty$.

(b) $\mathfrak{z}_n \to (\zeta, \varpi) \in \partial^*\mathbb{H} \times \{\varpi\}$ if $z_n \to \zeta$ in $\widehat{\mathbb{H}}$, that is, $x_n \to \zeta$ and $y_n \to 0$ as sequences in $\mathbb{R}$. In this case necessarily $w_n \to \varpi$.

(c) $\mathfrak{z}_n \to (\infty, w) \in \{\infty\} \times \mathbb{T}$ if $w_n \to w$ in $\mathbb{T}$ and $z_n \to \infty$ in $\widehat{\mathbb{H}}$, that is, $|x_n| \to +\infty$ and $y_n \to \mathsf{q}^{\mathfrak{h}(w)}$ as sequences in $\mathbb{R}$.

(d) $\mathfrak{z}_n \to (z, \varpi) \in \mathbb{H} \times \{\varpi\}$ if $z_n \to z$ in $\mathbb{H}$ and $w_n \to \varpi$ in $\widehat{\mathbb{T}}$, that is, $d(o, w_n \curlywedge o) \to +\infty$ and $\mathfrak{h}(w_n) \to \log_{\mathsf{q}}(Im\, z)$.

(e) $\mathfrak{z}_n \to (\infty, \varpi)$ if $z_n \to \infty$ and $w_n \to \varpi$. In this case, up to passing to a sub-sequence, we may assume in addition that there is $\tau \in [-\infty\,, +\infty]$ such that $\mathfrak{h}(w_n) \to \tau$ and $y_n \to \mathsf{q}^\tau \in [0\,, +\infty]$. (Each value τ can be attained in the limit by some sequence $\mathfrak{z}_n$.)

Theorem 5.5. *In the topology of $\widehat{\mathsf{HT}}$, the Brownian motion $X_t = (Z_t, W_t)$ on HT generated by $\Delta_{\alpha,\beta}$ converges almost surely to a boundary-valued limit random variable $X_\infty = (Z_\infty, W_\infty)$. Writing $\nu_{\mathfrak{z}}$ for its distribution when $X_0 = \mathfrak{z}$, we have the following.*

(i) *If $\ell(\alpha, \beta) > 0$, then $X_\infty \in \{\infty\} \times \partial^*\mathbb{T}$, and all of the latter set is charged by $\nu_{\mathfrak{z}}$.*

(ii) *If $\ell(\alpha, \beta) < 0$, then $X_\infty \in \partial^*\mathbb{H} \times \{\varpi\}$, and all of the latter set is charged by $\nu_{\mathfrak{z}}$.*

(iii) *If $\ell(\alpha, \beta) = 0$, then $X_\infty = (\infty, \varpi)$, a deterministic limit.*

The most useful tool for proving the last two theorems is the notion of regular sequences of Kaimanovich [28], which we formulate here just for hyperbolic plane and tree.

Definition 5.6. Let $\mathbb{X} = \mathbb{H}$ or $\mathbb{X} = \mathbb{T}$. A sequence (z_n) in $\mathbb{X}$ is called regular with *rate* $r \geq 0$ if there is a geodesic ray $(\pi_t)_{t\geq 0}$ in $\mathbb{X}$ (that is, $\mathsf{d}_{\mathbb{X}}(\pi_t, \pi_s) = |t - s|$ for all $s, t \geq 0$) such that

$$\mathsf{d}_{\mathbb{X}}(x_n\,, \pi_{rn})/n \to 0 \quad \text{as } n \to \infty\,.$$

The following was shown in [28].

Lemma 5.7. *A sequence* (z_n) *in* $\mathbb{H}$ *is regular if and only if there is* $b \in \mathbb{R}$ *such that*

$$\log Im(z_n)/n \to b \quad \text{and} \quad \mathsf{d}_{\mathbb{H}}(z_{n+1}, z_n)/n \to 0.$$

In this case, $r = |b|$ *and* $\mathsf{d}_{\mathbb{H}}(z_n, z_0)/n \to r$.

Furthermore, if $b > 0$ *then* $z_n \to \infty$ *in the topology of* $\widehat{\mathbb{H}}$*, while if* $b < 0$ *then there is some* $\zeta \in \partial^*\mathbb{H}$ *such that* $z_n \to \zeta$ *in the topology of* $\widehat{\mathbb{H}}$*. (There is no general statement of this form when* $b = 0$*.)*

The analogue for trees was proved in [15].

Lemma 5.8. *A sequence* (w_n) *in* $\mathbb{T}$ *is regular if and only if there is* $b \in \mathbb{R}$ *such that*

$$\mathfrak{h}(w_n)/n \to b \quad \text{and} \quad \mathsf{d}_{\mathbb{T}}(w_{n+1}, w_n)/n \to 0.$$

In this case, $r = |b|$ *and* $\mathsf{d}_{\mathbb{T}}(w_n, w_0)/n \to r$.

Furthermore, if $b > 0$ *then* $w_n \to \varpi$ *in the topology of* $\widehat{\mathbb{T}}$*, while if* $b < 0$ *then there is some* $\xi \in \partial^*\mathbb{T}$ *such that* $w_n \to \xi$ *in the topology of* $\widehat{\mathbb{T}}$*. (Again, there is no general statement of this form when* $b = 0$*.)*

Before embarking on the proofs of the above two theorems, we also need the following.

Lemma 5.9. $\lim\limits_{t\to\infty} Y_t/t = \ell(\alpha,\beta)/\log \mathsf{q}$ *almost surely, where (recall)* $Y_t = \pi^{\mathbb{R}}(X_t)$.

Proof. Corollary 4.9 and the law of large numbers imply that $\frac{1}{n}Y_{\tau(n)} \to \frac{\mathsf{a}-1}{\mathsf{a}+1}$ almost surely. Again by the law of large numbers, Proposition 4.6 tells us that $\tau(n)/n \to \mathsf{E}(\tau)$ almost surely. Combining these two facts, we get that $Y_{\tau(n)}/\tau(n) \to \frac{\mathsf{a}-1}{\mathsf{a}+1}/\mathsf{E}(\tau)$ almost surely. Given $t > 0$, let the random $\mathbf{n}_t \in \mathbb{N}$ be as in (4.2). Then $\mathbf{n}_t \to \infty$ and $\tau(\mathbf{n}_t)/t \to 1$ almost surely, as $t \to \infty$. By construction, Y_t lies between $Y_{\tau(\mathbf{n}_t)}$ and $Y_{\tau(\mathbf{n}_t+1)}$, which differ by 1. Therefore the almost sure limit

$$\lim_{t\to\infty}\frac{Y_t}{t} = \lim_{t\to\infty}\frac{Y_{\tau(\mathbf{n}_t)}}{t} = \lim_{t\to\infty}\frac{Y_{\tau(\mathbf{n}_t)}}{\tau(\mathbf{n}_t)}\,\frac{\tau(\mathbf{n}_t)}{t}$$

exists and has the proposed value. □

Let us now consider the process (W_t) on $\mathbb{T}$.

Proposition 5.10. *Let* a *be as in Proposition 4.6 (d) and* $\ell(\alpha,\beta)$ *as in Theorem 5.1. Then*

$$\lim_{t\to\infty}\frac{1}{t}\,\mathsf{d}_{\mathbb{T}}(W_t, W_0) = \frac{1}{\log \mathsf{q}}\,|\ell(\alpha,\beta)| \quad \textit{almost surely.}$$

If $\ell(\alpha,\beta) \le 0$ $(\iff \mathsf{a} \le 1)$ *then for any starting point* $w \in \mathbb{T}$,

$$\lim_{t\to\infty} W_t = \varpi \quad \textit{almost surely in the topology of } \widehat{\mathbb{T}}.$$

If $\ell(\alpha,\beta) > 0$ ($\iff \mathsf{a} > 1$) *then there is a* $\partial^*\mathbb{T}$*-valued random variable* W_∞ *such that for any starting point* $w \in \mathbb{T}$*, we have almost surely that*

$$\lim_{t\to\infty} W_t = W_\infty \quad \textit{in the topology of } \widehat{\mathbb{T}}\,.$$

In this case, let $\nu_w^{\partial\mathbb{T}}$ *be the distribution of* W_∞*, given that* $W_0 = w$*. This is a probability measure that has a strictly positive, continuous, bounded density with respect to the "Lebesgue" measure* λ^* *on* $\partial^*\mathbb{T}$ *explained in Remark* 2.23.

Proof. Consider first the random walk $(W_{\tau(n)})$ on $V(\mathbb{T})$. As $\mathsf{d}_{\mathbb{T}}(W_{\tau(n+1)}, W_{\tau(n)})=1$, lemmas 5.9 and 5.8 yield that the sequence $\big(W_{\tau(n)}\big)$ is almost surely regular. We obtain that first of all,

$$\frac{1}{n}\,\mathsf{d}_{\mathbb{T}}(W_{\tau(n)}\,, W_0) \to \left|\frac{\mathsf{a}-1}{\mathsf{a}+1}\right| \quad \text{almost surely.}$$

The proof now proceeds as the one of Lemma 5.9: with $\mathbf{n}_t$ as in (4.2), we have that W_t lies on the edge between $W_{\tau(\mathbf{n}_t)}$ and $W_{\tau(\mathbf{n}_t+1)}$, whence $\mathsf{d}_{\mathbb{T}}(W_t\,, W_{\tau(\mathbf{n}_t)}) \le 1$. Therefore

$$\lim_{t\to\infty}\frac{\mathsf{d}_{\mathbb{T}}(W_t\,, W_0)}{t} = \lim_{t\to\infty}\frac{\mathsf{d}_{\mathbb{T}}(W_{\tau(\mathbf{n}_t)}\,, W_0)}{\mathbf{n}_t}\,\frac{\mathbf{n}_t}{\tau(\mathbf{n}_t)}\,\frac{\tau(\mathbf{n}_t)}{t} = \left|\frac{\mathsf{a}-1}{\mathsf{a}+1}\right|\frac{1}{\mathsf{E}(\tau)}\,,$$

as proposed.

Second, again by Lemma 5.8, $(W_{\tau(n)})$ converges a.s. to ϖ, when $\mathsf{a} < 1$.

When $\mathsf{a} > 1$, it converges a.s. to a $\partial^*\mathbb{T}$-valued random variable W_∞. Using the formulas that are displayed in Proposition 9.23 of [34], one can compute the limit distribution $\nu_v^{\partial\mathbb{T}}$ of that random walk, when $W_0 = v \in V(\mathbb{T})$. Explicit computations can be found in [33]. One sees that $\nu_v^{\partial T}$ has the stated properties. In particular, it carries no point mass and is supported by the whole of $\partial^*\mathbb{T}$ (or equivalently, $\partial\mathbb{T}$).

If we replace the starting point $v \in V(\mathbb{T})$ by a starting point w that lies in the interior of some edge $[v^-, v]$ then the process starting at w also must converge to $\partial^*\mathbb{T}$, and we have

$$\nu_w^{\partial\mathbb{T}} = \mathsf{Pr}[W_{\tau(1)} = v \mid W_0 = w]\,\nu_v^{\partial\mathbb{T}} + \mathsf{Pr}[W_{\tau(1)} = v^- \mid W_0 = w]\,\nu_{v^-}^{\partial\mathbb{T}}\,.$$

We still have to show that $W_t \to \varpi$ almost surely, when $\mathsf{a} = 1$. This is obtained by the following simple argument. Being a transient nearest neighbour random walk, $(W_{\tau(n)})$ must converge almost surely to some random end of $\mathbb{T}$, see Theorem 9.18 in [34]. But the projection $\mathfrak{h}(W_{\tau(n)}) = Y_{\tau(n)}$ is a recurrent random walk on $\mathbb{Z}$, when $\mathsf{a} = 1$. Thus, there is a random subsequence (n') along which $\mathfrak{h}(W_{\tau(n')}) = 0$. This subsequence must converge to ϖ, whence ϖ is the limit of the entire sequence. □

In fact, the last proposition provides the simplest class of cases to which the results of [15] apply (but explaining how to apply those general results would consume more space and energy than the above direct arguments.) We next want to present the analogous proposition concerning the process (Z_t) on $\mathbb{H}$. Recall that we can interpret the random walk $(Z_{\tau(n)})$ on $\mathsf{L}\mathbb{H}$ as a right random walk on the group $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ which is identified with $\mathsf{L}\mathbb{H}$ via (4.16). With this identification, the law of that random walk is the probability $\widetilde{\mu}$ of (4.15). We know that in the notation of the group operation, the increments $Z_{\tau(n-1)}^{-1}Z_{\tau(n)}$, $n \geq 2$, are i.i.d. $\mathsf{Aff}(\mathbb{H},\mathsf{q})$-valued random variables with common distribution $\widetilde{\mu}$, so that they can be written as random affine transformations $\left(\begin{smallmatrix} A_n & B_n \\ 0 & 1 \end{smallmatrix}\right)$, where $A_n = \mathsf{q}^{Y_{\tau(n)}-Y_{\tau(n-1)}}$; the associated transformation of $\mathbb{H}$ is $z \mapsto A_n z + B_n$. While A_n only takes the two values q and $1/\mathsf{q}$, the common distribution of the real random variables B_n has a continuous density with respect to Lebesgue measure by Proposition 4.24. By Corollary 4.23, B_n satisfies an exponential moment condition.

Proposition 5.11. $\displaystyle\lim_{t\to\infty} \frac{1}{t}\,\mathsf{d}_{\mathbb{H}}(Z_t\,,Z_0) = |\ell(\alpha,\beta)|$ *almost surely.*

If $\ell(\alpha,\beta) \geq 0$ $(\iff \mathsf{a} \geq 1)$ *then for any starting point* $z \in \mathbb{H}$, *we have almost surely that*

$$\lim_{t\to\infty} Z_t = \infty \quad \textit{almost surely in the topology of } \widehat{\mathbb{H}}\,.$$

If $\ell(\alpha,\beta) < 0$ $(\iff \mathsf{a} < 1)$ *then there is a random variable* Z_∞ *taking values in* $\partial^*\mathbb{H} = \mathbb{R}$ *such that for any starting point* $z \in \mathbb{H}$, *we have almost surely that*

$$\lim_{t\to\infty} Z_t = Z_\infty \quad \textit{in the topology of } \widehat{\mathbb{H}}\,.$$

In this case, let $\nu_z^{\partial\mathbb{H}}$ *be the distribution of* Z_∞ , *given that* $Z_0 = z$. *This is a probability measure on* $\partial^*\mathbb{H} \equiv \mathbb{R}$ *that has a continuous, strictly positive density with respect to Lebesgue measure.*

Proof. By Corollary 4.20,

$$\frac{1}{n}\,\mathsf{d}_{\mathbb{H}}(Z_{\tau(n+1)}\,,Z_{\tau(n)})/n \to 0 \quad \text{almost surely.}$$

By Lemma 5.9,

$$\frac{1}{n}\log Im(Z_{\tau(n)}) = \frac{\log\mathsf{q}}{n}\,Y_{\tau(n)} \to \log\mathsf{q}\,\frac{\mathsf{a}-1}{\mathsf{a}+1} \quad \text{almost surely.} \tag{5.12}$$

Thus, by Lemma 5.7, the sequence $(Z_{\tau(n)})$ is almost surely regular in $\mathbb{H}$, with rate $\log\mathsf{q}\left|\frac{\mathsf{a}-1}{\mathsf{a}+1}\right|$. When $\mathsf{a} > 1$ it converges to ∞ in the topology of $\widehat{\mathbb{H}}$, while when $\mathsf{a} < 1$, it converges in that topology to a random element of $\partial^*\mathbb{H}$.

The more difficult situation is the one where the rate of the sequence is 0. In that case, it was proved by Brofferio [10] that $Z_{\tau(n)} \to \infty$ almost surely in the topology of $\widehat{\mathbb{H}}$. This is not yet enough to guarantee that also $Z_t \to \infty$ almost surely. We take inspiration from [10]. For any $g \in \mathsf{Aff}(\mathbb{H},\mathsf{q})$, a neighbourhood base of ∞ in $\widehat{\mathbb{H}}$ is given by the collection of all sets $\widehat{\mathbb{H}} \setminus g^{-1}V_r$, where

$$V_r = \{z = x + \mathsf{i}\,y : |x| \leq r \text{ and } 0 \leq y \leq \mathsf{q}^r\}\,, \quad r \in \mathbb{N}\,.$$

Our argument will not depend on the starting point, but only on what happens from time $\tau(1)$ onwards. Thus, we may assume that $Z_0 \in \mathsf{L}\mathbb{H}$, which can be identified with $\mathsf{Aff}(\mathbb{H},\mathsf{q})$. We know from [10] that for any r and for any starting point in $\mathsf{L}\mathbb{H}$, we have almost surely that $Z_{\tau(n)} \in \mathbb{H} \setminus V_r$ for all but finitely many n. Equivalently, for starting point $\mathfrak{i}$ and for some $g \in \mathsf{Aff}(\mathbb{H},\mathsf{q})$, for any r we have $Z_{\tau(n)} \in \mathbb{H} \setminus g^{-1}V_r$ for all but finitely many n .

Thus, we need an element $g \in \mathsf{Aff}(\mathbb{H},\mathsf{q})$ such that with probability 1, in between the times $\tau(n)$ and $\tau(n+1)$, the process (X_t) does not enter into $g^{-1}V_r$, if n is sufficiently large. This will follow from the Borel–Cantelli lemma after showing that

$$\sum_{n=1}^{\infty} \mathsf{Pr}_{\mathfrak{i}} \begin{bmatrix} Z_{\tau(n)} \in \mathbb{H} \setminus g^{-1}V_r \,, \;\; Z_t \in g^{-1}V_r \\ \text{for some } t \text{ with } \tau(n) < t < \tau(n+1) \end{bmatrix} < \infty\,. \tag{5.13}$$

Again, we use the identification (4.16) of $\mathsf{L}\mathbb{H}$ with $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ and consider the potential measure $\mathcal{U} = \sum_{n=0}^{\infty} \widetilde{\mu}^{(n)}$, where $\widetilde{\mu}^{(n)}$ is the nth convolution power of the measure $\widetilde{\mu}$ on $\mathsf{Aff}(\mathbb{H},\mathsf{q})$. By transience of $(Z_{\tau(n)})$, this $\mathcal{U}$ is a Radon measure on $\mathsf{Aff}(\mathbb{H},\mathsf{q}) \equiv \mathsf{L}\mathbb{H}$. For $z \in \mathsf{L}\mathbb{H}$, let

$$f_r(z) = \mathbf{1}_{\mathsf{L}\mathbb{H}\setminus V_r}(z)\,\mathsf{Pr}_z[Z_t \in V_r \text{ for some } t \text{ with } 0 < t < \tau(1)].$$

Then for any $g \in \mathsf{Aff}(\mathbb{H},\mathsf{q})$,

$$\sum_{n=1}^{\infty} \mathsf{Pr}_{\mathfrak{i}} \begin{bmatrix} Z_{\tau(n)} \in \mathbb{H} \setminus g^{-1}V_r \,, \;\; Z_t \in g^{-1}V_r \\ \text{for some } t \text{ with } \tau(n) < t < \tau(n+1) \end{bmatrix} = \int_{\mathsf{L}\mathbb{H}} f_r(gz)\,d\mathcal{U}(z)\,.$$

Let $z = b + \mathfrak{i}\,\mathsf{q}^m \in \mathsf{L}\mathbb{H} \setminus V_r$, with $m \in \mathbb{Z}$ and $b \in \mathbb{R}$. Write $z = g_z\mathfrak{i}$, where $g_z = \left(\begin{smallmatrix} \mathsf{p}^m & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathsf{Aff}(\mathbb{H},\mathsf{q})$. Then

$$f_r(z) = \mathbf{1}_{\mathsf{L}\mathbb{H}\setminus g_z^{-1}V_r}(\mathfrak{i})\,\mathsf{Pr}_{\mathfrak{i}}\,[Z_t \in g_z^{-1}V_r \text{ for some } t \text{ with } 0 < t < \tau(1)].$$

We have

$$g_z^{-1}V_r = \{x + \mathfrak{i}\,y : |x + \mathsf{q}^{-m}b| \le \mathsf{q}^{-m}r \text{ and } 0 \le y \le \mathsf{q}^{r-m}\}.$$

We must have $\mathfrak{i} \in \mathsf{L}\mathbb{H}\setminus g_z^{-1}V_r$. Starting at $\mathfrak{i}$, the process (Z_t) does not leave $\mathsf{S}_0 \cup \mathsf{S}_1$ before time $\tau(1)$. Compare with Figure 6. Thus, in order to be able to enter into $g_z^{-1}V_r$ before that time, we must have $r - m \ge 0$; otherwise $f_r(z) = 0$.

Suppose that we do have $r - m \ge 0$, and that $\mathfrak{i}$ stays to the left of $g_z^{-1}V_r$, so that $-r - b > 0$. Then in order to enter into $g_z^{-1}V_r$ before $\tau(1)$, the process must cross the vertical line where $x = -\mathsf{q}^{-m}(r + b)$. Setting $k = \lfloor -\mathsf{q}^{-m}(r + b)\rfloor$ (next lower integer), this means that Z_t must pass through the segment J_k of Figure 6. By Proposition 4.18, resp. (4.19) in its proof, $f_r(z) \le \rho^k$. Analogously, if $\mathfrak{i}$ stays to the right of $g_z^{-1}V_r$, which means that $r - b < 0$, then $f_r(z) \le \rho^k$, where $k = \lfloor \mathsf{q}^{-m}(b - r)\rfloor$. Setting $\lambda = -\log \rho$, we find that

$$f_r(b + \mathfrak{i}\,\mathsf{q}^m) \begin{cases} = 0\,, & \text{if } m > r \text{ or } |b| < r \\ \le \exp\Bigl(-\lambda\bigl(\mathsf{q}^{-m}(|b| - r) + 1\bigr)\Bigr) & \text{if } m \le r \text{ and } |b| \ge r. \end{cases}$$

The *right* Haar measure on $\mathsf{Aff}(\mathbb{H},\mathsf{q}) \equiv \mathsf{L}\mathbb{H}$ is one-dimensional Lebesgue measure on each of the lines L_k , compare with (2.12). Thus, the integral of f_r with respect to right Haar measure is

$$\sum_{m\le r}\int_{|b|\ge r} f_r(b+\mathrm{i}\,\mathsf{q}^m)\,db < \infty\,.$$

Lemma 1 in [10] yields that in this case, $\int_{\mathsf{L}\mathbb{H}} f_r(gz)\,d\mathcal{U}(z) < \infty$ for dg-almost all $g \in \mathsf{Aff}(\mathbb{H},\mathsf{q})$. This is true for all $r \in \mathbb{N}$. Thus, there is some fixed $g \in \mathsf{Aff}(\mathbb{H},\mathsf{q})$ such that the last integral is finite for every $r \in \mathbb{N}$. For this g, (5.13) holds for every $r \in \mathbb{N}$, so that $Z_t \to \boldsymbol{\infty}$ almost surely.

We finally have to explain that in the case $\mathsf{a} < 1$, the limit random variable on $\partial^*\mathbb{H}$ has a distribution with continuous, positive density with respect to Lebesgue measure.

Let us write $Z_{\tau(1)} = \left(\begin{smallmatrix} A_1 & B_1\\ 0 & 1\end{smallmatrix}\right)$, which is independent of the other $\left(\begin{smallmatrix} A_n & B_n\\ 0 & 1\end{smallmatrix}\right)$ but does in general not have the same distribution. We know from Proposition 4.24 that for arbitrary starting point $z \in \mathbb{H}$, the distribution of B_1 has a continuous density with respect to Lebesgue measure. We then have

$$Z_{\tau(n)} = \begin{pmatrix} A_1\cdots A_n & \sum_{k=1}^n A_1\cdots A_{k-1}B_k\\ 0 & 1\end{pmatrix}.$$

When $\mathsf{a} < 1$, it is very well known and quite easy to verify that in $\mathbb{R}$, the upper right matrix element of $Z_{\tau(n)}$ converges almost surely to

$$Z_\infty = \sum_{k=1}^\infty A_1\cdots A_{k-1}B_k\,,$$

as $n\to\infty$. Recalling the identification (4.16), we see that this is the limit of $Z_{\tau(n)}$ in $\widehat{\mathbb{H}}$, since $A_1\cdots A_n \to 0$ almost surely. It is now easy to verify that along with all the B_n (including B_1), for arbitrary starting point also the distribution of Z_∞ has a continuous density with respect to Lebesgue measure on $\mathbb{R}$. □

Remark 5.14. Our result on almost sure convergence of (Z_t) to $\boldsymbol{\infty}$ in the critical case $\mathsf{a} = 1$ also applies to Brownian motion with vertical drift on $\mathbb{H}$ without any bifurcation lines. Indeed, this corresponds just to the case when $\beta\mathsf{p} = 1$. This closes a small gap left open in the proof of Proposition 4.2 (iii) in [11], concerning the passage from discrete to continuous time.

Proof of Theorems 5.1 and 5.5. Theorem 5.1 regarding the rate of escape of (X_t) now follows by combining the inequalities of Proposition 2.8 with the rates of escape of (Y_t), (W_t) and (Z_t), as provided by Lemma 5.9 and Propositions 5.10 and 5.11, respectively.

Theorem 5.5 follows by combining those two propositions with the description (5.4) of convergence to the boundary in treebolic space. □

Theorem 5.5 provides the following, which (as mentioned) was only indicated in [4].

Corollary 5.15. *The processes* (X_t) *on* HT, (Z_t) *on* $\mathbb{H}$ *and* (W_t) *on* $\mathbb{T}_{\mathsf{p}}$ $(\mathsf{p} \geq 2)$, *as defined in Proposition 3.11, are transient.*

6. Central limit theorem

The proof of a CLT for $\mathsf{d}(X_t, X_0)$ $(t \to \infty)$ depends significantly on the sign of the drift $\ell(\alpha,\beta)$. It will follow from the CLT for the random walk $(X_{\tau(n)})$. Here we shall work with $\mathsf{d}(X_t, \mathfrak{o})$ instead of $\mathsf{d}(X_t, X_0)$, which makes no difference, as we divide by $\sqrt{t}$. In any case, before that we need the CLT for the vertical component Y_t of X_t.

Lemma 6.1. *With* $\mathsf{Var}(Y)$ *and* $\mathsf{Var}(\tau)$ *as in Proposition 4.6, set*

$$\sigma^2 = \sigma^2(\alpha,\beta) = \frac{1}{\mathsf{E}(\tau)}\,\mathsf{Var}(Y) + \frac{\ell(\alpha,\beta)^2}{\mathsf{E}(\tau)\log^2\mathsf{q}}\,\mathsf{Var}(\tau).$$

Then

$$\frac{1}{\sqrt{t}}\Big(Y_t - t\,\frac{\ell(\alpha,\beta)}{\log\mathsf{q}}\Big) \to N(0,\sigma^2) \quad \textit{in law, as } t \to \infty\,.$$

Proof. The $\mathbb{R}^2$-valued random variables $\big(Y_{\tau(n)} - Y_{\tau(n-1)}\,,\tau(n) - \tau(n-1)\big)_{n\geq 2}$ are i.i.d., see Proposition 4.6. By the two-dimensional CLT,

$$\frac{1}{\sqrt{n}}\left(Y_{\tau(n)} - n\,\frac{\mathsf{a}-1}{\mathsf{a}+1}\,,\,\tau(n) - n\mathsf{E}(\tau)\right) \to N(0,\Sigma^2) \quad \text{in law,} \tag{6.2}$$

where $\mathsf{E}(\tau)$ is as in Proposition 4.6 (e) and $N(0,\Sigma^2)$ is the two-dimensional normal distribution with mean vector 0 and Σ^2 is the covariance matrix of $\big(Y_{\tau(2)} - Y_{\tau(1)}\,,\tau(2)-\tau(1)\big)$, which is just the diagonal matrix with diagonal entries $\mathsf{Var}(Y)$ and $\mathsf{Var}(\tau)$.

As in the proof of Lemma 5.9, with the $\mathbf{n}_t$ of (4.2), we know that

$$\frac{\mathbf{n}_t}{t} = \frac{\mathbf{n}_t}{\tau(\mathbf{n}_t)}\,\frac{\tau(\mathbf{n}_t)}{t} \to \frac{1}{\mathsf{E}(\tau)} \quad \text{almost surely, as } t \to \infty\,, \tag{6.3}$$

and that $\big|Y_t - Y_{\tau(\mathbf{n}_t)}\big| < 1$. Now we decompose

$$\begin{aligned}\frac{Y_t - t\,\frac{\ell(\alpha,\beta)}{\log\mathsf{q}}}{\sqrt{t}} &= \frac{Y_t - Y_{\tau(\mathbf{n}_t)}}{\sqrt{t}} + \sqrt{\frac{\mathbf{n}_t}{t}}\cdot\frac{Y_{\tau(\mathbf{n}_t)} - \mathbf{n}_t\,\frac{\mathsf{a}-1}{\mathsf{a}+1}}{\sqrt{\mathbf{n}_t}}\\ &\quad - \frac{\ell(\alpha,\beta)}{\log\mathsf{q}}\cdot\frac{t-\tau(\mathbf{n}_t)}{\sqrt{t}} - \sqrt{\frac{\mathbf{n}_t}{t}}\,\frac{\ell(\alpha,\beta)}{\log\mathsf{q}}\cdot\frac{\tau(\mathbf{n}_t) - \mathbf{n}_t\mathsf{E}(\tau)}{\sqrt{\mathbf{n}_t}}\,.\end{aligned}$$

The first term of the sum on the right-hand side tends to 0 because $0 \le Y_t - Y_{\tau(\mathbf{n}_t)} < 1$ almost surely. The third term tends to 0 almost surely, because

$$\frac{t-\tau(\mathbf{n}_t)}{\sqrt{t}} \le \frac{\tau(\mathbf{n}_t+1)-\tau(\mathbf{n}_t)}{\sqrt{\mathbf{n}_t}}\sqrt{\frac{\mathbf{n}_t}{t}},$$

and $\big(\tau(n+1)-\tau(n)\big)/\sqrt{n} \to 0$ by Proposition 4.6 (d). Also, we know that $\mathbf{n}_t/t \to 1/\mathsf{E}(\tau)$ almost surely. Hence,

$$\frac{Y_t - t\,\frac{\ell(\alpha,\beta)}{\log \mathsf{q}}}{\sqrt{t}} \overset{\text{in law}}{\sim} \frac{1}{\sqrt{\mathsf{E}(\tau)}}\cdot\frac{Y_{\tau(\mathbf{n}_t)} - \mathbf{n}_t\,\frac{\mathsf{a}-1}{\mathsf{a}+1}}{\sqrt{\mathbf{n}_t}} - \frac{\ell(\alpha,\beta)}{\sqrt{\mathsf{E}(\tau)}\log \mathsf{q}}\cdot\frac{\tau(\mathbf{n}_t)-\mathbf{n}_t\mathsf{E}(\tau)}{\sqrt{\mathbf{n}_t}}$$

as $t\to\infty$. It follows from (6.2) that this converges in law to the centred normal distribution with variance $\sigma^2(\alpha,\beta)$, as proposed. □

Lemma 6.4. (a) *If* $\ell(\alpha,\beta)>0$, *then*

$$\limsup_{t\to\infty}\big(\mathsf{d}_{\mathsf{HT}}(X_t\,,o) - \mathsf{d}_{\mathbb{H}}(Z_t\,,\mathfrak{i})\big) < \infty \quad \textit{almost surely.}$$

(b) *If* $\ell(\alpha,\beta)<0$, *then*

$$\limsup_{t\to\infty}\big(\mathsf{d}_{\mathsf{HT}}(X_t\,,\mathfrak{o}) - (\log \mathsf{q})\mathsf{d}_{\mathbb{T}}(W_t,o)\big) < \infty \quad \textit{almost surely.}$$

(*The two appearing differences are always non-negative.*)

Proof. (a) By Proposition 5.10, $W_t \to W_\infty \in \partial^*\mathbb{T}$ almost surely. Therefore $\mathfrak{h}(o \curlywedge W_t) \to \mathfrak{h}(o \curlywedge W_\infty)$ a.s., that is, the two (finite!) random variables coincide from some random t_0 onwards, and in particular $\mathfrak{h}(W_t) = Y_t \ge 0$ for $t \ge t_0$. By (2.2),

$$\mathsf{d}_{\mathbb{T}}(W_t,o) = \mathfrak{h}(W_t) - 2\,\mathfrak{h}(o\curlywedge W_t) = Y_t - 2\,\mathfrak{h}(o\curlywedge W_\infty) \quad \text{for all } t\ge t_0\,,$$

and for those t, the first inequality of Proposition 2.8 yields

$$\mathsf{d}_{\mathsf{HT}}(X_t\,,\mathfrak{o}) \le \mathsf{d}_{\mathbb{H}}(Z_t\,,\mathfrak{i}) - 2\,(\log \mathsf{q})\,\mathfrak{h}(o\curlywedge W_\infty).$$

(We note here that $\mathfrak{h}(o\curlywedge W_\infty)\le 0$.) This yields (a).

(b) This time, we use Proposition 5.11 and get that $\mathfrak{i}\wedge Z_t \to \mathfrak{i}\wedge Z_\infty \in \mathbb{H}$ (a.s. convergence in $\widehat{\mathbb{H}}$). Therefore, by (2.3),

$$\limsup_{t\to\infty}\Big|\mathsf{d}_{\mathbb{H}}(Z_t\,,\mathfrak{i}) - \Big(2\log\big(\mathit{Im}(\mathfrak{i}\wedge Z_\infty)\big) - \log \mathit{Im}\,Z_t\Big)\Big| < \infty \quad \text{almost surely.}$$

Note that $\log \mathit{Im}\,Z_t < 0$ if t is sufficiently large. Thus, in the same way as in (a), Proposition 2.8 yields statement (b). □

We now consider $(X_{\tau(n)})$. The group $\mathcal{A}$ acts transitively on the set LT defined in (3.2) of all bifurcation lines in HT. In part (2) of the proof of Theorem 2.15, we have introduced the coordinates $[b,\gamma]$ for the elements of $\mathcal{A}$. In the same way, it will be useful to use coordinates $[x,v]$ for the elements of LT, such that $[x,v]$ is the point on L_v with horizontal coordinate x, that is, $[x,v] = (x + \mathfrak{i}\,\mathsf{q}^{\mathfrak{h}(v)}, v)$ in the notation of (2.4). In these coordinates, the natural $\mathcal{A}$-invariant measure on LT is given by $d[x,v] = q^{-\mathfrak{h}(v)} dx\, d_\sharp v$, where dx is standard Lebesgue measure and $d_\sharp v$ is the counting measure on $V(\mathbb{T})$.

By (natural) abuse of notation, we also write $\pi^{\mathbb{H}}$ and $\pi^{\mathbb{T}}$ for the projections $(g,\gamma) \mapsto g$ and $(g,\gamma) \mapsto \gamma$ from $\mathcal{A}$ onto $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ and $\mathsf{Aff}(\mathbb{T})$, respectively. This refers to the notation used in the statement of Theorem 2.15, while in the $[b,\gamma]$-coordinates, $\pi^{\mathbb{T}}[b,\gamma] = \gamma$ and $\pi^{\mathbb{H}}[b,\gamma] = \begin{pmatrix} \mathsf{q}^{\Phi(\gamma)} & b \\ 0 & 1 \end{pmatrix}$ as an affine transformation.

Now recall from §4 that $(X_{\tau(n)})$ is a Markov chain on LT whose transition probabilities are given by the probability measure μ, see (4.13). By Proposition 4.24, μ has a continuous density, which we denote by f_μ, with respect to $d[x,v]$. The projection $\widetilde{\mu}$ also has a continuous density on $\mathsf{L}\mathbb{H}$ with respect to the $\mathsf{Aff}(\mathbb{H},\mathsf{q})$-invariant measure which is analogous to $d[x,v]$. Furthermore, we note that for $v \in V(\mathbb{T})$,

$$\int_{\mathbb{R}} f_\mu[x,v]\, dx = p(o,v)\,,$$

the transition probabilities of $(W_{\tau(n)})$ that appeared in Corollary 4.9 (b). We now lift μ to a probability measure $\boldsymbol{\mu}$ on the group $\mathcal{A}$: it has density $\mathbf{f}$ with respect to the Haar measure (2.17) on $\mathcal{A}$, where $\mathbf{f}[b,\gamma] = f_\mu[b,\gamma o]$.

We then can construct (on a suitable probability space) a sequence $(\boldsymbol{X}_n)_{n\ge1}$ of i.i.d. $\mathcal{A}$-valued random variables with common distribution $\boldsymbol{\mu}$, and the associated right random walk on $\mathcal{A}$,

$$\boldsymbol{R}_n = \boldsymbol{X}_1\,\boldsymbol{X}_2 \cdots \boldsymbol{X}_n\,,\; n \ge 0\,.$$

The product is of course taken in the group $\mathcal{A}$, and $\boldsymbol{R}_0$ is the identity of that group. The (simple) proof of part (i) of the following lemma is omitted; it follows from Lemma 3.1 in [31] (see also [27], Remarque 6, p. 5). Statements (ii)–(iv) are immediate consequences.

Lemma 6.5. (i) *For any $\mathfrak{g} \in \mathcal{A}$, the sequence $(\mathfrak{g}\boldsymbol{R}_n\mathfrak{o})$ is a realisation of the induced random walk $(X_{\tau(n)})_{n\ge0}$ on LT starting at $\mathfrak{g}\mathfrak{o}$. That is, it is an LT-valued Markov chain with transition probabilities (4.11).*

(ii) *Via the identification (4.16) of $\mathsf{L}\mathbb{H}$ with $\mathsf{Aff}(\mathbb{H},\mathsf{q})$, the random walk $\pi^{\mathbb{H}}(\boldsymbol{R}_n)$ is a realisation of the process $(Z_{\tau(n)})$ on $\mathsf{L}\mathbb{H}$ starting at $\mathfrak{i}$.*

(iii) *$R_n = \pi^{\mathbb{T}}(\boldsymbol{R}_n)$ is a right random walk on the group $\mathsf{Aff}(\mathbb{T})$, and the process $(R_n o)_{n\ge0}$ is a realisation of the random walk $(W_{\tau(n)})_{n\ge0}$ on (the vertex set of) $\mathbb{T}$ as described in Corollary 4.9 (b), with $R_0 o = o$.*

(iv) *With Φ as in (2.14), the sequence $\big(\Phi(R_n)\big)_{n\ge0}$ is a realisation of the random walk $(Y_{\tau(n)})_{n\ge0}$ on $\mathbb{Z}$ as described in Corollary 4.9 (a), with starting point 0.*

Theorem 6.6. *If $\ell(\alpha,\beta) \ne 0$ and σ^2 is as in Lemma 6.1,*

$$\frac{1}{\sqrt{t}}\big(\mathsf{d}_{\mathsf{HT}}(X_t\,,\mathfrak{o}) - t\,|\ell(\alpha,\beta)|\big) \to N(0,\sigma^2) \quad \textit{in law, as } t \to \infty\,.$$

Proof. Case 1. $\ell(\alpha,\beta) > 0$.

Lemma 6.4 (a) tells us that we just have to consider $\mathsf{d}_{\mathbb{H}}(Z_t\,,\mathfrak{i}\,)$. Using the same notation as before Proposition 5.11, we write $Z_{\tau(n)}^{-1}Z_{\tau(n)} = \left(\begin{smallmatrix} A_n & B_n \\ 0 & 1 \end{smallmatrix}\right)$ as independent $\widetilde{\mu}$-distributed group elements of $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ for $n \geq 2$, as well as $Z_{\tau(1)} = \left(\begin{smallmatrix} A_1 & B_1 \\ 0 & 1 \end{smallmatrix}\right)$, which is independent of the other ones (but may have a different distribution, according to the starting point). The group inverses are $\left(\begin{smallmatrix} A_n & B_n \\ 0 & 1 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 1/A_n & -B_n/A_n \\ 0 & 1 \end{smallmatrix}\right)$. Since A_n only takes values q and $1/\mathsf{q}$, also $-B_n/A_n$ has exponential moments. Taking products in that group, $Z_{\tau(n)} = \left(\begin{smallmatrix} A_1 & B_1 \\ 0 & 1 \end{smallmatrix}\right)\cdots\left(\begin{smallmatrix} A_n & B_n \\ 0 & 1 \end{smallmatrix}\right)$, so that

$$Z_{\tau(n)}^{-1} = \begin{pmatrix} A_n & B_n \\ 0 & 1 \end{pmatrix}^{-1}\cdots\begin{pmatrix} A_2 & B_2 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} A_1 & B_1 \\ 0 & 1 \end{pmatrix}^{-1} \overset{\text{in law}}{=} \underbrace{\begin{pmatrix} A_2 & B_2 \\ 0 & 1 \end{pmatrix}^{-1}\cdots\begin{pmatrix} A_n & B_n \\ 0 & 1 \end{pmatrix}^{-1}}_{=:\, Z^*_{\tau(n)}}\begin{pmatrix} A_1 & B_1 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Note that in $\mathbb{R}$, we have $A_1\cdots A_n = \mathsf{q}^{Y_{\tau(n)}}$. Now $(Z^*_{\tau_n})$ is again a right random walk on $\mathsf{Aff}(\mathbb{H},\mathsf{q})$, and returning to the identification with $\mathsf{L}\mathbb{H}$, we have that

$$\mathsf{d}_{\mathbb{H}}(Z^*_{\tau(n)}, Z^*_{\tau(n-1)})/n \to 0 \quad \text{and}$$
$$\frac{1}{n}\log \mathit{Im}\, Z^*_{\tau(n)} = -\frac{(\log \mathsf{q})}{n}\big(Y_{\tau(n)} - Y_{\tau(1)}\big) \to -\log \mathsf{q}\frac{\mathsf{a}-1}{\mathsf{a}+1} \quad \text{almost surely.}$$

Since the last limit is < 0, by Lemma 5.7 our sequence is a.s. regular and converges to a random variable $Z^*_\infty \in \partial^*\mathbb{H} = \mathbb{R}$ almost surely in the topology of $\widehat{\mathbb{H}}$. But then, using (2.3) as in Lemma 6.4,

$$\begin{aligned}\mathsf{d}_{\mathbb{H}}(Z_{\tau(n)},\mathfrak{i}\,) = \mathsf{d}_{\mathbb{H}}(Z^{-1}_{\tau(n)},\mathfrak{i}\,) &\overset{\text{in law}}{\asymp} \mathsf{d}_{\mathbb{H}}(Z^*_{\tau(n)},\mathfrak{i}\,)\\ &\asymp 2\log \mathit{Im}(Z^*_\infty \wedge \mathfrak{i}\,) - \log \mathit{Im}\, Z^*_{\tau(n)} \asymp Y_{\tau(n)}\,.\end{aligned}$$

where $\asymp$ means that the difference between the left and right-hand sides is bounded in absolute value.

Therefore, combining Lemma 6.4 (a) with Corollary 4.20,

$$\frac{1}{\sqrt{t}}\mathsf{d}_{\mathsf{HT}}(X_t\,,o) \overset{\text{a.s.}}{\sim} \frac{1}{\sqrt{t}}\mathsf{d}_{\mathbb{H}}(Z_{\tau(\mathsf{n}_t)}\,,\mathfrak{i}\,) \overset{\text{in law}}{\sim} \frac{1}{\sqrt{t}}Y_{\tau(\mathsf{n}_t)} \overset{\text{a.s.}}{\sim} \frac{1}{\sqrt{t}}Y_t\,,$$

as $t \to \infty$. Now Lemma 6.1 yields the result, when $\ell(\alpha,\beta) > 0$.

Case 2. $\ell(\alpha,\beta) < 0$.

Here, Lemma 6.4 (b) tells us that $\mathsf{d}_{\mathsf{HT}}(X_t,\mathfrak{o})/\sqrt{t}$ behaves in law like $\mathsf{d}_{\mathbb{T}}(W_t,o)/\sqrt{t}$ on $\mathbb{T}$, which in turn in view of Lemma 6.5 behaves like $\mathsf{d}_{\mathbb{T}}(R_{\mathsf{n}_t}o,o)/\sqrt{t}$. One can proceed as in Case 1. This time we can use the proof of the CLT for (R_no) that is given in [15]; we get that

$$\mathsf{d}_{\mathbb{T}}(R_{\mathsf{n}_t}o,o)/\sqrt{t} \overset{\text{in law}}{\sim} -\Phi(R_{\mathsf{n}_t})/\sqrt{t} \sim -Y_t/\sqrt{t},$$

and the result follows. □

The central limit theorem in the drift-free case requires some subtle input from [24] and [25]; it will be modelled after [6], which in turn relies on [15]. (In [6], the weak limit is incorrect, due to a small error in the last step.)

We need standard Brownian motion $(\mathcal{B}_t)_{t\geq 0}$ on $\mathbb{R}$ starting at 0, and the associated random variables

$$\underline{\mathcal{M}} = \min\{\mathcal{B}_t : 0 \leq t \leq 1\}\,, \quad \overline{\mathcal{M}} = \max\{\mathcal{B}_t : 0 \leq t \leq 1\}\,, \quad \text{and} \quad \mathcal{N} = \mathcal{B}_1\,,$$

so that $\mathcal{N}$ has standard normal distribution.

Theorem 6.7. *If $\ell(\alpha,\beta) = 0$, then*

$$\frac{1}{\sqrt{t}}\,\mathsf{d}_{\mathsf{HT}}(X_t\,,\mathfrak{o}) \to \frac{\log \mathsf{q}}{\sqrt{\mathsf{E}(\tau)}}\,\big(2\overline{\mathcal{M}} - 2\underline{\mathcal{M}} - |\mathcal{N}|\big) \quad \textit{in law, as } t\to\infty\,.$$

Proof. In the proof, we suppose that (X_t) starts at $\mathfrak{o}$, so that $Z_0 = \mathfrak{i}$, $W_0 = o$ and $Y_0 = 0$. The passage to arbitrary starting point is a simple exercise that we leave to the reader. By Corollary 4.20 and (6.3),

$$\frac{1}{\sqrt{t}}\,\mathsf{d}_{\mathsf{HT}}(X_t\,,\mathfrak{o}) \sim \frac{1}{\sqrt{t}}\,\mathsf{d}_{\mathsf{HT}}(X_{\tau(n_t)}\,,\mathfrak{o}) \sim \frac{1}{\mathsf{E}(\tau)}\,\frac{1}{\sqrt{n_t}}\,\mathsf{d}_{\mathsf{HT}}(X_{\tau(n_t)}\,,\mathfrak{o}) \quad \text{almost surely,}$$

as $t\to\infty$. Thus, we want to show that $\mathsf{d}_{\mathsf{HT}}(X_{\tau(n)}\,,\mathfrak{o})/\sqrt{n} \to (\log \mathsf{q})\,U_0$ in law, as $n\to\infty$. By Proposition 2.8,

$$\frac{1}{\sqrt{n}}\,\mathsf{d}_{\mathsf{HT}}(X_{\tau(n)}\,,\mathfrak{o}) \sim \frac{1}{\sqrt{n}}\,\mathsf{d}_{\mathbb{H}}(Z_{\tau(n)}\,,\mathfrak{i}\,) + \frac{\log \mathsf{q}}{\sqrt{n}}\,\mathsf{d}_{\mathbb{T}}(W_{\tau(n)}\,,o) - \frac{\log \mathsf{q}}{\sqrt{n}}\,|Y_{\tau(n)}|. \tag{6.8}$$

By Lemma 6.5, we can identify

$$X_{\tau(n)} = \boldsymbol{R}_n\mathfrak{o}\,, \quad W_{\tau(n)} = R_n o\,, \quad \text{and} \quad Y_{\tau(n)} = \Phi(R_n).$$

In the drift-free case, $\big(\Phi(R_n)\big)$ is nothing but classical simple random walk on $\mathbb{Z}$ starting at 0. Define

$$\overline{M}_n = \max\{\Phi(R_k) : k = 0,\dots,n\} \quad \text{and} \quad \underline{M}_n = \min\{\Phi(R_k) : k = 0,\dots,n\}\,,$$
$$\overline{T}(n) = \max\{k\leq n : \Phi(R_k) = \overline{M}_n\} \quad \text{and} \quad \underline{T}(n) = \max\{k\leq n : \Phi(R_k) = \underline{M}_n\}$$

It is well known that

$$\frac{1}{\sqrt{n}}\,\big(\overline{M}_n\,,\underline{M}_n\,,\Phi(R_n)\big) \to \big(\overline{\mathcal{M}}\,,\underline{\mathcal{M}}\,,\mathcal{N}\big) \quad \text{in law.} \tag{6.9}$$

See e.g. Billingsley [7], (9.2)+(9.8), for this result and the joint distribution of the limiting triple, and Borodin and Salminen [8], 1.15.8 (2) on page 174, for the joint distribution of $\big(\overline{\mathcal{M}} - \underline{\mathcal{M}}\,,\mathcal{N}\big)$.

Each $Z_{\tau(n)}$ is an element of $\mathsf{Aff}(\mathbb{H},\mathsf{q})$ and can be inverted in that group. Since we are assuming that $Z_0 = \mathfrak{i}$, all increments $Z_{\tau(n-1)}^{-1}Z_{\tau(n)}$, $n\geq 1$, are i.i.d. and have the distribution $\widetilde{\mu}$ of (4.15). The support of $\widetilde{\mu}$ generates the whole of $\mathsf{Aff}(\mathbb{H},\mathsf{q})$, and it has finite moments of exponential order by Corollary 4.23. We can now invoke the method and result of [24]. Since its reformulation in our setting is not completely transparent, we provide a brief "translation". The reference [24] comprises the following, where we set $\overline{\boldsymbol{\tau}}(n) = \tau\big(\overline{T}(n)\big)$.

- The sequence of pairs of random variables

$$\big(Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}\,,\,Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}Z_{\tau(n)}\big)$$

with values in $\mathsf{Aff}(\mathbb{H},\mathsf{q})\times\mathsf{Aff}(\mathbb{H},\mathsf{q})\equiv\mathsf{L}\mathbb{H}\times\mathsf{L}\mathbb{H}$ converges in law (i.e., weakly) in $\widehat{\mathbb{H}}\times\widehat{\mathbb{H}}$ to a pair of independent random variables $(Z^\dagger,Z^\ddagger)$ with values in $\partial^*\mathbb{H}\times\partial^*\mathbb{H}\equiv\mathbb{R}^2$, both of which have continuous distributions (in fact, continuous densities with respect to Lebesgue measure). Thus, $Z^\dagger\neq Z^\ddagger$ with probability 1, so that

$$Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}\wedge Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}Z_{\tau(n)} \overset{\text{in law}}{\to} Z^\dagger\wedge Z^\ddagger\in\mathbb{H}\,.$$

Note that

$$\log Im\big(Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}\big) = -(\log\mathsf{q})\overline{M}_n \quad\text{and}\quad \log Im\big(Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}Z_{\tau(n)}\big) = (\log\mathsf{q})\big(\Phi(R_n)-\overline{M}_n\big).$$

Using (2.3), we get that

$$\begin{aligned}
\mathsf{d}_{\mathbb{H}}(Z_{\tau(n)}\,,\mathfrak{i}\,) &= \mathsf{d}_{\mathbb{H}}\big(Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}\,,\,Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}Z_{\tau(n)}\big)\\
&\asymp 2\log Im(Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}\wedge Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}Z_{\tau(n)}) - \log Im\big(Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}\big) - \log Im\big(Z_{\overline{\boldsymbol{\tau}}(n)}^{-1}Z_{\tau(n)}\big)\\
&\overset{\text{in law}}{\sim} Im(Z^\dagger\wedge Z^\ddagger) + (\log\mathsf{q})\big(2\overline{M}_n - \Phi(R_n)\big)\,.
\end{aligned}$$

On the tree, similarly to the above, the following is proved in [15].

- The sequence of pairs of random variables

$$\big(R_{\underline{T}(n)}^{-1}o\,,R_{\underline{T}(n)}^{-1}R_n o\big)$$

with values in $\mathbb{T}\times\mathbb{T}$ converges in law (i.e., weakly) in $\widehat{\mathbb{T}}\times\widehat{\mathbb{T}}$ to a pair of independent random variables $(R^\dagger,R^\ddagger)$ with values in $\partial^*\mathbb{T}\times\partial^*\mathbb{T}$, both of which have continuous distributions ($\equiv$ without point masses). Thus, $R^\dagger\neq R^\ddagger$ with probability 1, so that

$$R_{\underline{T}(n)}^{-1}o\curlywedge R_{\underline{T}(n)}^{-1}R_n o \overset{\text{in law}}{\to} R^\dagger\curlywedge R^\ddagger\in\mathbb{T}\,.$$

This time we use (2.2). Noting that

$$\mathfrak{h}(R_{\underline{T}(n)}^{-1}o) = -\underline{M}_n \quad\text{and}\quad \mathfrak{h}(R_{\underline{T}(n)}^{-1}R_n o) = \Phi(R_n)-\underline{M}_n\,,$$

we get

$$\begin{aligned}\mathsf{d}_{\mathbb{T}}(R_n o\,,o) &= \mathsf{d}_{\mathbb{T}}\big(R_{\underline{T}(n)}^{-1}o\,,\,R_{\underline{T}(n)}^{-1}R_n o\big)\\ &= \mathfrak{h}(R_{\underline{T}(n)}^{-1}o) + \mathfrak{h}(R_{\underline{T}(n)}^{-1}R_n o) - 2\,\mathfrak{h}(R_{\underline{T}(n)}^{-1}o \curlywedge R_{\underline{T}(n)}^{-1}R_n o)\\ &\overset{\text{in law}}{\sim} \Phi(R_n) - 2\,\underline{M}_n - 2\,\mathfrak{h}(R^\dagger \wedge R^\ddagger)\,.\end{aligned}$$

Putting things together (which is legitimate because our discrete time processes are all modelled via $\boldsymbol{R}_n$ on the same probability space), we get, from (6.8),

$$\frac{1}{\sqrt{n}}\,\mathsf{d}_{\mathsf{HT}}(X_{\tau(n)}\,,\mathfrak{o}) \overset{\text{in law}}{\sim} \frac{\log \mathsf{q}}{\sqrt{n}}\big(2\overline{M}_n - 2\underline{M}_n - |\Phi(R_n)|\big)\,.$$

Now (6.9) yields the theorem. □

Acknowledgement. We acknowledge helpful interaction with Sara Brofferio, as well as valuable hints and corrections by the referee.

References

[1] BARTHOLDI, L. AND WOESS, W.: Spectral computations on lamplighter groups and Diestel–Leader graphs. *J. Fourier Anal. Appl.* **11** (2005), 175–202.

[2] BARTHOLDI, L., NEUHAUSER, M. AND WOESS, W.: Horocyclic products of trees. *J. Eur. Math. Soc. (JEMS)* **10** (2008), 771–816.

[3] BENDIKOV, A. AND SALOFF-COSTE, L.: Smoothness and heat kernels on metric graphs. In preparation.

[4] BENDIKOV, A., SALOFF-COSTE, L., SALVATORI, M. AND WOESS, W.: The heat semigroup and Brownian motion on strip complexes. *Adv. Math.* **226** (2011), no. 1, 992–1055.

[5] BENDIKOV, A., SALOFF-COSTE, L., SALVATORI, M. AND WOESS, W.: Brownian motion on treebolic space: positive harmonic functions. Preprint, arXiv: 1412.2218, 2014.

[6] BERTACCHI, D.: Random walks on Diestel–Leader graphs. *Abh. Math. Sem. Univ. Hamburg* **71** (2001), 205–224.

[7] BILLINGSLEY, P.: *Convergence of probability measures.* Second edition. Wiley Series in Probability and Statistics, Wiley, New York, 1999.

[8] BORODIN, A. N. AND SALMINEN, P.: *Handbook of Brownian motion – facts and formulae.* Second edition. Probability and its Applications, Birkhäuser Verlag, Basel, 2002.

[9] BRIN, M. AND KIFER, Y.: Brownian motion, harmonic functions and hyperbolicity for Euclidean complexes. *Math. Z.* **237** (2001), 421–468.

[10] BROFFERIO, S.: How a centred random walk on the affine group goes to infinity. *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003), 371–384.

[11] BROFFERIO, S., SALVATORI, M. AND WOESS, W.: Brownian motion and harmonic functions on $\mathsf{Sol}(\mathsf{p},\mathsf{q})$. *Int. Math. Res. Not. (IMRN)* **22** (2012), 5182–5218.

[12] Brofferio, S. and Woess, W.: Green kernel estimates and the full Martin boundary for random walks on lamplighter groups and Diestel–Leader graphs. *Ann. Inst. H. Poincaré Probab. Statist.* **41** (2005), 1101–1123.

[13] Brofferio, S. and Woess, W.: Positive harmonic functions for semi-isotropic random walks on trees, lamplighter groups, and DL-graphs. *Potential Anal.* **24** (2006), 245–265.

[14] Cattaneo, C.: The spectrum of the continuous Laplacian on a graph. *Monatsh. Math.* **124** (1997), 215–235.

[15] Cartwright, D. I., Kaĭmanovich, V. A. and Woess, W.: Random walks on the affine group of local fields and of homogeneous trees. *Ann. Inst Fourier (Grenoble)* **44** (1994), 1243–1288.

[16] Coornaert, M., Delzant, T. and Papadopoulos, A.: *Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov.* Lecture Notes in Mathematics 1441, Springer-Verlag, Berlin, 1990.

[17] Diestel, R. and Leader, I.: A conjecture concerning a limit of non-Cayley graphs. *J. Algebraic Combin.* **14** (2001), 17–25.

[18] Eells, J. and Fuglede, B.: *Harmonic maps between Riemannian polyhedra.* Cambridge Tracts in Mathematics 142, Cambdrige University Press, Cambridge, 2001.

[19] Eskin, A., Fisher, D. and Whyte, K.: Quasi-isometries and rigidity of solvable groups. *Pure Appl. Math. Q.* **3** (2007), 927–947.

[20] Eskin, A., Fisher, D. and Whyte, K.: Coarse differentiation of quasi-isometries I: Spaces non quasi-isometric to Cayley graphs. *Ann. of Math. (2)* **176** (2012), 221–260.

[21] Eskin, A., Fisher, D. and Whyte, K.: Coarse differentiation of quasi-isometries II: Rigidity for Sol and Lamplighter groups. *Ann. of Math. (2)* **177** (2013), 869–910.

[22] Farb, B. and Mosher, L.: A rigidity theorem for the solvable Baumslag–Solitar groups. With an appendix by Daryl Cooper. *Invent. Math.* **131** (1998), 419–451.

[23] Ghys, E. and de la Harpe, P. (eds.): *Sur les groupes hyperboliques d'après Mikhael Gromov.* Progress in Mathematics 83, Birkhäuser, Basel, 1990.

[24] Grincevičjus, A. K.: A central limit theorem for the group of linear transformations of the line. *Dokl. Akad. Nauk SSSR* **219** (1974), 23–26 (Russian); English translation: *Soviet Math. Doklady* **15** (1974), 1512–1515.

[25] Grincevičjus, A. K.: Limit theorems for products of random linear transformations of the line. *Lithuanian Math. J.* **15** (1975), 61–77 (Russian).

[26] Gromov, M.: Hyperbolic groups. In *Essays in group theory,* 75–263. Math. Sci. Res. Inst. Publ. 8, Springer, New York, 1987.

[27] Guivarc'h, Y., Keane, M. and Roynette, B.: *Marches aléatoires sur les groupes de Lie.* Lecture Notes in Mathemathics 624, Springer-Verlag, Berlin-New York, 1977.

[28] Kaĭmanovich, V. A.: Lyapunov exponents, symmetric spaces and a multiplicative ergodic theorem for semisimple Lie groups. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **164** (1987), *Differentsialnaya Geom. Gruppy Li i Mekh. IX* 29–46, 196–197. (Russian); English translation: *J. Soviet Math.* **47** (1989), 2387–2398.

[29] Keller, M. and Lenz, D.: Dirichlet forms and stochastic completeness of graphs and subgraphs. *J. Reine Angew. Math.* **666** (2012), 189–223.

[30] Soardi, P. M. and Woess, W.: Amenability, unimodularity, and the spectral radius of random walks on infinite graphs. *Math. Z.* **205** (1990), 471–486.

[31] Woess, W.: Boundaries of random walks on graphs and groups with infinitely many ends. *Israel J. Math.* **68** (1989), 271–301.

[32] Woess, W.: Topological groups and recurrence of quasitransitive graphs. *Rend. Sem. Mat. Fis. Milano* **64** (1994), 185–213.

[33] Woess, W.: Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions. *Combin. Probab. Comput.* **14** (2005), 415–433.

[34] Woess, W.: *Denumerable Markov chains. Generating functions, boundary theory, random walks on trees.* EMS Textbooks in Mathematics, European Mathematical Society, Zürich, 2009.

Received September 15, 2013; revised June 14, 2014.

Alexander Bendikov: Institute of Mathematics, Wrocław University, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.
E-mail: bendikov@math.uni.wroc.pl

Laurent Saloff-Coste: Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA.
E-mail: lsc@math.cornell.edu

Maura Salvatori: Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy.
E-mail: maura.salvatori@unimi.it

Wolfgang Woess: Institut für Mathematische Strukturtheorie, Technische Universität Graz, Steyrergasse 30, A-8010 Graz, Austria.
E-mail: woess@TUGraz.at

Partially supported by Austrian Science Fund projects FWF-P19115, FWF-P24028 and W1230, by NAWI Graz, and by the ESF programme RGLIS.

A. Bendikov was supported by the Polish Government Scientific Research Fund, Grant 2012/05/B/ST 1/00613 and by SFB 701 of the German Research Council DFG.