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Dynamical dessins are dense

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Abstract. We apply a recent result of the first author to prove the following result: any continuum in the plane can be approximated arbitrarily closely in the Hausdorff topology by the Julia set of a postcritically finite polynomial with two finite postcritical points.

1. Introduction

Given compact subsets $A, B \subset \mathbb{C}$ their *Hausdorff distance* $d(A, B)$ is given by

$$d(A, B) := \inf\{r : A \subset N_r(B),\ B \subset N_r(A)\}$$

where $N_r(A), N_r(B)$ denote the r-neighborhoods of A and B, respectively. Given a polynomial $g \in \mathbb{C}[z]$, we denote by g^j the jth iterate of g, and define its

- *filled-in Julia set* $K(g) := \{z : g^j(z) \not\to \infty\}$, and
- *Julia set* $J(g) := \partial K(g)$.

K. Lindsey ([4], Theorem 2.2) has shown:

Theorem 1. *Given any Jordan curve $\mathcal{J}$ bounding a closed topological disk $\mathcal{K}$ and any $\epsilon > 0$, there exists a polynomial $g \in \mathbb{C}[z]$ such that*

(1) $d(K(g), \mathcal{K}) < \epsilon$,

(2) $d(J(g), \mathcal{J}) < \epsilon$.

The proof is constructive; the above paper illustrates the result of applying the method of proof to a Jordan domain $\mathcal{K}$ outlining the figure of a cat, yielding a polynomial g of degree 301.

In this note, a *continuum* is a compact connected subset of $\mathbb{C}$. It is elementary to show that any continuum can be approximated arbitrarily closely in the Hausdorff topolology by a Jordan curve. Conclusion (2) of Theorem 1 then implies:

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Corollary 1. *Given any continuum K and any $\epsilon > 0$, there exists a polynomial $g \in \mathbb{C}[z]$ such that $d(J(g), K) < \epsilon$.*

In this note, we generalize Corollary 1.

Before stating our main result, we recall some definitions. A continuum is a *dendrite* if it is locally connected and has empty interior. Given a complex polynomial $p \in \mathbb{C}[z]$, a complex number c is a *critical point* of p if $p'(c) = 0$; its image $p(c)$ is a *critical value*. We denote by $C(p) := \{c : p'(c) = 0\}$ the set of critical points of p. A polynomial f is a *Belyi polynomial* if $\deg(f) > 1$ and if its set of critical values $f(C(f))$ is contained in the set $\{0, 1\}$; these have been much studied from many points of view, see, e.g., [7]. We next introduce some dynamical notions. A polynomial $g \in \mathbb{C}[z]$ is *postcritically finite* if $P(g) := \{g^j(c) : c \in C(g), j > 0\}$ is finite. If g is postcritically finite, the following facts are known (see, e.g., [5]): $J(g)$ is connected and locally connected, and is a dendrite if and only if no element of $C(g)$ is periodic. In [6], a Belyi polynomial g is called an *extra-clean dynamical Belyi polynomial*[1] if $P(g) = \{0, 1\}$, $g(0) = g(1) = 0$, and $g'(0) \neq 0, g'(1) \neq 0$; we denote the set of such polynomials by $XDBP$. Note that if $g \in XDBP$ then $J(g)$ is a dendrite. Theorem 3.6 in [3] implies that each $g \in XDBP$ is naturally a point on a zero-dimensional variety defined over $\mathbb{Q}$. It follows that if $g \in XDBP$ then the coefficients of g lie in the field $\overline{\mathbb{Q}}$ of algebraic numbers. Two polynomials g_1, g_2 are *conjugate* as dynamical systems if there exists $A(z) = az + b, a, b \in \mathbb{C}$, $a \neq 0$, such that $g_2 = A \circ g_1 \circ A^{-1}$. We denote by

$$\mathcal{G} := \{A \circ g \circ A^{-1} : A(z) = az + b, a, b \in \overline{\mathbb{Q}}, a \neq 0, g \in XDBP\} \subset \overline{\mathbb{Q}}[z].$$

Since $\overline{\mathbb{Q}}[z]$ is countable, so is $\mathcal{G}$.

Our main result is:

Theorem 2. *Given any continuum $K \subset \mathbb{C}$ and any $\epsilon > 0$, there exists a polynomial $g \in \mathcal{G}$ with $d(J(g), K) < \epsilon$.*

A key ingredient in our proof is an approximation result of the first author wherein continua are approximated by sets of the form $f^{-1}([0, 1])$, where f is a Belyi polynomial and $[0, 1] \subset \mathbb{C}$ is the unit interval.

In this paragraph, we introduce some terminology and perspective related to Belyi polynomials; see [7]. We denote by BP the set of Belyi polynomials. If $f \in BP$, its *dessin* is $D(f) := f^{-1}([0, 1])$. By *ibid.* Lemma 3.4, $D(f)$ is a tree with vertices $V(f) := f^{-1}(\{0, 1\})$; an edge e of $D(f)$ is the closure of a component of $f^{-1}((0, 1))$. Thinking of $[0, 1]$ as a tree with a single edge and with two vertices $v_0 = 0, v_1 = 1$, the map $f\colon D(f) \to [0, 1]$ sends a closed edge e of $D(f)$ homeomorphically to the edge $[0, 1]$. Thus the valence of a vertex $\tilde{v}$ of $D(f)$, defined as the number of edges incident to $\tilde{v}$, coincides with the local degree $\deg(f, \tilde{v})$ of f at $\tilde{v}$, defined as the multiplicity of the zero of the polynomial $z \mapsto f(z) - f(\tilde{v})$. A *leaf* of $D(f)$ is a vertex $\tilde{v}$ of valence 1. Hence a vertex $\tilde{v}$ of $D(f)$ is a critical point of f if and only if it is not a leaf.

[1] The adjective 'clean' is inherited from a technical symmetry-breaking condition commonly assumed in the theory of dessins d'enfants; see [7]. The modifier 'extra' refers to the additional condition $g(0) = g(1) = 0$, and $g'(0) \neq 0, g'(1) \neq 0$.

The approximation result we use is the following theorem.

Theorem 3. *Given any continuum $K \subset \mathbb{C}$ and any $\epsilon > 0$, there exists $f \in BP$ for which* (i) $d(D(f), K) < \epsilon$, (ii) *for each* $\tilde{v} \in V(f)$, $\deg(f, \tilde{v}) \leq 4$, *and* (iii) *the coefficients of f belong to $\overline{\mathbb{Q}}$.*

Proof. Conclusion (i) is Theorem 1.1 in [2]; (ii) follows from its proof; see *op. cit.* § 3, paragraph 3. We now prove (iii). Let $f \in BP$ satisfy (i) with $d(D(f), K) < \epsilon/2$ and also (ii). Belyi's theorem and the Grothendieck correspondence [7] imply that there exists $h_0(z) = a_0 z + b_0$, $a_0, b_0 \in \mathbb{C}, a_0 \neq 0$, for which $f \circ h_0 \in \overline{\mathbb{Q}}[z]$. Using the density of $\overline{\mathbb{Q}}$ in $\mathbb{C}$, choose $a_1, b_1 \in \overline{\mathbb{Q}}$ with $a_1 \approx a_0, b_1 \approx b_0$ so that

$$\max\{|(h_1 \circ h_0^{-1})(z) - z| : z \in D(f)\} < \epsilon/2,$$

and put $f_1 := f \circ h_0 \circ h_1^{-1} \in \overline{\mathbb{Q}}[z]$. Then f_1 satisfies conditions (ii) and (iii), and (i) holds since $D(f_1) = (h_1 \circ h_0^{-1})(D(f))$ and

$$d(D(f_1), K) \leq d(D(f_1), D(f)) + d(D(f), K) < \epsilon. \qquad \square$$

The proof of our main result, Theorem 2, has two steps. Suppose $K \subset \mathbb{C}$ is a continuum and $\epsilon > 0$ is given.

(1) We apply Theorem 3 to obtain a polynomial $f \in BP \cap \overline{\mathbb{Q}}[z]$ satisfying both $d(D(f), K) < \epsilon/2$ and the valence condition (ii).

(2) We define a sequence of polynomials $g_n \in \mathcal{G}$ such that $d(J(g_n), D(f)) \to 0$ as $n \to \infty$. The convergence will be proven in Lemma 1; it is here we use the valence condition on f. Then, choosing n such that $d(J(g_n), D(f)) < \epsilon/2$ will establish that $d(J(g_n), K) < \epsilon$, completing the proof.

In the next two paragraphs, we construct the polynomials g_n.

Let $q(z) := 4z(1-z)$. Note that $q \in BP$, that $q([0,1]) = q^{-1}([0,1]) = [0,1]$, and that $q(0) = q(1) = 0$, with $C(q) = \{1/2\}$. For each $n \in \mathbb{N}$, $n \geq 1$, we have $q^n \circ f \in BP \cap \overline{\mathbb{Q}}[z]$ and $D(q^n \circ f) = D(f)$ as subsets of $\mathbb{C}$. Their tree structures differ: each edge of $D(f)$ is a union of 2^n edges of $D(q^n \circ f)$. It is easy to see that the set of leaves of $D(q^n \circ f)$ coincides with the set of leaves of $D(f)$, and that if $\tilde{v}$ is such a leaf then $(q^n \circ f)(\tilde{v}) = 0$. Lemma 2 will say that we can make edges of $q^n \circ f$ as small as we like by choosing n sufficiently large. Since $D(q^n \circ f) = D(f)$ as sets, the valence of the tree $D(q^n \circ f)$ remains bounded above by 4.

We now turn $q^n \circ f$ into a dynamical system; cf. [6]. Suppose $v_0, v_1 \in V(f)$ are leaves of $D(f)$, that is, vertices of valence 1. By replacing f with $q \circ f$, we may assume that $f(v_0) = f(v_1) = 0$. The assumption $f \in \overline{\mathbb{Q}}[z]$ implies $v_0, v_1 \in \overline{\mathbb{Q}}$. Let $A(z) = (v_1 - v_0)z + v_0$, so that $A(0) = v_0, A(1) = v_1$. Fix $n \in \mathbb{N}$. Let $g_n := A \circ q^n \circ f$.

The paragraph below discusses the properties of the polynomials g_n.

By construction, $g_n \in \overline{\mathbb{Q}}[z]$ and g_n has two critical values, namely v_0 and v_1. We have $D(f) = D(q^n \circ f) = g_n^{-1}([v_0, v_1])$ as sets. As trees, now an edge e of $D(q^n \circ f)$ is the closure of a component of $g_n^{-1}((v_0, v_1))$, where (v_0, v_1) is the interval $[v_0, v_1]$ minus its endpoints. Abusing notation slightly, we denote by $V(g_n) := g_n^{-1}(\{v_0, v_1\})$ the set of vertices of $D(q^n \circ f)$. Each critical point of g_n maps under g_n

either to v_0 or to v_1; by construction, $v_0 = g_n(v_0) = g_n(v_1)$, and $g_n'(v_0) \neq 0$, $g_n'(v_1) \neq 0$. It follows that $P(g_n) = \{v_0, v_1\} \subset \overline{\mathbb{Q}}$, so that g_n is postcritically finite, and that every critical point lands on the fixed point v_0 under iteration of g_n. It is a general fact that all fixed points of a postcritically finite map g_n are either critical points or they lie in the Julia set. We conclude $v_0 \in J(g_n)$. Since $g_n(v_1) = v_0$, we have $v_1 \in J(g_n)$ too. Hence $V(g_n) = g_n^{-1}(\{v_0, v_1\}) \subset J(g_n)$ by invariance of $J(g_n)$; moreover, $J(g_n)$ is a dendrite. The valence condition on f implies that the local degree of g_n at any point is at most 4. Since $A^{-1} \circ g_n \circ A \in XDBP$ and $A \in \overline{\mathbb{Q}}[z]$, we conclude $g_n \in \mathcal{G}$.

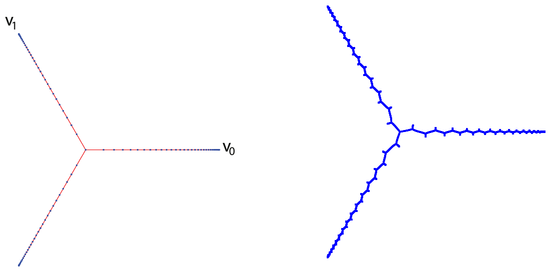

FIGURE 1. At left: the dessin $D(q^5 \circ f) = D(f)$ where $f(z) = z^3$, with leaves v_0, v_1 marked. At right: an approximation of $J(g_5)$ by the set $g_5^{-1}(D(f))$; its greater apparent thickness is an artifact of plotting the $3^2 \cdot 2^{10} - 1$ preimages of the vertices of $D(f)$. *Images courtesy of Don Marshall.*

The proof of Theorem 2 then rests upon establishing the closeness that Figure 1 suggests:

Lemma 1. *The Hausdorff distance $d(J(g_n), D(f)) \to 0$ as $n \to \infty$.*

2. Proof of Lemma 1

Suppose f, q, n, g_n are as in step 2 of the outline given in the Introduction.

Lemma 2. *The maximum diameter of an edge e of $D(q^n \circ f)$ tends to zero as $n \to \infty$.*

Proof. An easy exercise shows the conclusion holds when $f = q$. Now suppose $f \in BP$. Since the inverse branches of f are uniformly continuous on $(0, 1)$, the general conclusion holds. □

Let $D := D(f)$. We recall from step 2 the following: $D = g_n^{-1}([v_0, v_1])$; the set $g_n^{-1}(\{v_0, v_1\})$ is the set of vertices of the tree D; the edges of D are the closures of the components of $g_n^{-1}(v_0, v_1)$, where (v_0, v_1) is the Euclidean segment $[v_0, v_1]$ minus its endpoints.

We are going to cover D by a certain pair of Jordan domains W_i with the property that $W_i \cap \{v_0, v_1\} = v_i$, $i = 0, 1$. See Figure 2.

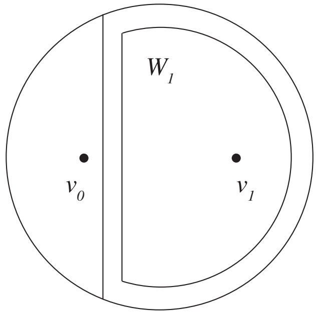

FIGURE 2. Caricature of W_1. The domain W_0 is similar. The disk shown is $B := B\left(\frac{v_1+v_0}{2}, 10M\right)$. The domain $\widehat{W}_1$ is the portion of the disk to the right of the longer vertical segment. The figure is not to scale; one should imagine that v_0, v_1 appear much closer together compared to the diameter of B, and that D is contained in the smaller disk $\frac{1}{10}B$ with the same center and $\frac{1}{10}$th the radius.

Their precise definition is a bit technical; we will give it later. Let W denote either of the domains W_0, W_1, and let $\widetilde{W}$ be a connected component of $g_n^{-1}(W)$; it will also be a Jordan domain. We will show diam $\widetilde{W} \to 0$ uniformly in n (Lemma 3). Lemma 1 will then follow easily.

In order to control the diameters of the domains $\widetilde{W}$, we will thicken the domains W_0, W_1 to Jordan domains $\widehat{W}_0, \widehat{W}_1$ so that $\overline{W}_i \subset \widehat{W}_i$ and in addition $\widehat{W}_i \cap \{v_0, v_1\} = W_i \cap \{v_0, v_1\} = v_i, i = 0, 1$. Now suppose $W, \widetilde{W}$ are as in the previous paragraph. Let $\widehat{W}$ be the thickening of W. There is a unique component $\widetilde{\widehat{W}}$ of $g_n^{-1}(\widehat{W})$ that contains $\widetilde{W}$; it is a thickening of $\widetilde{W}$. The "Koebe space" $\widetilde{\widehat{W}} \setminus \overline{\widetilde{W}}$ will allow us to control distortion and relate the diameter of $\widetilde{W}$ to the diameter of the edge it meets.

Suppose $W, \widetilde{W}, \widetilde{W}, \widetilde{\widehat{W}}$ are as in the previous two paragraphs. Choose a point $v := W \cap \{v_0, v_1\}$; it is a branch value of g_n. Since g_n is a polynomial, we obtain a map of pairs $g_n : (\widetilde{\widehat{W}}, \widetilde{W}) \to (\widehat{W}, W)$ in which each restriction is proper and each domain is a Jordan domain. Since $\widehat{W}$ contains exactly one branch value of g_n, the preimage $\widetilde{\widehat{W}} \cap g_n^{-1}(v)$ consists of a single point, which we will denote by $\tilde{v}$, which is a vertex of D. Since $v \in W$, we have $\tilde{v} \in \widetilde{W}$. Let $k := \deg(g_n, \tilde{v})$. Since the ramification of $g_n : \widetilde{\widehat{W}} \to \widehat{W}$, if there is any, occurs at the unique point $\tilde{v}$, we have $\deg(g_n : \widetilde{\widehat{W}} \to \widehat{W}) = k$ as well. The control on the local degrees of the polynomial f in Theorem 3 shows that $k \leq 4$. Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. Up to precomposition with a rotation about the origin, there exists a unique Riemann map $\phi : (\mathbb{D}, 0) \to (\widehat{W}, v)$. Since $g_n : \widetilde{\widehat{W}} \to \widehat{W}$ is ramified only possibly at $\tilde{v}$, we obtain a Riemann map $\widetilde{\phi} : (\mathbb{D}, 0) \to (\widetilde{\widehat{W}}, \tilde{v})$ such that the following diagram commutes:

$$\begin{array}{ccc} (\widetilde{\widehat{W}}, \tilde{v}) & \xrightarrow{g_n} & (\widehat{W}, v) \\ \uparrow{\scriptstyle \tilde{\phi}} & & \uparrow{\scriptstyle \phi} \\ (\mathbb{D}, 0) & \xrightarrow[z \mapsto z^k]{} & (\mathbb{D}, 0) \end{array}$$

We will apply the Koebe distortion principle to the map $\widetilde{\phi}$ and conclude that the diameter of $\widetilde{W}$ is bounded from above in terms of the diameters of the edges of D; by Lemma 2, these tend to zero as $n \to \infty$.

We now construct the domains W_0, W_1. First, denote $M := \operatorname{diam}(D)$ and $B(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$. Next see Figure 2.

We now give the definitions of the sets W_i and $\widehat{W}_i$. Let

$$\begin{aligned} v_0' &:= \frac{7v_0 + v_1}{8}, \ v_0'' = \frac{3v_0 + v_1}{4} \\ v_1' &:= \frac{v_0 + 7v_1}{8}, \ v_1'' = \frac{v_0 + 3v_1}{4} \\ \widehat{W}_{1-i} &:= B\Big(\frac{v_1 + v_0}{2}, 10M\Big) \bigcap \{|z - v_i'| < |z - v_i|\}, \ i = 0, 1 \\ W_{1-i} &:= B\Big(\frac{v_0 + v_1}{2}, 9M\Big) \bigcap \big\{|z - v_i''| < |z - v_i|\big\}, \ i = 0, 1. \end{aligned}$$

By construction,

- $\widehat{W}_i \cap \{v_0, v_1\} = W \cap \{v_0, v_1\} = v_i$, $i = 0, 1$;
- $D \subset W_0 \cup W_1$;
- $\widehat{W}_i \setminus \overline{W_i}$ is an annulus, $i = 0, 1$.

Lemma 3. *The maximum diameter of a component $\widetilde{W}$ tends to zero as $n \to \infty$.*

Proof. Suppose $g_n \colon (\widetilde{\widehat{W}}, \widetilde{W}) \to (\widehat{W}, W)$ is a map of pairs as in the preceding paragraphs; we adopt the notation used there. Up to precomposition with rotations about the origin, the map ϕ is one of only two possible Riemann maps. Hence there exist $0 < r < s < 1$ such that if $U := \phi^{-1}(W)$, then

$$B(0,r) \subset U \subset B(0,s) \subset \mathbb{D}.$$

Denote

$$\widetilde{U} := \{z \in \mathbb{D} \mid z^k \in U\}.$$

From the second part of Theorem 3 we have $1 \le k \le 4$. Hence

$$r \le \tilde{r} := r^{1/k}, \quad \tilde{s} := s^{1/k} \le s^{1/4},$$

and

$$B(0,r) \subset B(0,\tilde{r}) \subset \widetilde{U} \subset B(0,\tilde{s}) \subset B(0,s^{1/4}) \subset \mathbb{D}; \tag{2.1}$$

note that r and $s^{1/4}$ do not depend on the choice of component $\widetilde{\widetilde{W}}$. By definition, the following diagram commutes:

$$\begin{array}{ccc} (\widetilde{W}, \tilde{v}) & \xrightarrow{g_n} & (W, v) \\ \uparrow \widetilde{\phi} & & \uparrow \phi \\ (\widetilde{U}, 0) & \xrightarrow[z \mapsto z^k]{} & (U, 0) \end{array}$$

The rescaled map $\psi := |\widetilde{\phi}'(0)|^{-1}(\widetilde{\phi} - \widetilde{\phi}(0))$ is an element of the class of so-called *Schlicht functions*: injective holomorphic maps $\psi : \mathbb{D} \to \mathbb{C}$ with the normalization $\psi(0) = 0, \psi'(0) = 1$. By Theorem 5.3 in [1], for all $z \in \mathbb{D}$ and all Schlicht functions ψ,

$$|z|(1 + |z|)^{-2} \leq |\psi(z)| \leq |z|(1 - |z|)^{-2}.$$

Hence upon setting

$$\rho := r(1 + r)^{-2}, \quad \sigma := s^{1/4}(1 - s^{1/4})^{-2}, \quad \delta := |\widetilde{\phi}'(0)|$$

we have by (2.1) that

$$B(\tilde{v}, \rho\delta) \subset \widetilde{\phi}(\widetilde{U}) = \widetilde{W} \subset B(\tilde{v}, \sigma\delta).$$

Let e be any one of the k components of $g_n^{-1}((v_0, v_1))$ whose closure meets $\tilde{v}$; the closure of e is an edge of D containing $\tilde{v}$. Since $(v_0, v_1) \not\subset W$, we have $e \not\subset \widetilde{W}$, so

$$\rho\delta < \operatorname{diam}(e)$$

which implies

$$\sigma\delta < \operatorname{diam}(e)\frac{\sigma}{\rho}$$

and so

$$\operatorname{diam}(\widetilde{W}) \leq 2\sigma\delta < 2 \operatorname{diam}(e)\frac{\sigma}{\rho} \to 0$$

as $n \to \infty$, by Lemma 2. The constants ρ, σ are independent of n and of the choice of $\tilde{v}$, so the proof of Lemma 3 is complete. □

Proof of Lemma 1. Let W_0, W_1 be the domains as defined above, and let $\widetilde{W}_{\tilde{v}}$, $\tilde{v} \in V := g_n^{-1}(\{v_0, v_1\})$ denote the components of preimages $g_n^{-1}(W_i), i \in \{0, 1\}$. Denote $J := J(g_n)$. Pick $\epsilon < \frac{1}{2} \inf\{|a - b| : a \in D, b \in \mathbb{C} \setminus \overline{W_0 \cup W_1}\}$. Apply Lemma 3 to obtain n so that $\operatorname{diam}(\widetilde{W}_{\tilde{v}}) < \epsilon$ for all $\tilde{v} \in V(g_n)$. Each $\widetilde{W}_{\tilde{v}}$ is a Jordan domains, so it has the same diameter as its closure.

On the one hand, by our choice of ϵ,

$$g_n^{-1}(\overline{W_0 \cup W_1}) = \bigcup_{\tilde{v} \in V} \overline{\widetilde{W}_{\tilde{v}}} \underbrace{\subset}_{\text{Lemma 3}} N_\epsilon(D) \subset \overline{W_0 \cup W_1}$$

and so $\overline{W_0 \cup W_1}$ is backward-invariant under g_n. It is a general fact that J may be equivalently defined as the smallest closed subset of $\mathbb{C}$ satisfying $\#J > 1$ and $g_n^{-1}(J) \subset J$; see [5].

Thus $J \subset \overline{W_0 \cup W_1}$. By invariance of J we have then

$$J \subset g_n^{-1}(\overline{W_0 \cup W_1}) = \bigcup_{\tilde{v} \in V} \overline{\widetilde{W}_{\tilde{v}}} \subset N_\epsilon(D).$$

On the other hand, recalling the last sentence of Step 2, we have $V \subset J$, and $[v_0, v_1] \subset W_0 \cup W_1$ implies $D = g_n^{-1}([v_0, v_1]) \subset g_n^{-1}(W_0 \cup W_1) = \bigcup_{\tilde{v} \in V} \widetilde{W}_{\tilde{v}}$, so by our choice of ϵ and n, we have

$$N_\epsilon(J) \supset N_\epsilon(V) \supset \bigcup_{\tilde{v} \in V} \widetilde{W}_{\tilde{v}} \supset D.$$

This completes the proof of Lemma 1 and establishes Theorem 2. □

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