



Gauss words and the topology of map germs from \mathbb{R}^3 to \mathbb{R}^3

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Abstract. The link of a real analytic map germ $f: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ is obtained by taking the intersection of the image with a small enough sphere S_ϵ^2 centered at the origin in \mathbb{R}^3 . If f is finitely determined, then the link is a stable map γ from S^2 to S^2 . We define Gauss words which contains all the topological information of the link in the case that the singular set $S(\gamma)$ is connected and we prove that in this case they provide us with a complete topological invariant.

1. Introduction

Let $f: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be a finitely determined map germ. By the Mather–Gaffney criterion [11], we can consider a small enough representative $f: U \rightarrow V$ such that $f^{-1}(0) = \{0\}$ and f is stable on $U \setminus \{0\}$. Moreover, by shrinking U if necessary, we can also assume that f has no 0-dimensional stable singularities (A_3 , A_2A_1 or A_1^3) on $U \setminus \{0\}$. The topological structure of f is determined by the so-called link of f , which is obtained by taking the intersection of the image of f with a small enough sphere centered at the origin S_ϵ^2 . We use a theorem due to Fukuda [2], which ensures that the link of f is a stable map from S^2 to S^2 and that f is topologically equivalent to the cone on its link.

Here, we want to study the topological classification of finitely determined map germs, $f: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$, by looking at the topological type of their link. A natural open question is to determine whether given a stable map $\gamma: S^2 \rightarrow S^2$, there exists a finitely determined map germ $f: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ which is topologically equivalent to the cone on γ .

Given a stable map $\gamma: S^2 \rightarrow S^2$, then the singular set $S(\gamma)$ is a 1-dimensional closed submanifold of S^2 and its image, the discriminant $\Delta(\gamma)$, is a union of curves with only simple cusps or transverse double points. The restriction $\gamma: \gamma^{-1}(\Delta(\gamma)) \rightarrow \Delta(\gamma)$ contains all the topological information of γ , although in general we have also to take into account the embedding types of $\gamma^{-1}(\Delta(\gamma))$ and $\Delta(\gamma)$ in S^2 .

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In order to overcome that problem, we restrict ourselves to the case that $S(\gamma)$ is connected. Then, we will use an adapted version of Gauss words to classify such stable maps. We prove that, with this additional hypothesis, they become a complete topological invariant. In the case that $S(\gamma)$ is not connected, the Gauss words are not enough to classify stable maps and we need to use some other global type invariants (see [4], [12]). Combining the adapted version of Gauss words (that we call Gauss paragraphs) and Fukuda's theorem we prove that two finitely determined map germs $f, g : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ such that $S(f)$ and $S(g)$ are smooth and distinct from the origin, are topologically equivalent if and only if the Gauss paragraphs of their links are equivalent (see Theorem 3.9).

We should notice that the techniques used in this paper have been already used in previous papers [5], [6], [8]. We will use the results here in a forthcoming paper [9] to obtain topological classifications of corank 1 map germs from \mathbb{R}^3 to \mathbb{R}^3 . All map germs considered are real analytic except otherwise stated. We adopt the notation and basic definitions that are usual in singularity theory (e.g., \mathcal{A} -equivalence, stability, finite determinacy, etc.), as the reader can find in Wall's survey paper [11].

2. Stability, finite determinacy and the link of a germ

In this section we recall the basic definitions and results that we will need, including the characterization of stable maps from \mathbb{R}^3 to \mathbb{R}^3 , the Mather–Gaffney finite determinacy criterion and the link of a map germ.

Two smooth map germs $f, g : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ are \mathcal{A} -equivalent if there exist diffeomorphism germs $\phi, \psi : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then we say that f, g are topologically equivalent.

We say that $f : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ is k -determined if any map germ g with the same k -jet is \mathcal{A} -equivalent to f . We say that f is finitely determined if it is k -determined for some k .

Let $f : U \rightarrow V$ be a smooth proper map, where $U, V \subset \mathbb{R}^3$ are open subsets. We denote by $S(f) = \{p \in U : Jf_p = 0\}$ the singular set of f , where Jf is the Jacobian determinant. Following Mather's techniques of classification of stable maps, it is well known (see for instance [3]) that f is *stable* if and only if the following two conditions hold:

1. its only singularities are folds (A_1), cusps (A_2) and swallowtails (A_3);
2. $f|_{S_{1,0,0}(f)}$ is an immersion with normal crossings: curves of double points (A_1^2) and isolated triple points (A_1^3), $f|_{S_{1,1,0}(f)}$ is an injective immersion and the images of both restrictions intersect transversally (A_1A_2).

See Figure 1 for local pictures of the discriminant set of the stable singularities.

Both the stability criterion and the classification of the singular stable points are also true if we consider a holomorphic proper map $f : U \rightarrow V$, with U, V being open subsets of \mathbb{C}^3 . So we consider now a holomorphic map germ $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$ and we recall the Mather–Gaffney finite determinacy criterion [11].

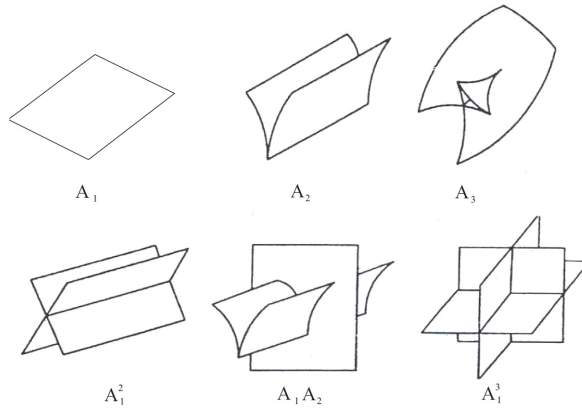


FIGURE 1.

Roughly speaking, f is finitely determined if and only if it has an isolated instability at the origin. To simplify the notation, we state the Mather–Gaffney theorem only in the case of map germs from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^3, 0)$, although it is true in a more general form for map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$.

Theorem 2.1. *Let $f : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$ be a holomorphic map germ. Then f is finitely determined if and only if there is a representative $f : U \rightarrow V$, where U and V are open subsets of \mathbb{C}^3 , such that*

- (1) $f^{-1}(0) = \{0\}$,
- (2) $f : U \rightarrow V$ is proper,
- (3) the restriction $f|_{U \setminus \{0\}}$ is stable.

From condition (3), the A_3 , A_1A_2 and A_1^3 singularities are isolated points in $U \setminus \{0\}$. By the curve selection lemma [7], we deduce that they are also isolated in U . Thus, we can shrink the neighbourhood U if necessary and get a representative such that $f|_{U \setminus \{0\}}$ is stable with only folds, cuspidal edges and double fold curves.

Coming back to the real case, we consider now an analytic map germ $f : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$. If we denote by $\hat{f} : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$ the complexification of f , it is well known that f is finitely determined if and only if \hat{f} is finitely determined. So, we have the following immediate consequence of the Mather–Gaffney finite determinacy criterion.

Corollary 2.2. *Let $f : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be a finitely determined map germ. Then there is a representative $f : U \rightarrow V$, with U, V being open subsets of \mathbb{R}^3 such that*

- (1) $f^{-1}(0) = \{0\}$,
- (2) $f : U \rightarrow V$ is proper,
- (3) the restriction $f|_{U \setminus \{0\}}$ is stable with only fold planes, cuspidal edges and double fold point curves.

We finish this section with an important result due to Fukuda, which tells us that any finitely determined map germ, $f: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$, with $n \leq p$, has a conic structure over its link. The link is obtained by intersecting the image of a representative of f with a small enough sphere centered at the origin of \mathbb{R}^p . In order to simplify the notation, we only state the result in our case $n = p = 3$.

We denote by $J^r(\mathbb{R}^3, \mathbb{R}^3)$ the r -jet space and if $s \geq r$ we have the natural projection $\pi_r^s: J^s(\mathbb{R}^3, \mathbb{R}^3) \rightarrow J^r(\mathbb{R}^3, \mathbb{R}^3)$.

Theorem 2.3. *Let $f: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be a finitely determined map germ. Then, up to \mathcal{A} -equivalence, there is a representative $f: U \rightarrow V$ and $\epsilon_0 > 0$, such that, for any ϵ with $0 < \epsilon \leq \epsilon_0$, we have:*

- (1) $\tilde{S}_\epsilon^2 = f^{-1}(S_\epsilon^2)$ is diffeomorphic to S^2 .
- (2) The map $f|_{\tilde{S}_\epsilon^2}: \tilde{S}_\epsilon^2 \rightarrow S_\epsilon^2$ is stable.
- (3) f is topologically equivalent to the cone on $f|_{\tilde{S}_\epsilon^2}$.

Proof. Assume that f is r -determined for some r and let $W = \{j^r f(0)\}$, where $j^r f(0)$ denotes the r -jet of f . By Fukuda’s theorem [2], there is $s \geq r$, and a closed semi-algebraic subset Σ_W of $(\pi_r^s)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ mapping $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $j^s g(0)$ belonging to $(\pi_r^s)^{-1}(W) \setminus \Sigma_W$, there exists $\epsilon_0 > 0$ such that (1), (2) and (3) hold, for any ϵ with $0 < \epsilon \leq \epsilon_0$. Since $(\pi_r^s)^{-1}(W) \setminus \Sigma_W \neq \emptyset$, we can take a map $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $j^s g(0) \in (\pi_r^s)^{-1}(W) \setminus \Sigma_W$. This implies that $j^r g(0) = j^r f(0)$ and g is \mathcal{A} -equivalent to f . □

Definition 2.4. Let $f: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be a finitely determined map germ. We say that the stable map $f|_{\tilde{S}_\epsilon^2}: \tilde{S}_\epsilon^2 \rightarrow S_\epsilon^2$ is the *link* of f , where f is a representative such that (1), (2) and (3) of Theorem 2.3 hold for any ϵ with $0 < \epsilon \leq \epsilon_0$. This link is well defined, up to \mathcal{A} -equivalence.

Since any finitely determined map germ is topologically equivalent to the cone on its link, we have the following immediate consequence.

Corollary 2.5. *Let $f, g: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be two finitely determined map germs whose associated links are topologically equivalent. Then f and g are topologically equivalent.*

We will see that the converse of this corollary is also true at the end of the following section, if we assume that the singular sets $S(f), S(g)$ are smooth.

3. Gauss words

We recall that a Gauss word is a word which contains each letter exactly twice, one with exponent $+1$ and another one with exponent -1 . They were introduced originally by Gauss to describe the topology of closed curves in the plane \mathbb{R}^2 or in the sphere S^2 (see for instance [10]). Here, we use the same terminology of Gauss word to represent a different type of word, adapted to our particular case of stable maps $S^2 \rightarrow S^2$.

Along this section, we assume that $\gamma: S^2 \rightarrow S^2$ is a stable map, that is, such that all its singularities are folds and cusp points and that $\gamma|_{S(\gamma)}$ only presents simple cusps and transverse double points. Moreover, we assume that $S(\gamma)$ and hence its image $\Delta(\gamma)$ are connected.

Lemma 3.1. *Let $\gamma: S^2 \rightarrow S^2$ be a stable map such that $S(\gamma)$ is connected. Then:*

- (1) $\gamma^{-1}(\Delta(\gamma))$ is also connected,
- (2) the restriction of γ to each connected component of $S^2 \setminus \gamma^{-1}(\Delta(\gamma))$ is a diffeomorphism.

Proof. If $S(\gamma)$ is empty, then $\gamma^{-1}(\Delta(\gamma))$ is also empty. Moreover, γ is a local diffeomorphism and hence a d -fold covering, for some $d \geq 1$. Then,

$$2 = \chi(S^2) = d\chi(S^2) = 2d,$$

so $d = 1$ and γ is a diffeomorphism.

Assume that $S(\gamma)$ is non empty, then $S(\gamma)$ and $\gamma^{-1}(\Delta(\gamma))$ are both non-empty graphs in S^2 . Since $S(\gamma)$ is connected, $\Delta(\gamma)$ is also connected and hence, $S^2 \setminus \Delta(\gamma)$ is a disjoint union of open discs. We show that $\gamma^{-1}(\Delta(\gamma))$ is connected by showing that $S^2 \setminus \gamma^{-1}(\Delta(\gamma))$ is also a disjoint union of open discs by Alexander duality.

Let C be a connected component of $S^2 \setminus \gamma^{-1}(\Delta(\gamma))$ and let $D = \gamma(C)$ be the corresponding connected component of $S^2 \setminus \Delta(\gamma)$. The restriction $\gamma|_C: C \rightarrow D$ is again a d -fold covering, for some $d \geq 1$. Therefore,

$$1 - \beta_1(C) = \chi(C) = d\chi(D) = d \geq 1,$$

where $\beta_1(C)$ is the first Betti number of C . Hence, $\beta_1(C) = 0$ and $d = 1$. We deduce that C is an open disc and $\gamma|_C: C \rightarrow D$ is a diffeomorphism. □

Now we look at the structure of the singular curves. We split $\gamma^{-1}(\Delta(\gamma))$ as $\gamma^{-1}(\Delta(\gamma)) = S(\gamma) \cup X(\gamma)$ where

$$X(\gamma) = \overline{\gamma^{-1}(\Delta(\gamma)) \setminus S(\gamma)}.$$

The local structure of these curves at a cusp or at a transverse double point is shown in Figure 2. In general $X(\gamma)$ may have several components, that is, it is equal to a finite union of closed curves with cusps or transverse double points. We denote such components by $X_1(\gamma), \dots, X_k(\gamma)$.

We now choose orientations on the spheres S^2 (we may take different orientations on each S^2). Then there are natural orientations induced on the singular curves:

- $S(\gamma)$: we have on the left the positive region (where γ preserves the orientation).
- $\Delta(\gamma)$: we have on the left the region of bigger multiplicity (the number of inverse images of a value).
- $X_j(\gamma)$: we have on the left the region of bigger premultiplicity (the premultiplicity of a point is the multiplicity of its image).

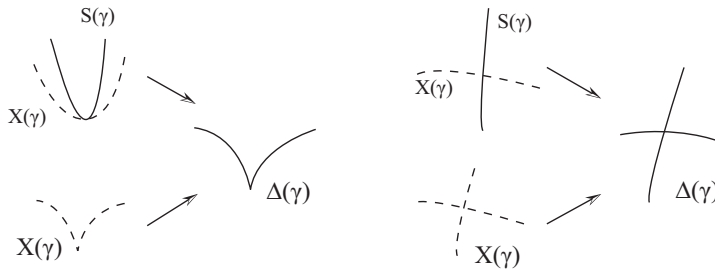


FIGURE 2.

At a transverse double point we have two oriented branches. One branch is called positive if the other branch crosses from right to left at the double point, otherwise we call it negative. We always have a positive and a negative branch meeting at a double point (see Figure 3).

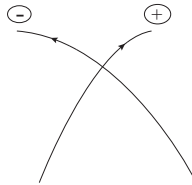


FIGURE 3.

The next step is to choose a base point on each curve $S(\gamma)$, $\Delta(\gamma)$ and $X_j(\gamma)$. We only need to choose a point in $S(\gamma)$; this point uniquely determines a base point on all the other curves: writing for simplicity, $X_0(\gamma) = S(\gamma)$, we fix a point $z_0 \in X_0(\gamma)$ which determines a point $\gamma(z_0) \in \Delta(\gamma)$. By following the orientation in $X_0(\gamma)$, we consider the first point z_1 appearing in the curves $X_1(\gamma), \dots, X_k(\gamma)$ and we reorder the curves in such a way that $z_1 \in X_1(\gamma)$. Now we proceed by induction. Assume we have chosen a base point z_i on each curve $X_i(\gamma)$, for $i = 0, \dots, \ell$, with $\ell < k$ (after reordering the curves). We consider the first curve $X_i(\gamma)$ which intersects one of the remaining curves $X_{\ell+1}(\gamma), \dots, X_k(\gamma)$ and take $z_{\ell+1}$ as the first point of intersection, following the base point and the orientation of $X_i(\gamma)$. We reorder the curves $X_{\ell+1}(\gamma), \dots, X_k(\gamma)$ in such a way that $z_{\ell+1} \in X_{\ell+1}(\gamma)$. Since $S(\gamma) \cup X(\gamma)$ is connected, this procedure will determine a unique base point z_i on each curve $X_i(\gamma)$, for $i = 1, \dots, k$.

We see in Figure 4 how to choose the base points in an example where $X(\gamma)$ has two disjoint components. This corresponds to the inverse image of the discriminant of example 3.3 (4). We also remark that the algorithm to choose the base points on the curves $X_1(\gamma), \dots, X_k(\gamma)$ is not unique.

Definition 3.2. Assume that $\Delta(\gamma)$ presents r double points and s simple cusps, which are labeled by $r + s$ letters $\{a_1, a_2, \dots, a_{r+s}\}$. The Gauss word of $\Delta(\gamma)$, denoted by W_0 , is the sequence of cusps and double points that appear when traveling around $\Delta(\gamma)$ starting from the base point and following the orientation.

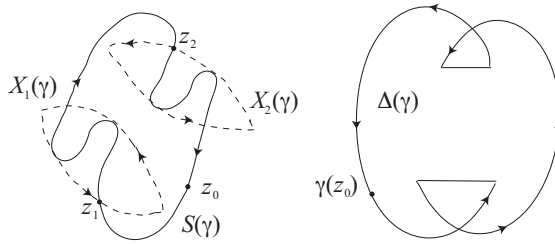


FIGURE 4.

If we arrive to a point a_i , then we put a_i^2 if it is a cusp, a_i if it corresponds to the positive branch of a double point or a_i^{-1} if it corresponds to the negative branch.

For each $j = 1, \dots, k$, the Gauss word of $X_j(\gamma)$ is denoted by W_j and is defined in an analogous way, but we have now more possibilities. Given a point which is an inverse image of a_i , if it belongs to $S(\gamma)$ we use the same letter a_i to label the point; otherwise we put $\overline{a_i}, \overline{\overline{a_i}}, \dots$ (we use multiple bars in order to distinguish between different inverse images). We also use the same convention with the exponents: $a_i^2, \overline{a_i}^2, \overline{\overline{a_i}}^2, \dots$ for a cusp, $a_i, \overline{a_i}, \overline{\overline{a_i}}, \dots$ for a positive branch of double point or $a_i^{-1}, \overline{a_i}^{-1}, \overline{\overline{a_i}}^{-1}, \dots$ for a negative branch of double point.

We call Gauss paragraph to the list of Gauss words $\{W_0, W_1, \dots, W_k\}$.

Example 3.3. Let us examine some examples of links, including those of the three stable singularities.

- (1) Let $\gamma: S^2 \rightarrow S^2$ be the link of the fold $f(x, y, z) = (x, y, z^2)$. Then $\Delta(\gamma)$ doesn't present any simple cusp or double point. The Gauss paragraph is just $\{\emptyset\}$ (Figure 5).

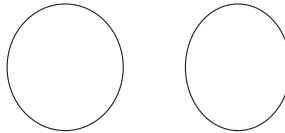


FIGURE 5.

- (2) Let $\gamma: S^2 \rightarrow S^2$ be the link of the cuspidal edge $f(x, y, z) = (x, y, xz + z^3)$. Then $\Delta(\gamma)$ presents 2 simple cusps, each one with a single inverse image. The Gauss paragraph in this case is $\{a^2b^2, a^2b^2\}$ (Figure 6).

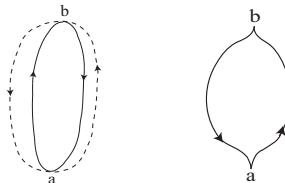


FIGURE 6.

- (3) Let $\gamma : S^2 \rightarrow S^2$ be the link of the swallowtail $f(x, y, z) = (x, y, z^4 + xz + yz^2)$. Then $\Delta(\gamma)$ present 2 simple cusps, each one with 2 inverse images, and a double fold point, with 2 inverse images. The Gauss paragraph is $\{a^{-1}b^2c^2a, a^{-1}\bar{b}^2c^2a\bar{c}^2b^2\}$ (Figure 7).

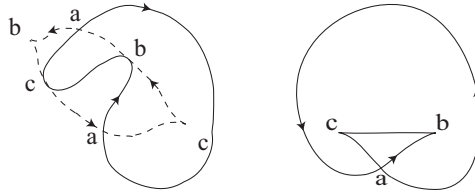


FIGURE 7.

- (4) Let $\gamma : S^2 \rightarrow S^2$ be the link of the germ $f(x, y, z) = (x, y, z^4 + xz - y^2z^2)$. Then $\Delta(\gamma)$ presents 4 simple cusps, each one with 2 inverse images, and 2 double fold points, each one with 2 inverse images. The Gauss paragraph in this case is (Figure 8):

$$\left\{ \begin{array}{l} a^{-1}b^2c^2ad^{-1}e^2f^2d, \\ a\bar{c}^2b^2a^{-1}\bar{b}^2c^2, \\ d\bar{f}^2e^2d^{-1}\bar{e}^2f^2. \end{array} \right.$$

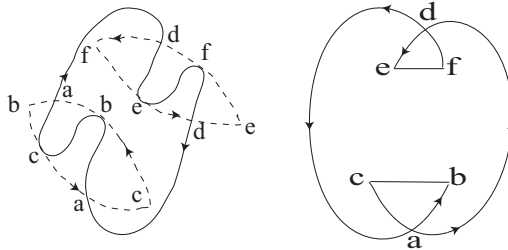


FIGURE 8.

It is obvious that the Gauss paragraph is not uniquely determined, since it depends on the labels a_1, \dots, a_{r+s} , the chosen orientations in each S^2 and the base point $z_0 \in S(\gamma)$. Different choices will produce the following changes in the Gauss paragraph:

- (1) a permutation in the set of the letters a_1, \dots, a_{r+s} ,
- (2) a reversion in the Gauss words together with a change in the exponents $+1$ to -1 and viceversa,
- (3) a cyclic permutation in the Gauss words.

We say that two Gauss paragraphs are equivalent if they are related through these three operations. Under this equivalence, the Gauss paragraph is now well defined.

In order to simplify the notation, given a stable map $\gamma : S^2 \rightarrow S^2$, we denote by $w(\gamma)$ the associated Gauss paragraph and by \simeq the equivalence relation between Gauss paragraphs.

As a consequence of this definition and previous remarks we have the following important result:

Theorem 3.4. *Let $\gamma, \delta : S^2 \rightarrow S^2$ be two stable maps such that $S(\gamma)$ and $S(\delta)$ are connected and non empty. Then γ and δ are topologically equivalent if and only if $w(\gamma) \simeq w(\delta)$.*

Proof. Let us denote by $w(\gamma) = \{W_0, W_1, \dots, W_k\}$ the Gauss paragraph of γ with respect to some labels $\{a_1, a_2, \dots, a_{r+s}\}$, some orientations in the source and the target S^2 and some base point $z_0 \in S(\gamma)$.

Suppose that δ is topologically equivalent to γ . Then, there are homeomorphisms $\phi, \psi : S^2 \rightarrow S^2$ such that $\delta = \psi \circ \gamma \circ \phi^{-1}$. We use the same labels $\{a_1, a_2, \dots, a_{r+s}\}$ in such a way that if a_i is the label of a cusp or double point of γ , then it is also the label of its image through ψ and if $a_i, \bar{a}_i, \overline{\bar{a}_i}, \dots$ is the label of an inverse image in γ , then we take the same label for its image through ϕ . We choose the orientations in the source and the target S^2 induced by the orientations of γ and the homeomorphisms ϕ, ψ . Finally, we set $\phi(z_0) \in S(\delta)$ as the base point. With these choices, we have that $w(\delta) = \{W_0, W_1, \dots, W_k\} = w(\gamma)$.

We show now the converse. We divide the proof into several cases.

Case 1: $w(\gamma) = w(\delta)$. We can assume that $w(\gamma) = w(\delta) \neq \emptyset$, since otherwise both maps should be topologically equivalent to the link of the fold.

We first observe that each stable map γ with $w(\gamma) \neq \emptyset$ induces a unique cellular structure on S^2 such that γ restricted to each cell is a homeomorphism. In the target, the 0-cells are the cusps and double folds and the 1-skeleton is $\Delta(\gamma)$; in the source, the 0-cells are the inverse images of the cusps and double folds and the 1-skeleton is $S(\gamma) \cup X(\gamma)$.

The second fact is that such cellular structure can be deduced in a unique way from the Gauss paragraph of γ . In the target, the 0-cells are labelled by the letters a_1, \dots, a_{r+s} , each 1-cell is an oriented edge given by two consecutive letters $a_i^\xi a_j^\eta$ in W_0 (including also the edge joining the last to the first letter) and each 2-cell is a face which is determined by a closed sequence of oriented edges or their inverses. In the source, we proceed analogously but this time we take into account all the Gauss words W_0, \dots, W_k .

If $w(\gamma) = w(\delta)$, we write $\gamma : M_1 \rightarrow P_1$ and $\delta : M_2 \rightarrow P_2$ where M_i, P_i denote S^2 with the associated cellular structure in the source or the target respectively. Since the Gauss word of $\Delta(\gamma)$ is equal to the Gauss word of $\Delta(\delta)$, we have that P_1, P_2 are isomorphic as CW-complexes. We choose a cellular homeomorphism $\beta : P_1 \rightarrow P_2$. Then we construct another cellular homeomorphism $\alpha : M_1 \rightarrow M_2$ such that $\delta \circ \alpha = \beta \circ \gamma$. Given a cell E in M_1 , then there is a unique cell E' in M_2 corresponding

to the same label in the Gauss word and such that $\beta(\gamma(E)) = \delta(E')$. We define $\alpha|_E : E \rightarrow E'$ as

$$\alpha|_E = (\delta|_{E'})^{-1} \circ \beta|_{\gamma(E)} \circ \gamma|_E.$$

Case 2: $w(\gamma) \simeq w(\delta)$.

1. Suppose that $w(\gamma), w(\delta)$ are related through a permutation τ in the set of the letters a_1, a_2, \dots, a_{r+s} . The proof is essentially the same as in case 1, but we construct the homeomorphisms α, β in such a way that a vertex with label a_i is mapped into a vertex with label $a_{\tau(i)}$, and so on.
2. Assume that $w(\gamma), w(\delta)$ are related through a reversion in the Gauss words together with a change in the exponents. We take $J : S^2 \rightarrow S^2$, with $J(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ such that either $w(\gamma) = w(\delta \circ J)$, $w(\gamma) = w(J \circ \delta)$ or $w(\gamma) = w(J \circ \delta \circ J)$. Then the result follows from case 1.
3. Assume that $w(\gamma), w(\delta)$ are related through cyclic permutations in the Gauss words. Then we can choose again a homeomorphism $T : S^2 \rightarrow S^2$ such that $w(\gamma) = w(\delta \circ T)$ and apply case 1. □

Remark 3.5. The equivalence between the Gauss words of $\Delta(\gamma)$ and $\Delta(\delta)$ is not enough to guarantee that γ and δ are topologically equivalent. In fact, even if γ and δ have isomorphic discriminants $\Delta(\gamma)$ and $\Delta(\delta)$, they are not necessarily topologically equivalent in general (see [1]).

Remark 3.6. Note that Theorem 3.4 is not true if $S(\gamma)$ is not connected. We find in [4], Figure 6, an example of two stable maps from S^2 to S^2 , both with empty Gauss words, which are not topologically equivalent. In that paper, the authors consider other global type invariants, for instance, the graph associated to the connected components of the complement of $S(\gamma)$, but again this is far from being a complete invariant.

Now, we are in position to state and prove the converse of Corollary 2.5 in the case that the singular sets are smooth. In fact, if $f : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ is a finitely determined map germ such that $S(f)$ is smooth and distinct from the origin, then the singular set of its link $f|_{\tilde{S}_\epsilon^2}$ is connected and non empty and hence, we can use Theorem 3.4.

Theorem 3.7. *Let $f, g : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be two finitely determined map germs such that $S(f)$ and $S(g)$ are smooth and distinct from the origin. Then, if f and g are topologically equivalent, their respective links are topologically equivalent.*

Proof. Since f and g are topologically equivalent, there are homeomorphisms $\phi, \psi : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ such that $\psi \circ f = g \circ \phi$. We take small enough representatives and $\epsilon > 0$ such that $f|_{\tilde{S}_\epsilon^2}$ is the link of f . Write $M = \phi(\tilde{S}_\epsilon^2)$ and $P = \psi(S_\epsilon^2)$; there is a commutative diagram

$$\begin{CD} \tilde{S}_\epsilon^2 @>f|_{\tilde{S}_\epsilon^2}>> S_\epsilon^2 \\ @V\phi VV @VV\psi V \\ M @>g|_M>> P \end{CD}$$

Let us denote by $w(f|_{\tilde{S}_\epsilon^2}) = \{W_0, W_1, \dots, W_k\}$ the Gauss paragraph with respect to some labels $\{a_1, a_2, \dots, a_{r+s}\}$, some orientations in \tilde{S}_ϵ^2 and S_ϵ^2 and a cusp $z_0 \in S(f|_{\tilde{S}_\epsilon^2})$ as a base point.

We also put $R = \phi(\tilde{D}_\epsilon^3)$ and $Q = \psi(D_\epsilon^3)$ and consider the restriction $g|_R : R \rightarrow Q$. We take $\delta > 0$ small enough such that $D_\delta^3 \subset Q$ and $g|_{\tilde{S}_\delta^2}$ is the link of g . Then we consider in R, Q the orientations induced by ϕ, ψ respectively, in $\tilde{D}_\delta^3, D_\delta^3$ the orientations induced as submanifolds of R, Q respectively and in $\tilde{S}_\delta^2, S_\delta^2$ the orientations induced as boundaries of $\tilde{D}_\delta^3, D_\delta^3$ respectively.

For each cusp or double fold in the target of $g|_{\tilde{S}_\delta^2}$ we can associate a unique letter a_i in the obvious way: consider the curve of cusps or double folds of g joining the origin to this point and take the point of such curve in P , which is the image of a cusp or double fold in the target of $f|_{\tilde{S}_\epsilon^2}$, labelled by a_i (see Figure 9). For cusps or double folds in the source of $g|_{\tilde{S}_\delta^2}$ we proceed analogously.

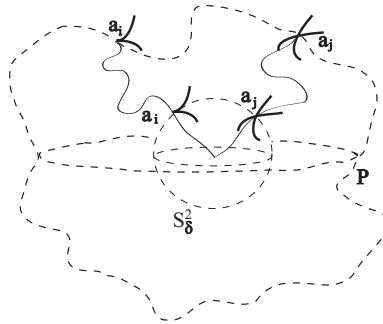


FIGURE 9.

By using the same procedure, we take as a base point the corresponding cusp $z'_0 \in S(g|_{\tilde{S}_\delta^2})$ coming from the cusp $z_0 \in S(f|_{\tilde{S}_\epsilon^2})$.

With these choices it becomes clear that $g|_{\tilde{S}_\delta^2}$ has the same Gauss paragraph $w(g|_{\tilde{S}_\delta^2}) = \{W_0, W_1, \dots, W_k\}$ and therefore, it is topologically equivalent to $f|_{\tilde{S}_\epsilon^2}$ by Theorem 3.4. □

Remark 3.8. If $S(f)$ is equal to the origin its associated link $\gamma : S^2 \rightarrow S^2$ becomes a regular map and hence a diffeomorphism by Lemma 3.1. Hence, in this case we only have one topological class, namely the regular map $f(x, y, z) = (x, y, z)$.

For example, if we consider the map germ $f(x, y, z) = (x, y, x^2z + y^2z + \frac{1}{3}z^3)$, its singular set $S(f)$ is given by the equation $x^2 + y^2 + z^2 = 0$, so, in the real case, it only contains the origin. As a consequence f is topologically equivalent to the regular map.

Putting together Theorems 3.4 and 3.7, and Corollary 2.5, we have the following result.

Theorem 3.9. *Let $f, g : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ be two finitely determined map germs such that $S(f)$ and $S(g)$ are smooth and distinct from the origin. Then f and g are topologically equivalent if and only if their links have equivalent Gauss paragraphs.*

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